EQUAL TEMPERAMENT BEST FITTED TO NATURAL SCALES

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Abstract. We optimise the number of steps t for the t-step *equaly tempered* musical system so that its *twelve* (or *seven*) keys reproduce the all 12 (or 7 white) keys of the the two famous *natural* musical scales, namely the 12-step *Pythagorean scale* and the 12-step *just intonation*.

1. The 12-scale Pythagorean and equally tempered systems

The well known *Pythagorean tuning system* assigns the following multipliers to each step of its *12-step scale*:

1	$\frac{256}{243}$	$\frac{9}{8}$	$\frac{32}{27}$	$\frac{81}{64}$	$\frac{4}{3}$	$\frac{729}{512}$	$\frac{3}{2}$	$\frac{128}{81}$	$\frac{27}{16}$	$\frac{16}{9}$	$\frac{243}{128}$	2

The analogous multipliers for the 12-step *just intonation* scale are:

1	$\frac{16}{15}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{64}{45}$	$\frac{3}{2}$	$\frac{8}{5}$	$\frac{5}{3}$	$\frac{16}{9}$	$\frac{15}{8}$	2

Cells in the above tables represent 12 keys of a piano keyboard spanning an *octave*. The frequencies of pure tones played by a given key expressed in Herzes are given by the frequency in Herzes played by the first key, multiplied by the multiplier from the cell correponding to this key. For example if, the key corresponding to the cell with multiplier $\frac{27}{16}$ plays a pure tone of 432Hz, then the first key plays a pure tone of frequency $f_0 = 256\text{Hz}$, so that $432\text{Hz} = \frac{27}{16} \times f_0 = \frac{27}{16} \times 256\text{Hz}$. The frequencies of the keys in other octaves of the piano are determined in the same way, but now the frequencies of the keys in the octave with a number $n = -3, -2, -1, 0, 1, 2, 3$ are multiplied by the frequency $2^n \times f_0 = 2^n \times 256\text{Hz}$.

The main reason for changing the mathematically beautiful Pythagorian tuning (with frequency multipliers $f = 3^p 2^q$) into mathematically irregular frequency patern in the just intonation, is that some major chords played in the just intonation sound more harmonious than in the Pythagorean tuning. For example, playing together a key with frequency multiplier 1 and frequency multiplier $\frac{81}{64}$ makes the sound of the superposed music a bit rough. This is due to the *overtones*. A hit in any piano key with a given frequency ν, produces secondary sounds of the piano string, called overtones, with ferquencies 2ν, 3ν, 4ν, 5ν, and so on. And, in particular the 5th overtone of the sound with multiplier 1, and the fourth overtone of the sound with multiplier $\frac{81}{64}$, produced by a simultaneous hiting of the keys 1 and $\frac{81}{64}$, makes the overtones with multipliers $5 \times 1 = 5$ and $4 \times \frac{81}{64} \simeq 5.0625$ to interfere, producing annoying *beats*. Here the physics of the real world triumphs over the mathematical beauty, and *humans* decided to change the perfect Pythagorean multiplier $\frac{81}{64}$ to the just intonation multiplier $\frac{5}{4}$; simply $5 \times 1 = 4 \times \frac{5}{4}$. The other changes on the just intonation scale, with respect to the Pythagorean scale, are due to similar reasons.

Both, the Pythagorean scale and the just intonation, suffer however from another problem. They are incompatibile with *transpositions*: a melody played starting at one key, in general, can not be played starting with another key; usually, starting with another key, there will be lack of keys in the scale to repeat the melody. For this reason musicians invented an *equally tempered* musical tuning system, in which the ratio between two multipliers of any two neigbouring keys is the same. In the 12-scale equally tempered system this ratio is $2^{\frac{1}{12}}$ and for it the corresponding frequency multipliers are:

Date: May 12, 2024.

Support: The research leading to these results has received funding from the Norwegian Financial Mechanism 2014-2021 with project registration number 2019/34/H/ST1/00636.

1	$2^{\frac{1}{12}}$	$2^{\frac{1}{6}}$	$2^{\frac{1}{4}}$	$2^{\frac{1}{3}}$	$2^{\frac{5}{12}}$	$2^{\frac{1}{2}}$	$2^{\frac{7}{12}}$	$2^{\frac{2}{3}}$	$2^{\frac{3}{4}}$	$2^{\frac{5}{6}}$	$2^{\frac{11}{12}}$	2

To avoid multiplications when passing from one key to the other, musicians, musicologists and tuners apply the logarithmic function for all of these multipliers. As a result they measure musical distances in a *linear scale*. They divide each of the 12 equally tempered intervals into 100 equally distant units, called *cents*, so that the entire scale spanning the *octave* has 1200 cents. In this way the first key in the equally tempered scale has 0 cents, the second 100 cents, the third 200 cents, and so on, until the thirteenth step has 1200 cents.

In terms of the cents the above 3 tables read respectively as follows:

- Pythagorean:

0	90.225	203.91	294.135	407.82	498.045	611.73	701.955	792.18	905.865	996.09	1109.78	1200

- Just intonation:

0	111.731	203.91	315.641	386.314	498.045	609.776	701.955	813.686	884.359	996.09	1088.27	1200

- Equal temperament:

0	100	200	300	400	500	600	700	800	900	1000	1100	1200

The exact formula, relating the multiplier f, being a number from the interval $f \in [1, 2]$, to the number $c(f)$ of cents corresponding to it, is

$$c(f) = 1200 \log_2 f.$$

The use of the $\log_2$ in it is very useful, because the logarithm function *transforms multiplication* of the multiplicities, needed to pass from one interval to the other, into the *addition of the corresponding cents*. The more intuitive nature of the *addition* than of the *multiplication* explains the advantage of using cents in the description of relations between sounds rathe than the multipliers, or frequencies.

The cents description of musical relations is perhaps as old as the dicovery of logarithms (see e.g. [1]). However, it turns out that mathematics serves yet another, related, but much more geometric description of the tuning intervals measure. This enables musicians to speak about tunings in a purely *visual* way. This description is in terms of the *complex numbers*. Although discovered at least as early as in the XVths century, they started to be really popular in the word of muscologists only in XIX-XX century; so musicians do not use them much. But it is a pity, since they particularly fit to the *musical equivalence* modulo an octave, i.e. a (perhaps psychological or cultural) phenomenon that *we perceive sounds musically distanced by an octave, as the same.*

In the complex plane, there is an embedded distinguished geometrical figure, the *unit circle*, which is the set of all points of the plane at distance one from the origin. In complex numbers terms, this circle is the set of all *complex numbers* z, whose *modulus* $|z|$ is equal to 1. To be more precise we say the following:

Instead of representing the tones of the octave in terms of the multipliers x, as in the first set of the tables above, or in terms of the cents $c(x)$ as in the second set of tables, it is much more convenient to represent these tones as the points on the unit circle $\mathbb{S} = \{\mathbb{C} \ni z \text{ s.t. } z = 1\}$. Even if one does not know anything about complex numbers, the formula for this representation is very simple. It transforms a multiplier $f \in [1, 2]$ of any frequency to an *angle*

$$\phi(f) = 2\pi \log_2 f$$

in *radians*. If one wants this angle in *degrees*, the formula is

$$\phi(f) = 360^\circ \log_2 f.$$

The point on the unit circle corresponding to the multiplier x has then the *Cartesian coordinates*

$$\big(x(f), y(f)\big) = \big(\cos\phi(f), \sin\phi(f)\big).$$

It correspond to a complex number

$$z(f) = e^{i\phi(f)}$$

from the unit circle $\mathbb{S}$. Note that the appearence of $2\pi\log_2$ in $\phi(f)$ and of the 2π-periodic trigonometric functions $\cos$ and $\sin$ in the formula for the tone point $(x(f), y(f))$, maps all the tones from the same *pitch class* to a single point in the circle, which is the mathematical manifestation of the *musical equivalence* modulo an octave. Indeed, if we have a multiplier f and another multiplier f_k in the same *pitch class*, i.e. if f and f_k are related by $f_k = 2^k f$, with k an integer, then

$$\big(x(f_k), y(f_k)\big) = \big(x(f), y(f)\big),$$

since e.g.

$$\begin{aligned}\cos(2\pi\log_2 f_k) =& \cos(2\pi\log_2(2^k f)) = \cos(2\pi\log_2 f + 2\pi\log_2 2^k) = \\ & \cos(2\pi\log_2 f + 2\pi k\log_2 2) = \cos(2\pi\log_2 f + 2k\pi) = \cos(2\pi\log_2 f).\end{aligned}$$

One can easily verify that the angles in radians attributed to each multiplier given in the three tables above, are as follows:

- Pythagorean:

0	0.472417	1.06767	1.54009	2.13534	2.60776	3.20301	3.67543	4.14784	4.7431	5.21552	5.81077	2π

- Just intonation:

0	0.585024	1.06767	1.65269	2.02273	2.60776	3.19278	3.67543	4.26045	4.63049	5.21552	5.69816	2π

- Equal temperament:

0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
0	0.523599	1.0472	1.5708	2.0944	2.61799	3.14159	3.66519	4.18879	4.71239	5.23599	5.75959	2π

Having these angles we can represent each of the three scales pictorially. Look at the picture below:

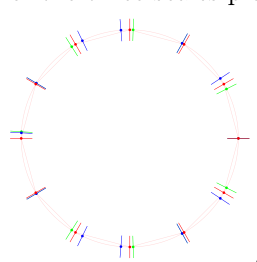

Figure 1

It depicts three dodecaphonic scales on one image. The *green points/lines* represent the 12 steps of the *Pythagorean scale*, the *blue points/lines* represent the 12 steps of the *just intonation*, and the *red points/lines* represent the 12 steps of the *equally tempered* 12 step scale. Here the *base* step/tone, the *unison*, is the same for all three scales, and corresponds to the red point/line at the extreme right of the light red circle. The other steps of each of the scales are represented by other point/lines, successively *clockwise* from the unison.

Except red point/line of the unison, the points/lines of other steps are grouped in two or three, always including the red point/line. If there are three points/lines with a given red point/line, it means that the three scales have slightly different multipliers at this step. If there are two point lines, as e.g close to the red point/line at hours 1, 5, 8 and 10, it means that at each of these steps, respectively at the minor second, the perfect fourth, the minor sixth and the minor seventh, the Pythagorean and just intonation tones coincide.

Note that the equal temperament scale has the property that its steps, as represented on the light red circle, are situated at the vertices of the regular dodecagon, which alsso is ploted in light red at the picture. The Pythagorean and just intonation steps are not that regularly distributed.

2. Quantitative comparison

2.1. **Aligning with 'C'.** Looking at the picture comparing the three 12 step scales we see that the equally tempered scale does not exactly fit to the Pythagorean or just intonation scales. It is handy to have a *quantitative measure* of these differences. This leads to the following definition.

Definition 2.1. Let $K = \{1, 2, \dots, \ell\}$ and $J = \{1, 2, \dots, n\}$ be two sets of indices, the first one from 1 to ℓ, and the second one from 1 to n.

Let $\{c_1, c_2, \dots, c_\ell\}$ be a set of increasing cent values $c_1 = 0 \leq c_k \leq 1200$, $k \in K$, of sounds in a *given* ℓ-step *scale*, and let $\{C_1 = 0, C_2, \dots, C_n\}$ be the set of cent values $C_j = 1200\frac{j-1}{n}$, $j \in J$, of sounds in the n-*step equally tempered* scale.

The *fitting index* $\delta(c, n)$ between the equally tempered scale $\{C_1, C_2, \dots, C_n\}$ and the ℓ-step scale $\{c_1, c_2, \dots, c_\ell\}$ is:

$$\delta(c, n) = \tfrac{1}{\ell} \sum_{k\in K} \min_{j\in J} |c_k - C_j|.$$

In words: the *fitting index* between these two scales is the *cent difference per key* of those ℓ steps from the n-step equally tempered scale, which are closest to ℓ different steps of a given ℓ-step scale.

Examples

- We calculate the fitting index between the 12-step equally tempered scale C^{12} and the 7-step Pythagorean scale c^7 consisting of the white keys of the Pythagorian tuned piano. We have: $K = \{1, 2, \dots, 7\}$, $J = \{1, 2, \dots, 12\}$,
$c^7 = \{0, 203.91, 407.82, 498.045, 701.955, 905.865, 1109.78\}$ and $C^{12} = \{C_j = 100(j-1), j \in J\}$. This gives

  $$\delta(c^7, 12) = \frac{|0-0|+|203.91-200|+|407.82-400|+|498.045-500|+|701.955-700|+|905.865-900|+|1109.78-1100|}{7} = 4.469.$$

- In the same way we calculate the fitting index between the 12-step equally tempered scale C^{12} and the 12-step Pythagorean scale c^{12}. We obtain

  $$\delta(c^{12}, 12) = 5.865.$$

- Another example is the fitting index between the 12-step equally tempered scale C^{12} and the 7-step just intonation scale $\tilde{c}^7$ consisting of the white keys of the just intonation tuned piano. This index is:

  $$\delta(\tilde{c}^7, 12) = 6.983.$$

- We close these set of example by giving the fitting index between the 12-step equally tempered scale C^{12} and the 12-step just intonation scale $\tilde{c}^{12}$. This is:

  $$\delta(\tilde{c}^{12}, 12) = 8.635.$$

Comparing these numbers with the pictorial presentation of these scales given in Figure 1, we see that the smaller the fitting index, the better is the fit between the two scales describing it. In particular, the 12-step equally tempered scale fits better to the 7-scale Pithagorian scale c^7 (index 4.469) than the 7-scale just intonation scale $\tilde{c}^7$ (index 6.983). It further follows that among all 4 scales, $c^7, \tilde{c}^7, c^{12}, \tilde{c}^{12}$, the one that fits best to the 12-step equally tempered scale is the 7-step Pythagorian scale of 7 wihite Pithagorian keys on the piano.

After these examples we are ready to formulate the main problem we solve in this note:

Find the smallest n for the n-step equal temperament to have the fitting index with
- a) *7-step scale of white keys in Pythagorean tuning*
- b) *12-step Pythagorean tuning*
- c) *7-step scale of white keys in just intonations*
- d) *12-step just intonation*

smaller than the fitting index between the 12-step equal temperament and the respective scales a), b), c) and d).

We have the following Theorems, which we have proven by inspection.

Theorem 2.2. *For $12 \le n \le 60$ the values of the fitting index δ between the n-step equal temperament and the 7-step scale of* white keys *in the* Pythagorean tuning *are given in the following table.*

n	δ	n	δ	n	δ	n	δ	n	δ
11	–	21	13.921	31	9.580	41	1.106	51	5.596
12	4.469	22	13.546	32	9.039	42	7.046	52	5.513
13	21.456	23	13.540	33	9.272	43	6.750	53	0.156
14	18.174	24	4.469	34	8.408	44	6.116	54	5.232
15	16.584	25	11.961	35	9.280	45	6.502	55	5.038
16	18.416	26	9.350	36	4.469	46	5.469	56	4.543
17	8.977	27	8.965	37	7.936	47	7.196	57	5.070
18	17.931	28	10.569	38	6.574	48	4.469	58	3.4132
19	15.210	29	3.413	39	6.630	49	5.855	59	5.663
20	15.403	30	10.311	40	6.896	50	5.130	60	4.469

Theorem 2.3. *For $12 \le n \le 60$ the values of the fitting index δ between the n-step equal temperament and the 7-step scale of* white keys *in the* just intonation *are given in the following table.*

n	δ	n	δ	n	δ	n	δ	n	δ
11	–	21	14.371	31	4.553	41	2.773	51	5.890
12	6.983	22	6.759	32	9.318	42	6.596	52	5.737
13	23.933	23	11.893	33	9.189	43	4.325	53	0.642
14	22.922	24	6.983	34	3.641	44	6.577	54	5.130
15	17.422	25	12.343	35	8.459	45	7.010	55	4.905
16	19.812	26	11.790	36	6.983	46	3.506	56	4.532
17	15.974	27	10.920	37	8.346	47	6.818	57	5.005
18	17.092	28	10.848	38	7.282	48	5.746	58	3.763
19	7.282	29	6.810	39	7.990	49	6.208	59	5.755
20	15.746	30	9.473	40	8.045	50	5.435	60	3.823

From these two tables we chose these ns for which the fitting index is smaller than the fitting index for the $n = 12$ step scale, and we do it in a way such that for each chosen n_0s the next chosen n has the fitting index smaller than the fitting index attroibuted to n_0. These correspond to the δs of the green collor in the tables.

We see that if we consider fits between the white keys in the just intonation scale and the equally tempered $n < 61$ step scale, the fitting indices distinguished by this method correspond to $n = 22$, $n = 31$, $n = 34$, $n = 41$ and $n = 53$.

For these distinguished ns we plot the comparison of the key-values of the corresponding n-step equally tempered system and the 12-step just intonation system. The relevant pictures are below:

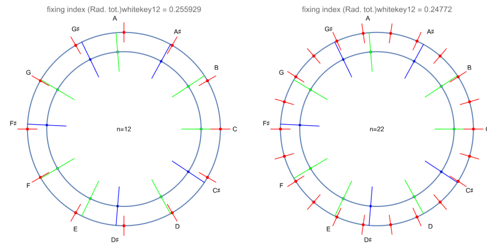

Figure 2. The left picture is for $n = 12$, and the right picture is for $n = 22$. The red lines correspond to the equally tempered intervals, the blue lines correspond to the white keys in the just intonation, and the green lines correspond to the black keys in the just intonation.

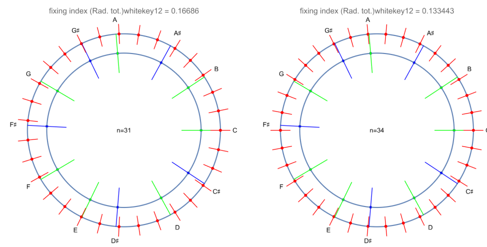

Figure 3. The $n = 31$ and $n = 34$ cases.

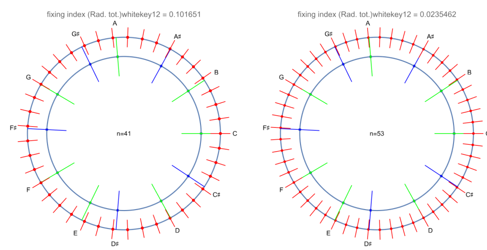

Figure 4. The $n = 41$ and $n = 53$ cases. The $n = 53$ equal temperament fits perfectly to the just inonation. Further imporovement of the fit makes no sense. And... the number $n = 53$ of piano keys is already too much.

In the above tables we distinguished in blue the number $n = 19$, because Roger Penrose in [2] proposes equally tempered 19-step scale as 'the best' extension of the just intonation scale. For comparison, we plot below his $n = 19$ case, and our proposed $n = 31$:

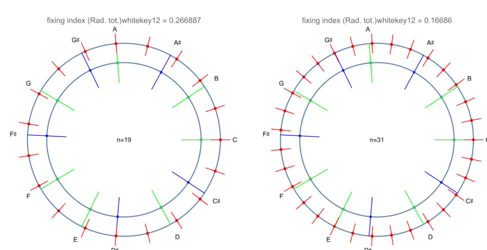

Figure 5

Visibly the $n = 31$-step scale fits better to the just intonation than the $n = 19$ scale.

And... **it sounds better!** I checked it with an electronic piano and the equal temperament simulation program Pianoteq8 PRO.

2.2. **Aligning issue with just intonation.** Notice that in the previous graphs, I aligned the note 'C' of the Pithagorian or just intonation scales with the first pitch (the unison) in the equal temperament. Later, I notice that the result is sligntly different from Roger Penrose's [2], where I guess, he aligned the unison of the equal temperament with the note 'A' in the just temperament. This rose the following questions:

(1) Does the fixing index (7-step scale of white keys) change when aligning the unison with a different-than-C note from the Pythagorean/just intonation scales?
(2) Which choice of a tone from the Pythagorean/Just intonation scales for the alignment with the unison 'C' is the best for minimizing the fitting index?

For (1), the answer is 'YES'. I plot the fixing index with different alignments in the following.

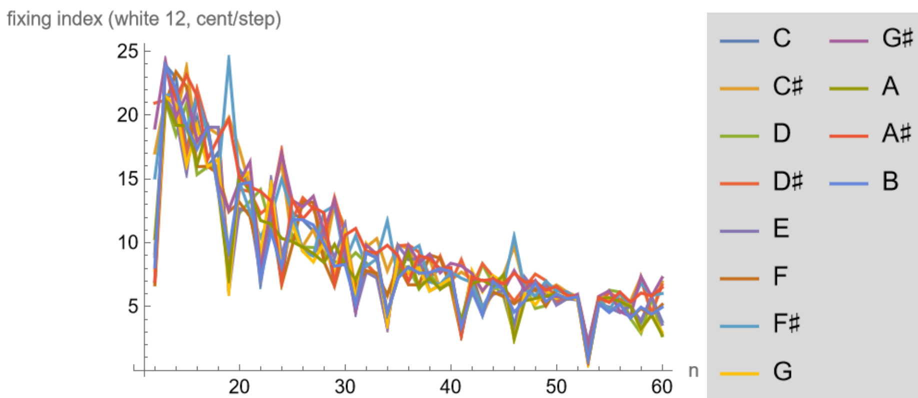

Figure 6. Fixing index of the just intonation fitted by n equal temperament

And according to calculation, it seems that the above mentioned scales, $n = 19, 22, 29, 31, 34, 41$ are good, but not for every possible alignment. For example, the fixing number corresponds to $n = 19$ and aligned with 'F♯' produce a huge number 24.3168. And $n = 53$ is surpricingly good for all alignment.

Also, according to the calculation, most of the anti-peaks are created by aligning with black keys. The following is the same graph except that the black key alignments are removed.

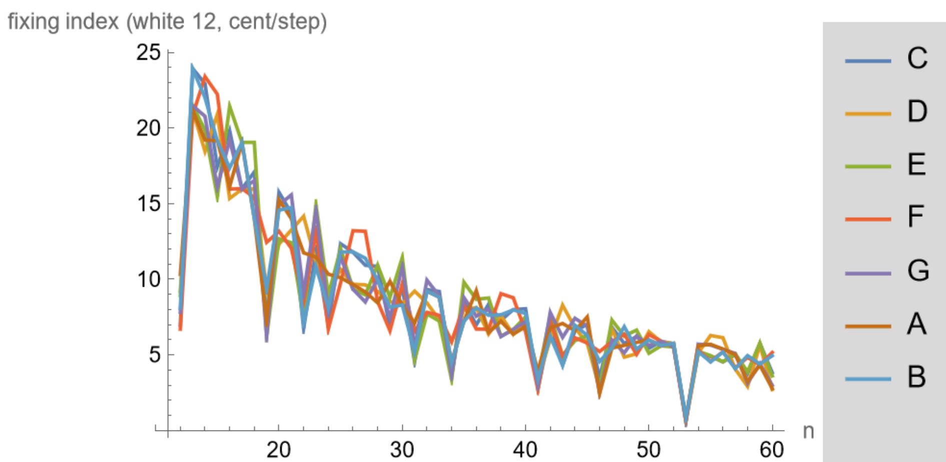

Figure 7. Fixing index of the just intonation fitted by n equal temperament

Now, for our question (2), from the calculated data, we conclude the following:

Theorem 2.4. *For $12 \leq n \leq 60$, for each alignment, better fitting index δ (compared to $n = 12$) are achieved by the following scales in the table. In it, $n(L)$ menas that the column is for the alingment with the L key of the just intonation.*

n(C)	δ	n(D)	δ	n(E)	δ	n(F)	δ	n(G)	δ
12	6.98269	12	9.21698	12	8.93788	12	6.70341	12	7.82055
22	6.75871	28	9.19478	19	6.22933	29	6.59708	19	6.20816
29	6.81041	29	8.51701	22	7.40093	31	6.54917	29	7.45038
31	4.55256	30	8.35624	29	8.79611	34	5.87299	31	5.5164
34	3.64081	32	8.42947	31	4.66442	37	6.6952	34	3.65284
41	2.7734	33	7.31728	32	7.70143	40	6.52018	36	7.55495
42	6.59539	34	5.90907	33	7.23364	41	2.70426	38	6.20816
43	4.32489	35	7.97029	34	3.3663	43	4.90686	39	6.68517
44	6.57715	36	8.1272	36	8.67227	44	6.13057	40	7.20696
46	3.50617	37	6.37476	37	8.7572	45	5.77839	41	2.98084
47	6.81841	38	7.58361	38	6.22933	46	5.21533	42	7.79835
48	5.74625	39	6.35079	39	6.66674	47	5.87356	43	6.15942
49	6.20808	40	7.17482	40	7.5182	48	6.36447	44	7.41501
50	5.43482	41	3.32657	41	3.60568	49	5.02964	45	6.89553
51	5.88961	42	6.03682	42	7.60469	50	6.33696	46	2.48068
52	5.73712	43	8.28206	43	4.95117	51	5.89668	47	5.97222
53	0.642428	44	6.78687	44	5.93493	52	5.53117	48	5.12802
54	5.12979	45	5.76471	45	6.27286	53	0.691148	49	6.24486
55	4.90477	46	2.76394	46	2.85179	54	5.32066	50	5.62447
56	4.53237	47	6.71974	47	7.29483	55	4.57213	51	5.87758
57	5.00504	48	4.84873	48	6.24535	56	5.19896	52	5.53117
58	3.76319	49	5.06642	49	6.6343	57	4.14583	53	0.613196
59	5.75497	50	6.52661	50	5.10429	58	4.82981	54	5.70204
60	3.82302	51	5.86059	51	5.60059	59	4.33934	55	5.65018
		52	5.73712	52	5.51275	60	5.15553	56	5.37022
		53	0.603451	53	0.804602			57	5.09102
		54-57	>4	54-57	>4			58	3.12321
		58	2.90989	58	3.87985			59	4.41365
		59	5.64438	59	5.77317			60	2.98516
		60	2.70588	60	3.60784				

n(A)	δ	n(B)	δ
12	10.3343	12	8.10002
19	7.21816	22	7.13591
25-28	>8	31	5.18081
29	9.86274	34	4.47638
30	8.29214	35-39	>7
31	7.1085	40	7.73357
32	9.20786	41	3.39824
33	8.81162	42	6.23047
34	4.50043	43	4.31012
35	7.21806	44	6.95435
36	9.24452	45	6.28654
37	6.4651	46	4.56095
38	7.21816	47	5.44646
39	6.36921	48	6.86357
40	6.86357	49	5.3823
41	3.95141	50	5.98071
42	6.78904	51	5.61262
43	7.07381	52	5.75555
44	6.59123	53	0.853322
45	7.50452	54	5.21838
46	2.45139	55	4.52758
47-52	>5	56	5.18786
53	0.77537	57	4.08102
54-57	>5	58	4.94647
58	3.2398	59	4.39551
59	4.3212	60	4.94035
60	2.76998		

It turns out that [2] was right, $n = 19$ is good when aligning with 'A'. But we also found $n = 19$ is not that good when aligning with 'C'. And $n = 19, 22, 29, 31$ are good for some alignments, $n = 34, 41, 53$ are good for all white key alignments.

2.3. **Other possibilities.** To sum up, how good is n will be affected by the following,

- the way we measure the fixing index
- the tuning system
- alignment of the first pitch

Finally, I present the fixing indices calculated by 60 randomly generated white key weights for Just Intontation.

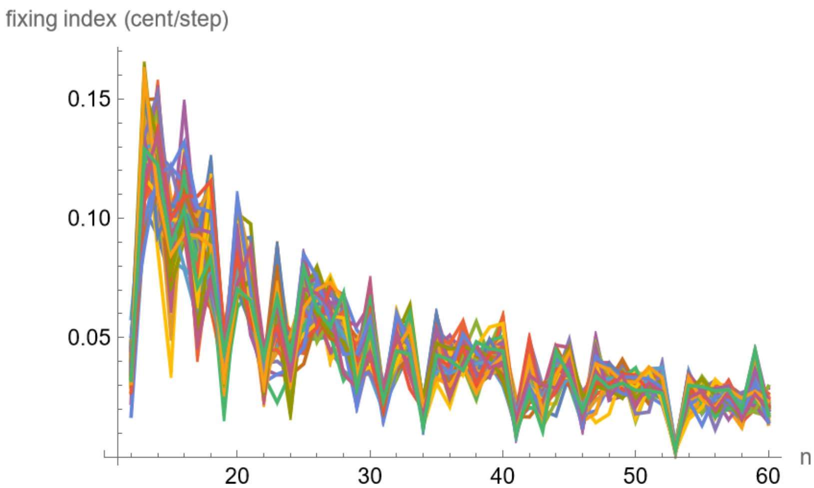

Figure 8. fixing index aligned with 'C' with random key weight

An interpretation of the graph is that the 'envelope' means that some of the notes in Just Intonation are far from the approximation and some are close (deviation is large); And for a valley, it means all of them are approximated well.

And this also shows that $n = 19, 22, 29, 31, 34, 41, 53$ are good.

So, Figure 5,8 together with the table in Theorem 2.4 show that indeed $n = 31$ behaves better than $n = 19$, although it has much more keys.

References

[1] Ellis, Alexander J. (1885), "On the Musical Scales of Various Nations", Journal of the Society of Arts: 485–527, retrieved 1 January 2020

[2] Penrose, Roger, "The Heritage of Pythagoras: Nineteen to the Dozen", in David Greer, Ian Rumbold, and Jonathan King (eds), *Musicology And Sister Disciplines Past,Present,Future: Proceedings of the 16th International Congress of the International Musicological Society London, 1997* (Oxford, 2000; online edn, Oxford Academic, 31 Oct. 2023), https://doi.org/10.1093/oso/9780198167341.003.0003..

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