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Homogeneous CR and Para-CR Structures in Dimensions 5 and 3

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Abstract

The classification of CR hypersurfaces $M^{2n+1} \subset \mathbb{C}^{n+1}$ up to biholomorphic equivalences, notably the *homogeneous* ones, is a *vast* problem, especially in dimension 5, *i.e.* for $n = 2$, even with the assistance of all existing mostly sophisticated mathematical tools: Lie-theoretic algebras of differential invariants; Exterior differential systems; Cartan connections; Parabolic geometries; Poincaré-Moser normal forms. As understood by *e.g.* Lie, Tresse, Segre, Cartan, such classification problems are tightly linked with point equivalences of completely integrable systems of partial differential equations in $n \geqslant 1$ independent variables and 1 dependent variable, over $\mathbb{C}$ or $\mathbb{R}$, so that those PDE systems that are associated to CR structures can rightly be called '*para-CR structures*'. In particular, the 3-dimensional case, *i.e.* $n = 1$, is linked with the well understood geometry of second order ODEs $y_{xx} = F(x, y, y_x)$. The present survey article: (1) focuses considerations on the study of (para-)CR structures in dimensions 3 and 5; (2) sketches relationships with affinely homogeneous submanifolds and their *tubifications;* (3) provides several concrete classification lists of various Lie symmetry algebras; (4) describes recent achievements due to Loboda and to Doubrov-Medvedev-The about *nondegenerate* homogeneous (para-)CR structures in 5D; (5) concludes by reviewing the recent classification arXiv:2003.08166, due to the two authors, of *degenerate* homogeneous para-CR structures in 5D, which is based on Cartan's method of equivalence and which is coherent with the CR classification due to Fels-Kaup.

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To Nessim, in Memoriam, in Admiration, in Friendship. Le'Haïm, Nessim!

H. Poincaré, en étudiant en 1907 le problème de la représentation analytique, ou *pseudo-conforme*, de deux domaines de l'espace de deux variables complexes x, y, a montré qu'une hypersurface analytique de cet espace admet une infinité d'invariants différentiels par rapport au groupe infini des transformations analytiques $x' = f(x, y)$, $y' = g(x, y)$. La détermination effective de ces invariants est en relation, comme l'a montré B. Segre en 1931, avec celle, effectuée par A. Tresse, des invariants d'une équation différentielle $\frac{d^2y}{dx^2} = \omega\left(x, y, \frac{dy}{dx}\right)$ par rapport au même groupe infini. Les deux problèmes ne sont cependant pas identiques. Dans un mémoire paru dans le dernier fascicule des *Annali di Matematica*, j'ai résolu directement le problème de Poincaré en lui appliquant une méthode générale remontant à 1908.

Élie Cartan, [8, p. 1305]

1 Introduction

In [76], Nurowski-Sparling explored in depth the close relationships between the geometry associated with second order ordinary differential equations defined modulo point transformations of variables, and the geometry of three-dimensional Cauchy-Riemann (CR) structures. The goal of this *expository* article is to explain how certain *degenerate* five-dimensional CR structures give rise, analogously, to certain closely tied pairs of PDEs, and then, to find all the concerned homogeneous geometries. On the way to this end, some detailed survey material will be offered to the readers. Sections 9 and 10 present original/new results.

Already in 1907, Poincaré [81] observed by a simple counting argument (quoted in [13, p. 2]) that there are *more* local real hypersurfaces $M^3 \subset \mathbb{C}^2$ than there are (local) *biholomorphisms* of $\mathbb{C}^2$. Thus, a classification problem was born.

2 Lie, Tresse, Segre, Cartan

Later, Beniamino Segre in 1931 [86, 87], inspired by Poincaré, observed that, to every *real analytic* hypersurface $M^3 \subset \mathbb{C}^2$ which is not locally biholomorphically equivalent to a hyperplane, one can associate an invariant 2-parameters family of *characteristic surfaces*, which are complex curves — called after Webster [92] *Segre varieties*. Precisely, if $0 = \rho(z, w, \overline{z}, \overline{w})$ is a real analytic (implicit) equation of M, with $\overline{\rho} = \rho$ real, of class $\mathcal{C}^\omega$, satisfying $d\rho \neq 0$ on $\{\rho = 0\}$, so that $\rho_w \neq 0$ after switching $z \leftrightarrow w$ if necessary, then Segre varieties are obtained as zero-sets $\{\rho(z, w, \overline{a}, \overline{b}) = 0\}$ by polarizing ρ and setting constant the antiholomorphic variables. Such curves can be graphed as $w = \Theta(z, \overline{a}, \overline{b})$.

In turn, Segre observed that by solving these two parameters $(\overline{a}, \overline{b})$ from $w = \Theta$ and $w_z = \Theta_z$, and by replacing them in $w_{zz} = \Theta_{zz}(z, \overline{a}, \overline{b})$, one can obtain a second order holomorphic ODE $\frac{d^2w}{dz^2} = \Phi\big(z, w, \frac{dw}{dz}\big)$, with $(z, w) \in \mathbb{C}^2$, provided that the expression of E.E. Levi [52]:

$$\mathsf{L}(\rho) := \rho_w \rho_{\overline{w}}\, \rho_{z\overline{z}} + \rho_z \rho_{\overline{z}}\, \rho_{w\overline{w}} - \rho_z \rho_{\overline{w}}\, \rho_{w\overline{z}} - \rho_w \rho_{\overline{z}}\, \rho_{z\overline{w}} \neq 0,$$

is nowhere vanishing. The relative invariancy of $\mathsf{L}(\rho)$ under biholomorphisms can be seen from a closed formula ([33, p. 217]). When $\mathsf{L}(\rho)(p) \neq 0$ at some point $p \in M$, one says that M is *Levi nondegenerate* at p.

We believe that such analogy links between second order ODEs $y_{xx} = F(x, y, y_x)$ and real hypersurfaces $M^3 \subset \mathbb{C}^2$ were most probably already known to Lie, *cf.* [25, chap. 23], long before having been reawoken by Segre.

In any case, Segre's note gave impetus to Élie Cartan, who undertook to study and completely settle Poincaré's classification problem. From the works of Lie and Tresse on second order ODEs, Cartan immediately deduced that an arbitrary hypersurface $M^3 \subset \mathbb{C}^2$, either is locally biholomorphic to the hypersphere $\frac{y-\overline{y}}{2i} = x\overline{x}$ having 8-dimensional Lie group consisting of fractional linear transformations:

$$\begin{pmatrix} z' \\ w' \end{pmatrix} = \frac{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} z \\ w \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}}{c_1 z + c_2 w + d}, \qquad \begin{pmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ c_1 & c_2 & d \end{pmatrix} \in SU_{2,1}(\mathbb{C}),$$

or has local group of biholomorphisms of (real) dimension $\leqslant 3$. Hypersurfaces locally biholomorphic to the model $\frac{y-\overline{y}}{2i} = x\overline{x}$ are called *spherical*.

Thus, on the quest for homogeneous models, Cartan could assume that $3 = \dim \mathfrak{hol}(M)$, and using Bianchi's (short) classification of real 3-dimensional (abstract) real Lie algebras, *see e.g.* [88, p. 107], he could determine a complete list, which we quote '*en français dans le texte*':

> *Si une hypersurface admettant un groupe pseudo-conforme transitif n'est pas localement équivalente à l'hypersphère, elle est globalement équivalente à l'une des hypersurfaces suivantes ou à l'une de leurs variétés de recouvrement:* [7, p. 1284]

1°	(E)	$\frac{y-\overline{y}}{2i} = \left(\frac{x-\overline{x}}{2i}\right)^m$, avec $\frac{x-\overline{x}}{2i} > 0$	$(\lvert m\rvert \geqslant 1,\ m \neq 1, 2)$;
2°	(F)	$\frac{y-\overline{y}}{2i} = e^{\frac{x-\overline{x}}{y-\overline{y}}}$;	
3°	(H)	$(x-\overline{x})^2 + (y-\overline{y})^2 + 4\,e^{2m \arctan \frac{x-\overline{x}}{y-\overline{y}}} = 0$;	
4°	(K)	$1 + x\overline{x} - y\overline{y} = \mu\,\lvert 1 + x^2 - y^2\rvert$, avec $\frac{x(1+\overline{y}) - \overline{x}(1+y)}{i} > 0$	$(\mu > 1)$;
5°	(K′)	$x\overline{x} + y\overline{y} - 1 = \mu\,\lvert x^2 + y^2 - 1\rvert$, saufles points réels $(\lvert\mu\rvert < 1,\ \mu \neq 0)$;	
6°	(L)	$x_1\overline{x}_1 + x_2\overline{x}_2 + x_3\overline{x}_3 = \mu\,\lvert x_1\overline{x}_1 + x_2\overline{x}_2 + x_3\overline{x}_3\rvert$	$(\mu > 1)$.

Nurowski-Tafel [77], motivated by algebraically special solutions to Einstein's field equations, rederived this classification using the fact that every Lie algebra of dimension $\geqslant 3$ contains a 3-dimensional Lie subalgebra. Cartan also gave global classification lists in $\mathbb{C}^2$, which we do not comment, because our focus is on local classifications. Let us nevertheless mention that in [43], Isaev explicitly determined all covers of Cartan's locally or globally homogeneous strongly pseudoconvex 3-dimensional hypersurfaces. To the best of our knowledge, such a task has not yet been endeavoured for 5-dimensional CR manifolds.

In [7, Chap. III] and in [10], Cartan applied his *method of equivalence* to set up a second, independent, alternative proof of the classification. It is *this method*, developed by the second-named author in several areas of group-theoretical differential geometry, that will be employed in the core of the paper for certain *degenerate* five-variables para-CR structures. Another method, based on Fels-Olver's *recurrence formulæ* [29], is upcoming.

Cartan's classification of homogeneous $M^3 \subset \mathbb{C}^2$ can be decomposed in two collections, the first — (E), (F), (H) — consisting of *tubes* $M^3 = C^1 \times i\mathbb{R}^2$ over certain *affinely homogeneous* curves $C^1 \subset \mathbb{R}^2 \ni (x, u)$, shown with their affine symmetries as:

(1) $u = x^s$, with $x > 0$, $s \in [-1, 0) \cup (1, 2) \cup (2, \infty)$, having symmetry $x\,\partial_x + su\partial_u$;
(2) $u = x \log x$, with $x > 0$, having symmetry $x\partial_x + (x + u)\partial_u$;
(3) logarithmic spirals $r = e^{a\varphi}$, with $a \geqslant 0$, where (r, φ) are polar coordinates on the (x, u)-plane, with $-\infty < \varphi < \infty$, having symmetry $(-u + ax)\,\partial_x + (x + au)\,\partial_u$.

Indeed, with $(z, w) = (x + iy, u + iv)$, every such curve $C^1 \subset \mathbb{R}^2$ gives rise to an associated *tube* hypersurface $C^1 \times i\mathbb{R}^2$, homogeneous under $\mathsf{Hol}(\mathbb{C}^2)$ since $2\,\mathrm{Im}\,\partial_z$ and $2\,\mathrm{Im}\,\partial_w$ obviously belong to $\mathfrak{hol}(M)$, of course together with:

$$(1)\ 2\,\mathrm{Re}\,\big(z\,\partial_z + sw\,\partial_w\big), \qquad (2)\ 2\,\mathrm{Re}\,\big(z\partial_z + (z + w)\,\partial_w\big),$$
$$(3)\ 2\,\mathrm{Re}\,\big((-w + az)\,\partial_z + (z + aw)\,\partial_w\big).$$

It is a matter of elementary computations to verify that there are *no* further symmetries, so that $\dim \mathfrak{hol}(M) = 3$, hence all such tubes are *also* simply homogeneous.

The second collection of other three items — (K), (K'), (L) — are *not* locally biholomorphic to tubes, and can be described as [43, 76]:

(5) $\big\{[z\colon w\colon \zeta] \in \mathbf{P}^2_{\mathbb{C}}\colon\ |z|^2 + |w|^2 + |\zeta|^2 = \alpha\,|z^2 + w^2 + \zeta^2|\big\}$, with $\alpha > 1$, having holomorphic symmetries $\big(z', w', \zeta'\big)^t = A\big(z, w, \zeta\big)^t$ with $A \in SO_3(\mathbb{R})$;
(6) $|z|^2 + |w|^2 - 1 = \alpha\,|z^2 + w^2 - 1|$ minus $\{x^2 + u^2 = 1\}$, with $-1 < \alpha < 1$, $\alpha \neq 0$, having holomorphic symmetries:

$$\begin{pmatrix} z' \\ w' \end{pmatrix} = \frac{\begin{pmatrix} \mathsf{a}_{11} & \mathsf{a}_{12} \\ \mathsf{a}_{21} & \mathsf{a}_{22} \end{pmatrix}\begin{pmatrix} z \\ w \end{pmatrix} + \begin{pmatrix} \mathsf{b}_1 \\ \mathsf{b}_2 \end{pmatrix}}{\mathsf{c}_1 z + \mathsf{c}_2 w + \mathsf{d}}, \qquad \begin{pmatrix} \mathsf{a}_{11} & \mathsf{a}_{12} & \mathsf{b}_1 \\ \mathsf{a}_{21} & \mathsf{a}_{22} & \mathsf{b}_2 \\ \mathsf{c}_1 & \mathsf{c}_2 & \mathsf{d} \end{pmatrix} \in SO_{2,1}(\mathbb{R})\mathfrak{I}$$

(7) $1 + |z|^2 - |w|^2 = \alpha\,|1 + z^2 - w^2|$, with $\mathrm{Im}\ z(1 + \overline{w}) > 0$, $\alpha > 1$, having holomorphic symmetries:

$$\begin{pmatrix} z' \\ w' \end{pmatrix} = \frac{\begin{pmatrix} \mathsf{a}_{22} & \mathsf{b}_2 \\ \mathsf{c}_2 & \mathsf{d} \end{pmatrix}\begin{pmatrix} z \\ w \end{pmatrix} + \begin{pmatrix} \mathsf{a}_{21} \\ \mathsf{c}_1 \end{pmatrix}}{\mathsf{a}_{12} z + \mathsf{b}_1 w + \mathsf{a}_{11}}, \qquad \begin{pmatrix} \mathsf{a}_{11} & \mathsf{a}_{12} & \mathsf{b}_1 \\ \mathsf{a}_{21} & \mathsf{a}_{22} & \mathsf{b}_2 \\ \mathsf{c}_1 & \mathsf{c}_2 & \mathsf{d} \end{pmatrix} \in SO^{c}_{2,1}(\mathbb{R}).$$

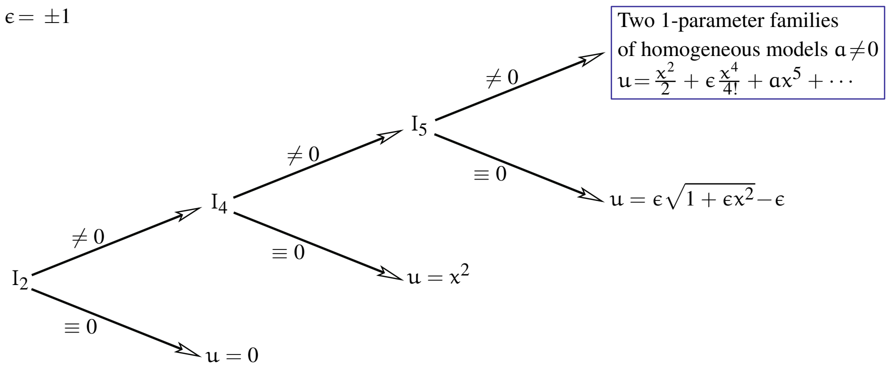
3 Affinely Homogeneous Curves $C^1 \subset \mathbb{R}^2$

It turns out that Cartan's CR models are quite *tied* with the classification of affinely homogeneous curves $C^1 \subset \mathbb{R}^2$ — and of surfaces $S^2 \subset \mathbb{R}^3$ as well, see below.

Indeed, according to [11, Sec. 16], a graphed curve $u = F(x)$ has, under $A_2(\mathbb{R})$, the *relative* differential invariant $I_2 := F_{xx}$. Obviously, $I_2 \equiv 0$ iff the curve is affinely equivalent to $u = 0$. In the branch $I_2 \neq 0$, the next *relative* differential invariant is $I_4 := \frac{1}{3} \frac{3F_{xx}F_{xxxx} - 5F_{xxx}^2}{F_{xx}^2}$. It is classical that $I_4 \equiv 0$ iff the curve is up to $A_2(\mathbb{R})$ the parabola $u = x^2$. Assuming therefore $I_4 \neq 0$, denoting its sign by ϵ, which is an invariant, there comes the first *true* differential invariant:

$$I_5 := \frac{1}{\sqrt{3}} \frac{9\,F_{xx}^2 F_{xxxxx} - 45\,F_{xx}\,F_{xxx}\,F_{xxxx} + 40\,F_{xxx}^3}{(\epsilon[3\,F_{xx}\,F_{xxxx} - 5\,F_{xxx}^3])^{3/2}},$$

sometimes called the *Monge* invariant. It is classical that $I_5 \equiv 0$ iff $u = F(x)$ is affinely equivalent to a (nondegenerate) conic. When $I_5 \neq 0$, it is not difficult to show that I_5 and its invariant derivatives generate the whole algebra of differential invariants [11, 79]. Furthermore, the curve is $A_2(\mathbb{R})$-homogeneous if and only if $I_5 =: a$ is constant, and any constant $a \in \mathbb{R}\backslash\{0\}$ works.



A first version, *incorporating information about branches created by differential invariants*, of the complete classification of curves $C^1 \subset \mathbb{R}^2$ homogeneous under $A_2(\mathbb{R})$ therefore states as:

(1) line $u = 0$, having symmetries ∂_x, $x\partial_x$, $u\partial_x$, $u\partial_u$;
(2) parabola $u = x^2$, having symmetries $\partial_x + 2x\partial_u$, $x\partial_x + 2u\partial_u$;
(3) ellipse and hyperbola $u = \epsilon\sqrt{1+\epsilon x^2} - \epsilon$, with $\epsilon = \pm 1$, having symmetry $(1+\epsilon u)\partial_x + x\partial_u$;
(4) two, depending on $\epsilon = \pm 1$, families of mutually inequivalent curves parametrized by any $a \in \mathbb{R}\backslash\{0\}$, whose power series are:

$$u = \frac{x^2}{2!} + \epsilon\frac{x^4}{4!} + a\frac{x^5}{5!} + \sum\nolimits_{k=6}^{\infty} F_k(a)\,\frac{x^k}{k!},$$

all coefficients $F_k(a)$ being uniquely determined by means of certain *recurrence formulæ* for differential invariants appearing in *e.g.* [11, 79], for instance $F_6 = (\pm\frac{5}{2} - 1)\, a^2 + 5$, and so on. For our degenerate para-CR structures, we will come up with a quite similar *first classification*, which, in our views, is the most natural one, because it *respects invariant branching*.

As shown *e.g.* by Eastwood-Ezhov in [23], when one tries to put these families in *closed* forms, one is conducted to gather (3) $\cup$ (4) and to split this union in two collections of 3 items, depending on $\epsilon = \pm 1$:

(a^+) the curves $x^2 + u^2 = e^{\beta \arctan \frac{u}{x}}$, for all $0 \leqslant \beta < \infty$, with $a = 4 \cdot 3^{-1/2} \beta (9 + \beta^2)^{-1/2}$ covering $0 \leqslant a < 4 \cdot 3^{-1/2}$;
(b^+) the curve $u = x \log x$, with $a = 4 \cdot 3^{-1/2}$;
(c^+) the curves $u = x^s$, for all $1 < s < 2$, with $a = 2 \cdot 3^{-1/2}(s+1)[(2-s)(2s-1)]^{-1/2}$ covering $4 \cdot 3^{-1/2} < a < \infty$;
(a^-) the curves $u = x^s$ for all $-1 \leqslant s < 0$, with $a = 2 \cdot 3^{-1/2}(s+1)[(2-s)(1-2s)]^{-1/2}$ covering $0 \leqslant a < 2^{1/2} 3^{-1/2}$;
(b^-) the curve $u = e^x$, with $a = 2^{1/2} 3^{-1/2}$;
(c^-) the curves $u = x^s$, for all $s > 2$, with $a = 2 \cdot 3^{-1/2}(s+1)[(s-2)(2s-1)]^{-1/2}$, covering finally $2^{1/2} 3^{-1/2} < a < \infty$.

By reorganizing all this, the first classification can be replaced by the more elegant and compact, closed, second version of the classification, in which one recognizes much of Cartan's classification for tubes $M^3 = C^1 \times i\mathbb{R}^2$:

(1') $u = x^s$ for $s \in [-1, 0) \cup [1, \infty)$;
(2') $u = e^x$;
(3') $u = x \log x$;
(4') $x^2 + u^2 = e^{\beta \arctan \frac{u}{x}}$, with $\beta \in [0, \infty)$.

However, in such a second classification, used in [54], the natural structuration of models by branches of differential invariants has been lost and mixed, since for instance $s = -1$ in (1') is the hyperbola while $\beta = 0$ in (4') is the ellipse; also (2') and (3') should join (1') for $s \neq -1, 1$ to be in the main branch; *etc.*

Lastly, in Cartan's items (E), (F), (H) — or (1), (2), (3) —, one recognizes all items of this second classification (1'), (2'), (3'), (4'), except that the two spherical tube $u = x^2$ and $u = e^x$ must be excluded, because according to Loboda [54], a tube $u = F(x)$ in $\mathbb{C}^2$ is spherical if and only if:

$$0 \equiv F_{xx}^3 F_{xxxxxx} - 7\, F_{xx}^2 F_{xxx} F_{xxxxx} + 25\, F_{xx} F_{xxx}^2 F_{xxxx} - 4\, F_{xx}^2 F_{xxxx}^2 - 15\, F_{xxx}^4.$$

4 Lie-Tresse Classification of Second Order ODEs

Although, according to Lie, Segre, Nurowski-Sparling, and others, there is a quite direct way from second order ODEs to 3-dimensional CR manifolds, *cf.* also Doubrov-Medvedev-The [21, App. D], Cartan's classification of homogeneous $M^3 \subset \mathbb{C}^2$ under biholomorphisms was done *after* the classification of second order ODEs under point transformations. Doubrov-Komrakov recently posted a complete memoir [18] on second order ODEs, from ancient notes.

At first, in 1883 ([53], *cf.* [62]), Lie showed that a second order ODE:

$$y_{xx} = F(x, y, y_x),$$

is equivalent to the flat one $y''_{x'x'} = 0$ by a point transformation $(x, y) \mapsto (x', y')$ if and only if $I_1 = I_2 = 0$, where, in terms of the total differentiation operator $D := \partial_x + y\,\partial_y + F\,\partial_p$, setting $p := y_x$:

$$\begin{aligned} I_1 := F_{pppp}, \ I_2 := \ & D^2F_{pp} - 4DF_{yp} - DF_{pp}F_p + 4F_pF_{yp} \\ & -3F_{pp}F_y + 6F_{yy}. \end{aligned}$$

In his systematic study [91], Tresse used higher order differential invariants to classify second order ODEs under point transformations.

Generally, equivalence classes of second order ODEs can be fully characterized by a number of (relative) invariants, generated by I_1 and I_2 and all their invariant derivatives. These invariants appear in a certain $\{e\}$-structure bundle $P^8 \longrightarrow J^1$ over the first jet space $J^1 \ni (x, y, p)$, *see e.g.* [78, Thm. 12.19], which is quite similar to the CR $\{e\}$-structure [7, 74, 76]

In his celebrated paper [6] on projective connections, Cartan used the class of second order ODEs for which the invariant I_1 vanishes as an example of a geometry that naturally gives rise to a *Cartan normal projective connection*. In fact, there is a *dual* second order ODE ([76]), say $y''_{x'x'} = F'(x', y', y'_{x'})$, whose relative invariants $I_1' \propto I_2$ and $I_2' \propto I_1$ are switched, up to a nonzero factor.

On the other hand, it is a matter of direct elementary computations to derive from Lie's list [26, Thm. 6, p. 71] of finite-dimensional continuous group actions on $\mathbb{R}^2$ all possible homogeneous second order ODEs. Because the Lie-Tresse classification of second order ODEs is strongly linked with our results, we show the list of mutually inequivalent (up to discrete switch) homogeneous non-flat second order ODEs over $\mathbb{R}$, taken from Doubrov-Komrakov [18, p. 31].

- $y_{xx} = y_x^{\alpha}$ where $\alpha \neq 0, 1, 2, 3$ (up to $\alpha \longleftrightarrow 3 - \alpha$), with symmetries ∂_x, ∂_y, $x\,\partial_x + \frac{\alpha-2}{\alpha-1}\,y\,\partial_y$.
- $y_{xx} = \big(1 + y_x^2\big)^{3/2}\, e^{-\alpha \arctan y_x}$, where $\alpha \neq 0$ (up to $\alpha \longleftrightarrow -\alpha$), with symmetries ∂_x, ∂_y, $(-y + \alpha\,x)\,\partial_x + (x + \alpha y)\,\partial_y$.
- $y_{xx} = e^{-y_x}$, with symmetries ∂_x, ∂_y, $x\,\partial_x + (x + y)\,\partial_y$.
- $2x\,y_{xx} = \pm y_x^3 - y_x$, with symmetries ∂_y, $x\,\partial_x + y\,\partial_y$, $2xy\,\partial_x + y^2\,\partial_y$.
- $x\,y_{xx} = y_x\big(1 - y_x^2\big) + \alpha\,\big|1 - y_x^2\big|^{3/2}$, where $\alpha \neq 0$ (up to $\alpha \longleftrightarrow -\alpha$), with symmetries ∂_y, $x\,\partial_x + y\,\partial_y$, $2xy\,\partial_x + (x^2 + y^2)\,\partial_y$.
- $x\,y_{xx} = y_x\big(1 + y_x^2\big) + \alpha\,\big(1 + y_x^2\big)^{3/2}$, where $\alpha \neq 0$ (up to $\alpha \longleftrightarrow -\alpha$), with symmetries ∂_y, $x\,\partial_x + y\,\partial_y$, $2xy\,\partial_x + (-x^2 + y^2)\,\partial_y$.

- $(1+x^2+y^2)\,y_{xx} = 2\left(1+y_x^2\right)(x\,y_x - y) + \alpha\left(1+y_x^2\right)^{3/2}$, where $\alpha \neq 0$ (up to $\alpha \longleftrightarrow -\alpha$), having symmetries:

$$-y\,\partial_x + x\,\partial_y, \quad (1+x^2-y^2)\,\partial_x + 2\,xy\,\partial_y, \quad 2\,xy\,\partial_x + (1-x^2+y^2)\,\partial_y.$$

Over $\mathbb{C}$, there are less inequivalent ODEs, *cf.* [78, p. 476].

5 Levi Nondegenerate CR and Para-CR Structures

Now, what about higher dimensional CR manifolds? The formal analogies between various CR structures and various systems of PDEs, emphasized *e.g.* in [42, 63], will hence be a guide in our future explorations. Earlier on, after the Chern-Moser celebrated article [16] on equivalence classes of Levi *nondegenerate* CR structures of hypersurface type of any dimension $2n+1 \geqslant 3$, Chern in [15], much inspired by Hachtroudi's Ph.D. [39], defended in 1937 in Paris under the direction of É. Cartan, studied completely integrable systems of second order PDEs of the form $y_{x^{i_1}x^{i_2}} = F^{i_1,i_2}\big(x^j, y, y_{x^k}\big)$, with $1 \leqslant i_1, i_2, j, k \leqslant n$, in dimension $n \geqslant 2$, for which reduction to a normal Cartan projective connection is very similar to the CR context.

For 5-dimensional hypersurfaces $M^5 \subset \mathbb{C}^3$ with nondegenerate Levi form, that is, when Chern-Moser tensors are available, there nowadays exist almost complete far-reaching classifications. At first, it is known that any Levi nondegenerate hypersurface $M^{2n+1} \subset \mathbb{C}^3$ has CR symmetry algebra $\mathfrak{aut}_{CR}(M) = 2\,\mathrm{Re}\ \mathfrak{hol}(M)$ of dimension $\leqslant 15$, this bound being attained when and only when M^5 is locally biholomorphic to one of the two hyperquadrics:

$$\mathrm{Im}\ w = z_1\overline{z}_1 \pm z_2\overline{z}_2,$$

depending on the signature of its Levi form.

In this context, the dimension drop is $15 \downarrow 8$, as the next largest possible dimension 8 for $\mathfrak{hol}(M)$ is achieved by the so-called *Winkelmann hypersurface* [93]:

$$\mathrm{Im}\ \big(w + \overline{z}_1 z_2\big) = |z_1|^4. \tag{1}$$

Locally homogeneous Levi nondegenerate hypersurfaces $M^5 \subset \mathbb{C}^3$ with isotropy Lie algebras $\mathfrak{hol}(M, p)$ for $p \in M$ of dimensions $\geqslant 1$, hence in the range $6 \leqslant \dim \mathfrak{hol}(M) \leqslant 7$, have been extensively classified by Loboda in [55–58], who handled local equations beyond the standard Moser normal form. In [58], Loboda also classified all strongly pseudoconvex (positive definite Levi form) hypersurfaces with $1 = \dim \mathfrak{hol}(M, p)$. Recently, both in the $(+,+)$ and $(-,+)$ signature cases, Loboda [59] terminated $0 = \dim \mathfrak{hol}(M, p)$.

Recently also, Doubrov-Medvedev-The developed an alternative approach, based on PDE systems $y_{x^{i_1}x^{i_2}} = F^{i_1,i_2}\big(x^j, y, y_{x^k}\big)$ with $1 \leqslant i_1, i_2, j, k \leqslant 2$ under point transformations (*cf.* [15, 63]). They almost completely classified the homogeneous models in [20], and they used Lie-theoretical methods to complete in [21] the classification of all multiply-transitive hypersurfaces in $\mathbb{C}^3$, by providing a new complete list of Levi indefinite hypersurfaces in $\mathbb{C}^3$ with 6-dimensional symmetry algebra $\mathfrak{hol}(M)$. They also confirmed Loboda's classifications as a whole, modulo one model.

Now, let us survey more precisely these achievements. Consider therefore a $\mathcal{C}^\omega$ hypersurface $M^5 \subset \mathbb{C}^3$, graphed as $w = \Theta\big(z_1, z_2, \overline{z}_1, \overline{z}_2, \overline{w}\big)$ with an analytic function Θ satisfying the condition:

$$w \equiv \Theta\big(z_1, z_2, \overline{z}_1, \overline{z}_2, \overline{\Theta}(\overline{z}_1, \overline{z}_2, z_1, z_2, w)\big),$$

which guarantees that M is *real*, namely of *real codimension* 1. View $\overline{z}_1, \overline{z}_2, \overline{w}$ as fixed parameters, differentiate once $w = \Theta$ to get $w_{z_1} = \Theta_{z_1}$ and $w_{z_2} = \Theta_{z_2}$, and observe that $\overline{z}_1, \overline{z}_2, \overline{w}$ can be eliminated from these 3 equations if and only if the corresponding *Levi* (Jacobian) *determinant* does not vanish:

$$\begin{vmatrix} \Theta_{\overline{z}_1} & \Theta_{\overline{z}_1} & \Theta_{\overline{w}} \\ \Theta_{z_1\overline{z}_1} & \Theta_{z_1\overline{z}_1} & \Theta_{z_1\overline{w}} \\ \Theta_{z_2\overline{z}_1} & \Theta_{z_2\overline{z}_1} & \Theta_{z_2\overline{w}} \end{vmatrix} \neq 0.$$

Lastly, replace the solved values for $\overline{z}_1, \overline{z}_2, \overline{w}$ in all three second derivatives $w_{z_{i_1} z_{i_2}} = \Theta_{z_{i_1} z_{i_2}}$ to obtain a system of second order $\mathbb{C}$-analytic partial differential equations:

$$w_{z_{i_1} z_{i_2}} = \Xi^{i_1, i_2}\big(z_1, z_2, w, w_{z_1}, w_{z_2}\big) \qquad (1 \leqslant i_1, i_2 \leqslant 2),$$

which is completely integrable by construction.

Because of the reality assumption, not all such systems over $\mathbb{C}$ are covered by this process. Hence, it is natural to relax the reality assumption, and to consider more generally arbitrary *submanifolds of solutions:*

$$z = Q(x, y, a, b, c) \qquad (\text{with } Q_c \neq 0),$$

modulo the infinite-dimensional group of local $\mathbb{C}$-analytic transformations separating variables and parameters:

$$\begin{aligned}(x, y, z,\ a, b, c) \quad\longmapsto\quad & \big(x'(x, y, z), y'(x, y, z), z'(x, y, z),\ a'(a, b, c), b'(a, b, c), \\ & c'(a, b, c)\big). \end{aligned} \tag{2}$$

Assuming similarly that the *generalized Levi form* is nonzero:

$$\begin{vmatrix} Q_a & Q_b & Q_c \\ Q_{xa} & Q_{xb} & Q_{xc} \\ Q_{ya} & Q_{yb} & Q_{yc} \end{vmatrix} \neq 0,$$

it can be verified [42, 63] that the study of equivalences of submanifolds of solutions amounts to the study of point equivalences of completely integrable systems of $\mathbb{C}$-analytic second order systems of PDEs:

$$z_{xx} = F\big(x, y, z, z_x, z_y\big), \quad z_{xy} = G\big(x, y, z, z_x, z_y\big), \quad z_{yy} = H\big(x, y, z, z_x, z_y\big). \tag{3}$$

Then the following lemma, known to Cartan [7] in the case of Levi nondegenerate hypersurfaces $M^3 \subset \mathbb{C}^2$, justifies the interest [42, 63] of classifying PDE systems *before* classifying (real) hypersurfaces, not to mention that most often, classifications over $\mathbb{C}$ are simpler than over $\mathbb{R}$.

Lemma 5.1 *For a Levi nondegenerate $\mathcal{C}^\omega$ hypersurface $\{w = \Theta(z_1, z_2, \overline{z}_1, \overline{z}_2, \overline{w})\}$ in $\mathbb{C}^3$, the* real *Lie algebra of its infinitesimal holomorphic automorphisms:*

$$\mathfrak{hol}(M) := \big\{ L = A_1(z_1, z_2, w)\partial_{z_1} + A_2(z_1, z_2, w)\partial_{z_2} + B(z_1, z_2, w)\partial_w : \\ (L + \overline{L})\big|_M \text{ is tangentto } M \big\},$$

is of dimension $\leqslant 15$.

Furthermore, the complex *Lie symmetry algebra [78] of its associated $\mathbb{C}$-analytic PDE system $w_{z_{i_1} z_{i_2}} = \Xi^{i_1, i_2}$ denoted (E_M) satisfies:*

$$\mathfrak{sym}(E_M) = \mathfrak{hol}(M) \otimes_{\mathbb{R}} \mathbb{C}. \qquad \square$$

Since Chern-Moser [16] in 1974 (at least), it is well known that the bound $\dim_{\mathbb{R}} \mathfrak{hol}(M) = 15$ is attained if and only if M is (locally) biholomorphic to one of the two *quadrics:*

$$\frac{w - \overline{w}}{2i} = z_1 \overline{z}_1 \pm z_2 \overline{z}_2,$$

which, in case of $(+, +)$ signature of the Levi form, has holomorphic automorphisms group given by:

$$\begin{pmatrix} z_1' \\ z_2' \\ w' \end{pmatrix} = \frac{\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ w \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}}{c_1 z_1 + c_2 z_2 + c_3 w + d}, \qquad \begin{pmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \\ c_1 & c_2 & c_3 & d \end{pmatrix} \in SU_{3,1}(\mathbb{C}),$$

with a similar formula in case of $(+, -)$ signature.

Next, abbreviating $p := z_x$ and $q := z_y$, with the two *total differentiation operators:*

$$D_x := \partial_x + p\,\partial_z + F\,\partial_p + G\,\partial_q, \qquad D_y := \partial_y + q\,\partial_z + G\,\partial_p + H\,\partial_q,$$

complete integrability of a general PDE system (3) as above holds if and only if $[D_x, D_y] = 0$, if and only if $D_x G = D_y F$ and $D_x H = D_y G$, if and only if the general solution is of the already seen form $z = Q(x, y, a, b, c)$.

Then on the first jet manifold $J^1 \ni (x, y, z, p, q)$ of dimension 5, the *horizontal* and *vertical* 2-dimensional distributions:

$$\mathcal{H} := \text{Span}\,\{\partial_p, \partial_q\}, \qquad \mathcal{V} := \text{Span}\,\{D_x, D_y\},$$

are invariant under point diffeomorphisms, and their sum $\mathcal{C} := \mathcal{H} \oplus \mathcal{V}$, of rank $2+2$, constitutes a *contact* distribution $\mathcal{C} \subset \mathsf{T}\mathsf{J}^1$. Following Hill-Nurowski [42], such PDE systems (3) are therefore called *nondegenerate para-CR structures of type* $(2, 2, 1)$.

Since Hachtroudi [39] in 1937 (at least), it is known that every PDE system (3) satisfies $\dim_{\mathbb{C}} \mathfrak{sym}\,(3) \leqslant 15$, and that equality is attained if and only if (3) is (locally) point equivalent to the *flat system:*

$$z_{xx} = 0, \qquad z_{xy} = 0, \qquad z_{yy} = 0. \tag{4}$$

In this case, the Lie symmetry group consists of all complex automorphisms of $\mathbb{P}^3(\mathbb{C})$, affinely represented as:

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \frac{\begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}}{\gamma_1 z_1 + \gamma_2 z_2 + \gamma_3 w + \delta}, \qquad \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \beta_1 \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \beta_2 \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 & \delta \end{pmatrix} \in \mathrm{SL}(4, \mathbb{C}).$$

These second order PDE structures belong to the class of so-called *parabolic geometries*. In particular, they enjoy a number of important properties derived from the general theory of parabolic geometries, developed *e.g.* in the monograph [4] of Čap-Slovak: existence of a natural Cartan connection; description of all primary relative differential invariants in terms of the representation theory of (semi-)simple Lie algebras; finite-dimensionality of all symmetry algebras; determination of *maximal* and *submaximal* symmetry algebras thanks to the methods of Kruglikov-The [51], which exhibit (and explain) a so-called '*gap phenomenon*' concerning their respective dimensions.

Recently, Doubrov-Medvedev-The [20] classified all *multiply-transitive* homogeneous nondegenerate para-CR structures of type $(2, 2, 1)$, which they call '*Integrable Legendrian contact structures*'. The term '*multiply-transitive*' means that the (local) Lie symmetry algebra is (locally) transitive and has isotropy subalgebras of dimension $\geqslant 1$ at all (local) points.

For second order ODEs $y_{xx} = F(x, y, y_x)$, it is known that multiple transitivity implies *flatness*, *i.e.* point equivalence to $y_{xx} = 0$, with symmetries the 8-dimensional group of projective automorphisms of $\mathbb{P}^2(\mathbb{C})$:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \frac{\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}}{\gamma_1 x + \gamma_2 y + \delta}, \qquad \begin{pmatrix} \alpha_{11} & \alpha_{12} & \beta_1 \\ \alpha_{21} & \alpha_{22} & \beta_2 \\ \gamma_1 & \gamma_2 & \delta \end{pmatrix} \in \mathrm{SL}(3, \mathbb{C}).$$

Consequently, the dimension gap is $8 \downarrow 3$, as was already implicitly asserted by the classification list (over $\mathbb{R}$ or $\mathbb{C}$) shown at the end of Sect. 4.

For second order PDE systems (3), multiple transitivity does *not* imply flatness. Indeed, Doubrov-Medvedev-The [20] showed that the dimension gap is $15 \downarrow 8$, with, up to equivalence, a unique submaximal model:

$$z_{xx} = z_y^2, \qquad z_{xy} = 0, \qquad z_{yy} = 0,$$

which is the PDE system associated to Winkelmann's homogeneous hypersurface (1), hence with isotropy algebras of dimension $8 - 5 = 3$.

As was discovered by Hachtroudi [39] and reproved by Chern [15], the main, fundamental, primary (relative) differential invariant of PDE systems (3) can be encoded as the following *relatively invariant* binary quartic, in which $[r\colon s] \in \mathbb{P}^1(\mathbb{C})$:

$$F_{qq}\, r^4 + 2\left(F_{pq} - G_{qq}\right) r^3 s + \left(F_{pp} - 4\, G_{pq} + H_{qq}\right) r^2 s^2 + 2\left(H_{pq} - G_{pp}\right) r s^3 + H_{pp}\, s^4.$$

Flatness, *i.e.* equivalence to (4), is known [15, 39] to hold if and only if this quartic is identically zero, namely:

$$0 \equiv F_{qq} \equiv F_{pq} - G_{qq} \equiv F_{pp} - 4\, G_{pq} + H_{qq} \equiv H_{pq} - G_{pp} \equiv H_{pp}.$$

Doubrov-Medvedev-The [20] classified multiply transitive PDE systems, namely those having isotropy Lie algebras of dimensions 3, 2, 1, according to the *root combinatorics* of the quartic. The type denominations below are inspired from Cartan's classification [5] of homogeneous $(2, 3, 5)$-distributions (complemented in [17]) and from the Petrov classification of the Weyl curvature tensor in 4-dimensional Lorentzian conformal geometry.

Type O: Quartic identically zero;
Type N: A single root of multiplicity 4;
Type D: Two distinct roots, each of multiplicity 2;
Type III: One root of multiplicity 3, another different root of multiplicity 1;
Type II: Three distinct roots, of respective multiplicities 2, 1, 1;
Type I: Four mutually distinct roots.

In the context of general parabolic geometries, Kruglikov-The gave in [51] a general method for finding the submaximal symmetry dimension, here $8 < 15$. These techniques have been pushed further in [20], to determine all the possible maximal symmetry dimensions of para-CR structures of type $(2, 2, 1)$ which possess *constant root type* (automatic in presence of homogeneity).

Root type	O	N	D	III	II	I
Maximal symmetry dimension	15	8	7	6	5	5

Leaving the simply transitive case, with dim $\mathfrak{sym}$ (3) = 5, to further explorations, let us provide an abbreviated description of the far-reaching classification of Doubrov-Medvedev-The [20], organized in 3 separate tables, each gathering models of the concerned root types N, D, III. We give the infinitesimal symmetries in the (x, y, z) space, and we refer to [20] for commutation tables. Equations for different items correspond to inequivalent para-CR structures. There may be some additional equivalence relations on parameters within the same item, *see* [20].

Item	Model	Parameters	Symmetries (Root type N)
N.8	$u_{xx} = u_y^2$ $u_{xy} = 0$ $u_{yy} = 0$		$\partial_x, \partial_y, \partial_u, x\partial_y, x\partial_u,$ $x\partial_x - 2u\partial_u, y\partial_y + 2u\partial_u,$ $x^2\partial_y - y\partial_u$
N.7-1a	$u_{xx} = x^\kappa u_y^2$ $u_{xy} = 0$ $u_{yy} = 0$	$\kappa \neq -1, -2, 0, -3$	$\partial_y, \partial_u, x\partial_y, x\partial_u,$ $y\partial_y + 2u\partial_u, x\partial_x + \kappa y\partial_y + (\kappa - 2)u\partial_u,$ $\frac{x^{\kappa+2}}{\kappa+2}\partial_y - \frac{\kappa+1}{2}y\partial_u$
N.7-1a	$u_{xx} = x^\kappa u_y^2$ $u_{xy} = 0$ $u_{yy} = 0$	$\kappa \neq -1, -2, 0, -3$	$\partial_y, \partial_u, x\partial_y, x\partial_u,$ $y\partial_y + 2u\partial_u, x\partial_x + \kappa y\partial_y + (\kappa - 2)u\partial_u,$ $\frac{x^{\kappa+2}}{\kappa+2}\partial_y - \frac{\kappa+1}{2}y\partial_u$
N.7-1b	$u_{xx} = x^{-1}u_y^2$ $u_{xy} = 0$ $u_{yy} = 0$		$\partial_y, \partial_u, x\partial_y, x\partial_u,$ $y\partial_y + 2u\partial_u, x\partial_x - y\partial_y - 3u\partial_u,$ $2x \log(x)\partial_y - y\partial_u$
N.7-1c	$u_{xx} = e^x u_y^2$ $u_{xy} = 0$ $u_{yy} = 0$		$\partial_y, \partial_u, x\partial_y, x\partial_u,$ $\partial_x + y\partial_y + u\partial_u, y\partial_y + 2u\partial_u,$ $2e^x\partial_y - y\partial_u$
N.7-2	$u_{xx} = \frac{1}{u_y}$ $u_{xy} = 1$ $u_{yy} = 0$		$\partial_y, \partial_u, \partial_x - \partial_u, \partial_y + 2x\partial_u,$ $2x\partial_x - \partial_y + 2u\partial_u, x\partial_y + x^2\partial_u,$ $x^2\partial_x + u\partial_y + x(x + 2u)\partial_u$
N.6-1a	$u_{xx} = u_y^\mu$ $u_{xy} = 1$ $u_{yy} = 0$	$\mu \neq -1, 2, 0, 1$	$\partial_x, \partial_y, \partial_u,$ $\partial_y + 2x\partial_u, x\partial_y + x^2\partial_u,$ $x\partial_x + (\mu + 1)y\partial_y + (\mu + 2)u\partial_u$
N.6-1b	$u_{xx} = \log u_y$ $u_{xy} = 1$ $u_{yy} = 0$		$\partial_x, \partial_y, \partial_u,$ $\partial_y + 2x\partial_u, x\partial_y + x^2\partial_u,$ $x\partial_x - (\frac{x}{2} - y)\partial_y + 2u\partial_u$
N.6-1c	$u_{xx} = u_y \log u_y$ $u_{xy} = 1$ $u_{yy} = 0$		$\partial_x, \partial_y, \partial_u,$ $\partial_y + 2x\partial_u, x\partial_y + x^2\partial_u,$ $x\partial_x - (\frac{x^2}{2} - 2y)\partial_y + (3u - \frac{x^3}{3})\partial_u$

Item	Model	Parameters	Symmetries	Root type N
N.6-2a	$u_{xx} = x^{\kappa} u_y^{\mu}$ $u_{xy} = 0$ $u_{yy} = 0$	$\mu \neq -1, 2, 0, 1$ $\kappa \neq 0, -3$	$\partial_y,\ \partial_u,\ x\partial_y,\ x\partial_u,$ $x\partial_x + (\kappa+2)y\partial_y + (\kappa+2)u\partial_u,$ $(\mu-1)y\partial_y + \mu u\partial_u$	
N.6-2b	$u_{xx} = x^{\kappa} e^{u_y}$ $u_{xy} = 0$ $u_{yy} = 0$	$\kappa \neq 0, -3$	$\partial_y,\ \partial_u,\ x\partial_y,\ x\partial_u,$ $x\partial_x + (\kappa+2)y\partial_y + (\kappa+2)u\partial_u,$ $y\partial_y + (y+u)\partial_u$	
N.6-2c	$u_{xx} = e^{x} e^{u_y}$ $u_{xy} = 0$ $u_{yy} = 0$		$\partial_y,\ \partial_u,\ x\partial_y,\ x\partial_u,$ $\partial_x + y\partial_y + u\partial_u,$ $y\partial_y + (y+u)\partial_u$	
N.6-2d	$u_{xx} = x^{\kappa} \log x$ $u_{xy} = 0$ $u_{yy} = 0$	$\kappa \neq -1, -2, 0, -3$	$\partial_y,\ \partial_u,\ x\partial_y,\ 00\ x\partial_u,$ $x\partial_x + (\kappa+2)y\partial_y + (\kappa+2)u\partial_u,$ $y\partial_y - \frac{x^{\kappa+2}}{(\kappa+1)(\kappa+2)}\partial_u$	
N.6-2e	$u_{xx} = x^{-2} \log u_y$ $u_{xy} = 0$ $u_{yy} = 0$		$\partial_y,\ \partial_u,\ x\partial_y,\ x\partial_u,$ $x\partial_x,$ $y\partial_y + \log x\, \partial_u$	
D.7a	$u_{xx} = u_x^2$ $u_{xy} = 0$ $u_{yy} = \lambda u_y^2$	$\lambda \neq 0, -1$	$\partial_x,\ \partial_y,\ \partial_u,$ $2x\partial_x - \partial_u,\ 2y\partial_y - \frac{1}{\lambda}\partial_u,$ $x^2\partial_x - x\partial_u,\ y^2\partial_y - \frac{1}{\lambda} y\partial_u$	
D.7b	$u_{xx} = u_x^2$ $u_{xy} = 0$ $u_{yy} = 0$		$\partial_x,\ \partial_y,\ \partial_u,$ $y\partial_y,\ y\partial_u,$ $2x\partial_x - \partial_u,\ x^2\partial_x - x\partial_u$	
D.6-1	$u_{xx} = u_x^2 - \frac{1}{4} u_y^4$ $u_{xy} = u_y (u_x - \frac{1}{2} u_y^2)$ $u_{yy} = u_x - \frac{1}{2} u_y^2$		$\partial_x,\ \partial_y,\ \partial_u,$ $x\partial_y - y\partial_u,\ 2x\partial_x + y\partial_y - \partial_u,$ $x^2\partial_x + xy\partial_y - (x + \frac{1}{2}y^2)\partial_u$	
D.6-2a	$u_{xx} = u_x^{\mu}$ $u_{xy} = 0$ $u_{yy} = 0$	$\mu \neq 0, 1, 2$	$\partial_x,\ \partial_y,\ \partial_u,$ $y\partial_u,\ y\partial_y,$ $\frac{\mu-1}{\mu-2} x\partial_x + y\partial_y + u\partial_u$	
D.6-2b	$u_{xx} = e^{u_x}$ $u_{xy} = 0$ $u_{yy} = 0$		$\partial_x,\ \partial_y,\ \partial_u,$ $y\partial_u,\ y\partial_y,$ $x\partial_x + y\partial_y + (u - x)\partial_u$	
D.6-3a	$u_{xx} = \lambda u_x^2 \frac{(u - u_x u_y)^{1/2}}{u^{3/2}}$ $u_{xy} = 1 + \lambda(u_x u_y - 2u) \frac{(u - u_x u_y)^{1/2}}{u^{3/2}}$ $u_{yy} = \lambda u_y^2 \frac{(u - u_x u_y)^{1/2}}{u^{3/2}}$	$\lambda \neq 0, \pm \frac{1}{2}$	$\partial_x,\ \partial_y,\ x\partial_x + u\partial_u,\ y\partial_y + u\partial_u,$ $x\partial_x + y^2\partial_y + 2yu\partial_u,$ $x^2\partial_x + u\partial_y + 2xu\partial_u$	
D.6-3b	$u_{xx} = u_x^2 (1 - 2u_x u_y)^{1/2}$ $u_{xy} = (u_x u_y - 1)(1 - 2u_x u_y)^{1/2}$ $u_{yy} = u_y^2 (1 - 2u_x u_y)^{1/2}$		$\partial_x,\ \partial_y,\ \partial_u,$ $x\partial_x - y\partial_y,$ $u\partial_y + x\partial_u,\ u\partial_x + y\partial_u$	
D.6-4	$u_{xx} = 0$ $u_{xy} = \frac{1 + u_x u_y}{u}$ $u_{yy} = 0$		$\partial_x,\ \partial_y,$ $2x\partial_x + u\partial_u,\ 2y\partial_y + u\partial_u,$ $x^2\partial_x + xu\partial_u,\ y^2\partial_y + yu\partial_y$	
III.6−1	$u_{xx} = \frac{u_x}{x - u_y}$ $u_{xy} = 0$ $u_{yy} = 0$		$\partial_y,\ \partial_u,$ $\partial_x + y\partial_u,\ x\partial_y + \frac{x^2}{2}\partial_u,$ $y\partial_y + u\partial_u,\ x\partial_x + u\partial_u$	
III.6−2	$u_{xx} = 2u_y (2u_x - u u_y)$ $u_{xy} = u_y^2$ $u_{yy} = 0$		$\partial_x,\ \partial_y,\ x\partial_y - \partial_u,$ $y\partial_y + u\partial_u,\ 2x\partial_x + y\partial_y - u\partial_u,$ $x^2\partial_x + xy\partial_y - (y + xu)\partial_u$	

Beyond, by classifying all real forms of these complex Lie algebras, Doubrov-Medvedev-The deduced a classification of multiply transitive Levi nondegenerate $\mathcal{C}^{\omega}$

hypersurfaces $M^5 \subset \mathbb{C}^3$, alternative to Loboda's works. Since our focus is more on PDE system, we skip the detailed presentation.

Before going to study Levi *degenerate* para-CR structures, we would like to point out that, in CR dimension $n \geqslant 2$, with the exception of [20, 21], Cartan's method has never been pushed beyond reduction to an $\{e\}$-structure or to the determination of submaximal groups, although Cartan himself fully classified all homogeneous models in CR dimension 1, with his method. Maybe the *computational complexity* is an obstacle to handle diffential invariants of high order in CR dimension $n \geqslant 2$.

Knowing this, we would like to mention that in the present article, for certain degenerate para-CR structures of dimension 5, *i.e.* with $n = 2$, we *do* manage to employ Cartan's method, notwithstanding its complexity. It would be nice to unify existing views on classification problems.

6 Degenerate CR Manifolds of Dimension 5

Now, to motivate our results, consider embedded real analytic 5-dimensional CR manifolds $M^5 \subset \mathbb{C}^3$ (hypersurfaces), of CR dimension 2, that are Levi *degenerate*, *i.e.* whose Levi form is of rank < 2. We will also handle *abstract* CR and even para-CR structures below. It is well known that the Levi rank equals 0 everywhere iff M is *Levi-flat*, biholomorphic to the hyperplane Re $w' = 0$. Levi rank 2 was commented briefly above. The study of constant Levi rank 1 has been initiated recently [31, 45, 60, 61, 71]. One has to exclude the degenerate product situation, where $M^5 \cong M^3 \times \mathbb{C}$, up to a local biholomorphism. How?

At first, it can be verified on any computer that, given a $\mathcal{C}^\omega$ hypersurface $M^5 \subset \mathbb{C}^3 \ni (z_1, z_2, w)$ having complex graphing equation $w = \Theta(z_1, z_2, \overline{z}_1, \overline{z}_2, \overline{w})$, then through a general biholomorphism:

$$(z_1, z_2, w) \longmapsto \Big(f^1(z_1, z_2, w),\, f^2(z_1, z_2, w),\, g(z_1, z_2, w)\Big) =: (z_1', z_2', w')$$

which sends $w = \Theta$ to some target hypersurface $w' = \Theta'(z_1', z_2', \overline{z}_1', \overline{z}_2', \overline{w}')$, in terms of the $(1,0)$-tangent vector fields:

$$\mathcal{L}_{z_1} := \frac{\partial}{\partial z_1} + \Theta_{z_1}\frac{\partial}{\partial w} \qquad \text{and} \qquad \mathcal{L}_{z_2} := \frac{\partial}{\partial z_2} + \Theta_{z_2}\frac{\partial}{\partial w},$$

the target Levi 3×3 determinant is a nonzero multiple of the source one:

$$\frac{\begin{vmatrix} \Theta'_{\overline{z}_1'} & \Theta'_{\overline{z}_2'} & \Theta'_{\overline{w}'} \\ \Theta'_{z_1'\overline{z}_1'} & \Theta'_{z_1'\overline{z}_2'} & \Theta'_{z_1'\overline{w}'} \\ \Theta'_{z_2'\overline{z}_1'} & \Theta'_{z_2'\overline{z}_2'} & \Theta'_{z_2'\overline{w}'} \end{vmatrix}}{\begin{vmatrix} \Theta_{\overline{z}_1} & \Theta_{\overline{z}_2} & \Theta_{\overline{w}} \\ \Theta_{z_1\overline{z}_1} & \Theta_{z_1\overline{z}_2} & \Theta_{z_1\overline{w}} \\ \Theta_{z_2\overline{z}_1} & \Theta_{z_2\overline{z}_2} & \Theta_{z_2\overline{w}} \end{vmatrix}} = \frac{\begin{vmatrix} f^1_{z_1} & f^1_{z_2} & f^1_w \\ f^2_{z_1} & f^2_{z_2} & f^2_w \\ g_{z_1} & g_{z_2} & g_w \end{vmatrix}^3}{\begin{vmatrix} \overline{f}^1_{\overline{z}_1} & \overline{f}^1_{\overline{z}_2} & \overline{f}^1_{\overline{w}} \\ \overline{f}^2_{\overline{z}_1} & \overline{f}^2_{\overline{z}_2} & \overline{f}^2_{\overline{w}} \\ \overline{g}_{\overline{z}_1} & \overline{g}_{\overline{z}_2} & \overline{g}_{\overline{w}} \end{vmatrix}^1} \frac{1}{\begin{vmatrix} \mathcal{L}_{z_1}(f^1) & \mathcal{L}_{z_2}(f^1) \\ \mathcal{L}_{z_1}(f^2) & \mathcal{L}_{z_2}(f^2) \end{vmatrix}^4}.$$

Thus, M having constant Levi rank 1 is an invariant property.

Furthermore, solely when $0 \equiv \det \, \mathrm{Levi}(\Theta)$, another determinant, which expresses 2-nondegeneracy satisfies the following invariant relation with a nowhere vanishing right-hand side when the Levi rank equals 1:

$$\frac{\begin{vmatrix} \Theta'_{\overline{z}'_1} & \Theta'_{\overline{z}'_2} & \Theta'_{\overline{w}'} \\ \Theta'_{z'_1\overline{z}'_1} & \Theta'_{z'_1\overline{z}'_2} & \Theta'_{z'_1\overline{w}'} \\ \Theta'_{z'_1z'_1\overline{z}'} & \Theta'_{z'_1z'_1\overline{z}'_2} & \Theta'_{z'_1z'_1\overline{w}'} \end{vmatrix}}{\begin{vmatrix} \Theta_{\overline{z}_1} & \Theta_{\overline{z}_2} & \Theta_{\overline{w}} \\ \Theta_{z_1\overline{z}_1} & \Theta_{z_1\overline{z}_2} & \Theta_{z_1\overline{w}} \\ \Theta_{z_1z_1\overline{z}_1} & \Theta_{z_1z_1\overline{z}_2} & \Theta_{z_1z_1\overline{w}} \end{vmatrix}} = \frac{\begin{vmatrix} f^1_{z_1} & f^1_{z_2} & f^1_{w} \\ f^2_{z_1} & f^2_{z_2} & f^2_{w} \\ g_{z_1} & g_{z_2} & g_{w} \end{vmatrix}^3}{\begin{vmatrix} \overline{f}^1_{\overline{z}_1} & \overline{f}^1_{\overline{z}_2} & \overline{f}^1_{\overline{w}} \\ \overline{f}^2_{\overline{z}_1} & \overline{f}^2_{\overline{z}_2} & \overline{f}^2_{\overline{w}} \\ \overline{g}_{\overline{z}_1} & \overline{g}_{\overline{z}_2} & \overline{g}_{\overline{w}} \end{vmatrix}^1} \; \frac{\left(\mathcal{L}_{z_2}(f^2) \begin{vmatrix} \Theta_{\overline{z}_1} & \Theta_{\overline{w}} \\ \Theta_{z_1\overline{z}_1} & \Theta_{z_1\overline{w}} \end{vmatrix} - \mathcal{L}_{z_1}(f^2) \begin{vmatrix} \Theta_{\overline{z}_1} & \Theta_{\overline{w}} \\ \Theta_{z_2\overline{z}_1} & \Theta_{z_2\overline{w}} \end{vmatrix} \right)^3}{\begin{vmatrix} \mathcal{L}_{z_1}(f^1) & \mathcal{L}_{z_2}(f^1) \\ \mathcal{L}_{z_1}(f^2) & \mathcal{L}_{z_2}(f^2) \end{vmatrix}^6 \; \begin{vmatrix} \Theta_{\overline{z}_1} & \Theta_{\overline{w}} \\ \Theta_{z_1\overline{z}_1} & \Theta_{z_1\overline{w}} \end{vmatrix}^3}.$$

It can be proved [72, p. 91] that an $M^5 \subset \mathbb{C}^3$ having constant Levi rank 1 is locally biholomorphically equivalent to a product $M^5 \cong M^3 \times \mathbb{C}$ of a Levi nondegenerate hypersurface $M^3 \subset \mathbb{C}^2$ times $\mathbb{C}$, if and only if this determinant vanishes *identically*. One then says that M is *everywhere* 2-*degenerate*, and certainly, one sets aside such an exceptional situation, because the equivalence problem reduces to that of an $M^3 \subset \mathbb{C}^2$. Thus, throughout, $M^5 \subset \mathbb{C}^3$ will be assumed *everywhere* 2-*nondegenerate*, in the sense that the above determinant is assumed nowhere vanishing.

The class of 2-nondegenerate constant Levi rank 1 hypersurfaces $M^5 \subset \mathbb{C}^3$ will be denoted by:

$$\mathfrak{C}_{2,1}.$$

This class is not empty, since it contains the tube in $\mathbb{C}^3$ over the future light cone in $\mathbb{R}^3$:

$$T^5_{\mathrm{LC}} := S^2_{\mathrm{LC}} \times i\mathbb{R}^3 \qquad \text{where} \qquad S^2_{\mathrm{LC}} := \big\{ x \in \mathbb{R}^3 \colon \; x_1^2 + x_2^2 = x_3^2, \; x_3 > 0 \big\}.$$

In fact, according to [45, 60, 71], T^5_{LC} is a *model* for such CR structures, having maximal CR symmetry group isomorphic to $SO_{3,2}(\mathbb{R})$. But it is not represented in graphed form.

With $w = u + i\,v$, a graphed representation $M^5_{\mathrm{LC}} \cong T^5_{\mathrm{LC}}$ was set up in [37] and [27]:

$$M_{\mathrm{LC}} \colon \quad w + \overline{w} = \frac{2\, z_1 \overline{z}_1 + z_1^2 \overline{z}_2 + \overline{z}_1^2 z_2}{1 - z_2 \overline{z}_2}. \tag{5}$$

How? Starting with $M^5 \subset \mathbb{C}^3$, with $0 \in M$, rigid, graphed as:

$$u = F(z_1, z_2, \overline{z}_1, \overline{z}_2),$$

constant Levi rank 1 means:

$$F_{z_1\overline{z}_1} \neq 0 \equiv \begin{vmatrix} F_{z_1\overline{z}_1} & F_{z_1\overline{z}_2} \\ F_{z_2\overline{z}_1} & F_{z_2\overline{z}_2} \end{vmatrix},$$

while 2-nondegeneracy means:

$$0 \neq \begin{vmatrix} F_{z_1\overline{z}_1} & F_{z_1\overline{z}_2} \\ F_{z_1z_1\overline{z}_1} & F_{z_1z_1\overline{z}_2} \end{vmatrix}.$$

After cleaning the terms up to order 3 included, with weights $[z_1] = [z_2] := 1$ and $[w] := 2$, any $M \in \mathfrak{C}_{2,1}$ graphed as:

$$u = F\big(z_1, z_2, \overline{z}_1, \overline{z}_2, v\big),$$

where $u = \mathrm{Re}\; w$ and $v = \mathrm{Im}\; w$, reads ([13, 37]):

$$w + \overline{w} = 2\, z_1\overline{z}_1 + z_1^2\overline{z}_2 + \overline{z}_1^2 z_2 + \mathrm{O}_{z_1,z_2,\overline{z}_1,\overline{z}_2,v}(4),$$

hence is 2-nondegenerate and of Levi rank 1 at the origin. *However*, higher order correction terms must be added to insure that the Levi form be of constant rank 1. Taking the simplest correction terms, one comes to M_{LC} above [13, 37].

The CR geometry of M_{LC} is as follows. The two natural $(1,0)$ vector fields tangent to M_{LC} are:

$$\mathcal{L}_1 := \frac{\partial}{\partial z_1} + \left[\frac{2\overline{z}_1 + 2z_1\overline{z}_2}{1 - z_2\overline{z}_2}\right]\frac{\partial}{\partial w}, \qquad \mathcal{L}_2 := \frac{\partial}{\partial z_2} + \left[\frac{(\overline{z}_1 + z_1\overline{z}_2)^2}{(1 - z_2\overline{z}_2)^2}\right]\frac{\partial}{\partial w}.$$

The kernel of the Levi form is generated by the $(1,0)$ vector field

$$\mathcal{K} := -\left[\frac{\overline{z}_1 + z_1\overline{z}_2}{1 - z_2\overline{z}_2}\right]\frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2} - \left[\frac{(\overline{z}_1 + z_1\overline{z}_2)^2}{(1 - z_2\overline{z}_2)^2}\right]\frac{\partial}{\partial w},$$

as one has:

$$[\mathcal{L}_1, \overline{\mathcal{K}}] = -\Big(\frac{1}{1 - z_2\overline{z}_2}\Big)\overline{\mathcal{L}_1}, \qquad [\mathcal{L}_2, \overline{\mathcal{K}}] = -\Big(\frac{\overline{z}_1 + z_1\overline{z}_2}{(1 - z_2\overline{z}_2)^2}\Big)\overline{\mathcal{L}}_1.$$

As predicted by the involutiveness of the Levi kernel, M_{LC} is necessarily foliated by complex curves. These are the *lines* $z_1 := z_0 - \overline{z}_0\zeta$, $z_2 := \zeta$, $w := z_0\overline{z}_0 + i\lambda - \zeta\overline{z}_0^2$, where $z_0 \in \mathbb{C}$, $\lambda \in \mathbb{R}$ and $\zeta \in \mathbb{C}$ satisfies $|\zeta| < 1$.

The determination in [37] of $\mathfrak{hol}(M_{\mathsf{LC}})$ was done in two steps. Firstly, by setting up the Segre-like PDE system satisfied by $w = w(z_1, z_2)$ considered in (5) as a holomorphic function of (z_1, z_2), while $\overline{z}_1$, $\overline{z}_2$, $\overline{w}$ are parameters, which goes by differentiating:

$$w_{z_1} = \frac{2\overline{z}_1 + 2z_1\overline{z}_2}{1 - z_2\overline{z}_2}, \qquad\qquad w_{z_1z_1} = \frac{2\,\overline{z}_2}{1 - z_2\overline{z}_2},$$

by solving for $(\overline{z}_1, \overline{z}_2)$, and by replacing, which yields:

$$w_{z_2} = \tfrac{1}{4}\,(w_{z_1})^2, \qquad\qquad w_{z_1z_1z_1} = 0.$$

Secondly, with the help of Lie's prolongation formulas — [25, Chap. 25] or [3, 78] — , by setting up and solving the linear differential system satisfied by the coefficients of a general vector field $X = \xi^1\,\partial_{z_1} + \xi^2\,\partial_{z_2} + \varphi\,\partial_w$, with ξ^1, ξ^2, φ holomorphic functions of (z_1, z_2, w), to be a CR symmetry. Nowadays, such calculations can be done instantly using the DifferentialGeometry package on Maple.

This gave a 10-dimensional real simple Lie algebra:

$$\mathfrak{g} := \mathfrak{aut}_{CR}\big(M_{LC}\big) \cong \mathfrak{so}_{2,3}(\mathbb{R}).$$

Assigning the weights:

$$[z] := 1 \qquad [\zeta] := 0, \qquad [w] := 2 \qquad \big[\partial_z\big] := -1 \qquad \big[\partial_\zeta\big] := 0$$
$$\big[\partial_w\big] := -2,$$

this real Lie algebra $\mathfrak{g}$ of holomorphic vector fields can be graded as [13, 34, 37]:

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \big(\mathfrak{g}_0^{\mathrm{trans}} \oplus \mathfrak{g}_0^{\mathrm{iso}}\big) \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2,$$

where:

$$\begin{aligned}
\mathfrak{g}_{-2} &:= \mathrm{Span}\,\big\{i\,\partial_w\big\},\\
\mathfrak{g}_{-1} &:= \mathrm{Span}\,\big\{(\zeta - 1)\,\partial_z - 2z\,\partial_w, \quad (i + i\zeta)\,\partial_z - 2iz\,\partial_w\big\},\\
\mathfrak{g}_0^{\mathrm{trans}} &:= \mathrm{Span}\,\Big\{z\zeta\,\partial_z + (\zeta^2 - 1)\,\partial_\zeta - z^2\,\partial_w, \quad iz\zeta\,\partial_z + (i + i\zeta^2)\,\partial_\zeta - iz^2\,\partial_w\Big\},\\
\mathfrak{g}_0^{\mathrm{iso}} &:= \mathrm{Span}\,\big\{z\,\partial_z + 2w\,\partial_w, \quad iz\,\partial_z + 2i\zeta\,\partial_\zeta\big\},\\
\mathfrak{g}_1 &:= \mathrm{Span}\,\big\{\big(z^2 - \zeta w - w\big)\,\partial_z + \big(2z\zeta + 2z\big)\,\partial_\zeta + 2zw\,\partial_w,\\
&\qquad \big(-iz^2 + i\zeta w - iw\big)\,\partial_z + \big(-2iz\zeta + 2iz\big)\,\partial_\zeta - 2izw\,\partial_w\big\},\\
\mathfrak{g}_2 &:= \mathrm{Span}\,\big\{izw\,\partial_z - iz^2\,\partial_\zeta + iw^2\,\partial_w\big\}.
\end{aligned}$$

In the breakthrough [28], Fels-Kaup developed a Lie-theoretical method for the computation of the Lie algebra $\mathfrak{hol}(M)$ of infinitesimal holomorphic automorphisms of any $M^5 \in \mathfrak{C}_{2,1}$. Mainly, they classified, up to local CR-equivalence, *all* locally homogeneous $M^5 \in \mathfrak{C}_{2,1}$.

Their starting point was the following simple observation. Suppose that $S^2 \subset \mathbb{R}^3$ is a surface which is homogeneous under the group $A_3(\mathbb{R})$ of affine transformations of $\mathbb{R}^3$. Then the tube $M^5 := S^2 \times i\mathbb{R}^3$ in $\mathbb{C}^3$ is clearly homogeneous under a group of complex-affine transformations, since every real-affine transformation leaving S^2 invariant extends to a complex-affine transformation leaving M^5 invariant and since M^5 is invariant under all translations along the three imaginary axes.

It is elementary to verify that such a tube M^5 does belong to $\mathfrak{C}_{2,1}$ if and only if S^2 is a *parabolic* surface, namely a surface whose Hessian is everywhere of rank 1. For $S^2 \subset \mathbb{R}^3$ locally graphed as $u = F(x, y)$, parabolicity expresses as:

$$F_{xx} \neq 0 \equiv \begin{vmatrix} F_{xx} & F_{xy} \\ F_{yx} & F_{yy} \end{vmatrix}. \tag{6}$$

Recently, Chen-Merker [11] studied the algebras of differential invariants of parabolic surfaces $S^2 \subset \mathbb{R}^3$ under the group $SA_3(\mathbb{R})$ of *special* affine transformations. Mild adaptations yield descriptions of such algebras valid for the *full* affine group, which we will present and use below.

Then Fels-Kaup raised and settled the crucial question whether $A_3(\mathbb{R})$-inequivalent surfaces $S^2 \not\cong S'^2$ always conduct to CR-inequivalent tubes $S^2 \times i\mathbb{R}^3 \not\cong^? S'^2 \times \mathbb{R}^3$? As we saw above, the same question existed about curves $C^1 \not\equiv C'^1$ in $\mathbb{R}^2$ under $A_2(\mathbb{R})$ and associated tubes $C^1 \times i\mathbb{R}^2 \not\cong^? C'^1 \times i\mathbb{R}^2$. As such, it was implicitly settled by Cartan [7, 10], and settled again by Loboda [54], who proved using Moser's method that all *nonspherical* Levi nondegenerate hypersurfaces over the homogeneous curves (1'), (2'), (3'), (4') are pairwise holomorphically inequivalent. Fels-Kaup did the same job about surfaces, as we review now.

To begin with, recall that the complete classification of $A_3(\mathbb{R})$-homogeneous surfaces $S^2 \subset \mathbb{R}^3$ was terminated by Doubrov-Komrakov-Rabinovich [19] after that Abdalla-Dillen-Vrancken [1] finished the delicate classification of affinely homogeneous surfaces in $\mathbb{R}^3$ having *vanishing Pick invariant*. The full classification, re-done by Eastwood-Ezhov in [23] who employed the power series method, includes the classification of $A_3(\mathbb{R})$-homogeneous *parabolic* surfaces, which can be presented as follows.

(1) $\{x_1^2 + x_2^2 = x_3^2,\ x_3 > 0\}$ the future light cone, having infinitesimal symmetries $x_1\partial_{x_1} + x_2\partial_{x_2} + x_3\partial_{x_3}, -x_2\partial_{x_1} + x_1\partial_{x_2}$;

(2a) $\{r(\cos t, \sin t, e^{\omega t}) \in \mathbb{R}^3 : r \in \mathbb{R}^+ \text{ and } t \in \mathbb{R}\}$ with $\omega > 0$ arbitrary, graphed as $u = \sqrt{x^2 + y^2}\, e^{\omega \arctan \frac{y}{x}}$, having symmetries $x\partial_x + y\partial_y + u\partial_u, -y\partial_x + x\partial_y + \omega u\partial_u$;

(2b) $\{r(1, t, e^t) \in \mathbb{R}^3 : r \in \mathbb{R}^+ \text{ and } t \in \mathbb{R}\}$, graphed as $u = xe^{\frac{y}{x}}$, having symmetries $x\partial_x + y\partial_y + u\partial_u, x\partial_y + u\partial_u$;

(2c) $\{r(1, e^t, e^{\theta t}) \in \mathbb{R}^3 : r \in \mathbb{R}^+ \text{ and } t \in \mathbb{R}\}$ with $\theta > 2$ arbitrary, graphed as $u = x\left(\frac{y}{x}\right)^\theta$, having symmetries $x\partial_x - (\theta - 1)u\partial_u, y\partial_y + \theta u\partial_u$;

(3) $\{c(t) + rc'(t) \in \mathbb{R}^3 : r \in \mathbb{R}^+ \text{ and } t \in \mathbb{R}\}$, where $c(t) := (t, t^2, t^3)$ parametrizes the *twisted cubic* $\{(t, t^2, t^3) : t \in \mathbb{R}\}$ in $\mathbb{R}^3$ and $c'(t) = (1, 2t, 3t^2)$, graphed

as $u = -2x^3 + 3xy - 2(x^2 - y)^{3/2}$, having symmetries $x\partial_x + 2y\partial_y + 3u\partial_u$, $\partial_x + 2x\partial_y + 3y\partial_u$.

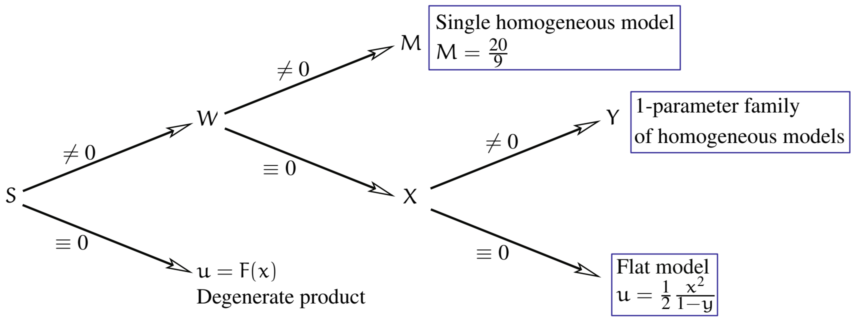
Fels-Kaup [28] established their Theorem I: *For every surface* S^2 *in* (1), (2a), (2b), (2c), (3), *the corresponding tube manifold* $M^5 := S^2 \times i\mathbb{R}^3$ *is a homogeneous* $\mathfrak{C}_{2,1}$ *hypersurface of* $\mathbb{C}^3$, *and any two of them are pairwise locally biholomorphically inequivalent*. Furthermore, they proved that *for every* S^2 *in* (2a), (2b), (2c), (3), *at every point* $p \in S^2 \times i\mathbb{R}^3$, *the isotropy subalgebra* $\mathfrak{hol}(M, p) = \{0\}$ *is trivial, and the full Lie algebra* $\mathfrak{hol}(M)$ *is* 5*-dimensional and solvable*. Fels-Kaup actually proved a version of these results in any $\mathbb{R}^{n \geqslant 3}$.

The hypersurfaces $S^2 \times i\mathbb{R}^3$ occurring here are quite special as they all are tube manifolds. From Cartan's list, in the Levi nondegenerate case, many (homogeneous) examples are known which are not locally CR-equivalent to any tube manifold, namely (K), (K'), (L). Therefore, the second main Theorem II of Fels-Kaup came as strikingly unexpected: *Every* 5*-dimensional locally homogeneous* 2*-nondegenerate hypersurface* $M^5 \subset \mathbb{C}^3$ *is locally biholomorphic to* $S^2 \times i\mathbb{R}^3$, *with* $S^2 \subset \mathbb{R}^3$ *being one of the parabolic surfaces* (1), (2a), (2b), (2c), (3).

Joint with Cartan's list, Fels-Kaup therefore deduced a full classification of *all* Levi degenerate homogeneous CR-manifolds of dimension $\leqslant 5$:

(i) $M^5 = S^2 \times i\mathbb{R}^3 \subset \mathbb{C}^3$, where $S^2 \subset \mathbb{R}^3$ is one of the surfaces (1), (2a), (2b), (2c), (3);
(ii) $M^5 = M^3 \times \mathbb{C}$, where M^3 is one of the 3-dimensional Levi nondegenerate homogeneous CR-manifolds from Cartan's list;
(iii) $M^5 = \mathbb{C}^2 \times \mathbb{R}$.

Now, an alternative view on the classification under $A_3(\mathbb{R})$ of homogeneous parabolic surfaces $\{u = F(x, y)\}$ in $\mathbb{R}^3$ satisfying (6) can be presented, by slightly modifying the results of Chen-Merker in [11] which concerned $SA_3(\mathbb{R})$. We use their notations.



The relative differential invariant $S \equiv 0$ vanishes identically if and only if the surface $S^2 \cong C^1 \times \mathbb{R}_y$ is affinely equivalent to the product of a curve $C^1 = \{u = F(x)\}$ in $\mathbb{R}^2_{x,u}$ times $\mathbb{R}_y$. In this degenerate case, the classification of curves under $A_2(\mathbb{R})$ has already been reviewed *supra*.

Assuming $S \neq 0$, the differential invariant W of [11] becomes a *relative* differential invariant. In the upper branch $W \neq 0$, one normalizes $W := 1$, and the power series of $u = F(x, y)$ can be shown to be normalizable to:

$$u = \frac{x^2}{2} + \frac{x^2y}{2} + 1\cdot\frac{x^3y}{6} + \frac{x^2y^2}{2} + M\,\frac{x^5}{5!} + 6\,\frac{x^3y^2}{3!2!} + \frac{x^2y^3}{2} + O_{x,y}(6)\mathcal{J}$$

here, the explicit expression of M in terms of $J^5_{x,y}u$ is a true differential invariant, whose numerator has 57 differential monomials.

Next, applying the Fels-Olver [29] recurrence formulas in this context, denoting $I_{6,0}$, $I_{5,1}$, $I_{7,0}$, $I_{6,1}$ the invariantized jets $\mathrm{inv}(u_{x^6})$, $\mathrm{inv}(u_{x^5y})$, $\mathrm{inv}(u_{x^7})$, $\mathrm{inv}(u_{x^6y})$, with of course $M := \mathrm{inv}(u_{x^5})$, one receives:

$$\begin{aligned}
D_1M &= I_{6,0} - 12\,M + \frac{10}{3}\,I_{5,1},\\
D_2M &= I_{5,1} - 7\,M + \tfrac{80}{9},\\
D_1I_{6,0} &= I_{7,0} + 4\,I_{6,0} - 42\,M^2 + \frac{45}{2}\,M\,I_{5,1} - 3\,I_{5,1}^2,\\
D_2I_{6,0} &= I_{6,1} - 9\,I_{6,0} + 21\,M - 8\,I_{5,1},\\
D_1I_{5,1} &= I_{6,1} + \frac{41}{3}\,I_{5,1} - I_{6,0} - 60\,M,\\
D_2I_{5,1} &= -40\,M + 12\,I_{5,1} + \tfrac{280}{9}.
\end{aligned}$$

These relations and the higher order ones show that the algebra of differential invariants in this branch is generated by M and all its invariant derivatives $D_1^{\nu_1}D_2^{\nu_2}M$.

Furthermore, when searching for $SA_3(\mathbb{R})$-homogeneous surfaces $\{u = F(x, y)\}$, the differential invariants are all by themselves constant, hence all left-hand sides vanish. Strikingly, these 6 equations force M to have only one specific value:

$$M := \tfrac{20}{9}, \qquad I_{6,0} := \tfrac{40}{9}, \qquad I_{5,1} := \tfrac{20}{3}, \qquad I_{7,0} := -\tfrac{280}{9}, \qquad I_{6,1} := \tfrac{140}{3},$$

and so on for all unwritten recurrence relations.

Lastly, from a Taylor expansion up to any finite order, by testing whether a general infinitesimal symmetry in $\mathfrak{sa}_3(\mathbb{R})$ is tangent to $\{u = F(x, y)\}$, one realizes that for $M = \frac{20}{9}$, one indeed obtains a *single* homogeneous model, equivalent to the developable surface generated by the twisted cubic (3) from Fels-Kaup's list above.

The other branch $W \equiv 0$ creates, under $SA_3(\mathbb{R})$, two differential invariants, denoted X and Y in [11]. For the full affine group $A_3(\mathbb{R})$, one degree of freedom is added, X becomes relative and when it is nonzero, it can be normalized to be 1, while, when $X \equiv 0 \equiv W$, it is easy to show that one comes to the flat model $u = \frac{1}{2}\frac{x^2}{1-y}$. Denoting $Y := I_{7,0} = \mathrm{inv}(u_{x^7})$, the recurrence relations are:

$$\begin{aligned}
D_1I_{7,0} &= I_{8,0} - \tfrac{35}{2}, & D_2I_{7,0} &= I_{7,1} - 6\,Y,\\
D_1I_{8,0} &= I_{9,0} - 4\,Y^2, & D_2I_{8,0} &= I_{8,1} - 7\,I_{8,0},
\end{aligned}$$

and so on, hence Y and its invariant derivatives $D_1^{\nu_1}D_2^{\nu_2}Y$ are generators.

In search for homogeneous models, denoting instead $a := Y$ which must be constant, one receives:

$$u = x^2 + \frac{x^2y}{2} + \frac{x^2y^2}{2} + \frac{x^5}{120} + \frac{x^2y^3}{2} + 4\,\frac{x^5y}{120} + \frac{x^2y^4}{2} + \mathsf{a}\,\frac{x^7}{5040} + \frac{x^2y^5}{2} + 20\,\frac{x^5y^2}{240}$$
$$+ I_{8,0}\,\frac{x^8}{8W} + 6\mathsf{a}\,\frac{x^7y}{7W} + 120\,\frac{x^5y^3}{5W3W} + 720\,\frac{x^2y^6}{2W6W} + O(9).$$

It is easy to see that this gives a 1-parameter family of homogeneous models, parametrized by any $\mathsf{a} \in \mathbb{R}$, which 'unifies' (2a), (2b), (2c) above. Thus, without trying to find closed forms for $u = F(x, y)$, the classification 'simplifies'.

This 'simplification', already discussed for the classification of curves $C^1 \subset \mathbb{R}^2$ will come again later in our results on degenerate para-CR structures.

7 Explicit Reduction to $\{e\}$-Structure for $M^5 \in \mathfrak{C}_{2,1}$

It was only five years after Fels-Kaup that three papers [45, 60, 80] achieved the constructions of 10-dimensional $\{e\}$-structure bundles (or Cartan connections) $P^{10} \longrightarrow M^5$. We only review Pocchiola's results [71, 80], following [31].

Consider therefore $M^5 \subset \mathbb{C}^3$ belonging to $\mathfrak{C}_{2,1}$, graphed as:

$$u = F\big(z_1, z_2, \overline{z}_1, \overline{z}_2, v\big).$$

Two generators of $T^{1,0}M$ and $T^{0,1}M$ are:

$$\mathcal{L}_k := \frac{\partial}{\partial z_1} + A^k \frac{\partial}{\partial v}, \qquad\qquad A^k := \frac{-i\,F_{z_k}}{1 + i\,F_v} \qquad (k = 1, 2).$$

he real 1-form $\rho_0 := dv - A^1\,dz_1 - A^2\,dz_2 - \overline{A}^1\,d\overline{z}_1 - \overline{A}^2\,d\overline{z}_2$ has kernel $\{\rho_0 = 0\} = T^{1,0}M \oplus T^{0,1}M$. The hypothesis that M has everywhere degenerate Levi form writes as:

$$0 \equiv = \begin{vmatrix} \rho_0\big(i\,[\mathcal{L}_1, \overline{\mathcal{L}}_1]\big) & \rho_0\big(i\,[\mathcal{L}_2, \overline{\mathcal{L}}_1]\big) \\ \rho_0\big(i\,[\mathcal{L}_1, \overline{\mathcal{L}}_2]\big) & \rho_0\big(i\,[\mathcal{L}_2, \overline{\mathcal{L}}_2]\big) \end{vmatrix}.$$

The hypothesis that the Levi form has constant rank equal to 1 reads as saying that the field:

$$\mathcal{T} := i\,\big[\mathcal{L}_1, \overline{\mathcal{L}}_1\big] = i\Big(\mathcal{L}_1\big(\overline{A}^1\big) - \overline{\mathcal{L}}_1\big(A^1\big)\Big)\frac{\partial}{\partial v} =: \ell\,\frac{\partial}{\partial v},$$

satisfies $\ell \neq 0$ everywhere. The Levi kernel subbundle $K^{1,0}M \subset T^{1,0}M$ has generator:

$$\mathcal{K} := k\,\mathcal{L}_1 + \mathcal{L}_2,$$

with slant function:

$$k := -\frac{\mathcal{L}_2\left(\overline{A}^1\right)-\overline{\mathcal{L}}_1\left(A^2\right)}{\mathcal{L}_1\left(\overline{A}^1\right)-\overline{\mathcal{L}}_1\left(A^1\right)}.$$

According to [71, 80], the hypothesis of 2-nondegeneracy states as:

$$0 \neq \overline{\mathcal{L}}_1(k).$$

There is a second fundamental function:

$$P := \frac{\ell_{z_1} + A^1\,\ell_v - \ell\,A^1_v}{\ell}.$$

For now, introduce the five 1-forms:

$$\begin{aligned}
\rho_0 &= \frac{dv - A^1 dz_1 - A^2 dz_2 - \overline{A}^1 d\overline{z}_1 - \overline{A}^2 d\overline{z}_2}{\ell}, \\
\kappa_0 &= dz_1 - k\, dz_2, \\
\zeta_0 &= dz_2, \\
\overline{\kappa}_0 &= d\overline{z}_1 - \overline{k}\, d\overline{z}_2, \\
\overline{\zeta}_0 &= d\overline{z}_2,
\end{aligned}$$

Pocchiola obtained modifications $\left\{\rho, \kappa, \zeta, \overline{\kappa}, \overline{\zeta}\right\}$ of these 1-forms $\left\{\rho_0, \kappa_0, \zeta_0, \overline{\kappa}_0, \overline{\zeta}_0\right\}$, together with four 1-forms π^1, π^2, $\overline{\pi}^1$, $\overline{\pi}^2$ which satisfy structure equations of the form:

$$\begin{aligned}
d\rho &= \left(\pi^1 + \overline{\pi}^1\right) \wedge \rho + i\,\kappa \wedge \overline{\kappa}, \\
d\kappa &= \pi^2 \wedge \rho + \pi^1 \wedge \kappa + \zeta \wedge \overline{\kappa}, \\
d\zeta &= \left(\pi^1 - \overline{\pi}^1\right) \wedge \zeta + i\,\pi^2 \wedge \kappa + \\
&\quad + R\,\rho \wedge \zeta + i\,\frac{1}{\overline{c}^3}\,\overline{J}_0\,\rho \wedge \overline{\kappa} + \frac{1}{c}\,W_0\,\kappa \wedge \zeta.
\end{aligned} \tag{7}$$

Here, there are four remaining group parameters c, e, $\overline{c}$, $\overline{e}$, and R is a secondary invariant:

$$R := \operatorname{Re}\left[i\,\tfrac{e}{c\overline{c}}\,W_0 + \tfrac{1}{c\overline{c}}\Big(-\tfrac{i}{2}\,\overline{\mathcal{L}}_1\big(W_0\big) + \tfrac{i}{2}\Big(-\tfrac{1}{3}\,\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)}{\overline{\mathcal{L}}_1(k)} + \tfrac{1}{3}\,\overline{P}\Big)\,W_0\Big)\right],$$

expressed in terms of Pocchiola's two primary relative invariants:

$$
\begin{aligned}
W_0 :=& -\tfrac{1}{3}\,\frac{\mathcal{K}\big(\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)\big)}{\overline{\mathcal{L}}_1(k)^2} + \tfrac{1}{3}\,\frac{\mathcal{K}\big(\overline{\mathcal{L}}_1(k)\big)\,\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)}{\overline{\mathcal{L}}_1(k)^3} + \\
& + \tfrac{2}{3}\,\frac{\mathcal{L}_1\big(\mathcal{L}_1(\overline{k})\big)}{\mathcal{L}_1(\overline{k})} + \tfrac{2}{3}\,\frac{\mathcal{L}_1\big(\overline{\mathcal{L}}_1(k)\big)}{\overline{\mathcal{L}}_1(k)} + \tfrac{i}{3}\,\frac{\mathcal{T}(k)}{\overline{\mathcal{L}}_1(k)}, \\
\overline{J}_0 :=& \tfrac{1}{6}\,\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)\big)\big)}{\overline{\mathcal{L}}_1(k)} - \tfrac{5}{6}\,\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)\big)\,\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)}{\overline{\mathcal{L}}_1(k)^2} - \tfrac{1}{6}\,\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)\big)}{\overline{\mathcal{L}}_1(k)}\,\overline{P} + \\
& + \tfrac{20}{27}\,\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)^3}{\overline{\mathcal{L}}_1(k)^3} + \tfrac{5}{18}\,\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)^2}{\overline{\mathcal{L}}_1(k)^2}\,\overline{P} + \tfrac{1}{6}\,\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)\,\overline{\mathcal{L}}_1\big(\overline{P}\big)}{\overline{\mathcal{L}}_1(k)} - \tfrac{1}{9}\,\frac{\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1(k)\big)}{\overline{\mathcal{L}}_1(k)}\,\overline{P}\,\overline{P} - \\
& - \tfrac{1}{6}\,\overline{\mathcal{L}}_1\big(\overline{\mathcal{L}}_1\big(\overline{P}\big)\big) + \tfrac{1}{3}\,\overline{\mathcal{L}}_1\big(\overline{P}\big)\,\overline{P} - \tfrac{2}{27}\,\overline{P}\,\overline{P}\,\overline{P}.
\end{aligned}
$$

The full $\{e\}$-structure obtained by Foo-Merker in [31] for nonrigid $M^5 \subset \mathbb{C}^3$ shows that a unique prolongation of G-structure is needed, introducing one further parameter $t \in \mathbb{R}$, together with a real modified Maurer-Cartan form $\Lambda = dt + \cdots$ and that all appearing torsion coefficients are *secondary invariants*.

8 Degenerate Para-CR Structures and Their Homogeneous Models

What precedes motivates the kinds of PDE systems studied in this article and in [69, 70], so let us summarize foundational considerations on such PDE systems, taken from the detailed elementary presentation [65].

Given a $\mathcal{C}^\omega$ real hypersurface $M^5 \subset \mathbb{C}^3$ of complex-graphed equation $w = \Theta(z_1, z_2, \overline{z}_1, \overline{z}_2, \overline{w})$ obtained by solving for w a real implicit equation $\rho(z_1, z_2, w, \overline{z}_1, \overline{z}_2, \overline{w}) = 0$, one can forget about complex conjugation, work over the fields $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, and consider instead, in coordinates (x, y, z, a, b, c) a so-called *submanifold of solutions* $\mathcal{M} \subset \mathbb{K}^{2+1}_{x,y,z} \times \mathbb{K}^{2+1}_{a,b,c}$ having implicit equation $\rho(x, y, z, a, b, c) = 0$, with symmetrically $d_{x,y,z}\rho \neq 0 \neq d_{a,b,c}\rho$.

One may therefore assume $\rho_z \neq 0 \neq \rho_c$, solve for z and for c by means of the implicit function theorem, and get two equivalent *graphed* equations:

$$
z = Q(x, y, a, b, c) \qquad \& \qquad c = P(a, b, x, y, z).
$$

In view of the intimate relationship with PDE systems, one may think that (x, y, z) are the *variables*, while (a, b, c) are the *parameters*. Two functional relations must be identically satisfied:

$$
z \equiv Q\big(x, y, a, b,\, P(a, b, x, y, z)\big) \qquad \& \qquad P\big(a, b, x, y,\, Q(x, y, a, b, c)\big) \equiv c. \tag{8}
$$

with $Q_c \neq 0 \neq P_z$ by hypothesis, and in fact $Q_c = \frac{1}{P_z}$. Two sets of five intrinsic coordinates may hence be considered:

$$
\big(x, y, a, b, c\big) \qquad \& \qquad \big(a, b, x, y, z\big).
$$

The infinite group of biholomorphic transformations of $\mathbb{C}^3$ would yield, by complex conjugation, the group of anti-biholomorphic transformations, in the $(\overline{z}_1, \overline{z}_2, \overline{w})$ variables. Also, as explained in [63], for a large class of completely integrable PDE systems, the natural infinite-dimensional group consists of *split-diffeomorphisms:*

$$\begin{aligned}(x, y, z,\ a, b, c) \longmapsto \big(&f(x,y,z), g(x,y,z), h(x,y,z),\ \varphi(a,b,c), \psi(a,b,c), \\ &\chi(a,b,c)\big) \\ =: \big(&x', y', z',\ a', b', c'\big),\end{aligned} \tag{9}$$

which are pairs of *uncoupled* diffeomorphisms both in the variables space and in the parameters space.

Through these transformations, both 2-dimensional foliations $\{a = a_0, b = b_0, c = c_0\}$ and $\{x = x_0, y = y_0, z = z_0\}$ are invariant. Their intersections with $\mathcal{M} = \{z = Q\} = \{c = P\}$ are spanned by *two pairs* of vector fields, firstly in coordinates (x, y, a, b, c):

$$\mathcal{L}_a := \frac{\partial}{\partial a} - \frac{Q_a}{Q_c}(x,y,a,b,c)\,\frac{\partial}{\partial c}, \qquad \mathcal{K}_x := \frac{\partial}{\partial x},$$
$$\mathcal{L}_b := \frac{\partial}{\partial b} - \frac{Q_b}{Q_c}(x,y,a,b,c)\,\frac{\partial}{\partial c}, \qquad \mathcal{K}_y := \frac{\partial}{\partial y},$$

and secondly in coordinates (a, b, x, y, z):

$$\mathcal{L}_a := \frac{\partial}{\partial a}, \qquad \mathcal{K}_x := \frac{\partial}{\partial x} - \frac{P_x}{P_z}(a,b,x,y,z)\,\frac{\partial}{\partial z},$$
$$\mathcal{L}_b := \frac{\partial}{\partial b}, \qquad \mathcal{K}_y := \frac{\partial}{\partial y} - \frac{P_y}{P_z}(a,b,x,y,z)\,\frac{\partial}{\partial z}.$$

However, in general, their sum:

$$\text{Span}\ \big\{\mathcal{L}_a, \mathcal{L}_b\big\} \oplus \text{Span}\ \big\{\mathcal{K}_x, \mathcal{K}_y\big\}$$

is *not* Frobenius-integrable, as show the four Lie brackets:

$$\big[\mathcal{K}_x, \mathcal{L}_a\big] = \frac{-Q_cQ_{xa} + Q_aQ_{xc}}{Q_cQ_c}\,\frac{\partial}{\partial c}, \qquad \big[\mathcal{K}_x, \mathcal{L}_b\big] = \frac{-Q_cQ_{xb} + Q_bQ_{xc}}{Q_cQ_c}\,\frac{\partial}{\partial c},$$
$$\big[\mathcal{K}_y, \mathcal{L}_a\big] = \frac{-Q_cQ_{ya} + Q_aQ_{yc}}{Q_cQ_c}\,\frac{\partial}{\partial c}, \qquad \big[\mathcal{K}_y, \mathcal{L}_b\big] = \frac{-Q_cQ_{yb} + Q_bQ_{yc}}{Q_cQ_c}\,\frac{\partial}{\partial c},$$

with similar formulas involving P in the other coordinates (a, b, x, y, z). This conducts to introduce *two* Levi forms, firstly with respect to parameters, having invariant matrix:

$$\text{Levi}_{\text{par}}(Q) := \begin{pmatrix} \frac{-Q_cQ_{xa}+Q_aQ_{xc}}{Q_c^2} & \frac{-Q_cQ_{xb}+Q_bQ_{xc}}{Q_c^2} \\ \frac{-Q_cQ_{ya}+Q_aQ_{yc}}{Q_c^2} & \frac{-Q_cQ_{yb}+Q_bQ_{yc}}{Q_c^2} \end{pmatrix},$$

and secondly with respect to variables, having invariant matrix:

$$\mathrm{Levi}_{\mathrm{var}}(P) := \begin{pmatrix} \frac{-P_z P_{ax} + P_x P_{az}}{P_z^2} & \frac{-P_z P_{ay} + P_y P_{az}}{P_z^2} \\ \frac{-P_z P_{bx} + P_x P_{bz}}{P_z^2} & \frac{-P_z P_{by} + P_y P_{bz}}{P_z^2} \end{pmatrix}.$$

Lemma 8.1 [65] *One has:*

$$\mathrm{Levi}_{\mathrm{par}}(Q) = -P_y \,{}^T\mathrm{Levi}_{\mathrm{var}}(P) \qquad \Longleftrightarrow \qquad -Q_c \,{}^T\mathrm{Levi}_{\mathrm{par}}(Q) = \mathrm{Levi}_{\mathrm{var}}(P).$$

Hint of proof Differentiate (8) up to order 2, perform suitable eliminations, and obtain for $1 \leqslant i, j \leqslant 2$, with $(a_1, a_2) := (a, b)$ and $(x_1, x_2) := (x, y)$:

$$\frac{-Q_c Q_{x_i a_j} + Q_{a_j} Q_{x_i c}}{Q_c\, Q_c} = -P_z \left(\frac{-P_z P_{x_i a_j} + P_{x_i} P_{a_j z}}{P_z\, P_z} \right).$$

□

As a corollary:

$$\mathrm{rank}\,\mathrm{Levi}_{\mathrm{par}}(Q) = \mathrm{rank}\,\mathrm{Levi}_{\mathrm{var}}(P).$$

So one can speak of *Levi nondegenerate*, or of *constant Levi rank* 1, submanifolds of solutions.

As already seen in Sect. 5, from the three equations:

$$z = Q, \qquad z_x = Q_x, \qquad z_y = Q_y,$$

one can solve the parameters (a, b, c) precisely when the Jacobian matrix is invertible:

$$0 \neq \begin{vmatrix} Q_a & Q_b & Q_c \\ Q_{xa} & Q_{xb} & Q_{xc} \\ Q_{ya} & Q_{yb} & Q_{yc} \end{vmatrix} = \det \mathrm{Levi}_{\mathrm{par}}(Q).$$

But when the Levi matrix is constantly of rank 1 [our current concern], one must examine '*higher order Levi forms*', for instance by differentiating up to order 3, which conducts to:

$$\mathrm{Freeman}_{\mathrm{par}}(Q) := \begin{vmatrix} Q_a & Q_b & Q_c \\ Q_{xa} & Q_{xb} & Q_{xc} \\ Q_{xxa} & Q_{xxb} & Q_{xxc} \end{vmatrix}, \qquad \mathrm{Freeman}_{\mathrm{var}}(P) := \begin{vmatrix} P_x & P_y & P_z \\ P_{ax} & P_{ay} & P_{az} \\ P_{aax} & P_{aay} & P_{aaz} \end{vmatrix}.$$

Indeed, under the assumption of constant Levi rank 1, and more precisely, under the following assumptions which can be met after a permutation of coordinates:

$$\begin{vmatrix} Q_a & Q_c \\ Q_{xa} & Q_{xc} \end{vmatrix} \neq 0 \equiv \begin{vmatrix} Q_a & Q_b & Q_c \\ Q_{xa} & Q_{xb} & Q_{xc} \\ Q_{ya} & Q_{yb} & Q_{yc} \end{vmatrix},$$

it can be verified as in [35, Prp. 2.2] that through a split-diffeomorphism (9), which transforms $\{z = Q(x, y, a, b, c)\}$ into $\{z' = Q'(x', y', a', b', c')\}$, one has:

$$\frac{\begin{vmatrix} Q'_{a'} & Q'_{b'} & Q'_{c'} \\ Q'_{z'a'} & Q'_{z'b'} & Q'_{z'c'} \\ Q'_{z'z'a'} & Q'_{z'z'b'} & Q'_{z'z'c'} \end{vmatrix}}{\begin{vmatrix} Q_a & Q_b & Q_c \\ Q_{za} & Q_{zb} & Q_{zc} \\ Q_{zza} & Q_{zzb} & Q_{zzc} \end{vmatrix}} = \frac{\begin{vmatrix} f_x & f_y & f_z \\ g_x & g_y & g_z \\ h_x & h_y & h_z \end{vmatrix}^3}{\begin{vmatrix} \varphi_a & \varphi_b & \varphi_c \\ \psi_a & \psi_b & \psi_c \\ \chi_a & \chi_b & \chi_c \end{vmatrix}^1} \frac{\left(\mathcal{K}_y(g)\begin{vmatrix} Q_a & Q_c \\ Q_{za} & Q_{zc} \end{vmatrix} - \mathcal{K}_x(g)\begin{vmatrix} Q_a & Q_c \\ Q_{ya} & Q_{yc} \end{vmatrix}\right)^3}{\begin{vmatrix} \mathcal{K}_x(f) & \mathcal{K}_y(f) \\ \mathcal{K}_x(g) & \mathcal{K}_y(g) \end{vmatrix}^6 \begin{vmatrix} Q_a & Q_c \\ Q_{za} & Q_{zc} \end{vmatrix}^3},$$

and this guarantees that the nonvanishing of $\mathrm{Freeman}_{\mathrm{par}}(Q)$ is an *invariant* condition.

Of course, there is a similar formula (by symmetry) satisfied by P which shows that the nonvanishing of $\mathrm{Freeman}_{\mathrm{var}}(P)$ is also invariant. But we would like to mention that such formulas would be *untrue* without the assumption that the Levi determinant vanishes identically.

When $z = Q$ is a real hypersurface $w = \Theta(z_1, z_2, \overline{z}_1, \overline{z}_2, \overline{w})$ in $\mathbb{C}^3$, with:

$$(x, y, z) := (z_1, z_2, w), \qquad\qquad (a, b, c) := (\overline{z}_1, \overline{z}_2, \overline{w}),$$

so that $Q := \Theta$ and $P := \overline{\Theta}$, it is clear that:

$$\mathrm{Freeman}_{\mathrm{var}}(\overline{\Theta}) = \overline{\mathrm{Freeman}_{\mathrm{par}}(\Theta)},$$

so that one determinant is nonzero if and only if the other is.

However, for general submanifolds of solutions, and even contrary to the 'equivalence' between the two Levi determinants expressed by Lemma 8.1, the two Freeman determinants are *totally unrelated*. Indeed, taking for instance:

$$\begin{aligned} z = Q &= c + xa + \beta\, xxb + \gamma\, yaa + O_4(x, y, a, b, c), \\ \iff \quad c = P &= z - ax - \gamma\, aay - \beta\, bxx + O_4(x, y, a, b, c), \end{aligned}$$

with two *uncoupled* [free, independent] constants β, γ, we have at the origin:

$$\mathrm{Freeman}_{\mathrm{par}}(Q)\big|_0 = \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 2\beta & 0 \end{vmatrix} = 2\beta,$$

$$\mathrm{Freeman}_{\mathrm{var}}(P)\big|_0 = \begin{vmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -2\gamma & 0 \end{vmatrix} = 2\gamma.$$

So the much studied concept of 2-nondegeneracy for CR manifolds [13, 27, 28, 31, 34–37, 45, 60, 61, 63, 69, 71, 80], when generalized to para-CR geometry, splits into *two* non-equivalent concepts.

Definition 8.2 [65] A submanifold of solutions $\{z = Q(x, y, a, b, c)\} = \{c = P(a, b, x, y, z)\}$ whose Levi form is everywhere of rank 1 will be called:

- 2-*nondegenerate with respect to parameters* if $0 \neq \begin{vmatrix} Q_a & Q_b & Q_c \\ Q_{xa} & Q_{xb} & Q_{xc} \\ Q_{xxa} & Q_{xxb} & Q_{xxc} \end{vmatrix} =: \Delta(Q)$;
- 2-*nondegenerate with respect to variables* if $0 \neq \begin{vmatrix} P_x & P_y & P_z \\ P_{ax} & P_{ay} & P_{az} \\ P_{aax} & P_{aay} & P_{aaz} \end{vmatrix} =: \square(P)$.

Thus, if we assume constant Levi rank 1 and 2-nondegeneracy with respect to parameters:

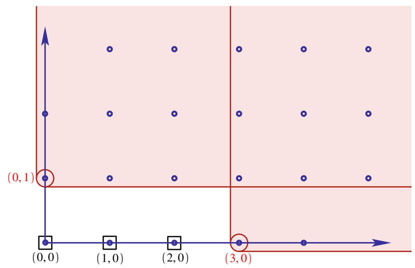
$$\begin{vmatrix} Q_a & Q_c \\ Q_{xa} & Q_{xc} \end{vmatrix} \neq 0 \equiv \begin{vmatrix} Q_a & Q_b & Q_c \\ Q_{xa} & Q_{xb} & Q_{xc} \\ Q_{ya} & Q_{yb} & Q_{yc} \end{vmatrix} \qquad \text{and} \qquad 0 \neq \begin{vmatrix} Q_a & Q_b & Q_c \\ Q_{xa} & Q_{xb} & Q_{xc} \\ Q_{xxa} & Q_{xxb} & Q_{xxc} \end{vmatrix},$$

then quite similarly to what Segre did [86, 87], from the three equations $z = Q$, $z_x = Q_x$, $z_{xx} = Q_{xx}$, we can solve, by means of the implicit function theorem, the three parameters (a, b, c), namely:

$$\left[\begin{aligned} z &= Q(x, y, a, b, c), \\ z_x &= Q_x(x, y, a, b, c), \\ z_y &= Q_y(x, y, a, b, c), \end{aligned}\right. \qquad \Longleftrightarrow \qquad \left[\begin{aligned} a &= A(x, y, z, z_x, z_{xx}), \\ b &= B(x, y, z, z_x, z_{xx}), \\ c &= C(x, y, z, z_x, z_{xx}), \end{aligned}\right.$$

and replace in other derivatives, so that we obtain a completely integrable system of two PDEs:

$$z_y = F(x, y, z, z_x, z_{xx}) \quad \& \quad z_{xxx} = H(x, y, z, z_x, z_{xx}). \tag{10}$$



The transfer of derivations:

$$
\begin{aligned}
\frac{\partial}{\partial z} &= A_z \frac{\partial}{\partial a} + B_z \frac{\partial}{\partial b} + C_z \frac{\partial}{\partial c}, \\
\frac{\partial}{\partial z_x} &= A_{z_x} \frac{\partial}{\partial a} + B_{z_x} \frac{\partial}{\partial b} + C_{z_x} \frac{\partial}{\partial c}, \\
\frac{\partial}{\partial z_{xx}} &= A_{z_{xx}} \frac{\partial}{\partial a} + B_{z_{xx}} \frac{\partial}{\partial b} + C_{z_{xx}} \frac{\partial}{\partial c},
\end{aligned}
$$

becomes after some elimination work [65]:

$$
\begin{aligned}
\frac{\partial}{\partial z} &= \frac{\begin{vmatrix} Q_{xb} & Q_{xxb} \\ Q_{xc} & Q_{xxc} \end{vmatrix}}{\Delta(Q)} \frac{\partial}{\partial a} - \frac{\begin{vmatrix} Q_{xa} & Q_{xxa} \\ Q_{xc} & Q_{xxc} \end{vmatrix}}{\Delta(Q)} \frac{\partial}{\partial b} + \frac{\begin{vmatrix} Q_{xa} & Q_{xxa} \\ Q_{xb} & Q_{xxb} \end{vmatrix}}{\Delta(Q)} \frac{\partial}{\partial c}, \\
\frac{\partial}{\partial z_x} &= -\frac{\begin{vmatrix} Q_b & Q_{xxb} \\ Q_c & Q_{xxc} \end{vmatrix}}{\Delta(Q)} \frac{\partial}{\partial a} + \frac{\begin{vmatrix} Q_a & Q_{xxa} \\ Q_c & Q_{xxc} \end{vmatrix}}{\Delta(Q)} \frac{\partial}{\partial b} - \frac{\begin{vmatrix} Q_a & Q_{xxa} \\ Q_b & Q_{xxb} \end{vmatrix}}{\Delta(Q)} \frac{\partial}{\partial c}, \\
\frac{\partial}{\partial z_{xx}} &= \frac{\begin{vmatrix} Q_b & Q_{xb} \\ Q_c & Q_{xc} \end{vmatrix}}{\Delta(Q)} \frac{\partial}{\partial a} - \frac{\begin{vmatrix} Q_a & Q_{xa} \\ Q_c & Q_{xc} \end{vmatrix}}{\Delta(Q)} \frac{\partial}{\partial b} + \frac{\begin{vmatrix} Q_a & Q_{xa} \\ Q_b & Q_{xb} \end{vmatrix}}{\Delta(Q)} \frac{\partial}{\partial c}.
\end{aligned}
$$

Lemma 8.3 *If the submanifold of solutions $z = Q$ has degenerate Levi form of constant rank 1 and if it is 2-nondegenerate with respect to parameters, then in its associated PDE system $z_y = F$, $z_{xxx} = H$, the function F is independent of z_{xx}:*

$$
0 \equiv F_{z_{xx}}.
$$

Proof. By construction:

$$
\begin{aligned}
F\big(x, y, z, z_x, z_{xx}\big) \;:=\; Q_y\Big(& x, y,\, A\big(x, y, z, z_x, z_{xx}\big),\, B\big(x, y, z, z_x, z_{xx}\big), \\
& C\big(x, y, z, z_x, z_{xx}\big)\Big),
\end{aligned}
$$

whence a differentiation with respect to z_{xx} make re-appear the Levi determinant:

$$
\begin{aligned}
F_{z_{xx}} &= A_{z_{xx}}\, Q_{ya} + B_{z_{xx}}\, Q_{yb} + C_{z_{xx}}\, Q_{yc} \\
&= \frac{\begin{vmatrix} Q_b & Q_{xb} \\ Q_c & Q_{xc} \end{vmatrix}}{\Delta(Q)} Q_{ya} - \frac{\begin{vmatrix} Q_a & Q_{xa} \\ Q_c & Q_{xc} \end{vmatrix}}{\Delta(Q)} Q_{yb} + \frac{\begin{vmatrix} Q_a & Q_{xa} \\ Q_b & Q_{xb} \end{vmatrix}}{\Delta(Q)} Q_{yc} \\
&= \frac{1}{\Delta(Q)} \begin{vmatrix} Q_a & Q_b & Q_c \\ Q_{xa} & Q_{xb} & Q_{xc} \\ Q_{ya} & Q_{yb} & Q_{yc} \end{vmatrix} \\
&\equiv 0.
\end{aligned}
$$

□

So we do assume that F is independent of z_{xx}. After a similar work, one gets

Proposition 8.4 [65] *The submanifold of solutions* $\{z = Q\}$ *is* 2*-nondegenerate with respect to* variables *if and only if:*

$$0 \neq F_{z_x z_x}. \qquad \square$$

Because it corresponds (exercise) to trivial products $\{z = Q(x, a, c)\} \times \mathbb{K}^1_y \times \mathbb{K}^1_b$, the degenerate branch $F_{z_x z_x} \equiv 0$ will not be studied, and we will constantly assume:

$$F_{z_{xx}} \equiv 0 \neq F_{z_x z_x}.$$

The graphed model (5) obtained in [37], rewritten $z+c = \frac{2xa+x^2b+a^2y}{1-yb}$, conducts, as we already saw, to the model PDE system:

$$z_y = \tfrac{1}{4}\,(z_x)^2 \quad \& \quad z_{xxx} = 0.$$

Introducing the two total differentiation operators pulled-back to the PDE system:

$$D := \partial_x + p\,\partial_z + r\,\partial_p + H\,\partial_r \quad \& \quad \Delta := \partial_y + F\,\partial_z + DF\,\partial_p + D^2F\,\partial_r,$$

the complete integrability expresses as $D^3F = \Delta H$, and guarantees [63, § 1] that the general solution is of the form $Q(x, y, a, b, c)$.

Forgetting about submanifolds of solutions, working now over $\mathbb{K} = \mathbb{R}$, we launch Cartan's method by defining a 2-nondegenerate para-CR structure on a real 5-manifold $M \ni (x, y, z, p, r)$ associated with the above two PDEs (10) as an equivalence class of 1-forms modulo point equivalences in terms of an *initial* coframe of (contact) 1-forms, together with *lifted* 1*-forms*, 'rotated' by an initial G-structure:

$$\begin{aligned}
\omega^1 &:= dz - p dx - F dy,\\
\omega^2 &:= dp - r dx - DF dy,\\
\omega^3 &:= dr - H dx - D^2F dy,\\
\omega^4 &:= dx,\\
\omega^5 &:= dy,
\end{aligned}
\qquad
\begin{pmatrix}\theta^1\\ \theta^2\\ \theta^3\\ \theta^4\\ \theta^5\end{pmatrix} := \begin{pmatrix} f^1 & 0 & 0 & 0 & 0\\ f^2 & \rho e^{\phi} & f^4 & 0 & 0\\ f^5 & f^6 & f^7 & 0 & 0\\ \bar f^2 & 0 & 0 & \rho e^{-\phi} & \bar f^4\\ \bar f^5 & 0 & 0 & \bar f^6 & \bar f^7\end{pmatrix} \begin{pmatrix}\omega^1\\ \omega^2\\ \omega^3\\ \omega^4\\ \omega^5\end{pmatrix}.$$

Similarly to the CR case ([31, 71, 80]), we perform several torsion normalizations, which lead us to change the initial coframe on M into:

$$\begin{pmatrix}\omega^1\\ \omega^2\\ \omega^3\\ \omega^4\\ \omega^5\end{pmatrix} \longmapsto \begin{pmatrix} -1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ \frac{(2H_r^2+9H_p-3DH_r)}{18} & \frac{H_r}{3} & -1 & 0 & 0\\ 0 & 0 & 0 & 1 & F_p\\ \frac{3F_{pp}F_{pppp}-5F_{ppp}^2}{18F_{pp}^2} & 0 & 0 & \frac{F_{ppp}}{3F_{pp}} & \frac{F_{ppp}F_p-3F_{pp}^2}{3F_{pp}} \end{pmatrix} \cdot \begin{pmatrix}\omega^1\\ \omega^2\\ \omega^3\\ \omega^4\\ \omega^5\end{pmatrix},$$

and we invariantly reduce the G-structure to only 4 parameters ρ, ϕ, f^2, $\bar{f}^2$, the bar having nothing to do with complex conjugation except some analogy link with Pocchiola's computations:

$$\begin{pmatrix} \theta^1 \\ \theta^2 \\ \theta^3 \\ \theta^4 \\ \theta^5 \end{pmatrix} := \begin{pmatrix} \rho^2 & 0 & 0 & 0 & 0 \\ f^2 & \rho e^{\phi} & 0 & 0 & 0 \\ \frac{(f^2)^2}{2\rho^2} & \frac{f^2 e^{\phi}}{\rho} & e^{2\phi} & 0 & 0 \\ \bar{f}^2 & 0 & 0 & \rho e^{-\phi} & 0 \\ -\frac{(\bar{f}^2)^2}{2\rho^2} & 0 & 0 & \frac{-\bar{f}^2 e^{-\phi}}{\rho} & e^{-2\phi} \end{pmatrix} \cdot \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \\ \omega^5 \end{pmatrix}.$$

After computational cleaning, we obtain our first result, which happens to be the para-CR analog of (7).

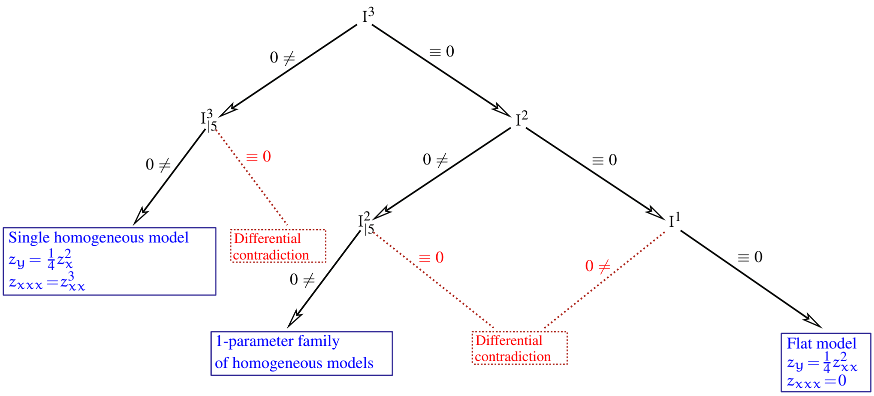
Theorem 8.5 *On the bundle $\mathcal{G}^9 = M^5 \times G^4$ with $M^5 \ni (x, y, z, p, r)$ times $\mathbb{R}^4 \ni (\rho, \phi, f_2, \bar{f}_2)$, there exist four 1-forms Ω_1, Ω_2, Ω_3, Ω_4 with θ^1, θ^2, θ^3, θ^4, θ^5, Ω_1, Ω_2, Ω_3, Ω_4 linearly independent at every point which satisfy the following para-CR invariant exterior differential system:*

$$\begin{aligned} d\theta^1 &= -\theta^1 \wedge \Omega_1 + \theta^2 \wedge \theta^4, \\ d\theta^2 &= \theta^2 \wedge (\Omega_2 - \tfrac{1}{2}\Omega_1) - \theta^1 \wedge \Omega_3 + \theta^3 \wedge \theta^4, \\ d\theta^3 &= 2\theta^3 \wedge \Omega_2 - \theta^2 \wedge \Omega_3 + \tfrac{e^{3\phi}}{\rho^3} I^1\, \theta^1 \wedge \theta^4 + \tfrac{e^{-\phi}}{\rho} I^3\, \theta^2 \wedge \theta^3 + \\ &\quad \tfrac{1}{8\rho^3}\Big(2e^{\phi}\bar{f}^2 I^3{}_{|5} + \rho(I^3{}_{|52} + 2I^3{}_{|4}) - 4e^{-\phi} f^2 I^3\Big)\, \theta^1 \wedge \theta^3, \\ d\theta^4 &= -\theta^2 \wedge \theta^5 - \theta^4 \wedge (\tfrac{1}{2}\Omega_1 + \Omega_2) - \theta^1 \wedge \Omega_4, \\ d\theta^5 &= -2\theta^5 \wedge \Omega_2 + \theta^4 \wedge \Omega_4 + \tfrac{e^{-3\phi}}{\rho^3} I^2\, \theta^1 \wedge \theta^2 - \tfrac{e^{\phi}}{2\rho} I^3{}_{|5}\, \theta^4 \wedge \theta^5 + \\ &\quad \tfrac{1}{8\rho^3}\Big(2e^{\phi}\bar{f}^2 I^3{}_{|5} + \rho(I^3{}_{|52} + 2I^3{}_{|4}) - 4e^{-\phi} f^2 I^3\Big)\, \theta^1 \wedge \theta^5, \end{aligned} \tag{11}$$

where I^1, I^2, I^3 are explicit relative differential invariants on the base M:

$$\begin{aligned} I^1 &:= -\tfrac{1}{54}\big(9D^2 H_r - 27 D H_p - 18 D H_r H_r + 18 H_p H_r + 4 H_r^3 + 54 H_z\big), \\ I^2 &:= \frac{40 F_{ppp}^3 - 45 F_{pp} F_{ppp} F_{pppp} + 9 F_{pp}^2 F_{ppppp}}{54\, F_{pp}^3}, \\ I^3 &:= \frac{2F_{ppp} + F_{pp} H_{rr}}{3\, F_{pp}}, \end{aligned}$$

and where $(\cdot)|_i$ for $i = 1, \ldots, 5$ denote directional derivatives along the vector fields X_i dual to θ^i.



We would like to mention that when $I^3 \equiv 0$, there are striking links with the geometry of 3rd order ODEs modulo contact transformations, *see* the recent [70].

Developing the technique of Cartan in *e.g.* [7, Chap. III], we split the study in two branches: $I^3 \neq 0$ and $I^3 \equiv 0$. When $I^3 \neq 0$, we show that one can normalize ρ, u_1, $\bar{f}^2$. Then in the obtained structure equations, $I^3|_5$ becomes a relative invariant. We show that $I^3|_5 \equiv 0$ conducts to a differential contradiction. When $I^3|_5 \neq 0$, we can also normalize ϕ, f^2, hence obtaining an $\{e\}$-structure on the base M, *cf.* [71, 80]. At first, certain 15 scalar constant curvatures appear, and by looking at differential consequences of $d \circ d = 0$, they reduce to *only one pair of solutions*, with $\epsilon = \pm 1$, and we come to Maurer-Cartan type equations:

$$\begin{aligned}
d\theta^1 &= \epsilon\Big(-6\theta^1 \wedge \theta^3 + \tfrac{1}{2}\theta^1 \wedge \theta^4 - \tfrac{3}{2}\theta^1 \wedge \theta^5 \Big) + \theta^2 \wedge \theta^4, \\
d\theta^2 &= \epsilon\Big(-\tfrac{1}{16}\theta^1 \wedge \theta^2 - 2\theta^2 \wedge \theta^3 + \tfrac{1}{2}\theta^2 \wedge \theta^4 - \theta^2 \wedge \theta^5 \Big) - \theta^1 \wedge \theta^3 + \\
&\quad \tfrac{1}{32}\theta^1 \wedge \theta^4 - \tfrac{1}{8}\theta^1 \wedge \theta^5 + \theta^3 \wedge \theta^4, \\
d\theta^3 &= \epsilon\Big(-\tfrac{3}{16}\theta^1 \wedge \theta^3 + \tfrac{1}{2}\theta^3 \wedge \theta^4 - \tfrac{1}{2}\theta^3 \wedge \theta^5 \Big) + \tfrac{1}{32}\theta^2 \wedge \theta^4 - \tfrac{1}{8}\theta^2 \wedge \theta^5, \\
d\theta^4 &= \epsilon\Big(-\tfrac{1}{8}\theta^1 \wedge \theta^4 + \tfrac{1}{4}\theta^1 \wedge \theta^5 + 4\theta^3 \wedge \theta^4 - \tfrac{1}{2}\theta^4 \wedge \theta^5 \Big) - \theta^2 \wedge \theta^5, \\
d\theta^5 &= \epsilon\Big(-\tfrac{1}{16}\theta^1 \wedge \theta^5 + 2\theta^3 \wedge \theta^5 - \tfrac{1}{4}\theta^4 \wedge \theta^5 \Big).
\end{aligned}$$

Next, in the branch $I^3 \equiv 0$, the equations (11) become:

$$\begin{aligned}
d\theta^1 &= -\theta^1 \wedge \Omega_1 + \theta^2 \wedge \theta^4, \\
d\theta^2 &= \theta^2 \wedge (\Omega_2 - \tfrac{1}{2}\Omega_1) - \theta^1 \wedge \Omega_3 + \theta^3 \wedge \theta^4, \\
d\theta^3 &= 2\theta^3 \wedge \Omega_2 - \theta^2 \wedge \Omega_3 + \frac{e^{3\phi}}{\rho^3} I^1\, \theta^1 \wedge \theta^4, \\
d\theta^4 &= -\theta^2 \wedge \theta^5 - \theta^4 \wedge (\tfrac{1}{2}\Omega_1 + \Omega_2) - \theta^1 \wedge \Omega_4, \\
d\theta^5 &= -2\theta^5 \wedge \Omega_2 + \theta^4 \wedge \Omega_4 + \frac{e^{-3\phi}}{\rho^3} I^2\, \theta^1 \wedge \theta^2.
\end{aligned}$$

Here, I^1 and I^2 are relative invariants.

In the sub-branch $I^2 \neq 0$, we first normalize ρ, u_1, $\bar{f}^2$. Then $I^2|_5$ becomes a relative invariant. We show that $I^2|_5 \equiv 0$ leads to a differential contradiction. When $I^2|_5 \neq 0$, we can also normalize ϕ, f^2, hence obtaining an $\{e\}$-structure on the base M, *cf.* [71, 80]. At first, certain 12 scalar constant curvatures appear, and by looking at differential consequences of $d \circ d = 0$, they reduce to *one pair of* 1-*parameter solutions* and we come to Maurer-Cartan type equations, parametrized by any $s \in \mathbb{R}$, again with $\epsilon = \pm 1$:

$$
\begin{aligned}
d\theta^1 &= -\,\epsilon\Big(\theta^1 \wedge \theta^3 + \theta^1 \wedge \theta^5\Big) + \theta^2 \wedge \theta^4,\\
d\theta^2 &= \epsilon\Big(s\theta^1 \wedge \theta^2 - \theta^2 \wedge \theta^5\Big) - s\theta^1 \wedge \theta^4 + \theta^3 \wedge \theta^4,\\
d\theta^3 &= \epsilon\Big(\theta^1 \wedge \theta^4 - \theta^3 \wedge \theta^5\Big) - \theta^1 \wedge \theta^2 - s\theta^2 \wedge \theta^4,\\
d\theta^4 &= \epsilon\Big(-s\theta^1 \wedge \theta^4 + \theta^3 \wedge \theta^4\Big) + s\theta^1 \wedge \theta^2 - \theta^2 \wedge \theta^5,\\
d\theta^5 &= \epsilon\Big(-\theta^1 \wedge \theta^4 + \theta^3 \wedge \theta^5\Big) + \theta^1 \wedge \theta^2 + s\theta^2 \wedge \theta^4.
\end{aligned}
$$

Lastly, when $I^2 \equiv 0$, we show that $I^1 \equiv 0$ too necessarily, and we show that the structure equations are those of the model $z_y = \frac{1}{4}\,(z_{xx})^2$ & $z_{xxx} = 0$. The diagram above summarizes these explanations.

By general features of Cartan's method, all obtained para-CR structures are pairwise not equivalent.

To conclude, by setting up the PDEs associated to para-CR submanifolds of solutions inspired from Fels-Kaup's list, we realize all these homogeneous models as stated in our main

Theorem 8.6 *Homogeneous models for* 2-*nondegenerate PDE five variables para-CR structures are classified by the following list of mutually inequivalent models:*

(i) $z_y = \frac{1}{4}(z_x)^2$ & $z_{xxx} = 0$*;*

(ii) $z_y = \frac{1}{4}(z_x)^2$ & $z_{xxx} = (z_{xx})^3$*;*

(iiia) $z_y = \frac{1}{4}(z_x)^b$ & $z_{xxx} = (2-b)\frac{(z_{xx})^2}{z_x}$ *with* $z_x > 0$ *for any real* $b \in [1, 2)$*;*

(iiib) $z_y = f(z_x)$ & $z_{xxx} = h(z_x)\big(z_{xx}\big)^2$, *where the function* f *is determined by the implicit equation:*

$$
\big(z_x^2 + f(z_x)^2\big)\exp\Big(2b \arctan\frac{bz_x - f(z_x)}{z_x + bf(z_x)}\Big) \;=\; 1 + b^2
$$

and where:

$$
h(z_x) := \frac{(b^2-3)z_x - 4bf(z_x)}{\big(f(z_x) - bz_x\big)^2},
$$

for any real $b > 0$.

The point automorphism groups for cases (i), (ii), (iiia), (iiib) can be determined infinitesimally. Indeed, a vector field with unknown coefficients $A^i = A^i(x, y, z, p, r)$, $i = 1, \dots, 5$:

$$X := A^1 \partial_x + A^2 \partial_y + A^3 \partial_z + A^4 \partial_p + A^5 \partial_r,$$

should act on 1-forms as the matrix (2.4), so that:

$$\begin{aligned} 0 &= \mathcal{L}_X(\omega^1) \wedge \omega^1, \\ 0 &= \mathcal{L}_X(\omega^2) \wedge \omega^1 \wedge \omega^2 \wedge \omega^3, \\ 0 &= \mathcal{L}_X(\omega^3) \wedge \omega^1 \wedge \omega^2 \wedge \omega^3, \\ 0 &= \mathcal{L}_X(\omega^4) \wedge \omega^1 \wedge \omega^4 \wedge \omega^5, \\ 0 &= \mathcal{L}_X(\omega^5) \wedge \omega^1 \wedge \omega^4 \wedge \omega^5. \end{aligned} \tag{12}$$

For instance, in case (ii), the first equation writes:

$$\begin{aligned} &\mathcal{L}_x(\omega^1) \wedge \omega^1 \\ &\quad = dx \wedge dy \Big[p \left(A^3_y - \tfrac{1}{4} p^2 A^2_y - p A^1_y - \tfrac{1}{4} p A^4 - \tfrac{1}{4} p A^3_x + \tfrac{1}{16} p^3 A^2_x + \tfrac{1}{4} A^1_x \right) \Big], \\ &\quad + dx \wedge dz \Big[p A^3_z - \tfrac{1}{4} p^3 A^2_z - p^2 A^1_z + A^3_x - \tfrac{1}{4} p^2 A^2_x - p A^1_x - A^4 \Big] \\ &\quad + dx \wedge dp \Big[p \left(A^3_p - \tfrac{1}{4} p^2 A^2_p - p A^1_p \right) \Big] + dx \wedge dr \Big[p \left(A^3_r - \tfrac{1}{4} p^2 A^2_r - p A^1_r \right) \Big] \\ &\quad + dy \wedge dz \Big[\tfrac{1}{4} p^2 A^3_z - \tfrac{1}{16} p^4 A^2_z - \tfrac{1}{4} p^3 A^1_z + A^3_y - \tfrac{1}{4} p^2 A^2_y - p A^1_y - \tfrac{1}{2} A^4 \Big] \\ &\quad + dy \wedge dp \Big[p^2 \left(\tfrac{1}{4} A^3_p - \tfrac{1}{16} p^2 A^2_p - \tfrac{1}{4} p A^1_p \right) \Big] \\ &\quad + dy \wedge dr \Big[p^2 \left(\tfrac{1}{4} A^3_r - \tfrac{1}{16} p^2 A^2_r - \tfrac{1}{4} p A^1_r \right) \Big] \\ &\quad + dz \wedge dp \Big[-A^3_p + \tfrac{1}{4} p^2 A^2_p + p A^1_p \Big] + dz \wedge dr \Big[-A^3_r + \tfrac{1}{4} p^2 A^2_r + p A^1_r \Big]. \end{aligned}$$

Solving this linear system of partial differential equations, we get

Corollary 8.7 *The Lie algebra of infinitesimal point automorphisms of the flat model (i) is simple, isomorphic to* $\mathfrak{so}_{3,2}(\mathbb{R})$*, with the* 10 *generators:*

$$\begin{aligned}
X_1 &:= xy\,\partial_x + y^2\,\partial_y - x^2\,\partial_z - (py + 2\,x)\,\partial_p - (2\,ry + 2)\,\partial_r,\\
X_2 &:= -(x^2 - yz)\,\partial_x - 2\,xy\,\partial_y - 2\,xz\,\partial_z - \big(\tfrac{1}{2}\,p^2 y + 2\,z\big)\,\partial_p\\
&\quad - \big(pry - 2\,rx + 2\,p\big)\,\partial_r,\\
X_3 &:= y\,\partial_x - 2\,x\,\partial_z - 2\,\partial_p,\\
X_4 &:= xz\,\partial_x - x^2\,\partial_y + z^2\,\partial_z - \big(\tfrac{1}{2}\,p^2 x - pz\big)\,\partial_p + \big(\tfrac{1}{2}\,p^2 - prx\big)\,\partial_r,\\
X_5 &:= z\,\partial_x - 2\,x\,\partial_y - \tfrac{1}{2}\,p^2\,\partial_p - pr\,\partial_r,\\
X_6 &:= x\,\partial_x + 2\,z\,\partial_z + p\,\partial_p,\\
X_7 &:= \partial_x,\\
X_8 &:= y\,\partial_y - z\,\partial_z - p\,\partial_p - r\,\partial_r,\\
X_9 &:= \partial_y,\\
X_{10} &:= \partial_z,
\end{aligned}$$

having commutator table:

	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}
X_1	0	0	0	0	$-X_2$	0	$-X_3$	$-X_1$	$-X_6-2X_8$	0
X_2	$*$	0	$2X_1$	0	$2X_4$	$-X_2$	$2X_6+2X_8$	0	$-X_5$	$-X_3$
X_3	$*$	$*$	0	X_2	$-2X_8$	X_3	$2X_{10}$	$-X_3$	$-X_7$	0
X_4	$*$	$*$	$*$	0	0	$-2X_4$	$-X_5$	X_4	0	$-X_6$
X_5	$*$	$*$	$*$	$*$	0	$-X_5$	$2X_9$	X_5	0	$-X_7$
X_6	$*$	$*$	$*$	$*$	$*$	0	$-X_7$	0	0	$-2X_{10}$
X_7	$*$	$*$	$*$	$*$	$*$	$*$	0	0	0	0
X_8	$*$	$*$	$*$	$*$	$*$	$*$	$*$	0	$-X_9$	X_{10}
X_9	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	0	0
X_{10}	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	0

In the CR context, observe that if $S^2 \subset \mathbb{R}^3$ is an affinely homogeneous parabolic surface, then the tube $M^5 := S^2 \times i\mathbb{R}^3$ has transitive holomorphic symmetry algebra $\mathfrak{hol}(M)$, with an *Abelian ideal* $\mathfrak{a} := \text{Span}\{i\partial_{z_1}, i\partial_{z_2}, i\partial_w\}$. Conversely, for an $M^5 \in \mathfrak{C}_{2,1}$, it is not difficult to show that if $\mathfrak{hol}(M) \supset \mathfrak{a}$ contains an Abelian ideal $\mathfrak{a}$ with $\text{rank}_{\mathbb{C}}\,\mathfrak{a} = 3$, then $M^5 \cong S^2 \times i\mathbb{R}^3$ is biholomorphically equivalent to the tube over an affinely homogeneous parabolic surface $S^2 \subset \mathbb{R}^3$.

In the para-CR context, all the Lie algebras in cases (i), (ii), (iiia), (iiib) have a 3-dimensional abelian ideal.

Corollary 8.8 *The Lie algebras of infinitesimal point automorphisms of the homogeneous models (ii), (iiia), (iiib) are all* 5*-dimensional and solvable, and are given in the* (x, y, z, p, r)*-space by the following generators together with their Lie brackets:*

(ii)

$$
\begin{aligned}
X_1 &:= x\,\partial_x + \tfrac{1}{2}\,y\,\partial_y + \tfrac{3}{2}\,z\,\partial_z + \tfrac{1}{2}\,p\,\partial_p - \tfrac{1}{2}\,r\,\partial_r,\\
X_2 &:= y\,\partial_x - 2\,x\,\partial_z - 2\,\partial_p,\\
X_3 &:= \partial_x,\\
X_4 &:= \partial_y,\\
X_5 &:= \partial_z,
\end{aligned}
$$

	X_1	X_2	X_3	X_4	X_5
X_1	0	$-\frac{1}{2}X_2$	$-X_3$	$-\frac{1}{2}X_4$	$-\frac{3}{2}X_5$
X_2	$*$	0	$2X_5$	$-X_3$	0
X_3	$*$	$*$	0	0	0
X_4	$*$	$*$	$*$	0	0
X_5	$*$	$*$	$*$	$*$	0

(iiia)

$$
\begin{aligned}
X_1 &:= x\,\partial_x + \frac{b\,z}{b-1}\,\partial_z + \frac{p}{b-1}\,\partial_p - \frac{r(b-2)}{b-1}\,\partial_r,\\
X_2 &:= y\,\partial_y - \frac{z}{b-1}\,\partial_z - \frac{p}{b-1}\,\partial_p - \frac{r}{b-1}\,\partial_z,\\
X_3 &:= \partial_x,\\
X_4 &:= \partial_y,\\
X_5 &:= \partial_z,
\end{aligned}
$$

	X_1	X_2	X_3	X_4	X_5
X_1	0	0	$-X_3$	0	$-\frac{b}{b-1}X_5$
X_2	$*$	0	0	$-X_4$	$\frac{1}{b-1}X_5$
X_3	$*$	$*$	0	0	0
X_4	$*$	$*$	$*$	0	0
X_5	$*$	$*$	$*$	$*$	0

(iiib)

$$
\begin{aligned}
X_1 &:= x\,\partial_x + y\,\partial_y + z\,\partial_z - r\,\partial_r,\\
X_2 &:= -y\,\partial_x + x\,\partial_y + \omega\,z\,\partial_z + \big(-F + \omega\,p\big)\,\partial_p + (-2DF + \omega\,r)\,\partial_r\\
X_3 &:= \partial_x,\\
X_4 &:= \partial_y,\\
X_5 &:= \partial_z,
\end{aligned}
$$

	X_1	X_2	X_3	X_4	X_5
X_1	0	0	$-X_3$	$-X_4$	$-X_5$
X_2	$*$	0	$-X_4$	X_3	$-\omega X_5$
X_3	$*$	$*$	0	0	0
X_4	$*$	$*$	$*$	0	0
X_5	$*$	$*$	$*$	$*$	0

To end this section, we would like to mention that Porter and Zelenko have made advances [82, 83, 85] on higher dimensional Levi-degenerate CR manifolds. Natural generalizations to para-CR geometry can be studied.

9 Homogeneous $\mathfrak{C}_{2,1}$ Models

Now, let us make a brief expository survey of [32]. Let $M \subset \mathbb{C}^{N \geqslant 2}$ be a local $\mathcal{C}^\omega$ CR hypersurface, in coordinates $z = (z_1, \dots, z_N) \in \mathbb{C}^N$, with $0 \in M$. Assume that M is

CR-homogeneous, so that the *real* Lie algebra:

$$\mathfrak{hol}(M) := \Big\{ L = \textstyle\sum_{i=1}^{N} a_i(Z)\,\frac{\partial}{\partial Z_i} \text{ holomorphic: } \ (L + \overline{L})\big|_M \text{ is tangent to } M \Big\},$$

is of dimension R with $\dim M \leqslant R \leqslant \infty$, due to $T_0 M = \mathrm{Span}\,\big\{(L+\overline{L})\big|_0 \colon L \in \mathfrak{hol}(M)\big\}$.

If $\mathfrak{hol}(M) \supset \mathfrak{a}$ contains an N-dimensional Abelian (real) Lie subalgebra $\mathfrak{a} = \mathrm{Span}\,\big(L_1, \dots, L_N\big)$ of holomorphic vector fields having *maximally real* span:

$$\mathrm{Span}\,\big(L_1 + \overline{L}_1\big|_0, \dots, L_N + \overline{L}_N\big|_0\big) \subset T_0\mathbb{C}^N,$$

then after a straightening biholomorphism, one has $L_1 = \sqrt{-1}\,\partial_{Z_1}$, ..., $L_N = \sqrt{-1}\,\partial_{Z_N}$.

Assume furthermore that $\mathfrak{a} \subset \mathfrak{hol}(M)$ is an *ideal*. Consider other $L_\nu \in \mathfrak{hol}(M)$ for $N+1 \leqslant \nu \leqslant R$ completing a basis. Since each $\big[\sqrt{-1}\,\partial_{Z_i}, L_\nu\big]$ must be a real linear combination of $\sqrt{-1}\,\partial_{Z_1}, \dots, \sqrt{-1}\,\partial_{Z_N}$, it comes:

$$L_\nu = \sum_{i=1}^{N} \Big(\sum_{j=1}^{N} a_{\nu,i,j}\, Z_j + b_{\nu,i} \Big) \frac{\partial}{\partial Z_i} \qquad (N+1 \leqslant \nu \leqslant R),$$

with constants $a_{\nu,i,j} \in \mathbb{R}$, and $b_{\nu,i} \in \mathbb{C}$; in fact $b_{\nu,i} \in \mathbb{R}$, after subtracting appropriate linear combinations of the $\sqrt{-1}\,\partial_{Z_i}$. Tangency to M of the real parts of the $\sqrt{-1}\,\partial_{Z_i}$ implies that $M = H \times i\mathbb{R}^N$ with $H \subset \mathbb{R}^N$ a hypersurface. Furthermore, writing $Z_i = X_i + \sqrt{-1}\,Y_i$, the vector fields

$$T_\nu := \sum_{i=1}^{N} \Big(\sum_{j=1}^{N} a_{\nu,i,j}\, X_j + b_{\nu,i} \Big) \frac{\partial}{\partial X_i} \qquad (N+1 \leqslant \nu \leqslant R),$$

are tangent to H, and their span at $0 \in H$ spans $T_0 H$. The converse is direct.

Focusing on $N = 3$, consider $\mathfrak{C}_{2,1}$ — *i.e.* 2-nondegenerate of constant Levi rank 1 — hypersurfaces $M^5 \subset \mathbb{C}^3$. They are CR analogs of parabolic surfaces $S^2 \subset \mathbb{R}^3$. Affinely homogeneous models have been presented at the end of Sect. 6.

Fels-Kaup's classification [28] of homogeneous $\mathfrak{C}_{2,1}$ hypersurfaces $M^5 \subset \mathbb{C}^3$ relies on expert knowledge of Lie structure theory. But only the equivalence method can reach information about CR invariants. The present objective is to explore the concerned CR invariants (either relative or absolute), since nothing about the branchings they create appears in [28, 31, 35, 47, 60].

In coordinates $\mathbb{C}^3 \ni \big(z, \zeta, w = u + \sqrt{-1}\,v\big)$, the graphed representation [14, 27, 34, 35, 37] of the flat model is:

$$u = \frac{z\overline{z} + \frac{1}{2}\,\overline{z}^2\zeta + \frac{1}{2}\,z^2\overline{\zeta}}{1 - \zeta\overline{\zeta}} =: m\big(z, \zeta, \overline{z}, \overline{\zeta}\big).$$

The 5-dimensional Lie group of its automorphisms fixing the origin writes:

$$z' := \lambda\,\frac{z + i\,\alpha\, z^2 + \big(i\,\alpha\,\zeta - i\,\overline{\alpha}\big)\,w}{1 + 2i\,\alpha\, z - \alpha^2 z^2 - \big(\alpha^2\zeta - \alpha\overline{\alpha} + i\,\rho\big)\,w},$$

$$\zeta' := \frac{\lambda}{\overline{\lambda}}\,\frac{\zeta + 2i\,\overline{\alpha}\, z - \big(\alpha\overline{\alpha} + i\,\rho\big)\,z^2 + \big(\overline{\alpha}^2 - i\,\rho\,\zeta - \alpha\overline{\alpha}\,\zeta\big)\,w}{1 + 2i\,\alpha\, z - \alpha^2 z^2 - \big(\alpha^2\zeta - \alpha\overline{\alpha} + i\,\rho\big)\,w},$$

$$w' := \lambda\overline{\lambda}\,\frac{w}{1 + 2i\,\alpha\, z - \alpha^2 z^2 - \big(\alpha^2\zeta - \alpha\overline{\alpha} + i\,\rho\big)\,w},$$

where $\lambda \in \mathbb{C}^*$, $\alpha \in \mathbb{C}$, $\rho \in \mathbb{R}$ are free.

A general $\mathfrak{C}_{2,1}$ hypersurface $M^5 \subset \mathbb{C}^3$ with $0 \in M$ writes as a perturbation of this model:

$$u \,=\, F\big(z,\zeta,\overline{z},\overline{\zeta},v\big) \,=\, m(z,\zeta,\overline{z},\overline{\zeta}) + G\big(z,\zeta,\overline{z},\overline{\zeta},v\big),$$

where:

$$F \,=\, \sum_{h,i,j,k,l} z^h\zeta^i\overline{z}^j\overline{\zeta}^k v^l\, F_{h,i,j,k,l} \,=\, \sum_{h,i,j,k} z^h\zeta^i\overline{z}^j\overline{\zeta}^k F_{h,i,j,k}(v),$$

with $\overline{F_{h,i,j,k,l}} = F_{j,k,h,i,l}$, with $0 = F_{0,0,0,0,0}$, and the same for G.

The Poincaré-Moser *convergent* normal form established in [35, 47] shows that, after some local biholomorphism fixing the origin, one can assume:

$$\begin{aligned}
0 &\equiv F_{h,i,0,0}(v), & 0 &\equiv F_{3,0,0,1}(v),\\
0 &\equiv F_{h,i,1,0}(v), & 0 &\equiv F_{4,0,0,1}(v) \equiv F_{3,0,1,1}(v),\\
0 &\equiv F_{h,i,2,0}(v), & 0 &\equiv F_{4,0,1,1}(v) \equiv F_{3,0,3,0}(v),
\end{aligned}$$

with the exceptions $1 \equiv F_{1,0,1,0}(v)$ and $\frac{1}{2} \equiv F_{2,0,0,1}(v)$.

Suppose $M'^5 \subset \mathbb{C}'^3$ is another such $\mathfrak{C}_{2,1}$ hypersurface, similarly normalized. If:

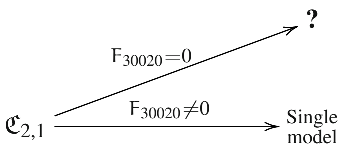
$$(z,\zeta,w) \longmapsto \big(f(z,\zeta,w),\, g(z,\zeta,w),\, h(z,\zeta,w)\big) =: (z',\zeta',w'),$$

is a local holomorphic map fixing the origin which sends M into M', then as follows from general Poincaré-Moser theory, it is of the form above for certain five real parameters $\lambda \in \mathbb{C}^*$, $\alpha \in \mathbb{C}$, $\rho \in \mathbb{R}$. Our goal is to normalize this remaining ambiguity, *cf.* Questions $\mathbf{Q}^{①}$ and $\mathbf{Q}^{④}$ in [35].

Attributing weights $[z] := 1$, $[\zeta] := 1$, $[w] := 2$, let us therefore show weighted order 5 terms:

$$\begin{aligned}
u \,=\, & z\overline{z} + \tfrac{1}{2}\,\overline{z}^2\zeta + \tfrac{1}{2}\,z^2\overline{\zeta} + z\overline{z}\zeta\overline{\zeta} + \tfrac{1}{2}\,\overline{z}^2\zeta\zeta\overline{\zeta} + \tfrac{1}{2}\,z^2\overline{\zeta}\zeta\overline{\zeta}\\
& + 2\,\mathrm{Re}\,\Big\{z^3\overline{\zeta}^2\,F_{3,0,0,2,0}\Big\} + O_{z,\zeta,\overline{z},\overline{\zeta},v}(6),
\end{aligned}$$

the remainder being *weighted* as well. This coefficient $\mathsf{F}_{3,0,0,2,0}$ is a relative invariant, hence it creates a branching.



Theorem 9.1 *In the branch $\mathsf{F}_{3,0,0,2,0} \neq 0$, one can normalize $\mathsf{F}_{3,0,0,2,0} := 1$, so $\lambda := 1$, and 3 supplementary (real) normalizations hold:*

$$\begin{aligned} \mathsf{F}_{4,0,0,2,0} &:= 0, & &\textit{so}\;\; \alpha := 0, \\ \mathrm{Im}\; \mathsf{F}_{3,0,2,1,0} &:= 0, & &\textit{so}\;\; \rho := 0, \end{aligned}$$

so that the isotropy is reduced to be zero-dimensional.

Furthermore, all coefficients $\mathsf{F}_{h,i,j,k,l} \in \mathbb{C}$ are uniquely determined to be specific constants, as shown in [32], and the related 5 holomorphic vector fields e_1, e_2, e_3, e_4, e_5 have structure:

$$\begin{array}{llll} [e_1, e_2] = -4\, e_4 - 4\, e_5, & [e_1, e_3] = -2 e_1, & [e_1, e_4] = 2\, e_2 + 4\, e_4, & [e_1, e_5] = 2\, e_2 - 4\, e_5, \\ & [e_2, e_3] = -4\, e_2 - 4\, e_4, & [e_2, e_4] = 0, & [e_2, e_5] = 0, \\ & & [e_3, e_4] = 2\, e_4, & [e_3, e_5] = -2\, e_2 + 6\, e_5, \\ & & & [e_4, e_5] = 0. \end{array}$$

This Lie algebra $\mathfrak{g}$ has the derived series of dimensions 5, 4, 2, 0, with:

$$[\mathfrak{g}, \mathfrak{g}] \;=\; \mathrm{Span}\; \big(\underline{-4\, e_4 - 4\, e_5}, \;\; -2\, e_1, \;\; \underline{2\, e_2 + 4\, e_4}, \;\; \underline{-4\, e_2 - 4\, e_4} \big).$$

The three underlined vector fields span a 3-dimensional Abelian ideal $\mathfrak{a} \subset \mathfrak{g}$, whose value at the origin $0 \in \mathbb{C}^3$ spans a maximally real 3-plane. This is coherent with Fels-Kaup's item (3) in Sect. 6.

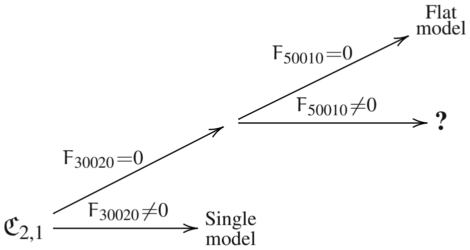
Next, assume $\mathsf{F}_{3,0,0,2,0} \equiv 0$, or equivalently, $\frac{1}{4} \overline{W}_0 \equiv 0$. Some differential consequences are:

$$\mathsf{F}_{4,0,0,2,0} \;=\; 0, \qquad\qquad \mathsf{F}_{3,0,1,2,0} \;=\; 0, \qquad\qquad \mathsf{F}_{3,0,0,3,0} \;=\; 0,$$

hence up to order 6:

$$\begin{aligned} u \;=\; & z\overline{z} + \tfrac{1}{2}\, \overline{z}^2 \zeta + \tfrac{1}{2}\, z^2 \overline{\zeta} + z\overline{z}\zeta\overline{\zeta} + \tfrac{1}{2}\, \overline{z}^2 \zeta\zeta\overline{\zeta} + \tfrac{1}{2}\, z^2 \overline{\zeta}\zeta\overline{\zeta} + z\overline{z}\zeta\overline{\zeta}\zeta\overline{\zeta} \\ & + 2\; \mathrm{Re}\, \Big\{ z^5 \overline{\zeta}\, \mathsf{F}_{5,0,0,1,0} + z^3 \overline{z}^2 \overline{\zeta}\, \mathsf{F}_{3,0,2,1,0} \Big\} + \mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta},v}(7). \end{aligned}$$

Suppose the graphed equation for M' is similar. Then $F_{5,0,0,1,0}$ is a relative invariant, and it creates a branching:



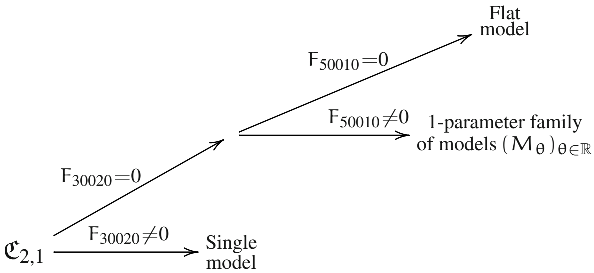
A further sub-branching could be created by the other relative invariant $F_{3,0,2,1,0}$, but this is not the case. The following result establishes, by normal forms techniques, Pocchiola's characterization of the flat model.

Theorem 9.2 *In the branch* $F_{3,0,0,2,0} = 0 = F_{5,0,0,1,0}$, *if* $M^5 \in \mathfrak{C}_{2,1}$ *is homogeneous, then all* $G_{h,i,j,k,l} = 0$, *and* M *coincides with the flat model:*

$$u = m + 0 = \frac{z\overline{z} + \frac{1}{2}\,\overline{z}^2\zeta + \frac{1}{2}\,z^2\overline{\zeta}}{1 - \zeta\overline{\zeta}}. \qquad \square$$

Thus, in this top-most (degenerate) branch, $F_{3,0,2,1,0} = 0$ is *implied*, suprisingly.

Next, in the branch $F_{3,0,0,2,0} = 0$ and $F_{5,0,0,1,0} \neq 0$, one can use $\lambda \in \mathbb{C}$ to normalize $F_{5,0,0,1,0} := 1$, so $\lambda = 1$. The final tree will be explained by the third theorem:



Theorem 9.3 *In the branch* $F_{3,0,0,2,0} = 0$ *and* $F_{5,0,0,1,0} = 1$, *three supplementary (real) normalizations hold:*

$$\begin{aligned} F_{6,0,0,1,0} &:= 0, && \textit{so}\ \ \alpha := 0, \\ \mathrm{Im}\ F_{4,0,3,0,0} &:= 0, && \textit{so}\ \ \rho := 0, \end{aligned}$$

so that the isotropy is reduced to be zero-dimensional. Notably, a constant value for $F_{3,0,2,1,0} = -15$ *is also implied.*

Furthermore, abbreviating:

$$\theta := \mathrm{Re}\ F_{4,0,3,0,0},$$

which is a free *absolute invariant, all coefficients* $F_{h,i,j,k,l} \in \mathbb{C}$ *are uniquely determined in terms of* $\theta \in \mathbb{R}$*, as shown in* [32]*, and the related* 5 *holomorphic vector fields* e_1*,* e_2*,* e_3*,* e_4*,* e_5 *have structure:*

$$\begin{aligned}
[e_1, e_2] &= -\tfrac{4}{5}\,\theta\, e_4 - 4\, e_5, & [e_1, e_3] &= 0, & [e_1, e_4] &= 2\, e_2, & [e_1, e_5] &= \tfrac{2}{5}\,\theta\; e_2 - 20\, e_4, \\
& & [e_2, e_3] &= -2\, e_2, & [e_2, e_4] &= 0, & [e_2, e_5] &= 0, \\
& & & & [e_3, e_4] &= 2\, e_4, & [e_3, e_5] &= 2\, e_5, \\
& & & & & & [e_4, e_5] &= 0.
\end{aligned}$$

This Lie algebra $\mathfrak{g}$ has the derived series of dimensions 5, 3, 0, with:

$$[\mathfrak{g}, \mathfrak{g}] = \text{Span}\,\Big(-\tfrac{4}{5}\,\theta\, e_4 - 4\, e_5,\;\; 2\, e_2,\;\; \tfrac{2}{5}\,\theta\, e_2 - 20\, e_4 \Big).$$

These three vector fields form a 3-dimensional Abelian ideal $\mathfrak{a} \subset \mathfrak{g}$, whose value at the origin $0 \in \mathbb{C}^3$ spans a maximally real 3-plane. This is coherent with Fels-Kaup's items (2a), (2b), (2c) in Sect. 6

10 Poincaré-Moser Normal Forms for Levi Degenerate Para-CR Structures

As an epilog to our survey, let us devote the remaining paragraphs to explain how the CR normal form of [35, 47] may be generalized to degenerate para-CR structures.

As before, consider a real or complex hypersurface $M \subset \mathbb{C}^3_{x,y,z} \times \mathbb{C}^3_{a,b,c}$ graphed as:

$$z = Q(x, y, a, b, c) \qquad {\scriptstyle (0 \neq Q_c)},$$

with Q analytic, *i.e.* expandable in converging power series. We may assume $0 \in M$, *i.e.* $0 = Q(0, 0, 0, 0, 0)$.

Also, consider the local infinite-dimensional Lie group of local biholomorphisms (9) which separate variables and parameters, but do not necessarily fix the origin. Define $\text{Sym}(M)$ to be those transformations which stabilize M, near the origin.

Infinitesimal generators of $\text{Sym}(M)$ constitute the following (local) Lie subalgebra of the infinite-dimensional Lie algebra associated to (9):

$$\begin{aligned}
\mathfrak{sym}(M) := \Big\{ L &= X(x, y, z)\, \partial_x + Y(x, y, z)\, \partial_y + Z(x, y, z)\, \partial_z \\
&\quad + A(a, b, c)\, \partial_a + B(a, b, c)\, \partial_b + C(a, b, c)\, \partial_c : \;\; L\big|_M \text{ istangentto } M \Big\}.
\end{aligned}$$

Our two main (invariant) hypotheses of Levi degeneracy and of *double* 2-nondegeneracy express in terms of Q and P as:

$$\begin{vmatrix} Q_a & Q_b & Q_c \\ Q_{xa} & Q_{xb} & Q_{xc} \\ Q_{ya} & Q_{yb} & Q_{yc} \end{vmatrix} \equiv 0 \neq \begin{vmatrix} Q_a & Q_b & Q_c \\ Q_{xa} & Q_{xb} & Q_{xc} \\ Q_{xxa} & Q_{xxb} & Q_{xxc} \end{vmatrix},$$

$$0 \neq \begin{vmatrix} P_x & P_y & P_z \\ P_{ax} & P_{ay} & P_{az} \\ P_{aax} & P_{aay} & P_{aaz} \end{vmatrix}.$$

We already know that in analogy with (5), the appropriate homogeneous model writes:

$$M_{LC}: \qquad z + c = \frac{2xa + x^2 b + a^2 y}{1 - yb} =: m(x, y, a, b).$$

The letter m here stands for $m$odel. Since this model is invariant under the scalings:

$$\big(x, y, z, \ a, b, c\big) \longmapsto \big(\lambda x, y, \lambda^2 z, \ \lambda a, b, \lambda^2 c\big) \qquad {\scriptstyle (\lambda \in \mathbb{C}^*)},$$

it is natural to assign the weights:

$$[x] := 1 =: [a], \qquad [y] := 0 =: [b], \qquad [z] := 2 =: [c],$$

whence coordinate vector fields inherit opposite weights:

$$\big[\partial_x\big] := -1 =: \big[\partial_a\big] \qquad \big[\partial_y\big] := 0 =: \big[\partial_b\big] \qquad \big[\partial_z\big] := -2 =: \big[\partial_c\big].$$

Then by taking inspiration from [34, 35, 37], the 10-dimensional simple Lie algebra of infinitesimal symmetries of the model:

$$\mathfrak{g} := \mathfrak{sym}\big(M_{LC}\big) \cong \mathfrak{so}(5, \mathbb{C}),$$

has 10 natural vector fields generators $\mathfrak{g} = \text{Span}\{L_1, \ldots, L_{10}\}$ tangent to M_{LC}, which may be organized in a graded Lie algebra:

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2,$$

whose components are:

$$\begin{aligned} \mathfrak{g}_{-2} &:= \text{Span}\,\big\{\partial_z - \partial_c\big\}, \\ \mathfrak{g}_{-1} &:= \text{Span}\,\big\{(y-1)\,\partial_x - 2x\,\partial_z + (b-1)\,\partial_a - 2a\,\partial_c, \\ &\qquad\qquad (1+y)\,\partial_x - 2x\,\partial_z - (1+b)\,\partial_a + 2a\,\partial_c\big\}, \end{aligned}$$

with $\mathfrak{g}_0 = \mathfrak{g}_0^{\mathsf{trans}} \oplus \mathfrak{g}_0^{\mathsf{iso}}$:

$$\begin{aligned}
\mathfrak{g}_0^{\mathsf{trans}} &:= \text{Span}\,\Big\{xy\,\partial_x + (y^2-1)\,\partial_y - x^2\,\partial_z + ab\,\partial_a + (b^2-1)\,\partial_b - a^2\,\partial_c, \\
&\qquad xy\,\partial_x + (y^2+1)\,\partial_y - x^2\,\partial_z - ab\,\partial_a - (b^2+1)\,\partial_b + a^2\,\partial_c\Big\}, \\
\mathfrak{g}_0^{\mathsf{iso}} &:= \text{Span}\,\{x\,\partial_x + 2z\,\partial_z + a\,\partial_a + 2c\,\partial_c, \\
&\qquad x\,\partial_x + 2y\,\partial_y - a\,\partial_a - 2b\,\partial_b\},
\end{aligned}$$

while:

$$\begin{aligned}
\mathfrak{g}_1 := \text{Span}\,\{&(x^2-yz-z)\partial_x + \big(2xy+2x\big)\partial_y + 2xz\,\partial_z + \big(a^2-bc-c\big)\partial_a \\
&+\big(2ab+2a\big)\partial_b + 2ac\,\partial_c, \\
&\big(-x^2+yz-z\big)\partial_x + \big(-2xy+2x\big)\partial_y - 2xz\,\partial_z - \big(-a^2+bc-c\big)\partial_a \\
&-\big(-2ab+2a\big)\partial_b + 2ac\,\partial_c\}, \\
\mathfrak{g}_2 := \text{Span}\,\{&xz\,\partial_x - x^2\,\partial_y + z^2\,\partial_z - ac\,\partial_a + a^2\,\partial_b - c^2\,\partial_c\}.
\end{aligned}$$

The objective is to normalize as much as possible the right-hand side power series:

$$z = \sum Q_{i,j,l,m,n}\, x^i y^j a^l b^m c^n \qquad (Q_{i,j,l,m,n} \in \mathbb{C}),$$

by means of split biholomorphisms (9). It is not difficult to show that any $z = Q$ can be put into the form:

$$z = -c + 2xa + a^2 y + x^2 b + xayb + O_{x,y,a,b,c}(5).$$

Theorem 10.1 *There exists a split-biholomorphism (9) fixing* 0 *which normalizes the submanifold of solutions to:*

$$\begin{aligned}
z = -\,c &+ \frac{2xa + a^2y + x^2b}{1-yb} \\
&+ 2\,\text{Re}\,\Big\{x^3b^2\,F_{3,0,0,2}(c) + yb\,\big(3\,x^2ab\,F_{3,0,0,2}(c)\big)\Big\} \\
&+ 2\,\text{Re}\,\Big\{x^5b\,F_{5,0,0,1}(c) + x^4b^2\,F_{4,0,0,2}(c) + x^3a^2b\,F_{3,0,2,1}(c) \\
&+ x^3ab^2\,F_{3,0,1,2}(c) + x^3b^3\,F_{3,0,0,3}(c)\Big\} \\
&+ x^3a^3\,O_{x,a}(1) + a^3y\,O_{x,y,a}(3) + x^3b\,O_{x,a,b}(3) + yb\,O_{x,a}(3)\,O_{x,y,a,b,c}(2).
\end{aligned}$$

Furthermore, the biholomorphism exists and is unique *if it is assumed to be of the form:*

$$\begin{aligned}
&x' := x + f_{\geqslant 2}(x,y,z)\; y' := y + g_{\geqslant 1}(x,y,z),\; z' := z + h_{\geqslant 3}(x,y,z), \\
&0 = f_z(0), \qquad\qquad\qquad\qquad 0 = h_{zz}(0),
\end{aligned}$$

with similar conditions on φ, ψ, χ. □

Here, $e_{\geqslant \nu}(x, y, z)$ denotes a holomorphic function near the origin all of whose monomials $x^i y^j z^k$ are of weight $i + 2\,k \geqslant \nu$.

Equivalently, writing:

$$z = Q = \sum_{i,j,l,m \geqslant 0} x^i y^j a^l b^m \, Q_{i,j,l,m}(c),$$

the normal form is defined by the general *prenormalization conditions:*

$$\begin{aligned} 0 &\equiv Q_{i,j,0,0}(c) \equiv Q_{0,0,l,m}(c), \\ 0 &\equiv Q_{i,j,1,0}(c) \equiv Q_{1,0,l,m}(c), \\ 0 &\equiv Q_{i,j,2,0}(c) \equiv Q_{2,0,l,m}(c), \end{aligned}$$

with the obvious exceptions $Q_{0,0,0,0}(c) \equiv -c$, $Q_{1,0,1,0}(c) \equiv 2$ and $Q_{0,1,2,0}(c) \equiv 1 \equiv Q_{2,0,0,1}(c)$, together with the *sporadic normalization conditions*, listed by increasing orders 4, 5, 6:

$$\begin{aligned} &0 \equiv Q_{3,0,0,1}(c) \equiv Q_{0,1,3,0}(c), \\ &0 \equiv Q_{4,0,0,1}(c) \equiv Q_{0,1,4,0}(c), && 0 \equiv Q_{3,0,1,1}(c) \equiv Q_{1,1,3,0}(c), \\ &0 \equiv Q_{4,0,1,1}(c) \equiv Q_{1,1,4,0}(c), && 0 \equiv Q_{3,0,3,0}(c). \end{aligned}$$

Without the above conditions $x' = x + f_{\geqslant 2}$, $y' = y + g_{\geqslant 1}$, $z' = z + h_{\geqslant 3}$ guaranteeing uniqueness, one can verify that a normalizing transformation is unique up to the right action of the 5-dimensional isotropy group (at the origin) of the model.

To terminate this survey as it started, namely with the 3-dimensional case, consider a submanifold of solutions $M^3 \subset \mathbb{C}^2_{x,y} \times \mathbb{C}^2_{a,b}$:

$$y = Q(x, a, b) \qquad \& \qquad b = P(a, x, y),$$

which is Levi nondegenerate:

$$0 \neq \begin{vmatrix} Q_a & Q_b \\ Q_{xa} & Q_{xb} \end{vmatrix} \qquad \Longleftrightarrow \qquad \begin{vmatrix} P_x & P_y \\ P_{ax} & P_{ay} \end{vmatrix} \neq 0,$$

modulo the split-biholomorphisms group:

$$(x, y,\, a, b) \longmapsto \big(f(x, y), g(x, y),\, \varphi(a, b), \psi(a, b)\big) =: \big(x', y',\, a', b'\big). \tag{13}$$

It is elementary to show that any such M can be put into the preliminary form:

$$y = -b + xa + O_{x,a,b}(3).$$

The *sphere* model has zero remainder:

$$M_S: \qquad y = -b + xa.$$

Natural weights being:

$$[x] := 1 =: [a], \qquad [y] := 2 =: [b],$$
$$[\partial_x] := -1 =: [\partial_a], \qquad [\partial_y] := -2 =: [\partial_b],$$

the Lie symmetry algebra of the model:

$$\mathfrak{g} := \mathfrak{sym}(M_S) = \mathfrak{pgl}(2, \mathbb{C}) = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2,$$

is 8-dimensional with components:

$$\begin{aligned}
\mathfrak{g}_{-2} &:= \text{Span}\,\{\partial_y - \partial_b\},\\
\mathfrak{g}_{-1} &:= \text{Span}\,\{\partial_x + x\partial_y + \partial_a + a\partial_b,\ \ \partial_x - x\partial_y - \partial_a + a\partial_b\},\\
\mathfrak{g}_0 &:= \text{Span}\,\{x\partial_x - a\partial_a,\ \ x\partial_x + 2y\partial_y + a\partial_a + 2b\partial_b\},\\
\mathfrak{g}_1 &:= \text{Span}\,\{(x^2 + y)\partial_x + xy\partial_y - (a^2 + b)\partial_a - ab\partial_b,\ \ (x^2 - y)\partial_x + xy\partial_y\\
&\qquad + (a^2 - b)\partial_a + ab\partial_b\},\\
\mathfrak{g}_2 &:= \text{Span}\,\{xy\partial_x + y^2\partial_y - ab\partial_a - b^2\partial_b\}.
\end{aligned}$$

Proceeding quite similarly as in [16, 46, 64], one can prove

Theorem 10.2 *There exists a split-biholomorphism (13) fixing* 0 *which normalizes the submanifold of solutions to:*

$$y = -b + xa + Q_{4,2}(b)\,x^4a^2 + Q_{2,4}(b)\,x^2a^4 + \sum_{\substack{i+l\geqslant 7\\ i\geqslant 2,\, l\geqslant 2}} x^i a^l\, Q_{i,l}(b).$$

Furthermore, the biholomorphism exists and is unique *if it is assumed to be of the form:*

$$x' := x + f(x, y), \qquad y' := y + g(x, y),$$
$$f_x(0) = f_y(0) = 0, \qquad g_x(0) = g_y(0) = g_{yy}(0) = 0.$$

Equivalently, writing:

$$y = Q = \sum_{i,l\geqslant 0} x^i a^l\, Q_{i,l}(b),$$

the normal form is defined by the general *prenormalization conditions:*

$$0 \equiv Q_{i,0}(b) \equiv Q_{0,l}(b),$$
$$0 \equiv Q_{i,1}(b) \equiv Q_{1,l}(b),$$

with the obvious exceptions $Q_{0,0}(b) \equiv -b$ and $Q_{1,1}(b) \equiv 1$, together with the *sporadic normalization conditions:*

$$0 \equiv Q_{2,2}(b) \equiv Q_{3,2}(b) \equiv Q_{2,3}(b).$$

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Declarations

Conflict of interest Authors declare they have no financial interests that are directly or indirectly related to the work submitted for publication.

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