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Five-dimensional para-CR manifolds and contact projective geometry in dimension three

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Abstract. We study invariant properties of 5-dimensional para-CR structures whose Levi form is degenerate in precisely one direction and which are 2-non-degenerate. We realize that *two*, out of three, primary (basic) para-CR invariants of such structures are the classical differential invariants known to *Monge* (1810) and to *Wünschmann* (1905):

$$M(G) := 40G_{ppp}^3 - 45G_{pp}G_{ppp}G_{pppp} + 9G_{pp}^2G_{ppppp},$$
$$W(H) := 9D^2H_r - 27DH_p - 18H_rDH_r + 18H_pH_r + 4H_r^3 + 54H_z.$$

The vanishing $M(G) \equiv 0$ provides a local necessary and sufficient condition for the graph of a function in the (p, G)-plane to be contained in a conic, while the vanishing $W(H) \equiv 0$ gives an *if-and-only-if* condition for a 3rd order ODE to define a natural Lorentzian geometry on the space of its solutions.

Mainly, we give a geometric interpretation of the *third* basic invariant of our class of para-CR structures, the simplest one, of lowest order, and of mixed nature $N(G, H) := 2G_{ppp} + G_{pp}H_{rr}$. We establish that the vanishing $N(G, H) \equiv 0$ gives an *if-and-only-if* condition for the *two* 3-dimensional quotients of the para-CR manifold by its two canonical integrable rank-2 distributions, to be equipped with contact projective geometries.

A curious transformation between the Wünschmann invariant and the Monge invariant, first noted by us in a recent publication [15], is also discussed, and its mysteries are further revealed.

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1. Introduction

Motivated by recent advances on *Levi degenerate* 5-dimensional CR manifolds [2, 7, 8, 10, 11, 16, 19–23], we classified in [15] all *homogeneous* Levi degenerate 5-

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dimensional para-CR manifolds, extending Fels-Kaup's spectacular CR classification [3]. Para-CR structures can be defined either as (smooth) *submanifolds of solutions* (see [9, 12, 13] for background), or as completely integrable (smooth) PDE systems, a computationally more accessible point of view.

Working over $\mathbb{R}$ (or $\mathbb{C}$), a Levi degenerate 5D para-CR structure which is 2-nondegenerate with respect to *variables* [13, Section 14] can be represented as a PDE system on the plane of the form

$$z_{xxx} = H(x, y, z, z_x, z_{xx}) \quad \text{and} \quad z_y = G(x, y, z, z_x, z_{xx}),$$

for a function $z = z(x, y)$ of two variables, with the compatibility $\Delta H = D^3 G$, where

$$D := \partial_x + p\partial_z + r\partial_p + H\partial_r, \qquad \triangle := \partial_y + G\partial_z + DG\partial_p + D^2G\partial_r$$

(abbreviate $p := z_x$, $r := z_{xx}$).

A general solution $z = z(x, y, \overline{x}, \overline{y}, \overline{z})$ exists [12], depending on 3 real parameters $\overline{x}$, $\overline{y}$, $\overline{z}$. The Levi form [13, Section 8] is of rank 1 iff $G_r \equiv 0$ [*ib.* Proposition 22.1]. The para-CR structure is *also* 2-nondegenerate with respect to *parameters* [*ib.* Proposition 25.2] iff $G_{pp} \neq 0$. Once the link with CR geometry through Levi form and 2-nondegeneracy has been understood, one can study "from scratch" PDE systems of this kind $z_{xxx} = H$, $z_y = G$, and re-discover H_r and G_{pp} as relative invariants.

Conducting non-straightforward Cartan-type computations in [15], we found the 3 primary relative differential invariants (*cf.* Theorem 5.1)

$$\begin{aligned}
I^1 &\sim 9D^2H_r - 27DH_p - 18H_rDH_r + 18H_pH_r + 4H_r^3 + 54H_z,\\
I^2 &\sim 40G_{ppp}^3 - 45G_{pp}G_{ppp}G_{pppp} + 9G_{pp}^2G_{ppppp},\\
I^3 &\sim 2G_{ppp} + G_{pp}H_{rr},
\end{aligned}$$

whose identical vanishing $0 \equiv I^1 \equiv I^2 \equiv I^3$ characterizes para-CR equivalence to the *flat model* $z_{xxx} = 0$, $z_y = \frac{1}{4} z_x^2$. The associated submanifold of solutions is the tube over the light cone $(x - \overline{x})^2 + (y - \overline{y})(z - \overline{z}) = 0$, see Section 2, which has Lie symmetry algebra isomorphic to $\mathfrak{so}(3, 2, \mathbf{R}) \cong \mathfrak{sp}(4, \mathbf{R})$.

Strikingly, the flat model of such 5D para-CR structures is the *same* as the flat model for the geometry of 3$^{\text{rd}}$ order ODEs considered modulo contact transformation of variables [1, 5, 6, 24], which is also given as an EDS satisfied by the Maurer-Cartan forms on $\mathrm{Sp}(4, \mathbb{R})$. A decade ago, the first author raised

Question 1.1. What are the links between 2-nondegenerate 5D para-CR structures and 3$^{\text{rd}}$ order ODEs?

Thus, consider a smooth 5D para-CR structure $z_y = G(x, y, z, z_x)$, $z_{xxx} = H(x, y, z, z_x, z_{xx})$, with $G_r \equiv 0 \neq G_{pp}$ and $\Delta H = D^3G$. There are at least *formal* links with 3$^{\text{rd}}$ order ODEs. Firstly, there is incorporated a family of third order ODEs $z_{xxx} = H(x, y, z, z_x, z_{xx})$, parametrized by y. Secondly, the para-CR invariant I^1 is the contact Wünschmann invariant appearing in the theory of 3$^{\text{rd}}$

order ODEs. Pushing forward our use of Cartan's method in [15], we will exhibit in Section 5 deeper *structural* links.

Theorem 1.2. *If the para-*CR *invariant* I^3 *vanishes:*

$$2G_{ppp} + G_{pp}H_{rr} \equiv 0,$$

*then associated to such a para-*CR *structure, there are* two *contact equivalence classes of* 3rd *order* ODEs, *both having respective Chern invariant zero:*

$$\mathbf{Z}(H) \equiv 0.$$

The other basic contact invariant of these (contact invariant) classes of ODEs, *namely the Wünschmann invariant* $\mathbf{A_1}$, *is proportional:*

(a) *To the Wünschmann para-*CR *invariant,* $\mathbf{A_1} \sim 9D^2H_r - 27DH_p - 18H_rDH_r + 18H_pH_r + 4H_r^3 + 54H_z$, *in the case of the first class of* ODEs*;*
(b) *To the Monge para-*CR *invariant,* $\mathbf{A_1} \sim 40G_{ppp}^3 - 45G_{pp}G_{ppp}G_{pppp} + 9G_{pp}^2G_{ppppp}$, *in the case of the second class of* ODEs.

Chern in 1940 [1] observed that $\mathbf{Z}(H) = H_{rrrr}$ is a contact relative invariant of any ODE $z''' = H(x, z, z', z'')$. The *geometric* meaning of $\mathbf{Z}(H) \equiv 0$ relies on a rather recent notion of a *contact projective structure* [4,5], see Definition 4.3 below. As a corollary, applying results of [5,6], we obtain

Theorem 1.3. *Under the same assumption* $I^3 \equiv 0$, *the para-*CR *structure defines* two *natural contact projective geometries on certain* two 3*-dimensional quotient spaces of the* 5D *para-*CR *manifold.*

Concept explanations being required to make the paper self contained, we start by briefly collecting:

(a) Basic facts about 5-dimensional para-CR structures ([9]; we follow exposition and notation from [15]);
(b) Rudiments of the theory of contact geometry of 3rd order ODEs ([1]; we follow [5,6]);
(c) Facts from the theory of contact projective structures ([4]; we follow [6]).

Then we will prove the above two theorems.

2. Degenerate 5-dimensional para-CR-structures

Recall from [9,12] that a *para-*CR *structure* is a geometric structure which a hypersurface $M^{2n-1} \subset (\mathbb{R}^n \times \mathbb{R}^n)$ acquires from the ambient *product* space $\mathbb{R}^n \times \mathbb{R}^n$. More specifically one considers a local hypersurface

$$M_{2n-1} = \big\{\mathbb{R}^n \times \mathbb{R}^n \ni (x, \bar{x}) \mid \Phi(x_1, \dots, x_n, \bar{x}_1, \dots, \bar{x}_n) = 0\big\},$$

with $d_x\Phi \neq 0 \neq d_{\bar{x}}\Phi$, modulo (local) diffeomorphisms $\varphi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$ preserving the splitting of $\mathbb{R}^{2n}$ into $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$, *i.e.*, $\varphi(x, \bar{x}) = (\psi(x), \bar{\psi}(x))$, where $\psi : \mathbb{R}^n \to \mathbb{R}^n$ and $\bar{\psi} : \mathbb{R}^n \to \mathbb{R}^n$ are (local) diffeomorphisms.

The lowest dimension where these structures are interesting is $n = 2$. If nondegenerate, such para-CR structures are in 1-1 correspondence with 2nd order ODEs considered modulo point transformations of variables [12,18]. In this article we will deal with the next dimension, $n = 3$, and will study 5-dimensional para-CR structures.

A 5-dimensional para-CR structure, *i.e.*, a hypersurface $M^5 \subset \mathbb{R}^3 \times \mathbb{R}^3$ considered modulo split transformations of the product $\mathbb{R}^3 \times \mathbb{R}^3$, can be defined in terms of a graph of a function z of five variables, $z = z(x, y, \bar{x}, \bar{y}, \bar{z})$, where $(x, y, z, \bar{x}, \bar{y}, \bar{z})$ are coordinates in $\mathbb{R}^6 = \mathbb{R}^3 \times \mathbb{R}^3$. This in turn, can be considered as a *general solution* to a completely integrable system of two PDEs on the plane (x, y) for a function $z = z(x, y)$, in which $(\bar{x}, \bar{y}, \bar{z})$ denote constants of integration and parametrize the solution space of the corresponding system of PDEs.

Example 2.1 (Model). Take $(x - \bar{x})^2 + (y - \bar{y})(z - \bar{z}) = 0$, and solve it for z obtaining: $z = -\frac{(x-\bar{x})^2}{y-\bar{y}} + \bar{z}$. Now think about (x, y) as independent variables, and $(\bar{x}, \bar{y}, \bar{z})$ as parameters. Obviously $z_{xxx} = 0$. Also, because $z_y = \frac{(x-\bar{x})^2}{(y-\bar{y})^2}$ and $z_x = \frac{-2(x-\bar{x})}{(y-\bar{y})}$, we have $z_y = \frac{1}{4}z_x^2$. So, a para-CR structure defined by the cone $(x - \bar{x})^2 + (y - \bar{y})(z - \bar{z}) = 0$ in $\mathbb{R}^3 \times \mathbb{R}^3$ defines a system of PDEs on the plane

$$\boxed{z_{xxx} = 0 \quad \text{and} \quad z_y = \tfrac{1}{4}z_x^2 \quad \text{for} \quad z = z(x, y)}.$$

Conversely, given this system of PDEs, $z_{xxx} = 0$ solves as $z = \alpha(y)x^2 + \beta(y)x + \gamma(y)$, and $z_y = \frac{1}{4}z_x^2$ gives sucessively: $\alpha' = \alpha^2$, hence $\alpha = \frac{-1}{y-\bar{y}}$, $\beta' = \frac{-\beta}{y-\bar{y}}$, hence $\beta = \frac{2\bar{x}}{y-\bar{y}}$, $\gamma' = \frac{\bar{x}^2}{(y-\bar{y})^2}$, hence $\gamma = \frac{-\bar{x}^2}{y-\bar{y}} + \bar{z}$. This finally gives $z = \frac{-\bar{x}^2}{y-\bar{y}} + \bar{z} + \frac{2x\bar{x}}{y-\bar{y}} - \frac{x^2}{y-\bar{y}}$, *i.e.*, the cone

$$\boxed{(x - \bar{x})^2 + (y - \bar{y})(z - \bar{z}) = 0}.$$

In general, we consider the following system of two PDEs on the plane

$$\boxed{z_{xxx} = H(x, y, z, z_x, z_{xx}) \text{ and } z_y = G(x, y, z, z_x, z_{xx}) \text{ for } z = z(x,y)}. \quad (2.1)$$

Lemma 2.2 ([12]). *The general solution of* (2.1) *depends on 3 parameters* $(\bar{x},\bar{y},\bar{z})$, *and has the form* $z = z(x, y; \bar{x}, \bar{y}, \bar{z})$ *if and only if*

$$\boxed{\triangle H = D^3 G}, \quad (2.2)$$

where, abbreviating $p = z_x$, $r = z_{xx}$,

$$D = \partial_x + p\partial_z + r\partial_p + H\partial_r, \qquad \triangle = \partial_y + G\partial_z + DG\partial_p + D^2G\partial_r.$$

General solutions of systems (2.1) give examples of 5-dimensional para-CR structures. We prefer the PDE point of view, and we will stick to this in the following. In particular, in this point of view, para-CR transformations for hypersurfaces in $(x, y, z, \bar{x}, \bar{y}, \bar{z})$, are the *point transformations of variables* of (2.1).

Thus, we can either describe our para-CR geometry as a geometry of hypersurfaces in the $(x, y, z, \bar{x}, \bar{y}, \bar{z})$ space (modulo appropriate diffeomorphisms), or as a geometry of PDEs (2.1) considered modulo point transformation of variables.

It is clear from the hypersurfaces picture, that a 5-dimensional para-CR manifold M^5 is equipped with *two integrable distributions* D_1 and D_2. These are tangent to the *foliations* of M^5 obtained by intersecting it with either the 3-planes $\{x = \text{const}, y = \text{const}, z = \text{const}\}$, or the 3-planes $\{\bar{x} = \text{const}, \bar{y} = \text{const}, \bar{z} = \text{const}\}$.

In the PDE picture, these two distributions are the respective *annihilators* of the following system of 1-forms:

$$D_1 = \begin{pmatrix} \omega^1 = \mathrm{d}z - p\mathrm{d}x - G\mathrm{d}y \\ \omega^2 = \mathrm{d}p - r\mathrm{d}x - DG\mathrm{d}y \\ \omega^3 = \mathrm{d}r - H\mathrm{d}x - D^2G\mathrm{d}y \end{pmatrix}^{\perp} \quad \text{and}$$
$$D_2 = \begin{pmatrix} \omega^1 = \mathrm{d}z - p\mathrm{d}x - G\mathrm{d}y \\ \omega^4 = \mathrm{d}x \\ \omega^5 = \mathrm{d}y \end{pmatrix}^{\perp}. \tag{2.3}$$

Actually, the condition that D_1 is integrable is precisely the integrability condition (2.2) guaranteeing that the PDE system (2.1) has a 3-parameter family of solutions [12]. Note that the rank 4 distribution $D = D_1 + D_2$ is also well defined.

This enables for a *definition* of a 5-dimensional para-CR structure, locally, "à la Élie Cartan".

Definition 2.3. A 5-dimensional para-CR structure is a structure consisting of an equivalence class $[\omega]$ of coframes $\omega = (\omega^1, \omega^2, \omega^3, \omega^4, \omega^5)$ on $\mathbb{R}^5$ parameterized by (x, y, z, p, r), with an equivalence relation $\sim$ given by

$$\bar{\omega} \sim \omega \qquad \Longleftrightarrow \qquad \begin{pmatrix} \bar{\omega}^1 \\ \bar{\omega}^2 \\ \bar{\omega}^3 \\ \bar{\omega}^4 \\ \bar{\omega}^5 \end{pmatrix} = \begin{pmatrix} f_1 & 0 & 0 & 0 & 0 \\ f_2 & \rho e^{\phi} & f_4 & 0 & 0 \\ f_5 & f_6 & f_7 & 0 & 0 \\ \bar{f}_2 & 0 & 0 & \rho e^{-\phi} & \bar{f}_4 \\ \bar{f}_5 & 0 & 0 & \bar{f}_6 & \bar{f}_7 \end{pmatrix} \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \\ \omega^5 \end{pmatrix},$$

with $\omega^1 = \mathrm{d}z - p\mathrm{d}x - G\mathrm{d}y$, $\omega^2 = \mathrm{d}p - r\mathrm{d}x - DG\mathrm{d}y$, $\omega^3 = \mathrm{d}r - H\mathrm{d}x - D^2G\mathrm{d}y$, $\omega^4 = \mathrm{d}x$, $\omega^5 = \mathrm{d}y$, being in the class $[\omega]$.

The integrabilities of the two distributions D_1 and D_2, as defined in (2.3), implies that

$$\Big(\mathrm{d}\omega^1 - L_{11}\omega^2 \wedge \omega^4 - L_{12}\omega^2 \wedge \omega^5 - L_{21}\omega^3 \wedge \omega^4 - L_{22}\omega^3 \wedge \omega^5\Big) \wedge \omega^1 \equiv 0,$$

with a certain 2×2 matrix L of functions L_{AB}, $A, B = 1, 2$, on M^5 defined by this condition.

The matrix L, called the *Levi form*, is *not* well defined by the equivalence class of ω, but *its signature is*. Hence $\det(L) = 0$, or $\det(L) \neq 0$, is a para-CR invariant condition at each point. If $\det(L) \neq 0$, the corresponding para-CR structure is *non*degenerate, and it defines one of the *parabolic* geometries in dimension 5 (flat model – a flying soucer in the *attacking mode*).

In this paper we consider para-CR structures with

$$L \neq 0 \qquad \text{but such that} \qquad \det(L) \equiv 0.$$

These are 5-dimensional para-CR structures *with Levi form L degenerate in 1 direction*.

In terms of our PDEs, this degeneracy means that

$$G_r \equiv 0, \quad \text{that is} \quad G = G(x, y, z, z_x). \tag{2.4}$$

We also *do not want* that our para-CR structure is locally para-CR-equivalent to a product of a 3-dimensional para-CR manifold M^3 and a product $\mathbb{R} \times \mathbb{R}$. This results in our further assumption that

$$G_{pp} \neq 0. \tag{2.5}$$

3. Basic invariants for degenerate para-CR structures

Summarizing, we study systems of PDEs on the plane:

$$\boxed{z_{xxx} = H(x, y, z, p, r) \qquad \text{and} \qquad z_y = G(x, y, z, p) \qquad \text{for} \qquad z(x, y)},$$

such that

$$\boxed{\triangle H = D^3 G} \qquad \text{and} \qquad \boxed{G_{pp} \neq 0},$$

with $D = \partial_x + p\partial_z + r\partial_p + H\partial_r$, $\triangle = \partial_y + G\partial_z + DG\partial_p + D^2G\partial_r$, and $p = z_x$, $r = z_{xx}$, considered modulo *point transformations of variables*. This is equivalent to study coframes $\omega^1 = \mathrm{d}z - p\mathrm{d}x - G\mathrm{d}y$, $\omega^2 = \mathrm{d}p - r\mathrm{d}x - DG\mathrm{d}y$, $\omega^3 = \mathrm{d}r - H\mathrm{d}x - D^2G\mathrm{d}y$, $\omega^4 = \mathrm{d}x$, $\omega^5 = \mathrm{d}y$, with $D^3G = \triangle H$, $G_{pp} \neq 0$, and $G_r \equiv 0$, given modulo

$$\begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \\ \omega^5 \end{pmatrix} \longmapsto \begin{pmatrix} f_1 & 0 & 0 & 0 & 0 \\ f_2 & \rho e^{\phi} & f_4 & 0 & 0 \\ f_5 & f_6 & f_7 & 0 & 0 \\ \bar{f}_2 & 0 & 0 & \rho e^{-\phi} & \bar{f}_4 \\ \bar{f}_5 & 0 & 0 & \bar{f}_6 & \bar{f}_7 \end{pmatrix} \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \\ \omega^5 \end{pmatrix}. \tag{3.1}$$

In reference [15], studying such structures, we established among other things, the following

Theorem 3.1. *It is always possible to invariantly force the lifted coframe* $\theta^1 = f_1\omega^1$, $\theta^2 = f_2\omega^1 + \rho e^{\phi}\omega^2 + f_4\omega^3$, $\theta^3 = f_5\omega^1 + f_6\omega^2 + f_7\omega^3$, $\theta^4 = \bar{f}_2\omega^1 + \rho e^{-\phi}\omega^4 + \bar{f}_4\omega^5$, $\theta^5 = \bar{f}_5\omega^1 + \bar{f}_6\omega^2 + \bar{f}_7\omega^3$ *to satisfy the following* EDS*:*

$$
\begin{aligned}
d\theta^1 &= \Omega_1 \wedge \theta^1 + \theta^2 \wedge \theta^4,\\
d\theta^2 &= \theta^2 \wedge \left(\Omega_2 - \tfrac{1}{2}\Omega_1\right) - \theta^1 \wedge \Omega_3 + \theta^3 \wedge \theta^4,\\
d\theta^3 &= 2\theta^3 \wedge \Omega_2 - \theta^2 \wedge \Omega_3 + Q\theta^1 \wedge \theta^3 - \tfrac{1}{2}\left(\tfrac{e^{\phi}}{3\rho}\right)^3 \mathbf{A}\theta^1 \wedge \theta^4 + \tfrac{e^{-\phi}}{3\rho}\mathbf{C}\theta^2 \wedge \theta^3,\\
d\theta^4 &= -\theta^2 \wedge \theta^5 - \theta^4 \wedge \left(\tfrac{1}{2}\Omega_1 + \Omega_2\right) - \theta^1 \wedge \Omega_4,\\
d\theta^5 &= -2\theta^5 \wedge \Omega_2 + \theta^4 \wedge \Omega_2 + \left(\tfrac{e^{\phi}}{3\rho}\right)^3 \mathbf{B}\theta^1 \wedge \theta^2 + Q\theta^1 \wedge \theta^5 + \tfrac{e^{\phi}}{3\rho}\tilde{\mathbf{C}}\theta^4 \wedge \theta^5,
\end{aligned}
$$

in which three primary relative differential invariants are

$$
\begin{aligned}
\mathbf{A} &= 9D^2H_r - 27DH_p - 18H_rDH_r + 18H_pH_r + 4H_r^3 + 54H_z,\\
\mathbf{B} &= \left(\tfrac{1}{2G_{pp}^3}\right)\left[40G_{ppp}^3 - 45G_{pp}G_{ppp}G_{pppp} + 9G_{pp}^2G_{ppppp}\right],\\
\mathbf{C} &= \left(\tfrac{1}{G_{pp}}\right)\left[2G_{ppp} + G_{pp}H_{rr}\right],
\end{aligned}
$$

that is, the vanishing or not of each of **A**, **B**, **C** *is* an invariant property *of the corresponding para-*CR *structure. Lastly,* $\tilde{\mathbf{C}}$ *vanishes identically when* $\mathbf{C} \equiv 0$.

Remarks 3.2.

- *Flat model*: $\mathbf{A} = \mathbf{B} = \mathbf{C} = 0$, and this is locally equivalent to $z_{xxx} = 0$, $z_y = \frac{1}{4}z_x^2$, *i.e.*, to the para-CR structure from our Example 2.1 in the beginning, *cf.* [15];
- *Symmetries*: A vector field X on $M^5 \ni (x, y, z, p, r)$ is a *symmetry* of the para-CR structure as defined in (2.1)-(2.3) if and only if

$$
\begin{aligned}
&(\mathcal{L}_X\omega^1) \wedge \omega^1 = 0,\\
&(\mathcal{L}_X\omega^2) \wedge \omega^1 \wedge \omega^2 \wedge \omega^3 = 0, \qquad (\mathcal{L}_X\omega^4) \wedge \omega^1 \wedge \omega^4 \wedge \omega^5 = 0,\\
&(\mathcal{L}_X\omega^3) \wedge \omega^1 \wedge \omega^2 \wedge \omega^3 = 0, \qquad (\mathcal{L}_X\omega^5) \wedge \omega^1 \wedge \omega^4 \wedge \omega^5 = 0.
\end{aligned}
$$

  Any Lie bracket of two symmetries is a symmetry, which brings the notion of a *symmetry algebra* of a para-CR-structure: the Lie algebra over the reals of all symmetries;
- For our flat model with $\mathbf{A} = \mathbf{B} = \mathbf{C} = 0$, the *symmetry algebra* is $\mathfrak{sp}(4, \mathbb{R}) \simeq \mathfrak{so}(2, 3)$.

4. Geometry of Wünschmann and Monge Invariants

The explicit expressions for the *relative invariants* **A** and **B** of the considered para-CR structures redirect us to the *theory of* 3rd *order* ODEs *considered mod-*

ulo contact transformations of variables and to *differential geometry of conics on the plane*. We therefore make the following interlude in our main theme now.

4.1. 3rd order ODEs considered modulo contact transformation of variables

We formulate a theorem [5, 6] about the main structure which is associated with third-order ODEs modulo contact transformations of variables, namely about an $\mathfrak{sp}(4,\mathbb{R})$-valued Cartan connection on the bundle $P^{10} \to \mathsf{J}^2$. This structure will serve as a starting point for analyzing further geometries of ODEs.

Theorem 4.1. *To every third order ODE $z''' = H(x,z,z',z'')$, there is associated a (principal) fibre bundle $H_6 \to P^{10} \to \mathsf{J}^2$, over the space of second jets, where* $\dim P^{10} = 10$ *and H_6 is an appropriate six-dimensional subgroup of $SP(4,\mathbb{R})$, with the group parameters u_i, $i = 1, 2\dots, 6$, and a unique coframe of 1-forms $(\theta^1, \theta^2, \theta^3, \theta^4, \theta^5, \Omega_1, \Omega_2, \Omega_3, \Omega_4, \Omega_5)$ on P^{10}, which satisfies the following* EDS*:*

$$
\begin{aligned}
\mathrm{d}\theta^1 &= \Omega_1\wedge\theta^1 + \theta^4\wedge\theta^2,\\
\mathrm{d}\theta^2 &= \Omega_2\wedge\theta^1 + \Omega_3\wedge\theta^2 + \theta^4\wedge\theta^3,\\
\mathrm{d}\theta^3 &= \Omega_2\wedge\theta^2 + (2\Omega_3 - \Omega_1)\wedge\theta^3 + \mathbf{A_2}\theta^2\wedge\theta^1 + \mathbf{A_1}\theta^4\wedge\theta^1,\\
\mathrm{d}\theta^4 &= \Omega_4\wedge\theta^1 + (\Omega_1 - \Omega_3)\wedge\theta^4 + \theta^5\wedge\theta^2,\\
\mathrm{d}\theta^5 &= \Omega_4\wedge\theta^4 + (\Omega_1 - 2\Omega_3)\wedge\theta^5 + (\mathbf{A_7} + \mathbf{Z_3})\theta^1\wedge\theta^2 + \mathbf{Z_4}\theta^1\wedge\theta^3\\
&\quad - \mathbf{A_5}\theta^1\wedge\theta^4 + \mathbf{Z_1}\theta^2\wedge\theta^3,\\
\mathrm{d}\Omega_1 &= \Omega_5\wedge\theta^1 + \Omega_4\wedge\theta^2 - \Omega_2\wedge\theta^4,\\
\mathrm{d}\Omega_2 &= (\Omega_3 - \Omega_1)\wedge\Omega_2 + \tfrac{1}{2}\Omega_5\wedge\theta^2 + \Omega_4\wedge\theta^3 + \mathbf{A_3}\,\theta^1\wedge\theta^2 + \mathbf{A_4}\theta^1\wedge\theta^4,\\
\mathrm{d}\Omega_3 &= [2pt]\tfrac{1}{2}\Omega_5\wedge\theta^1 + \Omega_4\wedge\theta^2 + \theta^5\wedge\theta^3 + \mathbf{A_5}\theta^1\wedge\theta^2 + \mathbf{A_2}\theta^1\wedge\theta^4,\\
\mathrm{d}\Omega_4 &= \theta^5\wedge\Omega_2 + \Omega_4\wedge\Omega_3 + \tfrac{1}{2}\Omega_5\wedge\theta^4 + (\mathbf{A_6} + \mathbf{Z_2})\theta^1\wedge\theta^2 + 2\mathbf{Z_3}\theta^1\wedge\theta^3\\
&\quad - \mathbf{A_3}\theta^1\wedge\theta^4 + \mathbf{Z_4}\theta^2\wedge\theta^3\\
\mathrm{d}\Omega_5 &= \Omega_5\wedge\Omega_1 + 2\Omega_4\wedge\Omega_2 + \mathbf{C_1}\theta^1\wedge\theta^2 + 2\mathbf{Z_2}\theta^1\wedge\theta^3\\
&\quad + \mathbf{A_8}\theta^1\wedge\theta^4 + 2\mathbf{Z_3}\theta^2\wedge\theta^3.
\end{aligned}
\tag{4.1}
$$

Here $\mathbf{A_1}, \dots, \mathbf{A_8}, \mathbf{Z_1}, \dots, \mathbf{Z_4}, \mathbf{C_1}$ are functions on P^{10}.

The $8 + 4 + 1$ *functions $\mathbf{A_1}, \dots, \mathbf{Z_1}, \dots, \mathbf{C_1}$ are contact relative invariants of the underlying ODE and the full set of contact invariants can be constructed by consecutive differentiations of $\mathbf{A_1}, \dots, \mathbf{Z_1}, \dots, \mathbf{C_1}$ with respect to the frame $(X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8, X_9, X_{10})$ dual to $(\theta^1, \theta^2, \theta^3, \theta^4, \Omega_1, \Omega_2, \Omega_3, \Omega_4, \Omega_5, \Omega_6)$.*

The coframe $(\theta^1, \theta^2, \theta^3, \theta^4, \Omega_1, \Omega_2, \Omega_3, \Omega_4, \Omega_5, \Omega_6)$ *defines the* $\mathfrak{sp}(4,\mathbb{R})$*-valued Cartan normal connection* $\widehat{\omega}$ *on* P^{10} *by*

$$\widehat{\omega} = \begin{pmatrix} \frac{1}{2}\Omega_1 & \frac{1}{2}\Omega_2 & -\frac{1}{2}\Omega_4 & -\frac{1}{4}\Omega_5 \\ \theta^4 & \Omega_3 - \frac{1}{2}\Omega_1 & -\theta^5 & -\frac{1}{2}\Omega_4 \\ \theta^2 & \theta^3 & \frac{1}{2}\Omega_1 - \Omega_3 & -\frac{1}{2}\Omega_2 \\ 2\theta^1 & \theta^2 & -\theta^4 & -\frac{1}{2}\Omega_1 \end{pmatrix}. \tag{4.2}$$

The EDS (4.1) *gives explicit formulas for the* curvature $\widehat{K} = \mathrm{d}\widehat{\omega} + \widehat{\omega} \wedge \widehat{\omega}$ *of this Cartan normal connection, with the invariant functions* $\mathbf{A_{ff}}, \mathbf{Z_{fi}}, \mathbf{C_1}$*, being the apropriate entries in the coframe components matrices* $\widehat{K}_{ij}$ *of* $\widehat{K} = \frac{1}{2}\widehat{K}_{ij}\theta^i \wedge \theta^j$.

Two 3rd *order* ODEs $y''' = F(x, y, y', y'')$ *and* $\bar{y}''' = \bar{F}(\bar{x}, \bar{y}, \bar{y}', \bar{y}'')$ *are locally contact equivalent if and only if their associated Cartan connections are locally diffeomorphic, that is, there exists a local bundle diffeomorphism* $\Phi\colon \bar{P} \to P$ *such that*

$$\Phi^* \widehat{\omega} = \overline{\widehat{\omega}}.$$

It further follows that:

- $\mathbf{A_2}, \ldots, \mathbf{A_8}$ express in terms of coframe derivatives of $\mathbf{A_1}$;
- $\mathbf{Z_2}, \ldots, \mathbf{Z_4}$ express in terms of coframe derivatives of $\mathbf{Z_1}$;
- $\mathbf{C_1}$ is a function of coframe derivatives of both $\mathbf{A_1}$ and $\mathbf{Z_1}$.

So only $\mathbf{A_1}$ and $\mathbf{Z_1}$ are *basic* (primary) invariants, namely all other (secondary) invariants are deduced by differentiation. Their remarkable explicit expressions are given by

Proposition 4.2. *Letting* $D = \partial_x + p\partial_z + r\partial_p + H\partial_r$*, and* u_1 *and* u_3 *be the parameters along the gauge group* H_6 *mentioned in Theorem* 4.1*, one has:*

$$\begin{aligned} \mathbf{A_1} &= \tfrac{1}{2}\left(\frac{u_3}{3u_1}\right)^3 \left[9D^2 H_r - 27 D H_p - 18 H_r D H_r + 18 H_p H_r + 4 H_r^3 + 54 H_z\right] \\ &=: \tfrac{1}{2}\left(\frac{u_3}{3u_1}\right)^3 \mathbf{A}, \\ \mathbf{Z_1} &= \frac{u_1^2}{6u_3^5}\, H_{rrrr} =: \frac{u_1^2}{6u_3^5}\, \mathbf{Z}. \end{aligned}$$

Thus, the *contact* relative invariant $\mathbf{A_1}$ for a contact equivalence class of ODEs $z''' = H(x, z, z', z'')$ is given, modulo a nonvanishing scaling factor, by the same expression as one of our basic para-CR invariants $\mathbf{A}$ for the 5-dimensional para-CR manifolds with Levi form degenerate in one direction.[1]

[1] It is not a big surprise, though, since our PDEs on the plane (2.1) include a one parameter family of ODEs $z''' = H(x, y, z, z', z'')$, parametrized by the variable y.

The expression $\mathbf{A} = 9D^2H_r - 27DH_p - 18H_rDH_r + 18H_pH_r + 4H_r^3 + 54H_z$ was for the first time obtained in 1905 by Wünschmann [24], who observed that its vanishing or not is a contact invariant property of an ODE $z''' = H(x, z, z', z'')$. More importantly, he also established the *geometric interpretation* of the *vanishing* of $\mathbf{A}$. According to Wünschmann, if $\mathbf{A} \equiv 0$, the 3-dimensional solution space of the ODE $z''' = H(x, z, z', z'')$ is naturally equipped with a conformal Lorentzian structure; moreover, there is a *local one-to-one correspondence* between *3-dimensional conformal Lorentzian structures* and *contact equivalence classes of* ODEs $z''' = H(x, z, z', z'')$ *satisfying* $\mathbf{A} \equiv 0$.

The first person who observed that the vanishing or not of $\mathbf{Z} = H_{rrrr}$ is a contact invariant property of the ODE $z''' = H(x, z, z', z'')$ was Chern in 1940 [1]. The *geometric meaning of the condition that* $\mathbf{Z}$ *vanishes* is less known [5]. To fully apreciate it, one needs a rather recent notion of a *contact projective structure* [4]. Here is its definition, adapted to our case of a 3-dimensional manifold of first jets J^1 of the equation $z''' = H(x, z, z', z'')$.

Definition 4.3. A *contact projective structure* on the first jet space $J^1 \ni (x, z, p)$ consists of:

(i) The contact distribution $\mathcal{C}$, that is the distribution annihilated by $\omega^1 = \mathrm{d}z - p\mathrm{d}x$;
(ii) A family of unparameterized curves in J^1, which are everywhere tangent to $\mathcal{C}$ and such that:
   (a) For any given point and direction in $\mathcal{C}$, there is exactly one curve passing through that point and tangent to that direction;
   (b) Curves of the family are among unparameterized geodesics for some linear connection on J^1.

In other words, the idea of this geometry in the context of ODEs is as follows[2]: *Consider the solutions of the ODE* $z''' = H(x, z, z', z'')$ *as a family of curves in* J^1 *and ask whether these curves are among geodesics of a linear connection. The answer to this question is positive if and only if* $H_{rrrr} \equiv 0$*, and in this case there is a whole family of connections for which the solutions are geodesics.*

This information about the Wünschmann, $\mathbf{A}$, and the Chern, $\mathbf{Z}$, invariants can be nicely phrased in terms of the natural *double fibration*

$$\begin{array}{ccc} & \mathrm{J}^2 & \\ {}^{\pi_2}\swarrow & & \searrow^{\pi_1} \\ \mathrm{S} & & \mathrm{J}^1 \end{array} \qquad (4.3)$$

of the space of second jets for the ODE $z''' = H(x, z, z', z'')$ over (a) the solution space S and (b) the space of first jets J^1. Here, π_1 is the natural projection from J^2

[2] Here we quote from the PhD Thesis [5] of Godliński, who was the first to observe this.

to J^1, $\pi_1(x,z,z',z'') = (x,z,z')$, and π_2 is a projection from J^2 to the space of solutions S identifying points on the integral curves of the total differential vector field $D = \partial_x + z'\partial_z + z''\partial_{z'} + H\partial_{z''}$ on J^2. In terms of this double fibration, we have the following proposition, in which $z' = p$, $z'' = r$, and $D = \partial_x + p\partial_z + r\partial_p + H\partial_r$.

Proposition 4.4. *Two basic (primary) local contact relative invariants for* 3rd *order* ODEs $z''' = H(x,z,z',z'')$ *are the* Wünschmann invariant, $\mathbf{A} = 9D^2H_r - 27DH_p - 18H_rDH_r + 18H_pH_r + 4H_r^3 + 54H_z$, *and the Chern invariant,* $\mathbf{Z} = H_{rrrr}$.

The vanishing of the Wünschmann invariant, $\mathbf{A} \equiv 0$, *is equivalent to having a* conformal Lorentzian structure *on the solution space S, while the vanishing of the Chern invariant,* $\mathbf{Z} \equiv 0$, *is equivalent to having a* contact projective structure *on the space of first jets* J^1.

4.2. Conics on the plane

Consider the most general *conic* on the plane $\mathbb{R}^2$ parameterized by $(p,G) \in \mathbb{R}^2$. Such a conic is a curve in $\mathbb{R}^2$ given by the equation

$$a_1G^2 + 2a_2pG + a_3p^2 + a_4G + a_5p + a_6 = 0,$$

and $a_1,\dots,a_6$ are real constants. One can think about the equation $a_1G^2 + 2a_2pG + a_3p^2 + a_4G + a_5p + a_6 = 0$ as an implicit relation for a function $G = G(p)$, whose graph on the plane is a conic. It was Monge [17], who in 1810 found a differential equation satisfied by this function. To get this equation one eliminates $a_2,\dots,a_6$ from the system of linear equations

$$\frac{\mathrm{d}^k}{\mathrm{d}p^k}\Big(a_1G^2+2a_2pG+a_3p^2+a_4G+a_5p+a_6\Big)=0, \quad \text{for all} \quad k=0,1,2,3,4,5.$$

The result is

$$a_1G_{pp}\left(40G_{ppp}^3 - 45G_{pp}G_{ppp}G_{pppp} + 9G_{pp}^2G_{ppppp}\right) = 0.$$

Excluding the nongeneric case when $a_1G_{pp} = 0$, one obtains the Monge 5th order ODE

$$40G_{ppp}^3 - 45G_{pp}G_{ppp}G_{pppp} + 9G_{pp}^2G_{ppppp} = 0$$

for a local function $G = G(p)$ to have a graph contained in a general conic.

In the context of this paper it is necessary to note, that the left-hand side of this expression $\mathbf{M} := 40G_{ppp}^3 - 45G_{pp}G_{ppp}G_{pppp} + 9G_{pp}^2G_{ppppp}$ is, modulo a nonvanishing factor, the same as the relative para-CR invariant $\mathbf{B}$ for 5-dimensional para-CR structures given by (2.1)-(2.2), (2.4)-(2.5). More precisely, the vanishing of $\mathbf{B}$ is equivalent to the vanishing of a 3-parameter family of Monge 5th order ODEs $\mathbf{M} = 0$, with parameters x, y, z.

This justifies our terminology, which we adopt from now on, that the relative para-CR invariant

$$\mathbf{B} = \frac{1}{2G_{pp}^3}\mathbf{M},$$

or its core

$$\mathbf{M} = 40G_{ppp}^3 - 45G_{pp}G_{ppp}G_{pppp} + 9G_{pp}^2 G_{ppppp},$$

will be called the *Monge invariant*.

In this way we have a nice geometric interpretation of the vanishing of the para-CR invariant $\mathbf{B}$: it vanishes if and only if $G = G(x, y, p, z)$ defines a (general) conic on the plane (p, G).

We close this section with a remark that we have yet another geometric interpretation of the vanishing of the invariant $\mathbf{B}$. This is described in our recent paper [14], and is related to the single PDE $z_y = G(x, y, z, z_x)$ for a function $z = z(x, y)$ considered modulo point transformations of variables.

5. 5-Dimensional Para-CR Structures as 3rd Order ODEs

Theorem 3.1, which we invoked in Section 2 of the present paper, has its more technical, but also more refined, version which we need now. We quote it from reference [15].

Theorem 5.1. *Given the* 1*-forms*

$$\begin{aligned}
\omega^1 &= \mathrm{d}z - p\mathrm{d}x - G\mathrm{d}y,\\
\omega^2 &= \mathrm{d}p - r\mathrm{d}x - DG\mathrm{d}y,\\
\omega^3 &= \mathrm{d}r - H\mathrm{d}x - D^2G\mathrm{d}y,\\
\omega^4 &= \mathrm{d}x, \qquad \omega^5 = \mathrm{d}y,
\end{aligned}$$

representing a 5*-dimensional para-*CR *manifold with* $G_r = 0$ *and* $G_{pp} \neq 0$ *one can always find a para-*CR *equivalent set of* 1*-forms*

$$\begin{aligned}
\bar\omega^1 &= f_1\omega^1,\\
\bar\omega^2 &= f_2\omega^1 + \rho \mathrm{e}^{\phi}\omega^2 + f_4\omega^3,\\
\bar\omega^3 &= f_5\omega^1 + f_6\omega^2 + f_7\omega^3,\\
\bar\omega^4 &= \bar f_2\omega^1 + \rho \mathrm{e}^{-\phi}\omega^4 + \bar f_4\omega^5,\\
\bar\omega^5 &= \bar f_5\omega^1 + \bar f_6\omega^4 + \bar f_7\omega^5,
\end{aligned}$$

and additional 1*-forms* ϖ_1*,* ϖ_2*,* ϖ_3*,* ϖ_4 *with*

$$\bar\omega^1 \wedge \bar\omega^2 \wedge \bar\omega^3 \wedge \bar\omega^4 \wedge \bar\omega^5 \wedge \varpi_1 \wedge \varpi_2 \wedge \varpi_3 \wedge \varpi_4 \neq 0,$$

such that the nine 1-forms $(\bar{\omega}^1, \bar{\omega}^2, \bar{\omega}^3, \bar{\omega}^4, \bar{\omega}^5, \varpi_1, \varpi_2, \varpi_3, \varpi_4)$ *satisfy the following* EDS*:*

$$
\begin{aligned}
\mathrm{d}\bar{\omega}^1 &= -\,\bar{\omega}^1 \wedge \varpi_1 + \bar{\omega}^2 \wedge \bar{\omega}^4, \\
\mathrm{d}\bar{\omega}^2 &= -\,\bar{\omega}^1 \wedge \varpi_3 + \bar{\omega}^2 \wedge \big(\varpi_2 - \tfrac{1}{2}\varpi_1\big) + \bar{\omega}^3 \wedge \bar{\omega}^4, \\
\mathrm{d}\bar{\omega}^3 &= -\,\bar{\omega}^2 \wedge \varpi_3 + 2\bar{\omega}^3 \wedge \varpi_2 + \tfrac{1}{8}\big(2I^3{}_{|4} + I^3{}_{|52}\big)\bar{\omega}^1 \wedge \bar{\omega}^3 \\
&\qquad + I^1\,\bar{\omega}^1 \wedge \bar{\omega}^4 + I^3\,\bar{\omega}^2 \wedge \bar{\omega}^3, \\
\mathrm{d}\bar{\omega}^4 &= -\,\bar{\omega}^1 \wedge \varpi_4 - \bar{\omega}^4 \wedge \big(\varpi_2 + \tfrac{1}{2}\varpi_1\big) - \bar{\omega}^2 \wedge \bar{\omega}^5, \\
\mathrm{d}\bar{\omega}^5 &= \bar{\omega}^4 \wedge \varpi_4 - 2\bar{\omega}^5 \wedge \varpi_2 + I^2\,\bar{\omega}^1 \wedge \bar{\omega}^2 + \tfrac{1}{8}\big(2I^3{}_{|4} + I^3{}_{|52}\big)\bar{\omega}^1 \wedge \bar{\omega}^5 \\
&\qquad - \tfrac{1}{2}I^3{}_{|5}\,\bar{\omega}^4 \wedge \bar{\omega}^5,
\end{aligned}
\tag{5.1}
$$

$$
\begin{aligned}
\mathrm{d}I^1 &= I^1{}_{|1}\bar{\omega}^1 + I^1{}_{|2}\bar{\omega}^2 + I^1{}_{|3}\bar{\omega}^3 + I^1{}_{|4}\bar{\omega}^4 - \tfrac{3}{2}I^1\varpi_1 - 3I^1\varpi_2, \\
\mathrm{d}I^2 &= I^2{}_{|1}\bar{\omega}^1 + I^2{}_{|2}\bar{\omega}^2 + I^2{}_{|4}\bar{\omega}^4 + I^2{}_{|5}\bar{\omega}^5 - \tfrac{3}{2}I^2\varpi_1 + 3I^2\varpi_2, \\
\mathrm{d}I^3 &= I^3{}_{|1}\bar{\omega}^1 + I^3{}_{|2}\bar{\omega}^2 + I^3{}_{|3}\bar{\omega}^3 + I^3{}_{|4}\bar{\omega}^4 + I^3{}_{|5}\bar{\omega}^5 - \tfrac{1}{2}I^3\varpi_1 + I^3\varpi_2.
\end{aligned}
$$

Integrability conditions ($\mathrm{d}^2 \equiv 0$) *of these equations imply the existence of a* 1*-form* ϖ_5 *such that:*

$$
\begin{aligned}
\mathrm{d}\varpi_1 &= \bar{\omega}^1 \wedge \varpi_5 + \bar{\omega}^2 \wedge \varpi_4 - \bar{\omega}^4 \wedge \varpi_3, \\
\mathrm{d}\varpi_2 &= -\,\tfrac{1}{4}I^3\bar{\omega}^1 \wedge \varpi_3 - \tfrac{1}{8}I^3{}_{|5}\bar{\omega}^1 \wedge \varpi_4 - \tfrac{1}{2}\bar{\omega}^2 \wedge \varpi_4 - \tfrac{1}{2}\bar{\omega}^4 \wedge \varpi_3 \\
&\quad + \tfrac{1}{16}\big(I^3{}_{|522} + 2I^3{}_{|42} - 8I^2{}_{|5}\big)\bar{\omega}^1 \wedge \bar{\omega}^2 + \tfrac{1}{16}\big(I^3{}_{|523} + 2I^3{}_{|43}\big)\bar{\omega}^1 \wedge \bar{\omega}^3 \\
&\quad + \tfrac{1}{16}\big(8I^1{}_{|3} - I^3{}_{|524} - 2I^3{}_{|44}\big)\bar{\omega}^1 \wedge \bar{\omega}^4 - \tfrac{1}{16}\big(I^3{}_{|525} + 2I^3{}_{|45}\big)\bar{\omega}^1 \wedge \bar{\omega}^5 \\
&\quad + \tfrac{1}{8}\big(I^3{}_{|52} - 2I^3{}_{|4}\big)\bar{\omega}^2 \wedge \bar{\omega}^4 - \tfrac{1}{2}I^3{}_{|5}\bar{\omega}^2 \wedge \bar{\omega}^5 + I^3\bar{\omega}^3 \wedge \bar{\omega}^4 - \bar{\omega}^3 \wedge \bar{\omega}^5, \\
\mathrm{d}\varpi_3 &= \varpi_3 \wedge \big(\tfrac{1}{2}\varpi_1 + \varpi_2\big) + \tfrac{1}{8}\big(2I^3{}_{|4} + I^3{}_{|52}\big)\bar{\omega}^1 \wedge \varpi_3 + \tfrac{1}{4}I^3\bar{\omega}^2 \wedge \varpi_3 \\
&\quad + \tfrac{1}{8}I^3{}_{|5}\bar{\omega}^2 \wedge \varpi_4 + \tfrac{1}{2}\bar{\omega}^2 \wedge \varpi_5 + \bar{\omega}^3 \wedge \varpi_4 + J^1\bar{\omega}^1 \wedge \bar{\omega}^2 \\
&\quad + \tfrac{1}{4}\big(4I^2{}_{|5} + 4I^3{}_{|1} - 2I^3{}_{|42} - I^3{}_{|522}\big)\bar{\omega}^1 \wedge \bar{\omega}^3 + \big(I^1I^3 - I^1{}_{|2}\big)\bar{\omega}^1 \wedge \bar{\omega}^4 \\
&\quad + I^1\bar{\omega}^1 \wedge \bar{\omega}^5 - \tfrac{1}{16}\big(2I^3{}_{|43} + I^3{}_{|523}\big)\bar{\omega}^2 \wedge \bar{\omega}^3 \\
&\quad + \tfrac{1}{16}\big(I^3{}_{|524} - 8I^1{}_{|3} + 2I^3{}_{|44}\big)\bar{\omega}^2 \wedge \bar{\omega}^4 + \tfrac{1}{16}\big(2I^3{}_{|45} + I^3{}_{|525}\big)\bar{\omega}^2 \wedge \bar{\omega}^5 \\
&\quad - \tfrac{1}{8}\big(2I^3{}_{|4} + I^3{}_{|52}\big)\bar{\omega}^3 \wedge \bar{\omega}^4, \\
\mathrm{d}\varpi_4 &= \varpi_4 \wedge \big(\tfrac{1}{2}\varpi_1 - \varpi_2\big) + \tfrac{1}{8}\big(2I^3{}_{|4} + I^3{}_{|52}\big)\bar{\omega}^1 \wedge \varpi_4 - \tfrac{1}{4}I^3\bar{\omega}^4 \wedge \varpi_3 \\
&\quad - \tfrac{1}{8}I^3{}_{|5}\bar{\omega}^4 \wedge \varpi_4 + \tfrac{1}{2}\bar{\omega}^4 \wedge \varpi_5 + \bar{\omega}^5 \wedge \varpi_3 \\
&\quad + \tfrac{1}{128}\Big(16\big(I^3{}_{|14} - I^1I^3{}_{|3}\big) + 8\big(I^3{}_{|521} - I^1{}_{|3}I^3\big) \\
&\qquad\qquad + 2I^3I^3{}_{|44} + I^3I^3{}_{|524}\Big)\bar{\omega}^1 \wedge \bar{\omega}^4 \\
&\quad + \tfrac{1}{2}\big(2I^2{}_{|4} + I^2I^3{}_{|5}\big)\bar{\omega}^1 \wedge \bar{\omega}^2 - I^2\bar{\omega}^1 \wedge \bar{\omega}^3
\end{aligned}
$$

$$
\begin{aligned}
&+\tfrac{1}{16}\big(8I^2{}_{|5}-2I^3{}_{|42}-I^3{}_{|522}\big)\bar{\omega}^2\wedge\bar{\omega}^4\\
&+\tfrac{1}{4}\big(I^3{}_{|524}-4I^1{}_{|3}+2I^3{}_{|44}+2I^3{}_{|51}\big)\bar{\omega}^1\wedge\bar{\omega}^5\\
&+\tfrac{1}{8}\big(2I^3{}_{|4}+I^3{}_{|52}\big)\bar{\omega}^2\wedge\bar{\omega}^5\\
&-\tfrac{1}{16}\big(2I^3{}_{|43}+I^3{}_{|523}\big)\bar{\omega}^3\wedge\bar{\omega}^4-\tfrac{1}{16}\big(2I^3{}_{|45}+I^3{}_{|525}\big)\bar{\omega}^4\wedge\bar{\omega}^5,\\
\mathrm{d}\varpi_5=&\,\varpi_5\wedge\varpi_1+2\varpi_4\wedge\varpi_3+J^2\bar{\omega}^1\wedge\varpi_3+J^3\bar{\omega}^1\wedge\varpi_4\\
&+\tfrac{1}{4}\big(2I^3{}_{|4}+I^3{}_{|52}\big)\bar{\omega}^1\wedge\varpi_5\\
&+\tfrac{1}{8}\big(2I^3{}_{|4}+I^3{}_{|52}\big)\bar{\omega}^2\wedge\varpi_4-\tfrac{1}{8}\big(2I^3{}_{|4}+I^3{}_{|52}\big)\bar{\omega}^4\wedge\varpi_4\\
&+J^4\bar{\omega}^1\wedge\bar{\omega}^2+J^5\bar{\omega}^1\wedge\bar{\omega}^3\\
&+J^6\bar{\omega}^1\wedge\bar{\omega}^4+J^7\bar{\omega}^1\wedge\bar{\omega}^5-I^2\bar{\omega}^2\wedge\bar{\omega}^3+J^8\bar{\omega}^2\wedge\bar{\omega}^4\\
&+\tfrac{1}{4}\big(I^3{}_{|524}-4I^1{}_{|3}+2I^3{}_{|44}+2I^3{}_{|51}\big)\bar{\omega}^2\wedge\bar{\omega}^5\\
&+\tfrac{1}{4}\big(4I^2{}_{|5}+4I^3{}_{|1}-2I^3{}_{|42}-I^3{}_{|522}\big)\bar{\omega}^3\wedge\bar{\omega}^4-I^1\bar{\omega}^4\wedge\bar{\omega}^5,\\
\mathrm{d}I^3{}_{|2}=&\,\tfrac{1}{16}\Big(16\big(I^3{}_{|12}-I^2I^3{}_{|5}\big)+I^3\big(8I^2{}_{|5}-2I^3{}_{|42}-I^3{}_{|522}\big)\Big)\bar{\omega}^1\\
&+I^3{}_{|22}\bar{\omega}^2+I^3{}_{|23}\bar{\omega}^3+\tfrac{1}{8}\Big(8(I^3{}_{|42}+I^3{}_{|1})+I^3(I^3{}_{|52}-2I^3{}_{|4})\Big)\bar{\omega}^4\\
&+\tfrac{1}{2}\Big(2(I^3{}_{|52}-I^3{}_{|4})-I^3I^3{}_{|5}\Big)\bar{\omega}^5\\
&-I^3{}_{|2}\varpi_1+2I^3{}_{|2}\varpi_1-I^3{}_{|3}\varpi_3-I^3\varpi_4,\\
\mathrm{d}I^3{}_{|3}=&\,\tfrac{1}{16}\Big(16I^3{}_{|13}-2I^3{}_{|3}(2I^3{}_{|4}+I^3{}_{|52})-I^3(I^3{}_{|523}+2I^3{}_{|43})\Big)\bar{\omega}^1\\
&+(I^3{}_{|23}-I^3I^3{}_{|3})\bar{\omega}^2\\
&+I^3{}_{|33}\bar{\omega}^3+\tfrac{1}{2}\Big(I^3{}_{|523}+2I^3{}_{|43}-2(I^3{}_{|2}+(I^3)^2)\Big)\bar{\omega}^4+3I^3\bar{\omega}^5\\
&-\tfrac{1}{2}I^3{}_{|3}\varpi_1+3I^3{}_{|3}\varpi_2,\\
\mathrm{d}I^3{}_{|5}=&\,I^3{}_{|51}\bar{\omega}^1+I^3{}_{|52}\bar{\omega}^2+4I^3\bar{\omega}^3+I^3{}_{|54}\bar{\omega}^4+I^3{}_{|55}\bar{\omega}^5\\
&-\tfrac{1}{2}I^3{}_{|5}\varpi_1-I^3{}_{|5}\varpi_2,\\
\mathrm{d}I^3{}_{|52}=&\,I^3{}_{|521}\bar{\omega}^1+I^3{}_{|522}\bar{\omega}^2+4\big((I^3)^2+I^3{}_{|2}\big)\bar{\omega}^3+I^3{}_{|524}\bar{\omega}^4\\
&+\big(2I^3{}_{|45}+(I^3{}_{|5})^2+I^3{}_{|525}-2I^3{}_{|54}\big)\bar{\omega}^5-I^3{}_{|52}\varpi_1-4I^3\varpi_3,\\
\mathrm{d}I^3{}_{|55}=&\,\tfrac{1}{16}\big(16I^3{}_{|515}-I^3{}_{|5}(2I^3{}_{|45}+I^3{}_{|525})-2I^3{}_{|55}(I^3{}_{|4}+I^3{}_{|52})\big)\bar{\omega}^1\\
&+\tfrac{1}{2}\big(4I^3{}_{|45}+(I^3{}_{|5})^2+2I^3{}_{|525}-2I^3{}_{|54}\big)\bar{\omega}^2+3I^3{}_{|5}\bar{\omega}^3\\
&+\tfrac{1}{2}\big(2I^3{}_{|545}+I^3{}_{|5}I^3{}_{|55}\big)\bar{\omega}^4\\
&+I^3{}_{|555}\bar{\omega}^5-\tfrac{1}{2}I^3{}_{|55}\varpi_1-3I^3{}_{|55}\varpi_2.
\end{aligned}
$$

*Here, the coefficients I^1, I^2, I^3 are the respective incarnations of the basic para-*CR *relative invariants* **A**, **B** *and* **C** *from Theorem* 3.1. *Each of them is a nonzero* *multiple of the respective* **A**, **B**, **C**, *as follows:*

$$
\begin{aligned}
I^1 &\sim 9D^2H_r - 27DH_p - 18H_rDH_r + 18H_pH_r + 4H_r^3 + 54H_z,\\
I^2 &\sim 40G_{ppp}^3 - 45G_{pp}G_{ppp}G_{pppp} + 9G_{pp}^2G_{ppppp},\\
I^3 &\sim 2G_{ppp} + G_{pp}H_{rr}.
\end{aligned}
$$

The other functions, such as, e.g., ${I^3}_{|5}$, *are coframe derivatives of the basic invariants* I^1, I^2 *and* I^3, *with the convention that, for a function* f:

$$
\begin{aligned}
\mathrm{d}f = {} & f_{|1}\omega^1 + f_{|2}\omega^2 + f_{|3}\omega_3 + f_{|4}\omega^4 + f_{|5}\omega^5 + (\dots)\varpi_1\\
& + (\dots)\varpi_2 + (\dots)\varpi_3 + (\dots)\varpi_4.
\end{aligned}
$$

The dotted coeffcients in this expression follow from $\mathrm{d}^2 = 0$ *applied to the above* EDS *and to* f. *The coefficients* $J^1, J^2, \dots, J^8$ *are not important here.*

In this section we have an *a priori* "crazy idea" of relating the EDS of Theorem 5.1 to the EDS (4.1) from Theorem 4.1 describing 3rd order ODEs. There are several reasons indicating that this idea is not so weird as it looks at first glance.

- As we already noticed, in our 5-dimensional para-CR structure theory, there is a family of third order ODEs $z_{xxx} = H(x, y, z, z_x, z_{xx})$ incorporated;
- One of our para-CR invariants **A** is the contact (therefore also point) Wünschmann invariant appearing in the theory of 3rd order ODEs;
- The flat model of our 5-dimensional para-CR structures is described in terms of the Maurer-Cartan forms on the Lie group $Sp(4,\mathbb{R})$, which is the same as the description of the flat model for the geometry of third order ODEs considered modulo contact transformation of variables, which is also given as an EDS satisfied by the Maurer-Cartan forms on $Sp(4,\mathbb{R})$.

This motivates our "crazy question", which actually, due to the discrete symmetry $D_1 \longleftrightarrow D_2$ between the two integrable para-CR distributions D_1 and D_2, consists of two questions:

Q1. Can we bring the EDS of Theorem 5.1, by only using para-CR transformations of forms $(\bar\omega^1, \bar\omega^2, \bar\omega^3, \bar\omega^4, \bar\omega^5)$, to the EDS (4.1) describing contact equivalence classes of 3rd order ODEs? More specifically, can we force the system of 1-forms

$$
\begin{aligned}
\theta^1 &= f_1\bar\omega^1,\\
\theta^2 &= f_2\bar\omega^1 + \rho e^{\phi}\bar\omega^2 + f_4\bar\omega^3,\\
\theta^3 &= f_5\bar\omega^1 + f_6\bar\omega^2 + f_7\bar\omega^3,\\
\theta^4 &= \bar f_2\bar\omega^1 + \rho e^{-\phi}\bar\omega^4 + \bar f_4\bar\omega^5,\\
\theta^5 &= \bar f_5\bar\omega^1 + \bar f_6\bar\omega^4 + \bar f_7\bar\omega^5,
\end{aligned}
\tag{5.2}
$$

to satisfy the EDS (4.1), by an appropriate choice of the fiber parameters $(f_1, f_2, \rho, \phi, f_4, f_5, f_6, f_7, \bar f_2, \bar f_4, \bar f_5, \bar f_6, \bar f_7)$?

Q2. The same question as **Q1**, but now with the flip $(\bar\omega^2, \bar\omega^3) \longleftrightarrow (\bar\omega^4, \bar\omega^5)$, namely: can we force the system of 1-forms

$$\begin{aligned}
\theta^1 &= f_1\bar\omega^1,\\
\theta^2 &= f_2\bar\omega^1 + \rho e^{\phi}\bar\omega^4 + f_4\bar\omega^5,\\
\theta^3 &= f_5\bar\omega^1 + f_6\bar\omega^4 + f_7\bar\omega^5,\\
\theta^4 &= \bar f_2\bar\omega^1 + \rho e^{-\phi}\bar\omega^2 + \bar f_4\bar\omega^3,\\
\theta^5 &= \bar f_5\bar\omega^1 + \bar f_6\bar\omega^2 + \bar f_7\bar\omega^3,
\end{aligned} \tag{5.3}$$

to satisfy the EDS (4.1), by an appropriate choice of the fiber parameters $(f_1, f_2, \rho, \phi, f_4, f_5, f_6, f_7, \bar f_2, \bar f_4, \bar f_5, \bar f_6, \bar f_7)$?

The next theorem gives the *if-and-only-if* answer for these questions, as well as the obstructions to achive the goals specified in questions **Q1** and **Q2**, in terms of the para-CR invariants.

Theorem 5.2.

- *Question* **Q1** *above has a positive answer if and only if* $I^3{}_{|3} \equiv 0$. *The para-*CR *structures related to* $I^3{}_{|3} \equiv 0$ *can be distinguished by the* $\mathfrak{sp}(4,\mathbb{R})$*-valued Cartan connection* (4.2) *whose curvature* $\hat K$ *has the basic invariant* $\mathbf{Z_1} \equiv 0$ *and the basic invariant* $\mathbf{A_1}$ *proportional to the Wünschmann invariant*

$$\mathbf{A_1} \sim 9D^2H_r - 27DH_p - 18H_rDH_r + 18H_pH_r + 4H_r^3 + 54H_z;$$

- *Question* **Q2** *above has a positive answer if and only if* $I^3{}_{|55} \equiv 0$. *The para-*CR *structures related to* $I^3{}_{|55} \equiv 0$ *can be distinguished by the* $\mathfrak{sp}(4,\mathbb{R})$*-valued Cartan connection* (4.2) *whose curvature* $\hat K$ *has the basic invariant* $\mathbf{Z_1} \equiv 0$ *and the basic invariant* $\mathbf{A_1}$ *proportional to the Wünschmann invariant*

$$\mathbf{A_1} \sim 40G_{ppp}^3 - 45G_{pp}G_{ppp}G_{pppp} + 9G_{pp}^2G_{ppppp};$$

- *Furthermore, each condition* $I^3{}_{|3} \equiv 0$ *and* $I^3{}_{|55} \equiv 0$, *considered separately, implies that the relative fundamental para-*CR *invariant* $I^3 \equiv 0$. *So there is only* one *"if and only if" condition for a positive answer to questions* **Q1** *or* **Q2**: *any of them has a positive answer if and only if the para-*CR *invariant* **C** *vanishes:*

$$2G_{ppp} + G_{pp}H_{rr} \equiv 0.$$

Remark 5.3. Before starting the proof, we remark that this theorem provides a way of *transforming two classical invariants, the Wünschmann one and the Monge one, into each other*. This can be achieved by passing from the third order ODE corresponding to the Cartan connection related to the question **Q1**, to its *dual* 3rd order ODE, described by the Cartan connection related to question **Q2**.

Proof of Theorem 5.2. We first answer question **Q1**. We start with the forms (θ^1, θ^2, θ^3, θ^4, θ^5) as in (5.2), and we try to make normalizations on $d\theta^i$ as in (4.1). For full generality, we will not use (4.1) with the 1-form θ^5 in it. We will call this 1-form Ω_0 for a while. As we will see in the proof, the procedure we apply now, which is an adaptation of Cartan's equivalence method, is powerfull enough to determine the relation between Ω_0 and θ^5.

The first normalizations coming from (4.1), namely $d\theta^1 \wedge \theta^1 \wedge \theta^2 = 0$ and $d\theta^1 \wedge \theta^1 \wedge \theta^4 = 0$, give

$$f_4 = \bar{f}_4 = 0,$$

and then, $d\theta^1 \wedge \theta^1 = -\theta^1 \wedge \theta^2 \wedge \theta^4$, gives

$$f_1 = -\rho^2.$$

Now the first condition in (4.1) enables to determine

$$\Omega_1 = \varpi_1 - \frac{\bar{f}_2}{\rho^2}\theta^2 + \frac{f_2}{\rho^2}\theta^4 + d\log(\rho^2) - u_1\theta^1,$$

up to the term with θ^1, which requires to introduce a new variable u_1.

We now make the normalization $d\theta^2 \wedge \theta^1 \wedge \theta^2 = -\theta^1 \wedge \theta^2 \wedge \theta^3 \wedge \theta^4$, which results in

$$f_7 = -e^{2\phi}.$$

After this normalization, the second equation in (4.1) solves for Ω_2 and Ω_3 as follows:

$$\begin{aligned}\Omega_2 =& -\frac{\bar{f}_2}{\rho^2}\theta^3 + \frac{f_5\rho^2 - f_2^2 - f_2 f_6 \rho e^{-\phi}}{\rho^4}\theta^4 - \frac{f_2}{\rho^2}\left(\tfrac{1}{2}\varpi_1 + \varpi_2\right) - \frac{e^{\phi}}{\rho}\varpi_3 \\ &+ \frac{f_2}{\rho^2} d\log\left(\frac{\rho e^{\phi}}{f_2}\right) + \frac{2\rho^2 u_2 - f_2 u_1}{2\rho^2}\theta^1 + \frac{2\rho^4 u_3 - f_2\bar{f}_2}{2\rho^4}\theta^2, \\ \Omega_3 =& -\frac{f_2 + f_6\rho e^{-\phi}}{\rho^2}\theta^4 + \tfrac{1}{2}\varpi_1 - \varpi_2 + d\log\left(\rho e^{\phi}\right) \\ &+ \frac{2\rho^4 u_3 - 3 f_2\bar{f}_2 - 2\bar{f}_2 f_6 \rho e^{-\phi}}{2\rho^4}\theta^1 - \frac{\bar{f}_2 + 2\rho^2 u_4}{2\rho^2}\theta^2,\end{aligned}$$

where u_2, u_3, u_4 are new variables taking account on how indeterminate are Ω_2 and Ω_3.

Now $d\theta^4 \wedge \theta^2 \wedge \theta^4 = \Omega_4 \wedge \theta^1 \wedge \theta^2 \wedge \theta^4$ gives

$$\begin{aligned}\Omega_4 =& \frac{f_2 e^{-2\phi}}{\bar{f}_7\rho^2}\theta^5 + \frac{\bar{f}_2}{\rho^2}\left(\varpi_2 - \tfrac{1}{2}\varpi_1\right) - \frac{e^{-\phi}}{\rho}\varpi_4 + \frac{\bar{f}_2}{\rho^2} d\log\left(\frac{\rho e^{-\phi}}{\bar{f}_2}\right) \\ &+ \frac{3 f_2\bar{f}_2^2 - 2\bar{f}_2\rho^4(u_1 + u_3) - 2\rho^6 u_5 + 2\bar{f}_2^2 f_6 \rho e^{-\phi}}{2\rho^6}\theta^1 \\ &+ \frac{2\bar{f}_2\rho^2 u_4 - \bar{f}_2^2 - 2\rho^4 u_6}{2\rho^4}\theta^2 + \frac{2 f_2\bar{f}_2 - \rho^4 u_7 + \bar{f}_2 f_6 \rho e^{-\phi}}{\rho^4}\theta^4,\end{aligned}$$

and $d\theta^4 \wedge \theta^4 = (\Omega_4 \wedge \theta^1 + \Omega_0 \wedge \theta^2) \wedge \theta^4$, shows that the ODE 1-form Ω_0 must be expressed in terms of θ^1, θ^2, θ^4, θ^5 as follows:

$$\Omega_0 = \frac{e^{-2\phi}}{\bar{f}_7}\theta^5 + \frac{2\bar{f}_7\rho^2(\bar{f}_2 u_4 - \rho^2 u_6) - 3\bar{f}_2^2\bar{f}_7 - 2\bar{f}_2\bar{f}_6\rho e^{-\phi} + 2\bar{f}_5\rho^2 e^{-2\phi}}{2\bar{f}_7\rho^4}\theta^1 + u_8\theta^2 + u_9\theta^4.$$

All of this is true with new undetermined variables $u_5, \dots, u_9$. Please note that in this formula there are *no terms consistsing of the differentials of the group parameters*!

Now to get the fourth equation (4.1) satisfied we have to put

$$u_7 = \frac{3f_2\bar{f}_2\bar{f}_7 + 2\bar{f}_7\rho^4(u_1 + u_3) + 2f_2\bar{f}_6\rho e^{-\phi}}{2\bar{f}_7\rho^4} \quad \text{and}$$

$$u_9 = \frac{2\bar{f}_7\rho^2 u_4 - 3\bar{f}_2\bar{f}_7 - 2\bar{f}_6\rho e^{-\phi}}{2\bar{f}_7\rho^2}.$$

With these normalizations and definitions of Ω_μ's, the differentials $d\theta^1$, $d\theta^2$, $d\theta^4$ are precisely as in (4.1).

The third equation in (4.1) is achieved by a unique choice of f_5, f_6, u_4 and u_3 as follows:

$$f_5 = -\frac{f_2^2}{2\rho^2}, \quad f_6 = -\frac{f_2 e^{\phi}}{\rho}, \quad u_4 = \frac{\bar{f}_2 - I^3\rho e^{-\phi}}{2\rho^2},$$

$$u_3 = \frac{8f_2 I^3 e^{-\phi} - \rho(2I^3{}_{|4} + I^3{}_{|52} + 8u_1\rho^2)}{16\rho^3}.$$

With these normalizations, we achieve that $d\theta^3$ is precisely as in the third equation in (4.1) with

$$\mathbf{A_1} = -\left(\frac{e^{\phi}}{\rho}\right)^3 I^1 \quad \text{and}$$

$$\mathbf{A_2} = \frac{f_2\big(4f_2\bar{f}_2 + \rho^2(2I^3{}_{|4} + I^3{}_{|52}) - 4\rho^4 u_1\big) + 8\rho^6 u_2 - 4f_2^2\rho e^{-\phi} I^3}{8\rho^6}.$$

Now there is a unique way to bring the differential $d\Omega_0$ to the form of the ninth equation (4.1). For this we have:

$$u_8 = \frac{I^3{}_{|3} e^{-3\phi}}{2\rho} \quad \text{and} \quad u_6 = \frac{-4\bar{f}_2^2 + \rho^2 e^{-2\phi}(2I^3{}_{|43} + I^3{}_{|523}) - 4f_2\rho e^{-3\phi} I^3{}_{|3}}{8\rho^4}.$$

After this normalization the formula for $d\Omega_0$ is as in (4.1). In particular, we get explicit expressions for $\mathbf{A_5}$, $\mathbf{Z_4}$, $\mathbf{A_7} + \mathbf{Z_3}$, which are not important here, but also the formula for $\mathbf{Z_1}$ which is:

$$\mathbf{Z_1} = \frac{e^{-5\phi}}{2\rho} I^3{}_{|33}.$$

The last 3rd order ODE invariant 1-form Ω_5 is now determined from $\mathrm{d}\Omega_1\wedge\theta^2\wedge\theta^4 = \Omega_5\wedge\theta^1\wedge\theta^2\wedge\theta^4$, as

$$\begin{aligned}\Omega_5 =& -u_1\Omega_1 - \frac{\bar{f}_2}{\rho^2}\Omega_2 + \frac{f_2}{\rho^2}\Omega_4 + \frac{\bar{f}_2 \mathrm{e}^{\phi}}{\rho^3}\varpi_3 - \frac{f_2\mathrm{e}^{-\phi}}{\rho^3}\varpi_4 + \frac{1}{\rho^2}\varpi_5 - u_{10}\theta^1\\ &- u_{11}\theta^2 - u_{12}\theta^4 - \mathrm{d}u_1\end{aligned}$$

up to θ^1, θ^2, θ^4 terms, which require introduction of new parameters u_{10}, u_{11}, u_{12}.

Now there is a unique way of killing all the unwanted terms in $\mathrm{d}\Omega_\mu$, $\mu = 0,\dots,5$, to achieve the full system (4.1). It turns out that now, this involves solving *linear* equations for all the remaining auxiliary variables u_2, u_5, u_{10}, u_{11}, u_{12} – except u_1. They are determined successively, as follows: u_5 is determined by killing the unwanted terms in $\mathrm{d}\Omega_1\wedge\theta^4$, u_2 is determined by killing the unwanted terms in $\mathrm{d}\Omega_1$, u_{12} is deteremined by killing the unwanted terms in $\mathrm{d}\Omega_2\wedge\theta^1$, u_{11} is determined by killing the unwanted terms in $\mathrm{d}\Omega_2$, and finally u_{10} is deteremined by killing the unwanted terms in $\mathrm{d}\Omega_5\wedge\theta^1\wedge\theta^3$. The explicit expressions for these auxiliary variables are not relevant here.

The final result of these normalizations is:

$$\begin{aligned}\theta^1 &= -\rho^2\bar{\omega}^1,\\ \theta^2 &= f_2\bar{\omega}^1 + \rho\mathrm{e}^{\phi}\bar{\omega}^2,\\ \theta^3 &= -\frac{f_2^2}{2\rho^2}\bar{\omega}^1 - \frac{f_2\mathrm{e}^{\phi}}{\rho}\bar{\omega}^2 - \mathrm{e}^{2\phi}\bar{\omega}^3,\\ \theta^4 &= \bar{f}_2\bar{\omega}^1 + \rho\mathrm{e}^{-\phi}\bar{\omega}^4,\\ \theta^5 &= \bar{f}_5\bar{\omega}^1 + \bar{f}_6\bar{\omega}^4 + \bar{f}_7\omega^5,\end{aligned}$$

$$\boxed{\Omega_0} = \boxed{s_4\theta^1 - \frac{2\bar{f}_2\bar{f}_7 + \rho\mathrm{e}^{-\phi}(2\bar{f}_6 + \bar{f}_7 I^3)}{2\bar{f}_7\rho^2}\theta^4 + \frac{\mathrm{e}^{-2\phi}}{\bar{f}_7}\theta^5} + \frac{\mathrm{e}^{-3\phi}}{2\rho}\,I^3{}_{|3}\,\theta^2,$$

$$\Omega_1 = -u_1\theta^1 - \frac{\bar{f}_2}{\rho^2}\theta^2 + \frac{f_2}{\rho^2}\theta^4 + \varpi_1 + \mathrm{d}\log(\rho^2),$$

$$\begin{aligned}\Omega_2 =& \frac{8I^1{}_{|3}\rho^3\mathrm{e}^{\phi} - 3f_2\big(4f_2\bar{f}_2 + \rho^2(2I^3{}_{|4} + I^3{}_{|52})\big) + 12f_2^2\rho\mathrm{e}^{-\phi}I^3}{24\rho^6}\theta^1\\ &+ \frac{8f_2\rho\mathrm{e}^{-\phi}I^3 - 8f_2\bar{f}_2 - \rho^2(2I^3{}_{|4} + I^3{}_{|52}) - 8\rho^4 u_1}{16\rho^4}\theta^2\\ &- \frac{\bar{f}_2}{\rho^2}\theta^3 - \frac{f_2^2}{2\rho^4}\theta^4 - \frac{f_2}{\rho^2}\left(\tfrac{1}{2}\varpi_1 + \varpi_2\right) - \frac{\mathrm{e}^{\phi}}{\rho}\varpi_3 + \frac{f_2}{\rho^2}\mathrm{d}\log\left(\frac{\rho\mathrm{e}^{\phi}}{f_2}\right),\end{aligned}$$

$$
\begin{aligned}
\Omega_3 =& \frac{8f_2\rho \mathrm{e}^{-\phi} I^3 - 8f_2\bar{f}_2 - \rho^2(2I^3{}_{|4} + I^3{}_{|52}) - 8\rho^4 u_1}{16\rho^4}\theta^1 \\
&+ \frac{\rho \mathrm{e}^{-\phi} I^3 - 2\bar{f}_2}{2\rho^2}\theta^2 - \frac{f_2 + f_6\rho \mathrm{e}^{-\phi}}{\rho^2}\theta^4 + \tfrac{1}{2}\varpi_1 - \varpi_2 + \mathrm{d}\log(\rho \mathrm{e}^{\phi}), \\
\Omega_4 =& s_1\theta^1 + s_2\theta^2 + s_3\theta^4 + \frac{f_2 \mathrm{e}^{-2\phi}}{\bar{f}_7\rho^2}\theta^5 + \frac{\bar{f}_2}{\rho^2}\big(\varpi_2 - \tfrac{1}{2}\varpi_1\big) \\
&- \frac{\mathrm{e}^{-\phi}}{\rho}\varpi_4 + \frac{\bar{f}_2}{\rho^2}\mathrm{d}\log\left(\frac{\rho \mathrm{e}^{-\phi}}{\bar{f}_2}\right), \\
\Omega_5 =& s_5\theta^1 + s_6\theta^2 + \frac{\bar{f}_2^2}{\rho^4}\theta^3 + s_7\theta^4 + \frac{f_2^2 \mathrm{e}^{-2\phi}}{\bar{f}_7\rho^4}\theta^5 - u_1\varpi_1 \\
&+ \frac{2f_2\bar{f}_2}{\rho^4}\varpi_2 + \frac{2\bar{f}_2 \mathrm{e}^{\phi}}{\rho^3}\varpi_3 - \frac{2f_2 \mathrm{e}^{-\phi}}{\rho^3}\varpi_4 + \frac{1}{\rho^2}\varpi_5 - \mathrm{d}u_1 \\
&- \frac{2u_1}{\rho}\mathrm{d}\rho + \frac{\bar{f}_2}{\rho^4}\mathrm{d}f_2 - \frac{f_2}{\rho^4}\mathrm{d}\bar{f}_2 - \frac{2f_2\bar{f}_2}{\rho^4}\mathrm{d}\phi.
\end{aligned}
$$

Here

$$
s_4 = \frac{-4\bar{f}_2^2\bar{f}_7 - 4\bar{f}_2(2\bar{f}_6 + \bar{f}_7 I^3)\rho \mathrm{e}^{-\phi} + \big(8\bar{f}_5 - \bar{f}_7(2I^3{}_{|43} + I^3{}_{|523})\big)\rho^2 \mathrm{e}^{-2\phi} + 4f_2\bar{f}_7 I^3{}_{|3}\rho \mathrm{e}^{-3\phi}}{8\bar{f}_7\rho^4},
$$

and the coefficients $s_1, s_2, s_3, s_5, s_6, s_7$ although explicitely determined, are totally irrelevant for the sequel.

The above 1-forms $(\theta^1, \dots, \theta^4, \Omega_0, \dots, \Omega_5)$ satisfy the 3rd order ODE system (4.1) with

$$
\mathbf{A_1} = -\left(\frac{\mathrm{e}^{\phi}}{\rho}\right)^3 I^1 \quad \text{and} \quad \mathbf{Z_1} = \frac{\mathrm{e}^{-5\phi}}{2\rho} I^3{}_{|33}.
$$

Also all other coefficients $\mathbf{A_i}$, $\mathbf{Z_j}$ and $\mathbf{C_1}$ are totally determined, as for example $\mathbf{A_2} = \frac{\mathrm{e}^{\phi}}{3\rho^3} I^1{}_{|3}$, or $\mathbf{A_5} = -\frac{\mathrm{e}^{-\phi}}{6\rho^3} I^1{}_{|33}$, but they are not that illuminating to quote them here.

This explicitly shows that *every* 5-dimensional para-CR structure, which has Levi form degenerate in one direction and which is not locally a trivial extension of a 3-dimensional nondegenerate para-CR structure, defines an invariant EDS for a contact equivalence class of 3rd ODEs for which the classical Wünschmann invariant is the para-CR invariant $\mathbf{A}$.

A problem arises if this obtained EDS is para-CR invariant. At first glance yes, but it is really *not*.

The reason for that is that the form θ^5 disappeared from the description. It was replaced by the form Ω_0. Looking at the explicit form of Ω_0, one observes that the forms $(\theta^1, \theta^2, \theta^3, \theta^4, \theta^5)$ and $(\theta^1, \theta^2, \theta^3, \theta^4, \Omega_0)$ are *not* para-CR equivalent, *because* the 1-form θ^2 appears in the formula relating Ω_0 and θ^5. But θ^2 should *not* be there! Only the "boxed" part of this formula consists of some para-CR transformation between θ^5 and Ω_0. The appearence of the term $\frac{e^{-3\phi}I^3{}_{|3}}{2\rho}\theta^2$ in this formula breaks the para-CR equivalence.

There is only one way to restore the para-CR invariance of the obtained EDS: one has to *restrict to para-*CR *structures for which*

$$I^3{}_{|3} \equiv 0.$$

In such a case one can use the remaining para-CR transformations to achieve

$$\Omega_0 = \theta^5$$

reducing all the auxiliary parameters from $(f_1, f_2, \rho, \phi, f_4, f_5, f_6, f_7, \bar{f}_2, \bar{f}_4, \bar{f}_5, \bar{f}_6, \bar{f}_7, u_1, \dots, u_{12})$ to only five $(\rho, \phi, f_2, \bar{f}_2, u_1)$, This makes the resulting EDS really 10-dimensional, as it should be for it to describe a curvature of a Cartan $\mathfrak{sp}(4, \mathbb{R})$-connection.

The proof of the answer to the question **Q2** is essentially the same as above. We start with the lifted coframe (5.3), and impose the normalizations required by the system (4.1) in the same order as in the case of question **Q1**. We skip the details, reporting here the important differences only. The first of them is that now the normalizations result in $d\theta^3$ as in (4.1), but with

$$\mathbf{A_1} = -\left(\frac{e^{\phi}}{\rho}\right)^3 I^2.$$

The next difference is that now, in the induced EDS (4.1), the coefficient $\mathbf{Z_1}$ is

$$\mathbf{Z_1} = -\frac{e^{-5\phi}}{4\rho} I^3{}_{|555}.$$

As the last important difference we mention that now the 1-form Ω_0 appearing in the induced EDS (4.1) is related to θ^5 via:

$$\boxed{\Omega_0} = \boxed{s_4\theta^1 + \frac{4\bar{f}_2\bar{f}_7 - \rho e^{-\phi}(4\bar{f}_6 - \bar{f}_7 I^3{}_{|5})}{4\bar{f}_7\rho^2}\theta^4 + \frac{e^{-2\phi}}{\bar{f}_7}\theta^5} - \frac{e^{-3\phi}}{4\rho} I^3{}_{|55}\,\theta^2.$$

So now, the term $\frac{e^{-3\phi}I^3{}_{|55}}{4\rho}\theta^2$ brakes the para-CR equivalence, and to answer the question **Q2** in positive, we are forced to restrict to para-CR structures with

$$I^3{}_{|55} \equiv 0.$$

Consequently, if we assume that $I^3{}_{|55} \equiv 0$, we finally use the remaining para-CR transformations to achieve

$$\Omega_0 = \theta^5.$$

This again reduces all the auxiliary parameters from $(f_1, f_2, \rho, \phi, f_4, f_5, f_6, f_7, \bar{f}_2, \bar{f}_4, \bar{f}_5, \bar{f}_6, \bar{f}_7, u_1, \dots, u_{12})$ to only five $(\rho, \phi, f_2, \bar{f}_2, u_1)$, and makes the resulting EDS the curvature conditions of a Cartan $\mathfrak{sp}(4, \mathbb{R})$-connection in 10 dimensions.

As the final step of the proof, we remark that if we insert any of the conditions ${I^3}_{|3} \equiv 0$ or ${I^3}_{|55} \equiv 0$ into the EDS (5.1) then its integrability conditions ($\mathrm{d}^2 \equiv 0$) very quickly show that each of them is equivalent to

$$I^3 \equiv 0.$$

For this, observe that if ${I^3}_{|3} \equiv 0$, then the equation for $\mathrm{d}{I^3}_{|3}$ in Theorem 5.1 gives immediately $I^3 \equiv 0$. Likewise, if ${I^3}_{|55} \equiv 0$ then the equation for $\mathrm{d}{I^3}_{|55}$ in Theorem 5.1 gives ${I^3}_{|5} \equiv 0$, and then the equation for $\mathrm{d}{I^3}_{|5}$ in gives eventually $I^3 \equiv 0$.

This finishes the proof of Theorem 5.2. □

Theorem 5.1, and the calculations done in its proof, have an interesting

Corollary 5.4. *Consider a 5-dimensional para-*CR *structure given by the system of* PDEs (2.1) *in terms of functions* $H = H(x, y, z, p, r)$ *and* $G = G(x, y, z, p, r)$ *satisfying conditions* (2.2)*,* (2.4)*,* (2.5)*. Assume for this structure that the para-*CR *invariant* $\mathbf{C}$ *vanishes:*

$$2G_{ppp} + G_{pp}H_{rr} \equiv 0.$$

*Then, associated to such a para-*CR *structure, there are* two *contact equivalence classes of third order* ODEs*. Both of these classes of* ODEs *have their respective Chern invariants zero:*

$$\mathbf{Z} \equiv 0.$$

The other basic contact invariant of these (contact invariant) classes of ODEs*, namely the Wünschmann invariant* $\mathbf{A_1}$ *is proportional:*

(a) *To the Wünschmann para-*CR *invariant,* $\mathbf{A_1} \sim 9D^2H_r - 27DH_p - 18H_rDH_r + 18H_pH_r + 4H_r^3 + 54H_z$*, in the case of the first class of* ODEs*;*
(b) *To the Monge para-*CR *invariant,* $\mathbf{A_1} \sim 40G_{ppp}^3 - 45G_{pp}G_{ppp}G_{pppp} + 9G_{pp}^2G_{ppppp}$*, in the case of the second class of* ODEs.

Proof of Corollary 5.4. In the language of Theorem 5.2, the assumption that $\mathbf{C} \equiv 0$ means that $I^3 \equiv 0$. This, in particular means that ${I^3}_{|33} \equiv 0$ and that ${I^3}_{|555} \equiv 0$. Thus, the quantity $\mathbf{Z_1}$ vanishes in the EDS obatained from the normalizations of the lifted coframe (5.2) as well as of the lifted coframe (5.3).

Moreover, since $I^3 \equiv 0$ implies also ${I^3}_{|3} \equiv 0$ and ${I^3}_{|55} \equiv 0$, we know from Theorem 5.2 that both EDSs with $\mathbf{Z_1} \equiv 0$ are para-CR invariant. According to Chern's theory of 3rd order ODEs considered modulo contact transformation [1, 6], both EDS's, considered separately, describe a contact equivalence class of 3rd order ODEs on the 4-dimensional leaf space J^2 of the rank 6 integrable distribution annihilating 1-forms $\theta^1, \theta^2, \theta^3, \theta^4$. This space J^2 can be locally identified with the space of second jets of the corresponding class of 3rd order ODEs. This class, in both EDSs, has Chern invariant equal to zero (because $\mathbf{C} \equiv 0$ implies $\mathbf{Z_1} \equiv 0$ in the EDSs), and as it visible from the proof of Theorem 5.2, the classical Wünschmann invariant $\mathbf{A_1}$ either proportional to $9D^2H_r - 27DH_p - 18H_rDH_r + 18H_pH_r + 4H_r^3 + 54H_z$, or to $40G_{ppp}^3 - 45G_{pp}G_{ppp}G_{pppp} + 9G_{pp}^2G_{ppppp}$, depending which of the two EDSs we are considering.

This finishes the proof of Corollary 5.4. For further details about it, consult our Appendix in Section 6. □

End of the proof of Theorem 1.3. Since in both of the 10-dimensional para-CR invariant EDSs we have $\mathbf{Z_1} \equiv 0$, then according to the result of Godliński [5, 6], the image of the projection $\pi_1 : \mathrm{J}^2 \to \mathrm{J}^1$ from the second jet space J^2 appearing in the proof of the above corollary, which can be identified with the 3-dimensional leaf space of the rank 7 integrable distribution annihilating 1-forms $\theta^1, \theta^2, \theta^4$, acquires a natural 3-dimensional contact projective geometry. This proves our Theorem 1.3 from the introduction. □

To illustrate the phenomena described in this section we consider the following example.

Example 5.5. Our starting point in this example is a para-CR structure defined in terms of PDEs

$$\boxed{z_y = f(z_x) \quad \text{and} \quad z_{xxx} = -z_{xx}^2 \frac{f^{(3)}(z_x)}{f''(z_x)}, \quad \text{for} \quad z = z(x,y),} \tag{5.4}$$

with $f = f(p)$ being a differentiable function such that $f''(p) \neq 0$. In other words we have

$$G = f(p) \quad \text{and} \quad H = -r^2 \frac{f^{(3)}(p)}{f''(p)}. \tag{5.5}$$

It is straightforward to check that $\triangle H = D^3 G$, $G_{pp} \neq 0$ and, more importantly, that

$$2G_{ppp} + G_{pp}H_{rr} \equiv 0.$$

Therefore the Theorem 5.2 and Corolary 5.4 apply, and we should see two equivalence classes of 3rd order ODEs associated with these para-CR structures as well as two contact projective structures on the spaces of first jets for these ODEs.

Before passing to show how these structure are explicitly visible here, we calculate the Wünschmann invariant $\mathbf{A}$ for the function H from (5.5). We have:

$$\begin{aligned}\mathbf{A} &= 9D^2H_r - 27DH_p - 18H_rDH_r + 18H_pH_r + 4H_r^3 + 54H_z \\ &= \frac{r^3}{f''^3}\Big(40f^{(3)3} - 45f''f^{(3)}f^{(4)} + 9f''^2f^{(5)}\Big) \\ &= \left(\frac{r^3}{G_{pp}^3}\right)\Big(40G_{ppp}^3 - 45G_{pp}G_{ppp}G_{pppp} + 9G_{pp}^2G_{ppppp}\Big) = 2r^3\mathbf{B}.\end{aligned}$$

Thus, for our para-CR structure, represented by the functions G and H, the Wünschmann invariant $\mathbf{A}$ is a nonvanishing multiple of the Monge invariant $\mathbf{B}$.

This is a *special case* of the phenomenon mentioned in Remark 5.3: in this example we found the *explicit* transformation between the Wünschmann invariant for H and Monge invariant of G. It was possible explicitly because this example is so special that, as we see in a minute, the two *a'priori different* contact equivalent classes of 3rd order ODEs naturally associated to our para-CR structure, are actualy *the same*.

To see this we first write the coframe on M^5 encoding our para-CR structure. This is given by:

$$\begin{aligned}
\omega^1 &= \mathrm{d}z - p\mathrm{d}x - f\mathrm{d}y,\\
\omega^2 &= \mathrm{d}p - r\mathrm{d}x - rf'\mathrm{d}y,\\
\omega^3 &= \mathrm{d}r + r^2\frac{f^{(3)}}{f''}\mathrm{d}x - r^2\frac{f''^2 - f'f^{(3)}}{f''}\mathrm{d}y,\\
\omega^4 &= \mathrm{d}x, \quad \omega^5 = \mathrm{d}y.
\end{aligned}$$

Now, it is convenient to introduce *new coordinates* (X, Y, P, Q, R) on M^5 related to the coordinates (x, y, z, p, r) via:

$$\begin{aligned}
x &= -P + \frac{qf'}{f''},\\
y &= -\frac{q}{f''},\\
z &= Y - PX + q\frac{Xf' - f}{f''},\\
p &= X,\\
r &= \frac{1}{q - Q},
\end{aligned}$$

where now, due to $p = X$, we have $f = f(X)$, $f' = f'(X)$ and $f'' = f''(X)$. In these new coordinates the coframe $(\omega^1, \dots, \omega^5)$ defining our para-CR structures reads:

$$\begin{aligned}
\omega^1 &= \mathrm{d}Y - P\mathrm{d}X,\\
\omega^2 &= \frac{1}{q - Q}(\mathrm{d}P - Q\mathrm{d}X),\\
\omega^3 &= \frac{1}{(q - Q)^2}\left(\mathrm{d}Q - \frac{f^{(3)}}{f''}\mathrm{d}P\right),\\
\omega^4 &= -\mathrm{d}P + \mathrm{d}\left(q\frac{f'}{f''}\right),\\
\omega^5 &= -\frac{1}{f''}\left(\mathrm{d}q - q\frac{f^{(3)}}{f''}\mathrm{d}X\right).
\end{aligned}$$

Now, a special para-CR transformation

$$
\begin{aligned}
&\omega^1 \to \omega^1, \\
&\omega^2 \to (q - Q)\,\omega^2, \\
&\omega^3 \to (q - Q)^2 \left(\omega^3 + \frac{f^{(3)}}{(q - Q)\,f''}\omega^2\right), \\
&\omega^4 \to -\omega^4 - f'\omega^5, \\
&\omega^5 \to -f''\,\omega^5
\end{aligned}
$$

as in (3.1), brings this para-CR coframe to

$$
\begin{aligned}
\omega^1 &= \mathrm{d}Y - P\mathrm{d}X, \\
\omega^2 &= \mathrm{d}P - Q\mathrm{d}X, \\
\omega^3 &= \mathrm{d}Q - Q\frac{f^{(3)}}{f''}\mathrm{d}X, \\
\omega^4 &= \mathrm{d}P - q\mathrm{d}X, \\
\omega^5 &= \mathrm{d}q - q\frac{f^{(3)}}{f''}\mathrm{d}X.
\end{aligned}
$$

Note the remarkable similarity of the 1-forms (ω^2, ω^3) to the 1-forms (ω^4, ω^5); they merely differ by the flip $Q \leftrightarrow q$.

Now, let us consider two foliations of M^5 by two families of hypersurfaces; M^5 is foliated by

$$
\mathrm{J}^2{}_{q_0} = \big\{M^5 \in (X, Y, P, Q, q) \,:\, q = q_0 = \mathrm{const}\big\}
$$

and by

$$
\mathrm{j}^2{}_{Q_0} = \big\{M^5 \in (X, Y, P, Q, q) \,:\, Q = Q_0 = \mathrm{const}\big\}.
$$

It follows from our calculations above that every hypersurface $\mathrm{J}^2{}_{q_0}$ in the first family has a structure of the space of second jets J^2 coordinatized by (X, Y, P, Q) for the 3rd order ODE

$$
Y''' = Y''\frac{f^{(3)}(X)}{f''(X)} \tag{5.6}
$$

for a function $Y = Y(X)$, with $Y' = P$, $Y'' = Q$. Similarly, every hypersurface $\mathrm{j}^2{}_{Q_0}$ in the second family has a structure of the space of second jets J^2 coordinatized by (X, Y, P, q) for the *same* 3rd order ODE $Y''' = Y''\frac{f^{(3)}(X)}{f''(X)}$ for a function $Y = Y(X)$, with $Y' = P$, $Y'' = q$. Note that the passage from the first family of the second jet spaces to the second family of the second jet spaces corresponds to the flip $(\omega^2, \omega^3) \leftrightarrow (\omega^4, \omega^5)$ between the original coframe forms of the considered para-CR structure (5.4).

Since in our notation from Theorem 4.1 and Proposition 4.2 the ODE (5.6) has $H = r\frac{f^{(3)}(x)}{f''(x)}$, then its Chern invariant $\mathbf{Z} = H_{rrrr} \equiv 0$. Thus, according to Proposition 4.4 each of the corresponding first jet spaces J^1 and j^1, which curiously are both parametrized by (X, Y, P), has a natural projective contact structure.

To see this structure, we restrict to the case of J^2; the case of j^2 is the same, modulo the replacement $q \to Q$. Fortunately the ODE is easy to solve; its general solution is

$$Y = c_1 f(X) + c_2 X + c3.$$

This general solution defines a 3-parameter family of curves

$$\gamma(t; c_1, c_2, c_3) = (X(t), Y(t), P(t)) = (t, c_1 f(t) + c_2 t + c_3, c_1 f'(t) + c_2)$$

in J^1. Now we fix a frame $(e_1, e_2, e_3) = (\partial_Y, \partial_P, \partial_X + P\partial_Y)$ in J^1, and consider tangent vectors $\dot\gamma(t)$ to each of these curves. Straightforward differentiation gives:

$$\dot\gamma(t) = \sum_{i=1}^{3} \dot\gamma^i e_i = c_1 f''(t) e_2 + e_3.$$

Since the contact distribution $\mathcal{C}$ in J^1 is given by

$$\mathcal{C} = (\omega^1)^\perp = \mathrm{Span}(e_2, e_3)$$

we see that our 3-parameter family of curves $\gamma(t; c_1, c_2, c_3)$ is always *tangent* to $\mathcal{C}$. And now, writing the geodesic equations for the curves $\gamma(t)$ in the coframe (e_1, e_2, e_3)

$$\frac{\mathrm{d}\dot\gamma^i}{\mathrm{d}t} + \sum_{j,k=1}^{3} \Gamma^i{}_{jk} \dot\gamma^j \dot\gamma^k = 0,$$

one can easily see that our 3-parameter family of curves $\gamma(t; c_1, c_2, c_3)$ satisfies these equations with a torsionless connection ∇, such that $\nabla_{e_i} e_j = \sum_{k=1}^{3} \Gamma^k{}_{ji} e_k$, in which all the coeffcients $\Gamma^k{}_{ij}$ vanish, except $\Gamma^2{}_{23} = \Gamma^2_{32} = -\frac{f^{(3)}(X)}{2f''(X)}$.

Thus we have a 3-parameter family of curves $\gamma(t; c_1, c_2, c_3)$ in J^1, which are (a) tangent to the contact distribution $\mathcal{C}$ and (b) are geodesics with respect to the torsionless connection ∇. This shows that J^1 is equipped with a *contact projective structure*.

We thus have shown on an example, how a PDE system (2.1)-(2.2), (2.4)-(2.5), with $2G_{ppp} + G_{pp}H_{rr} \equiv 0$ defines two contact equivalence classes of 3[rd] odrer ODEs and a contact projective structure on their space of first jets.

Finally, note that the quotient 3-manifolds on which the contact projective structures associated with our para-CR structure resides are just the quotients of the M^5 by the respective *integrable para-*CR *distributions* D_1 and D_2 in M^5.

6. Appendix

It is instructive to show the result of Cartan's equivalence procedure applied to the 1-forms (5.2) or (5.3) when we have $I^3 \equiv 0$. We do it here for the 1-forms (5.2).

For this, we need the system (5.1) and its integrability conditions, as in Theorem 5.1, adapted to $I^3 \equiv 0$. This restricted to $I^3 \equiv 0$ system reads:

$$\begin{aligned}
\mathrm{d}\bar\omega^1 &= -\bar\omega^1 \wedge \varpi_1 + \bar\omega^2 \wedge \bar\omega^4,\\
\mathrm{d}\bar\omega^2 &= -\bar\omega^1 \wedge \varpi_3 + \bar\omega^2 \wedge \big(\varpi_2 - \tfrac12\varpi_1\big) + \bar\omega^3 \wedge \bar\omega^4,\\
\mathrm{d}\bar\omega^3 &= -\bar\omega^2 \wedge \varpi_3 + 2\bar\omega^3 \wedge \varpi_2 + I^1\,\bar\omega^1 \wedge \bar\omega^4,\\
\mathrm{d}\bar\omega^4 &= -\bar\omega^1 \wedge \varpi_4 - \bar\omega^4 \wedge \big(\varpi_2 + \tfrac12\varpi_1\big) - \bar\omega^2 \wedge \bar\omega^5,\\
\mathrm{d}\bar\omega^5 &= \bar\omega^4 \wedge \varpi_4 - 2\bar\omega^5 \wedge \varpi_2 + I^2\,\bar\omega^1 \wedge \bar\omega^2,
\end{aligned} \tag{6.1}$$

$$\begin{aligned}
\mathrm{d}I^1 &= I^1{}_{|1}\bar\omega^1 + I^1{}_{|2}\bar\omega^2 + I^1{}_{|3}\bar\omega^3 + I^1{}_{|4}\bar\omega^4 - \tfrac32 I^1\varpi_1 - 3I^1\varpi_2,\\
\mathrm{d}I^2 &= I^2{}_{|1}\bar\omega^1 + I^2{}_{|2}\bar\omega^2 + I^2{}_{|4}\bar\omega^4 + I^2{}_{|5}\bar\omega^5 - \tfrac32 I^2\varpi_1 + 3I^2\varpi_2.
\end{aligned}$$

Integrability conditions of these equations imply an existence of a 1-form ϖ_5 such that:

$$\begin{aligned}
\mathrm{d}\varpi_1 =& \bar\omega^1 \wedge \varpi_5 + \bar\omega^2 \wedge \varpi_4 - \bar\omega^4 \wedge \varpi_3,\\
\mathrm{d}\varpi_2 =& \tfrac12\bar\omega^2 \wedge \varpi_4 - \tfrac12\bar\omega^4 \wedge \varpi_3 - \tfrac12 I^2{}_{|5}\bar\omega^1 \wedge \bar\omega^2 + \tfrac12 I^1{}_{|3}\bar\omega^1 \wedge \bar\omega^4 - \bar\omega^3 \wedge \bar\omega^5,\\
\mathrm{d}\varpi_3 =& \varpi_3 \wedge \big(\tfrac12\varpi_1 + \varpi_2\big) + \tfrac12\bar\omega^2 \wedge \varpi_5 + \bar\omega^3 \wedge \varpi_4 + \big(I^1{}_{|23} + I^2{}_{|45}\big)\bar\omega^1 \wedge \bar\omega^2\\
&+ I^2{}_{|5}\bar\omega^1 \wedge \bar\omega^3 - I^1{}_{|2}\bar\omega^1 \wedge \bar\omega^4 + I^1\bar\omega^1 \wedge \bar\omega^5 - \tfrac12 I^1{}_{|3}\bar\omega^2 \wedge \bar\omega^4,\\
\mathrm{d}\varpi_4 =& \varpi_4 \wedge \big(\tfrac12\varpi_1 - \varpi_2\big) + \tfrac12\bar\omega^4 \wedge \varpi_5 + \bar\omega^5 \wedge \varpi_3\\
&+ I^2{}_{|4}\bar\omega^1 \wedge \bar\omega^2 - I^2\bar\omega^1 \wedge \bar\omega^3 + \tfrac12 I^2{}_{|5}\bar\omega^2 \wedge \bar\omega^4 - I^1{}_{|3}\bar\omega^1 \wedge \bar\omega^5,\\
\mathrm{d}\varpi_5 =& \varpi_5 \wedge \varpi_1 + 2\varpi_4 \wedge \varpi_3 - I^2{}_{|5}\bar\omega^1 \wedge \varpi_3 - 3I^1{}_{|3}\bar\omega^1 \wedge \varpi_4\\
&+ \big(I^2{}_{|15} + 2I^2{}_{|44}\big)\bar\omega^1 \wedge \bar\omega^2 - 4I^2{}_{|4}\bar\omega^1 \wedge \bar\omega^3\\
&+ \big(I^1{}_{|31} - 2I^1{}_{|22} - 2I^1{}_{|234} - 2I^2{}_{|445}\big)\bar\omega^1 \wedge \bar\omega^4\\
&- 2\big(I^1{}_{|2} + I^1{}_{|34}\big)\bar\omega^1 \wedge \bar\omega^5 - I^2\bar\omega^2 \wedge \bar\omega^3\\
&+ \big(I^1{}_{|23} + I^2{}_{|45}\big)\bar\omega^2 \wedge \bar\omega^4 - I^1{}_{|3}\bar\omega^2 \wedge \bar\omega^5 + I^2{}_{|5}\bar\omega^3 \wedge \bar\omega^4 - I^1\bar\omega^4 \wedge \bar\omega^5,
\end{aligned} \tag{6.2}$$

and as before, the coefficients I^1 and I^2 are, modulo a scale, the respective basic para-CR relative invariants **A** and **B** from Theorem 3.1:

$$\begin{aligned}
I^1 &\sim 9D^2H_r - 27DH_p - 18H_rDH_r + 18H_pH_r + 4H_r^3 + 54H_z,\\
I^2 &\sim 40G_{ppp}^3 - 45G_{pp}G_{ppp}G_{pppp} + 9G_{pp}^2G_{ppppp}.
\end{aligned}$$

There is only one way of forcing the forms (5.2), with 1-forms $(\bar\omega^1, \bar\omega^2, \bar\omega^3, \bar\omega^4, \bar\omega^5)$ described by the EDS (6.1)-(6.2), to satisfy the system (4.1). Such a requirement determines all θ^is and Ω_μs uniquely. Explicitly

$$\theta^i = g^i{}_j \bar{\omega}^j, \qquad \text{for all} \qquad i = 1, 2, \dots, 5,$$

with the reduced matrix $g = (g^i{}_j)$ equal to

$$g = \begin{pmatrix} -\rho^2 & 0 & 0 & 0 & 0 \\ f_2 & \rho e^\phi & 0 & 0 & 0 \\ -\frac{f_2^2}{2\rho^2} & -\frac{f_2 e^\phi}{\rho} & -e^{2\phi} & 0 & 0 \\ \bar{f}_2 & 0 & 0 & \rho e^{-\phi} & 0 \\ -\frac{\bar{f}_2^2}{2\rho^2} & 0 & 0 & -\frac{\bar{f}_2 e^{-\phi}}{\rho} & e^{-2\phi} \end{pmatrix},$$

and the remaining forms $\Omega_1, \dots, \Omega_5$ are as follows:

$$\Omega_1 = -u_1\theta^1 - \frac{\bar{f}_2}{\rho^2}\theta^2 + \frac{f_2}{\rho^2}\theta^4 + \varpi_1 + d\log(\rho^2),$$

$$\begin{aligned}
\Omega_2 = {} & \frac{2I^1{}_{|3}\rho^3 e^\phi - 3f_2^2\bar{f}_2}{6\rho^6}\theta^1 - \frac{f_2\bar{f}_2 + \rho^4 u_1}{2\rho^4}\theta^2 \\
& - \frac{\bar{f}_2}{\rho^2}\theta^3 - \frac{f_2^2}{2\rho^4}\theta^4 - \frac{f_2}{\rho^2}\left(\tfrac{1}{2}\varpi_1 + \varpi_2\right) - \frac{e^\phi}{\rho}\varpi_3 + \frac{f_2}{\rho^2} d\log\left(\frac{\rho e^\phi}{f_2}\right), \\
\Omega_3 = {} & -\frac{f_2\bar{f}_2 + \rho^4 u_1}{2\rho^4}\theta^1 - \frac{\bar{f}_2}{\rho^2}\theta^2 + \tfrac{1}{2}\varpi_1 - \varpi_2 + d\log(\rho e^\phi), \\
\Omega_4 = {} & \frac{3f_2\bar{f}_2^2 - 2I^2{}_{|5}\rho^3 e^{-\phi}}{6\rho^6}\theta^1 + \frac{\bar{f}_2^2}{2\rho^4}\theta^2 + \frac{f_2\bar{f}_2 - \rho^4 u_1}{2\rho^4}\theta^4 + \frac{f_2}{\rho^2}\theta^5 \\
& + \frac{\bar{f}_2}{\rho^2}(\varpi_2 - \tfrac{1}{2}\varpi_1) - \frac{e^{-\phi}}{\rho}\varpi_4 + \frac{\bar{f}_2}{\rho^2} d\log\left(\frac{\rho e^{-\phi}}{\bar{f}_2}\right), \\
\Omega_5 = {} & \left(\tfrac{1}{2}u_1^2 + \frac{2I^1{}_{|23} + 4I^2{}_{|45}}{3\rho^4} - \frac{I^2{}_{|5} f_2 e^{-\phi} + I^1{}_{|3}\bar{f}_2 e^\phi}{\rho^5} + \frac{f_2^2\bar{f}_2^2}{\rho^8}\right)\theta^1 \\
& + \left(\frac{\bar{f}_2 u_1}{\rho^2} - \frac{e^{-\phi} I^2{}_{|5}}{3\rho^3} + \frac{f_2\bar{f}_2^2}{\rho^6}\right)\theta^2 + \frac{\bar{f}_2^2}{\rho^4}\theta^3 \\
& + \left(\frac{f_2^2\bar{f}_2}{\rho^6} - \frac{f_2 u_1}{\rho^2} - \frac{e^\phi I^1{}_{|3}}{3\rho^3}\right)\theta^4 + \frac{f_2^2}{\rho^4}\theta^5 - u_1\varpi_1 + \frac{2f_2\bar{f}_2}{\rho^4}\varpi_2 \\
& + \frac{2\bar{f}_2 e^\phi}{\rho^3}\varpi_3 - \frac{2f_2 e^{-\phi}}{\rho^3}\varpi_4 + \frac{1}{\rho^2}\varpi_5 - du_1 - \frac{2u_1}{\rho}d\rho + \frac{\bar{f}_2}{\rho^4}df_2 \\
& - \frac{f_2}{\rho^4}d\bar{f}_2 - \frac{2f_2\bar{f}_2}{\rho^4}d\phi.
\end{aligned}$$

The resulting EDS (4.1) for these forms reads:

$$\mathrm{d}\theta^1 = \Omega_1 \wedge \theta^1 + \theta^4 \wedge \theta^2,$$

$$\mathrm{d}\theta^2 = \Omega_2 \wedge \theta^1 + \Omega_3 \wedge \theta^2 + \theta^4 \wedge \theta^3,$$

$$\mathrm{d}\theta^3 = \Omega_2 \wedge \theta^2 + (2\Omega_3 - \Omega_1) \wedge \theta^3 + \frac{\mathrm{e}^{\phi}}{3\rho^3} I^1{}_{|3}\, \theta^2 \wedge \theta^1 - \left(\frac{\mathrm{e}^{\phi}}{\rho}\right)^3 I^1\, \theta^4 \wedge \theta^1,$$

$$\mathrm{d}\theta^4 = \Omega_4 \wedge \theta^1 + (\Omega_1 - \Omega_3) \wedge \theta^4 + \theta^5 \wedge \theta^2,$$

$$\mathrm{d}\theta^5 = \Omega_4 \wedge \theta^4 + (\Omega_1 - 2\Omega_3) \wedge \theta^5 - \left(\frac{\mathrm{e}^{-\phi}}{\rho}\right)^3 I^2\, \theta^1 \wedge \theta^2 + \frac{\mathrm{e}^{-\phi}}{3\rho^3} I^2{}_{|5}\, \theta^1 \wedge \theta^4,$$

$$\mathrm{d}\Omega_1 = \Omega_5 \wedge \theta^1 + \Omega_4 \wedge \theta^2 - \Omega_2 \wedge \theta^4,$$

$$\mathrm{d}\Omega_2 = (\Omega_3 - \Omega_1) \wedge \Omega_2 + \tfrac{1}{2}\Omega_5 \wedge \theta^2 + \Omega_4 \wedge \theta^3 + \mathbf{A_3}\, \theta^1 \wedge \theta^2 + \mathbf{A_4} \theta^1 \wedge \theta^4,$$

$$\mathrm{d}\Omega_3 = \tfrac{1}{2}\Omega_5 \wedge \theta^1 + \Omega_4 \wedge \theta^2 + \theta^5 \wedge \theta^3 - \frac{\mathrm{e}^{-\phi}}{3\rho^3} I^2{}_{|5}\, \theta^1 \wedge \theta^2 + \frac{\mathrm{e}^{\phi}}{3\rho^3} I^1{}_{|3}\, \theta^1 \wedge \theta^4,$$

$$\mathrm{d}\Omega_4 = \theta^5 \wedge \Omega_2 + \Omega_4 \wedge \Omega_3 + \tfrac{1}{2}\Omega_5 \wedge \theta^4 + \mathbf{A_6} \theta^1 \wedge \theta^2 - \mathbf{A_3} \theta^1 \wedge \theta^4,$$

$$\mathrm{d}\Omega_5 = \Omega_5 \wedge \Omega_1 + 2\Omega_4 \wedge \Omega_2 + \mathbf{C_1} \theta^1 \wedge \theta^2 + \mathbf{A_8} \theta^1 \wedge \theta^4.$$

Here

$$\mathbf{A_3} = \frac{\mathrm{e}^{\phi} \bar{f}_2 I^1{}_{|3}}{3\rho^5} - \frac{\mathrm{e}^{-\phi} f_2 I^2{}_{|5}}{3\rho^5} + \frac{I^1{}_{|23}}{3\rho^4} + \frac{I^2{}_{|45}}{3\rho^4},$$

$$\mathbf{A_4} = \frac{\mathrm{e}^{\phi}}{\rho^4} \left(\frac{f_2 I^1{}_{|3}}{3\rho} - \frac{\mathrm{e}^{2\phi} \bar{f}_2 I^1}{\rho} - \tfrac{1}{3}\mathrm{e}^{\phi}\big(I^1{}_{|34} + 3I^1{}_{|2}\big) \right),$$

$$\mathbf{A_6} = \frac{\mathrm{e}^{-\phi}}{\rho^4} \left(\frac{\bar{f}_2 I^2{}_{|5}}{3\rho} - \frac{\mathrm{e}^{-2\phi} f_2 I^2}{\rho} + \tfrac{1}{3}\mathrm{e}^{-\phi}\big(I^2{}_{|25} + 4I^2{}_{|4}\big) \right),$$

and we will not display $\mathbf{C_1}$ and $\mathbf{A_8}$ as not important.

Since already here the symmetry $I^1 \leftrightarrow I^2$, corresponding to the change $(\theta^2, \theta^3) \leftrightarrow (\theta^4, \theta^5)$, is clearly visible, we skip writing down the analogous formulas for the 1-forms (5.3).

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