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Conformal Walker metrics and linear Fefferman-Graham equations

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The conformal Fefferman-Graham ambient metric construction is one of the most fundamental constructions in conformal geometry. It provides an embedding of a manifold of dimension n with a conformal structure into a semi-Riemannian manifold of dimension $n+2$ whose Ricci tensor vanishes up to a certain order along the original manifold. Despite the general existence result of such ambient metrics by Fefferman and Graham, not many explicit examples of conformal structures with smooth Ricci-flat ambient metrics are known. Motivated by previous examples, for which the Fefferman-Graham equations for the ambient metric to be Ricci-flat reduce to a system of linear PDEs, in the present article we develop a method to find ambient metrics for conformal classes of metrics with two-step nilpotent Schouten tensor. Using this method, for metrics for which the image of the Schouten tensor is invariant under parallel transport, i.e., certain types of Walker metrics, we obtain explicit ambient metrics. This includes certain left-invariant Walker metrics as well as pp-waves.

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1. Introduction and main results

This paper is a follow-up of our papers [3, 25, 26], where we presented several examples of semi-Riemannian conformal structures, not conformally Einstein, with explicit Ricci-flat Fefferman-Graham ambient metrics. The Fefferman-Graham ambient metric is a fundamental construction in conformal geometry that is defined as follows:

Given a conformal class represented by a metric g on a smooth n-dimensional manifold M, a *Fefferman-Graham ambient metric* or just an *ambient metric* is a metric

$$\widetilde{g} = 2\,\mathrm{d}t\mathrm{d}(\rho t) + t^2(g(x^i) + h(x^i, \rho)), \tag{1.1}$$

defined on $\widetilde{M} = (0, \infty) \times M \times (-\epsilon, \epsilon)$, with coordinates x^i on M, $t \in (0, \infty)$ and $\rho \in (-\epsilon, \epsilon)$, such that $h(x^i, \rho)$ is smooth, $h(x^i, \rho)_{|\rho=0} = 0$, and

$$Ric(\widetilde{g}) = O(\rho^m), \text{ with } m = \infty \text{ if } n \text{ is odd and } m = \tfrac{n-2}{2} \text{ if } n \text{ is even.} \tag{1.2}$$

Fefferman and Graham [15, 16] have shown that an ambient metric always exists and is unique in a certain sense, which justifies it to call it *the* ambient metric (for details see Section 2.1).

Moreover, when n is even, there is a conformally covariant, symmetric, divergence and trace free $(0, 2)$-tensor $\mathcal{O}$, the *Fefferman-Graham obstruction tensor*, which vanishes whenever (1.2) holds also for $m \geq \frac{n}{2}$.

We will refer to the equations (1.2) for a metric of the form (1.1) as the *Fefferman-Graham equations.* Sometimes we will say that a solution of (1.2) is given by h, by which we mean that h defines a metric $\widetilde{g}$ via the formula (1.1) such that $Ric(\widetilde{g}) = O(\rho^m)$. Moreover if equation (1.2) holds for all m when n is even, we emphasise this by calling $\widetilde{g}$ a *Ricci-flat ambient metric.*

Finding explicit (Ricci-flat) ambient metrics amounts to solving a system of second order PDEs for the unknown symmetric ρ-dependent $(0, 2)$-tensor field h. In general, these PDE are nonlinear in h, however, for the examples presented in [3, 25, 26], we were able to solve these PDEs explicitly, by the following approach: we found an ansatz for h such that the operator $Ric(\widetilde{g})$ became linear in h, which allowed us to solve the equation $Ric(\widetilde{g}) = 0$. This raises the immediate question: *Which features of the conformal class are responsible for this phenomenon?* In the present paper we will identify one of these features as a property of the conformal holonomy. To formulate our results we have to define the following linear differential operator $\mathcal{A}$: if g is a semi-Riemannian metric and h a symmetric ρ-dependent symmetric bilinear form, we define $\mathcal{A}(h) = \mathcal{A}_{ij}(h)$ as

$$\mathcal{A}_{ij}(h) = 2\rho\ddot{h}_{ij} + (2-n)\dot{h}_{ij} + 2R^k{}_{ij}{}^l h_{kl} - \Box h_{ij}, \tag{1.3}$$

where R_{ijkl} is the curvature tensor, $\Box = \nabla^k\nabla_k$ the tensor Laplacian of g and the dot denotes the derivative with respect to ρ. The following theorem provides a partial answer to the above question. It is a consequence of several key results in this paper. In the remainder of the introduction we will explain how to derive it from the other results.

Theorem 1.1. *Let $(M,[g])$ be a conformal manifold such that the conformal holonomy admits an invariant subspace that is totally null and of dimension greater than* 1*. Then, locally on an open dense subset of M, there is a metric g in the conformal class defining the linear differential operator $\mathcal{A}$ in equation (1.3), such that a solution of equation (1.2) is given via (1.1) by a smooth, divergence free, symmetric bilinear form h that solves the equation*

$$h^{kl}\nabla_k\nabla_l h_{ij} + \nabla_k h_{li}\nabla^l h^k{}_j + \mathcal{A}_{ij}(h) + 2R_{ij} = O(\rho^m), \tag{1.4}$$

with $m=\infty$ if n is odd and $m=\frac{n-2}{2}$ if n is even, and where R_{ij} is the Ricci tensor of g.

Although the appearance of the quadratic terms in equation (1.4) is somewhat unsatisfactory, in many cases there is an ansatz for h such that the quadratic terms vanish and the resulting linear equation for h can be solved explicitly. We will come back to this.

Recall that the conformal holonomy is defined as follows. To a conformal class of signature (p,q) one can assign the normal conformal $\mathfrak{so}(p+1,q+1)$-valued Cartan connection which induces a principal connection and in turn a connection on the vector bundle of conformal standard tractors that is compatible with a bundle metric of signature $(p+1,q+1)$. The conformal holonomy group is the holonomy group of this connection and its natural representation on $\mathbb{R}^{p+1,q+1}$ is the *conformal holonomy representation* or simply the *conformal holonomy*. In analogy to Riemannian geometry, the reduction of the conformal holonomy to a proper subgroup of $\mathbf{SO}(p+1,q+1)$ is related to the existence of special structure of the conformal manifold, such as the existence of Einstein scales [6, 18], the structure of a Fefferman space [17], twistor spinors [7, 29], or exceptional conformal structures [9, 31]. One feature that is very different to Riemannian holonomy reductions is that the same conformal holonomy reductions can induce different structures along different *curved orbits* [11]. There is however a close relationship between the conformal holonomy and the holonomy of the Levi-Civita connection of the Fefferman-Graham ambient metric [10] and in the present paper we will analyse this relationship further for a specific class of conformal structure that plays an important role for the classification of conformal holonomies.

If the conformal holonomy representation is irreducible, several classification results are known [1, 2, 14]. In the case when the holonomy representation is not irreducible, three essentially different situations have to be distinguished: the invariant subspace is a) of dimension one, b) of dimension greater than one and non-degenerate, or c) of dimension greater than one and degenerate with respect to the tractor metric. For case a) it is well known that, locally on an open and dense set in M, there is an Einstein metric in the conformal class. The open dense set is in fact the complement of the zero set of a smooth function σ with the property that σ and $\mathrm{d}\sigma$ have no common zeros. This restriction is a feature of all the other cases that follow. Hence, in case (a) there is an explicit Ricci-flat ambient metric, see Remark 2.1 below. Case b) is similar, here there is a metric in the conformal class that is a product of Einstein metrics (with related Einstein constants), [4, 5, 28]. Again, such conformal structures admit Ricci-flat ambient metrics [19]. The last case c), when the invariant subspace is degenerate, can be reduced to the situation in Theorem 1.1: intersecting the invariant subspace with its orthogonal space gives a holonomy invariant totally null space. It was shown in a series of papers [22, 27, 30] that the assumption in Theorem 1.1 — that the conformal holonomy admits an invariant totally null subspace of dimension $k+1>1$ — is equivalent to the existence, locally and outside a singular set with dense complement, of a totally null distribution $\mathcal{N}$ of rank k and a metric g in the conformal class, such that:

(A) The image of the Schouten tensor P of g, considered as an endomorphism field $\mathsf{P}^\sharp$, $\mathrm{Im}(\mathsf{P}^\sharp)=\{\mathsf{P}^\sharp(X)\mid X\in TM\}$, is contained in $\mathcal{N}$ (which implies $(\mathsf{P}^\sharp)^2=0$),

(B) $\mathcal{N}$ is parallel (with respect to the Levi-Civita connection of g).

In the present paper we will deal with the problem of finding ambient metrics for such conformal classes.

Metrics with a parallel totally null distribution $\mathcal{N}$ are called *Walker metrics* [36]. Metrics with properties (A) and (B) are special Walker metrics, for which the image of the Schouten tensor is contained in the parallel null distribution $\mathcal{N}$. This implies that these metrics are scalar flat and hence the Schouten tensor is a constant multiple of the Ricci-tensor. In particular, their Ricci and Schouten tensors are divergence free. In the following we will call metrics that have both properties (A) and (B) *null Ricci Walker metrics*, referring to the property that image of the Ricci tensor is totally null. The case $k = 1$ was considered in [27], where the metrics were called *pure radiation metrics with parallel rays.* There are many known examples of null Ricci Walker metrics. This includes Lorentzian pp-waves but also the examples of metrics we gave in [3], which are of signature $(3, 3)$ and lie in Bryant's conformal classes [9]. Recently, in [21] the ambient metric for *Patterson–Walker metrics* was computed. Patterson–Walker metrics are null Ricci Walker metrics in neutral signature (n, n) that arise from projective structures in dimension n. In Section 5 we will give more examples of null Ricci Walker metrics including left-invariant metrics.

With the above characterisation of the assumption, Theorem 1.1 is a consequence of several results we will prove in this paper. To explain these results, we recall that property (A) is satisfied by all all the examples in [3], the generalised pp-waves (see more in Sections 5.3 and 5.4), the conformal structures given by generic distributions in dimension five and six (neither of which are directly the subject of the current paper). This observation combined with the fact that $\partial_\rho h|_{\rho=0} = 2\mathsf{P}$, suggested our ansatz for h as a tensor satisfying $\mathrm{Im}(h^\sharp) \subset \mathcal{N}$. If not only (A) but also (B) is satisfied, which is the case for most but not all of the examples in [3], then we can show that the condition $\mathrm{Im}(h^\sharp) \subset \mathcal{N}$ is necessary.

Theorem 1.2. *Let (M, g) be a semi-Riemannian null Ricci Walker metric with parallel null distribution $\mathcal{N}$. Then for every ambient metric $\widetilde{g} = 2\,\mathrm{d}t\mathrm{d}(\rho t) + t^2(g(x^i) + h(x^i, \rho))$, i.e., a solution to the equations (1.2) with smooth h, it holds*

$$\mathrm{div}^g(h) = O(\rho^m), \quad \mathrm{Im}(h^\sharp) \subset \mathcal{N} \mod O(\rho^m) \tag{1.5}$$

with $m = \infty$ when n is odd and $m = \frac{n}{2}$ when n is even. Moreover, when n is even, the obstruction tensor satisfies $\mathrm{Im}(\mathcal{O}^\sharp) \subset \mathcal{N}$ and there is an ambient metric for which h satisfies equations (1.5) for $m = \infty$.

We will prove this theorem in Section 4.2. Note that the statement about the obstruction tensor can also be obtained from results in [24]. Theorem 1.2 leads us to study the equation (1.2) for $\widetilde{g}$ as in (1.1) defined by a Walker metric g with parallel null distribution $\mathcal{N}$ and with a tensor h with $\mathrm{Im}(h^\sharp) \subset \mathcal{N}$. From the results and computations in Section 3 and Section 4 we obtain the following statement, which together with Theorem 1.2 implies Theorem 1.1:

Theorem 1.3. *Let (M, g) be a null Ricci Walker metric with parallel null distribution $\mathcal{N}$ and assume that h is a divergence-free symmetric $(0, 2)$-tensor field such that $\mathrm{Im}(h^\sharp) \subset \mathcal{N}$. Then the metric $\widetilde{g}$ defined by h via equation (1.1) satisfies (1.2) if and only if h satisfies equation (1.4).*

In the case when the parallel null distribution $\mathcal{N}$ has rank one or satisfies an additional condition on the curvature, we can strengthen this result in the sense that the quadratic terms in equation (1.4) will vanish:

Corollary 1.1. *Let $(M, [g])$ be a conformal manifold given by a null Ricci Walker metric g with parallel null distribution $\mathcal{N}$ that has rank one or satisfies $\mathcal{N} \lrcorner R = 0$, for R the curvature tensor of g. Then there is an ambient metric, i.e., a solution of (1.2), that is given via (1.1) by a divergence free symmetric bilinear form h that solves the linear PDE system*

$$\mathcal{A}_{ij}(h) + 2R_{ij} = O(\rho^m), \tag{1.6}$$

with $m = \infty$ if n is odd and $m = \frac{n-2}{2}$ if n is even, and where R_{ij} is the Ricci tensor of g. When n is even, the obstruction tensor satisfies $\mathrm{Im}(\mathcal{O}^\sharp) \subset \mathcal{N}$ and $\mathcal{N} \lrcorner \nabla \mathcal{O} = 0$ and is given by

$$\mathcal{O}_{ij} = c_n \,\Box^m R_{ij},$$

where c_n is a nonzero constant and $\Box^m$ is the m-th power of the tensor Laplacian.

This corollary follows from the previous results by the following considerations: if the rank of $\mathcal{N}$ is one, then h being divergence free implies that

$$\mathcal{L}_X h = 0, \qquad \text{for all } X \in \mathcal{N}, \tag{1.7}$$

where $\mathcal{L}_X$ denotes the Lie derivative in direction X, and hence that $\nabla_X h = 0$ for all X in $\mathcal{N}$, which in turn yields to the vanishing of the quadratic terms in (1.4). Similarly if the rank of $\mathcal{N}$ is larger than one, one can show that the curvature condition $\mathcal{N} \lrcorner R = 0$ implies condition

$$\mathcal{L}_X \mathsf{P} = 0, \qquad \text{for all } X \in \mathcal{N}, \tag{1.8}$$

and consequently that $\nabla_X \mathsf{P} = 0$ for all $X \in \mathcal{N}$. This can then be used to show that h has to satisfy the condition (1.7) and which again implies the vanishing of the quadratic terms.

Let us point out that Corollary 1.1 is sharp in the sense that there are null Ricci Walker metrics with $\mathcal{N} \lrcorner R \neq 0$ for which the Fefferman-Graham equations remain quadratic in h. We make this explicit in Example 4.1. It turns out that for the linearisation of the Fefferman-Graham equations the conditions (1.7) and (1.8) are crucial. In fact, when (1.8) is satisfied, the ansatz (1.7) enables us to reduce the Fefferman-Graham equations to linear equations in a much larger class than the one that satisfies the assumptions of Corollary 1.1. In Section 3 we show that for $\widetilde{g}$ as in (1.1) the Ricci tensor $Ric(\widetilde{g})$ becomes at most in h if we assume that

(1) the image of P is contained in a totally null distribution $\mathcal{N}$,

(2) and that $\mathcal{N}^{\perp}$ is *involutive* (but not necessarily parallel).

The form of the Fefferman-Graham equations in this more general situation, although being at most quadratic in h, is however more complicated than equations (1.6). Nevertheless, we find this more general class noteworthy: the examples of $\mathbf{G}_2$-conformal metrics in [3, 26], for which the linear Fefferman-Graham equation were reduced to linear PDEs, are *not* null Ricci Walker metrics but rather from this more general class. The reduction was possible because these metrics satisfy the additional property (1.8), which suggested the ansatz (1.7).

Based on Corollary 1.1, we are able to construct explicit ambient metrics for several examples of null Ricci Walker metrics, including left-invariant metrics on Lie groups and generalised pp-waves. Our main results in Section 5 are the following:

Theorem 1.4. *Let $\mathfrak{k}$ be a two-step nilpotent Lie algebra, H be a Lie group with Lie algebra $\mathfrak{h}$, and $\phi : \mathfrak{h} \to \mathfrak{der}(\mathfrak{k})$ a Lie algebra homomorphism to the derivations of $\mathfrak{k}$. Let G be the Lie group corresponding to the Lie algebra $\mathfrak{g}$ that is given as the semi-direct sum*

$$\mathfrak{g} = \mathfrak{h} \ltimes_{\phi} \mathfrak{k}.$$

Moreover, let g be a semi-Riemannian left-invariant metric on G such that $\mathfrak{z}^{\perp} = \mathfrak{k}$ and $\mathfrak{g} = \mathfrak{h}^{\perp} \oplus \mathfrak{z}$, where $\mathfrak{z}$ is the centre of $\mathfrak{k}$. Then the conformal class of g on G admits a Ricci-flat ambient metric

$$\widetilde{g} = 2\mathrm{d}(\rho t)\mathrm{d}t + t^2\Big(g + \frac{2\rho}{n-2} Ric(g)\Big), \tag{1.9}$$

where n is the dimension of G and $Ric(g)$ is the Ricci tensor of g.

We should point out that the ambient metric in (1.9) is not unique (when n even or when non-analytic ambient metrics are allowed). In fact, in Theorem 5.1 we find the most general form for Ricci-flat ambient metrics for the left-invariant metrics in Theorem 1.4 and show that the ambiguity is parametrised by $\frac{k(k+1)}{2}$ functions of $n-k$ variables, where k is the dimension of $\mathfrak{h}$ and n the dimension of G.

Finally, amongst other results, in Section 5 we extend our results in [25] and [3]. The first paper [25] dealt with *Lorentzian pp-waves*, that is, metrics on $\mathbb{R}^n$ of the form

$$g = 2\mathrm{d}u\mathrm{d}v + H\,\mathrm{d}u^2 + \sum_{i=1}^{n-2}(\mathrm{d}x^i)^2, \quad H \in C^\infty(\mathbb{R}^n) \text{ with } \tfrac{\partial H}{\partial v} = 0. \tag{1.10}$$

In general, these metrics are not conformally Einstein. In [25] a formula for a Ricci-flat ambient metric is given when n is odd or in the case when n is even and $\Delta^{\frac{n}{2}} H = 0$, where Δ is the flat Laplacian in the coordinates $x^1, \ldots, x^{n-2}$. In the odd case, this is the unique analytic ambient metric, in the even case however, we were not able to rigorously prove that the obstruction tensor is given by $\Delta^{\frac{n}{2}}(H)\mathrm{d}u^2$. This result was generalised in [3] in two ways: it was generalised to analogues of Lorentzian pp-waves to other signatures and, more importantly, in even dimensions we provided the general solution to the Fefferman Graham solutions including log-terms and the explicit form of the ambiguity. Here, in the current paper, we improve this result by deriving a formula for the ambient metric in the more general setting in which we allow for a non-flat Riemannian metric G_{ij} in the $x^1, \ldots, x^{n-2}$ coordinates. Moreover, for Lorentzian pp-waves we can use Corollary 1.1 to show that the obstruction tensor is in fact given by $\Delta^{\frac{n}{2}} H\mathrm{d}u^2$. We summarise our results of Sections 5.3 and 5.4 in the case of Lorentzian manifolds:

Theorem 1.5. *Let*

$$g = 2\mathrm{d}u\mathrm{d}v + H\,\mathrm{d}u^2 + \sum_{i,j=1}^{n-2} G_{ij}\mathrm{d}x^i\mathrm{d}x^j, \quad \text{with } \tfrac{\partial H}{\partial v} = \tfrac{\partial G_{ij}}{\partial v} = \tfrac{\partial G_{ij}}{\partial u} = 0,$$

be a Lorentzian generalised pp-wave metric. For Δ_G the Laplacian for the Riemannian metric $G_{ij}\mathrm{d}x^i\mathrm{d}x^j$ on $\mathbb{R}^{n-2}$ with coordinates $x^1, \ldots, x^{n-2}$ and $f = f(x^1, \ldots, x^{n-2}, u)$ a smooth function of the x^i's and u, consider the *metric*

$$(1.11)\quad \widetilde{g} = 2\mathrm{d}(\rho t)\mathrm{d}t + t^2 g + \\ + t^2\left(\sum_{k=1}^{m} \frac{\Delta_G^k(H)}{k!\prod_{i=1}^{k}(2i-n)}\rho^k + \sum_{k=0}^{\infty} \frac{\Delta_G^k(f)}{k!\prod_{i=1}^{k}(2i+n)}\rho^{\frac{n}{2}+k}\right)\mathrm{d}u^2,$$

where $m = \infty$ when n is odd and $m = \frac{n-2}{2}$ when n is even. Then, when n is odd, the metric in (1.11) with $f \equiv 0$ gives the unique Ricci-flat ambient metric that is analytic in ρ. When n is even, we have the following:

(1) *If $\Delta_G^{\frac{n}{2}}(H) = 0$, then the metrics in (1.11) are Ricci-flat ambient metrics that are analytic in ρ. Otherwise the metrics are solutions to (1.2), that is $Ric(\widetilde{g}) = O(\rho^{\frac{n}{2}})$.*

(2) *If $G_{ij} \equiv \delta_{ij}$ is the Euclidean metric, then the obstruction tensor $\mathcal{O}$ is given by*

$$\mathcal{O} = c_n\, \Delta^{\frac{n}{2}}(H)\, \mathrm{d}u^2,$$

for some non-zero constant c_n. If $\mathcal{O}$ vanishes, the metrics in (1.11) are Ricci-flat.

In addition to this, we obtain non-analytic Ricci-flat ambient metrics with $h \downarrow 0$ if $\rho \to 0$ from formulas (5.9) and (5.11) in Theorems 5.2 and 5.3, in particular in the case when n is even and the obstruction tensor does not vanish.

We believe that the formulas we provide in this paper turn out to be useful for obtaining explicit solutions to the Fefferman-Graham equations for new examples beyond the ones given in Theorems 1.4 and 1.5.

2. The Fefferman-Graham ambient construction

2.1. The Fefferman-Graham ambient metric construction

A *conformal structure* $(M, [g])$ on an $n = p + q$ dimensional manifold M is an equivalence $[g]$ class of (p, q)-signature metrics on M, such that two metrics g and $\hat{g}$ are in the same class $[g]$ if and only if there exists a function ϕ on M, such that $\hat{g} = \mathrm{e}^{2\phi} g$.

Let us focus on a given conformal structure $(M, [g])$. In the following definition of an ambient metric we will refer to a manifold $\widetilde{M}$ that is a product

$$\widetilde{M} = (0,\infty) \ \times \ M \ \times \ (-\epsilon,\epsilon), \qquad \epsilon > 0,$$

with respective coordinates (t, x^i, ρ).

Definition 2.1. An *ambient metric* $\widetilde{g}$ *for* $(M,[g])$ *(that is in normal form with respect to g)* is a metric on $\widetilde{M}$ given by

$$\widetilde{g} = 2\,\mathrm{d}t\,\mathrm{d}(\rho t) + t^2 g(x^i,\rho), \tag{2.1}$$

with a 1-parameter family of symmetric, non-degenerate smooth bilinear forms $g(x^i,\rho)$ on M, parametrized by ρ, such that

$$g(x^i,\rho)_{|\rho=0} = g(x^i),$$

for some metric $g = g(x^i)$ from the conformal structure $[g]$ and such that

- $Ric(\widetilde{g}) = O(\rho^\infty)$ if n is odd, and
- $Ric(\widetilde{g}) = O(\rho^{\frac{n}{2}-1})$ and $\mathrm{tr}_g\left(\rho^{1-\frac{n}{2}} Ric(\widetilde{g})_{|TM\otimes TM}\right) = 0$ along $\rho = 0$, if n is even.

Here, using the usual convention, for a smooth tensor field tensor S on $\widetilde{M}$ we write $S = O(\rho^k)$ if $S = \rho^k T$ for a smooth tensor field T. The existence and uniqueness result for ambient metrics in [15, 16] states that for each choice of $g = g(x^i)$ there is an ambient metric w.r.t. g. In all dimensions $n \geq 3$, $g(x^i,\rho)$ has an expansion of the form

$$g(x^i,\rho) = \sum_{k\geq 0} g^{(k)}(x^i)\rho^k$$

starting with

$$g(x^i,\rho) = g(x^i) + 2\rho\,\mathsf{P}(x^i) + O(\rho^2),$$

where $\mathsf{P} = \frac{1}{n-2}(Ric - \frac{Scal}{2(n-1)}g)$ is the Schouten tensor of $g = g(x^i)$. In odd dimensions the Ricci-flatness condition determines $g^{(k)}$ uniquely for all k, whereas in even dimensions only the $g^{\left(k<\frac{n}{2}\right)}$ and the trace of $g^{\left(\frac{n}{2}\right)}$ are determined uniquely. The ambient metric construction is conformally invariant in the sense that ambient metrics for different metrics in the conformal class are diffeomorphic to each other (modulo $O(\rho^{\frac{n}{2}})$ when n is even).

For n even a conformally invariant symmetric $(0,2)$-tensor on M, the *ambient obstruction tensor* $\mathcal{O}$, obstructs the existence of smooth solutions to $Ric(\widetilde{g}) = O(\rho^{\frac{n}{2}})$. For $\widetilde{g}$ in normal form w.r.t. g as in Definition 2.1 it is given by

$$\mathcal{O} = c_n \left(\rho^{1-\frac{n}{2}} (Ric(\widetilde{g})_{|TM \otimes TM}) \right) \Big|_{\rho=0}, \tag{2.2}$$

where c_n is some known nonzero constant [16]. From this one can deduce that $\mathcal{O}$ is trace- and divergence free.

Remark 2.1. If $[g]$ contains the *flat* metric g_0, then the corresponding ambient metric is

$$\widetilde{g} = 2\,\mathrm{d}t\,\mathrm{d}(\rho t) + t^2 g_0.$$

Similarly, if $[g]$ contains an *Einstein* metric g_Λ, $Ric(g_\Lambda) = \Lambda g_\Lambda$, then

$$\widetilde{g} = 2\,\mathrm{d}t\,\mathrm{d}(\rho t) + t^2 \left(1 + \frac{\Lambda \rho}{2(n-1)}\right)^2 g_\Lambda$$

is an ambient metric for $[g_\Lambda]$ that is Ricci-flat.

2.2. Poincaré-Einstein metrics

As noticed by Fefferman and Graham in [15, 16], their ambient metric construction associating a $(p+1, q+1)$-signature *Ricci-flat* metric $\widetilde{g}$ to a (p, q)-signature conformal structure $[g]$, is very closely related to another construction, called the Poincaré-Einstein construction, which associates a certain $(p+1, q)$- or $(p, q+1)$-signature *Einstein* metric g_{PE} to the conformal class $[g]$. This construction goes back to Penrose [33] and is widely used by mathematical physicists[1] in such fields as AdS/CFT correspondence [13, 34] and General Relativity, where it describes the geometry near horizons of black holes [12] or the evolution of conformal data from hypersurfaces formed by the starting points of all null geodesics of a spacetime, which is of importance in Penrose's conformal cyclic cosmology, [32, 35].

As the ambient metric construction is a generalisation of the relation between the Minkowski spacetime $(\mathbb{R}^{n+1,1}, \eta)$ and the flat conformal structure on the Euclidean sphere $\mathbb{S}^n$, which is its light cone cut, the Poincaré-Einstein

[1]There is a confusion of terminology: mathematical physicists refer to the Poincaré-Einstein construction as 'Fefferman-Graham construction', whereas in mathematics, since the first Fefferman-Graham paper [15], this term has been reserved for the 'ambient metric construction'.

construction is the generalisation of the relation between the usual hyperbolic metric on the interior of a ball B^{n+1} and the conformally flat structure on the Euclidean sphere $\mathbb{S}^n$ which is the boundary of this ball. In the lowest dimension this simple ball-sphere relation is just the relation describing the classical Poincaré disk model of the 2-dimensional geometry. To be more explicit, the passage from an ambient metric $\widetilde{g}$ in $\widetilde{M}$ to the corresponding Poincaré-Einstein metric g_{PE} in one dimension lower is as follows.

Let $[g]$ be a conformal class of signature (p, q), with p denoting the number of spacelike vectors in an orthonormal basis, and let $(\widetilde{M}, \widetilde{g})$ be its ambient space, expressed in the ambient coordinates (t, x^i, ρ) as in (2.1). Then consider a hypersurface in $\widetilde{M}$ defined in the following steps:

- Let M_{PE} be an open set in $\mathbb{R}^{n+1}$ parametrized by the coordinates $(r > 0, x^i)$.
- Imbed M_{PE} into $\widetilde{M}$, via the map $\iota : M_{PE} \to \widetilde{M}$, given by

$$\iota(r, x^i) = (t = \tfrac{1}{r}, x^i, \rho = -\tfrac{\epsilon}{2} r^2).$$

  Choose the parameter ϵ to be either 1 or -1.
- For the choice of the parameter ϵ, pull the ambient metric $\widetilde{g}$ back by ι^* from $\widetilde{M}$ to M_{PE} obtaining:

$$g_{PE} := \iota^*(\widetilde{g}) = \tfrac{1}{r^2}\big(\epsilon \mathrm{d}r^2 + g(x^i, -\tfrac{\epsilon}{2} r^2)\big).$$

- The metric g_{PE} on M_{PE} has signature $(p + 1, q)$ if $\epsilon = 1$ and $(p, q + 1)$ if $\epsilon = -1$.
- More importantly, the metric g_{PE} on M_{PE} is Einstein,

$$Ric(g_{PE}) = -\epsilon n g_{PE},$$

  since the ambient metric $\widetilde{g}$ is Ricci-flat.

For each chosen value of the parameter ϵ, the pseudo-Riemannian manifold (M_{PE}, g_{PE}) is the *Poincaré-Einstein manifold* associated with the conformal structure $[g]$ on M. The metric g_{PE} is called the *Poincaré-Einstein metric* for $[g]$. Note that the metric $\hat{g} = r^2 g_{PE}$, conformally related to g_{PE}, defines a regular conformal class of metrics on the boundary ∂M_{PE} of M_{PE}, given by $\partial M_{PE} = \{(r, x^i) \mid r = 0\}$. The conformal manifold $(\partial M_{PE}, [\hat{g}])$ is (locally) conformally equivalent to the original conformal structure $(M, [g])$.

We remark, that via the procedure described above, all explicit ambient metrics appearing in this paper have their Poincaré-Einstein counterparts and we will not comment any further on this relation in what follows.

2.3. The Fefferman-Graham equations

Given a conformal structure and having its representative $\overset{0}{g}$, the search for a corresponding Fefferman-Graham ambient metric

$$\widetilde{g} = 2\mathrm{d}(\rho t)\mathrm{d}t + t^2 g(x,\rho),$$

consists in finding a 1-parameter family $g(x,\rho)$ of metrics on M with $g|_{\rho=0} = \overset{0}{g}$ and such that the Ricci tensor of the metric $\widetilde{g}$ satisfies equations (1.2). In Ref. [16, Eq. 3.17] the components of $Ric(\widetilde{g})$ for (2.1) were written explicitly for the unknown tensor $g = g(x^i,\rho)$. Writing g as $g = g_{ij}\mathrm{d}x^i\mathrm{d}x^j$, with $g_{ij} = g_{ij}(x^k,\rho)$ (or in abstract index notation), equation (1.2) then reads as:

$$\rho\ddot{g}_{ij} - \rho g^{kl}\dot{g}_{ik}\dot{g}_{jl} + \tfrac{1}{2}\rho g^{kl}\dot{g}_{kl}\dot{g}_{ij} - \tfrac{n-2}{2}\dot{g}_{ij} - \tfrac{1}{2}g^{kl}\dot{g}_{kl}g_{ij} + R_{ij} = O(\rho^m), \tag{2.3}$$

$$g^{kl}\left(\nabla_k\dot{g}_{il} - \nabla_i\dot{g}_{kl}\right) = O(\rho^m), \tag{2.4}$$

$$g^{kl}\ddot{g}_{kl} + \tfrac{1}{2}g^{kl}g^{pq}\dot{g}_{pk}\dot{g}_{ql} = O(\rho^m), \tag{2.5}$$

for $m = \infty$ when n is odd and $m = \frac{n-2}{2}$ when n is even. Here for each ρ, ∇ is the Levi-Civita connection of the metric $g(x^k,\rho) = g_{ij}(x^k,\rho)\mathrm{d}x^i\mathrm{d}x^j$, R_{ij} is the Ricci tensor of $g(x^i,\rho)$, and the dot denotes partial derivative of g_{ij} with respect to ρ. The left-hand sides of these equations are the components of the Ricci-tensor $Ric(\widetilde{g})$ of $\widetilde{g}$.

The first of the Fefferman-Graham equations above is a system of nonlinear 2nd order PDEs for the coefficients g_{ij}. It is also obvious that finding the general solution for this system with a given initial condition $g_{ij|\rho=0} = \overset{0}{g}_{ij}$ is rather hopeless. One can search for Fefferman-Graham metrics assuming that the metric $g(x,\rho)$ admits a power series expansion with integer powers in ρ. Fefferman and Graham [16] gave expressions for the first few terms in the power series expansion in ρ of $g(x,\rho)$ so that $\widetilde{g}$ is Ricci-flat up to the order 3. Up to this order, their expansion reads:

$$g = \overset{0}{g} + 2\mathsf{P}\rho + \mu\rho^2 + \ldots,$$

with P being the Schouten tensor for $\overset{0}{g}$, and

$$(4-n)\mu_{ij} = B_{ij} + (4-n)\mathsf{P}_i{}^k\mathsf{P}_{kj}.$$

Here B is the Bach tensor of the metric $\overset{0}{g}$ defined by

$$B_{ij} = \overset{0}{\nabla}{}^k A_{ijk} - \mathsf{P}^{kl} W_{kijl},$$

with

$$A_{ijk} = \overset{0}{\nabla}_j \mathsf{P}_{ki} - \overset{0}{\nabla}_k \mathsf{P}_{ji}$$

the Cotton tensor. The symbol $\overset{0}{\nabla}$ denotes the Levi-Civita connection for $\overset{0}{g}$ and $W^i{}_{jkl}$ is the Weyl tensor for $\overset{0}{g}$.

2.4. Our approach

Our approach in this paper will be the following: We will write the unknown family of semi-Riemannian metrics $g(x^i, \rho)$ in the Fefferman-Graham metric as

$$g(x^i, \rho) = \overset{0}{g}(x^i) + h(x^i, \rho),$$

where $\overset{0}{g} = \overset{0}{g}(x^i)$ is a suitable metric from the conformal class (independent of ρ) and $h = h(x^i, \rho)$ is symmetric, ρ-dependent symmetric bilinear form on M. For our approach we will express the Levi-Civita connection and the Ricci tensor of $g(x^i, \rho)$, which is needed in equations (2.3, 2.4, 2.5), in terms of the Levi-Civita connection and the Ricci tensor of $\overset{0}{g}$. For this, recall the formulas relating the Levi-Civita connections and the curvatures of two given metrics g_{ij} and $\overset{0}{g}_{ij}$. The difference of both Levi-Civita connections is given by a tensor field $C^k{}_{ij}$,

$$\nabla_i X_j - \overset{0}{\nabla}_i X_j = C^k{}_{ij} X_k, \tag{2.6}$$

where X_k is a one-form. For vector fields we have

$$\nabla_i X^j - \overset{0}{\nabla}_i X^j = -C^j{}_{ik} X^k.$$

Since both connections are torsion-free, it is $C^k{}_{ij} = C^k{}_{ji}$, which, together with $\nabla_i g_{jk} = 0$, implies

$$C^k{}_{ij} = \tfrac{1}{2} g^{kl} \left(\overset{0}{\nabla}_l g_{ij} - \overset{0}{\nabla}_i g_{jl} - \overset{0}{\nabla}_j g_{il} \right) \tag{2.7}$$

For the curvature tensors, defined by $R_{ijk}{}^l v_l = 2\nabla_{[i} \nabla_{j]} v_k$ we obtain

$$R_{ijk}{}^l = \overset{0}{R}_{ijk}{}^l + 2\overset{0}{\nabla}{}_{[i} C^l{}_{j]k} + 2C^p{}_{k[i} C^l{}_{j]p},$$

and hence for the Ricci tensor

$$R_{ij} = R_{ikj}{}^{k} = \overset{0}{R}_{ij} + \overset{0}{\nabla}_i C^k{}_{kj} - \overset{0}{\nabla}_k C^k{}_{ij} + C^p{}_{ij} C^k{}_{kp} - C^p{}_{jk} C^k{}_{ip}. \tag{2.8}$$

We will use these formulas later on.

Unless we use indices, for a symmetric $(0,2)$-tensor h we denote by $h^\sharp$ the corresponding endomorphism defined by the metric g via $h(X,Y) = g(h^\sharp X, Y)$. In the following it will be clear from the context which metric will be used to perform this dualisation. We will say that h is two-step nilpotent if $(h^\sharp)^2 = 0$. When making statements about the image of a symmetric $(0,2)$-tensor, we refer to the image of $h^\sharp$ as an endomorphism of TM. It is immediate that a symmetric $(0,2)$-tensor is 2-step nilpotent if and only if its image is trivial or totally null. In particular, we have that $\mathrm{Im}(h^\sharp) \subset \mathrm{Ker}(h^\sharp)$. We summarise the situation in case of the Schouten tensor:

Lemma 2.1. *Let* $(M, \overset{0}{g})$ *be a semi-Riemannian manifold with Ricci tensor* Ric *and Schouten tensor* P. *Then the following are equivalent:*

(1) $(\mathsf{P}^\sharp)^2 = 0$,

(2) $(Ric^\sharp)^2 = 0$,

(3) $\mathrm{Im}(\mathsf{P}^\sharp)$ *is totally null or trivial,*

(4) $\mathrm{Im}(Ric^\sharp)$ *is totally null or trivial.*

If any of these conditions is satisfied, then $(M, \overset{0}{g})$ *has vanishing scalar curvature.*

Hence, since we aim to find ambient metrics for metrics with two-step nilpotent Schouten tensor P, it is reasonable to assume that there is a totally null vector distribution $\mathcal{N}$ that contains the image of the Schouten tensor (which will be the case for the null Ricci Walker manifolds of the Introduction and Section 4). On the other hand from [16] we know that $\dot{\mathbf{h}}|_{\rho=0} = 2\mathsf{P}$, which leads to our ansatz for the ambient metric to assume that $\mathrm{Im}(\mathbf{h}(\rho)) \subset \mathcal{N}$ for all ρ. It will turn out that further conditions on $\mathcal{N}$ and on $\mathbf{h}$ will be needed to ensure that the Fefferman-Graham equations become linear in $\mathbf{h}$.

3. Towards linear Fefferman-Graham equations

In this section we will compute the Ricci tensor for metrics of the form

$$\widetilde{\mathbf{g}} = 2\mathrm{d}(\rho t)\mathrm{d}t + t^2(\overset{0}{\mathbf{g}} + \mathbf{h}),$$

where $\mathbf{h} = \mathbf{h}(\rho)$ is a ρ-dependent family of symmetric bilinear forms with $\mathbf{h}|_{\rho=0} = 0$ and moreover with the property that

$$\mathrm{Im}(\mathbf{h}^\sharp) \subset \mathcal{N},$$

for a totally null distribution $\mathcal{N}$. We will then successively impose further conditions on $\mathcal{N}$ and on $\mathbf{h}$ so that the Fefferman-Graham equations become at most quadratic and eventually linear in $\mathbf{h}$.

3.1. Conventions

In this and in the following sections we work with specific (co)-frames. Hence we will distinguish between tensors (written in boldface letters) $\overset{0}{\mathbf{g}}$, $\mathbf{h}$ and later $\mathbf{g}$, and their components $\overset{0}{g}_{ij}$, h_{ij} and g_{ij} in a *specific* (co)-frame that is adapted to $\mathcal{N}$ and later on satisfies additional properties. Some of the statements in the next sections will only hold for the components h_{ij} of $\mathbf{h}$ in such a basis.

Let $\overset{0}{\mathbf{g}}$ be a semi-Riemannian metric and $\mathcal{N}$ be a vector distribution that is totally null and of rank $p \geq 1$. We fix a local frame

(3.1)
$$\mathbf{e}_1, \ldots, \mathbf{e}_n,$$
such that $\mathrm{span}\{\mathbf{e}_1, \ldots \mathbf{e}_p\} = \mathcal{N}$ and $\mathrm{span}\{\mathbf{e}_1, \ldots, \mathbf{e}_{n-p}\} = \mathcal{K} := \mathcal{N}^\perp$.

Note that $p \leq n - p$. We will use the following index conventions:

$$\begin{aligned} i, j, k, \ldots &\in \{1, \ldots, n\} \\ a, b, c, \ldots &\in \{1, \ldots, p\} \\ A, B, C, \ldots &\in \{p+1, \ldots, n-p\} \\ \bar{a}, \bar{b}, \bar{c} \ldots &\in \{n-p+1, \ldots, n\}. \end{aligned} \tag{3.2}$$

We use the indices $i, j, k, \ldots$ as abstract indices (or with respect to an arbitrary frame), whereas indices $\bar{a}, B, \bar{c}$ will refer to components in a frame

$\mathbf{e}_a, \mathbf{e}_B, \mathbf{e}\bar{c}$, such that

$$\begin{aligned} \overset{0}{\mathbf{g}}(\mathbf{e}_{\bar{a}}, \mathbf{e}_b) = \overset{0}{\mathbf{g}}(\mathbf{e}_b, \mathbf{e}_{\bar{a}}) = \overset{0}{g}_{\bar{a}b} = \overset{0}{g}_{b\bar{a}} \text{ constant and non degenerate,} \\ \overset{0}{\mathbf{g}}(\mathbf{e}_A, \mathbf{e}_B) = \overset{0}{\mathbf{g}}(\mathbf{e}_B, \mathbf{e}_A) = \overset{0}{g}_{AB} = \overset{0}{g}_{BA} \text{ constant and non degenerate,} \\ \overset{0}{\mathbf{g}}(\mathbf{e}_i, \mathbf{e}_j) = 0 \text{ otherwise.} \end{aligned} \tag{3.3}$$

In other words, if $\Theta^1, \dots, \Theta^n$ denote the algebraic duals to the $\mathbf{e}_i$'s, i.e.

$$\Theta^i(\mathbf{e}_j) = \delta^i{}_j$$

then the metric is

$$\overset{0}{\mathbf{g}} = \overset{0}{g}_{ij}\Theta^i\Theta^j = 2\overset{0}{g}_{a\bar{c}}\Theta^a\Theta^{\bar{c}} + \overset{0}{g}_{AB}\Theta^A\Theta^B. \tag{3.4}$$

Note that the inverse $\overset{0}{g}{}^{ij}$ of the matrix $\overset{0}{g}_{ij}$ is given by $\overset{0}{g}{}^{a\bar{b}} = \overset{0}{g}{}^{\bar{b}a}$ and $\overset{0}{g}{}^{AB}$ satisfying

$$\overset{0}{g}_{a\bar{b}}\overset{0}{g}{}^{\bar{b}c} = \delta_a{}^c, \qquad \overset{0}{g}_{\bar{a}b}\overset{0}{g}{}^{b\bar{c}} = \delta_{\bar{a}}{}^{\bar{c}}, \qquad \overset{0}{g}_{AB}\overset{0}{g}{}^{BC} = \delta_A^{\ C}.$$

This relates the algebraic duals Θ^i to the metric duals $\overset{0}{\mathbf{g}}(\mathbf{e}_i, .)$ of $\mathbf{e}_i$ as follows

$$\Theta^a = \overset{0}{g}{}^{a\bar{c}}\overset{0}{\mathbf{g}}(\mathbf{e}_{\bar{c}}, .), \qquad \Theta^{\bar{a}} = \overset{0}{g}{}^{\bar{a}c}\overset{0}{\mathbf{g}}(\mathbf{e}_c, .), \qquad \Theta^A = \overset{0}{g}{}^{AB}\overset{0}{\mathbf{g}}(\mathbf{e}_B, .)$$

Now we consider a symmetric bilinear form $\mathbf{h}$ (depending on a parameter ρ) that satisfies

$$\mathrm{Im}(\mathbf{h}^\sharp) \subset \mathcal{N}.$$

This is equivalent for $\mathbf{h}$ to be of the form

$$\mathbf{h} := h_{\bar{a}\bar{c}}\,\Theta^{\bar{a}}\Theta^{\bar{c}} = h_{ij}\Theta^i\Theta^j, \tag{3.5}$$

i.e., $h_{ij} = 0$ unless $i, j = \bar{a}, \bar{c}$, for smooth functions $h_{\bar{a}\bar{c}} = h_{\bar{a}\bar{c}}(\rho, x)$ with $h_{\bar{a}\bar{c}} = h_{\bar{c}\bar{a}}$, The corresponding $(1,1)$ tensor $\mathbf{h}^\sharp$ has components

$$h_{\bar{a}}{}^b = h_{\bar{a}\bar{c}}\overset{0}{g}{}^{b\bar{c}}$$

and all others zero, i.e.

$$\mathbf{h}^\sharp = h_{\bar{a}}{}^b\Theta^{\bar{a}} \otimes \mathbf{e}_b.$$

and satisfies

$$(\mathbf{h}^\sharp)^2 = 0, \quad \text{i.e. } h_{\bar{a}}{}^k h_k{}^b = 0.$$

It holds that

$$\mathcal{K} = \mathcal{N}^{\perp} \subset \ker(\mathbf{h}^{\sharp}).$$

Finally, we obtain the $(2,0)$-tensor defined by $h^{ij} = \overset{0}{g}{}^{ik}\overset{0}{g}{}^{jl}h_{kl}$, i.e., with

$$h^{bd} = \overset{0}{g}{}^{b\bar{a}}\overset{0}{g}{}^{d\bar{c}}h_{\bar{a}\bar{c}}$$

and all other components zero. From now on the components of all the tensor are given in the frame (3.1) with the index conventions as in (3.3).

Lemma 3.1. *For* $\mathbf{h}$ *as in (3.5) denote by* $\mathbf{h}^{(r)} = (h^{(r)}_{ij})$ *the tensor whose components are given by the* r*-th* ∂_ρ*-derivative of the components of* h_{ij}*, i.e.,* $\mathbf{h}^{(r)} := \partial^r_\rho(h_{ij})\Theta^i \circ \Theta^j$*. Then*

$$\overset{0}{g}{}^{ij}h^{(r)}_{ij} = 0 \quad \text{and} \quad h^{(r)}_{ik}h^{(s)k}{}_j = 0 \text{ for all } 0 \le r, s. \tag{3.6}$$

Moreover, if $\overset{0}{\nabla}$ *is the Levi-Civita connection of* $\overset{0}{\mathbf{g}}$*, then*

$$\overset{0}{\nabla}_k h^{(r)}_{ij} = 0, \quad \text{unless } i = \bar{a} \text{ or } j = \bar{a}, \tag{3.7}$$

as well as

$$\overset{0}{g}{}^{kl}\overset{0}{\nabla}_i h^{(r)}_{kl} = 0, \tag{3.8}$$

and

$$h^{(r)}{}_i{}^l\,\overset{0}{\nabla}_k h^{(s)}_{jl} = -h^{(s)}{}_j{}^l\,\overset{0}{\nabla}_k h^{(r)}_{il} \tag{3.9}$$

for all $r, s = 0, 1, \dots$.

Proof. Equations (3.6) follow from the fact that $h_i{}^j$ squares to zero and is trace free. Indeed, since $h_{ij} = 0$ unless $i, j = \bar{a}, \bar{c}$, we also have for the derivatives that $h^{(r)}_{ij} = 0$ unless $i, j = \bar{a}, \bar{c}$, for all $r \ge 0$. Therefore $h^{(r)}_{ik}h^{(s)k}{}_j = 0$ unless $i, j = \bar{a}, \bar{c}$ and in this case we have

$$h^{(r)}_{\bar{a}k}h^{(s)k}{}_{\bar{c}} = h^{(r)}_{\bar{a}\bar{b}}h^{(s)\bar{b}}{}_{\bar{c}} = h^{(r)}_{\bar{a}\bar{b}}\,\overset{0}{g}{}^{\bar{b}\bar{d}}\,h^{(s)}_{\bar{d}\bar{c}} = 0,$$

because $\overset{0}{g}{}^{\bar{b}\bar{d}} = 0$. Equation (3.7) follows from

$$\overset{0}{\nabla}_X\mathbf{h}(\mathbf{e}_i, \mathbf{e}_j) = X(\mathbf{h}(\mathbf{e}_i, \mathbf{e}_j)) - \mathbf{h}(\overset{0}{\nabla}_X\mathbf{e}_i, \mathbf{e}_j) - \mathbf{h}(\mathbf{e}_i, \overset{0}{\nabla}_X\mathbf{e}_j) = 0$$

unless $\mathbf{e}_i$ or $\mathbf{e}_j$ is equal to $\mathbf{e}_{\bar{a}}$.

The last equation (3.9) follows from (3.6),

$$0 = \nabla_k \left(h^{(r)}{}_i{}^l h^{(s)}_{jl} \right) = h^{(r)}{}_i{}^l \nabla_k h^{(s)}_{jl} + h^{(s)}{}_j{}^l \nabla_k h^{(r)}_{il},$$

by the Leibniz rule. □

3.2. The Ricci tensor of a 2-step nilpotent pertubation

In the following, for a semi-Riemannian metric $\overset{0}{\mathbf{g}}$ we will consider perturbations by a 2-step nilpotent, symmetric bilinear form $\mathbf{h}$ depending on a parameter ρ. By the results in the previous section we can write this perturbation as

$$\mathbf{g} = \overset{0}{\mathbf{g}} + \mathbf{h}, \text{ where } \mathbf{h} = h_{\bar{a}\bar{c}} \Theta^{\bar{a}} \circ \Theta^{\bar{c}} \text{ and } \overset{0}{\mathbf{g}} = \overset{0}{g}_{a\bar{b}} \Theta^a \Theta^{\bar{b}} + \overset{0}{g}_{AB} \Theta^A \Theta^B, \tag{3.10}$$

where we use the conventions in Section 3.1 and with smooth functions $h_{\bar{a}\bar{c}} = h_{\bar{a}\bar{c}}(\rho, x)$ with $h_{\bar{a}\bar{c}} = h_{\bar{c}\bar{a}}$. The metric coefficients of $\mathbf{g}$ are $g_{ij}(\rho, x) := \overset{0}{g}_{ij}(x) + h_{ij}(\rho, x)$. The perturbed metric $\mathbf{g}$ has the property that the inverse of $\mathbf{g}$ is linear in the perturbation $\mathbf{h}$, i.e., if g^{ij} are the coefficients of the inverse of g_{ij}, then

$$g^{ij} = \overset{0}{g}{}^{ij} - h^{ij}. \tag{3.11}$$

In the following we will raise the indices with $\overset{0}{g}_{ij}$. First we observe:

Proposition 3.1. *Let $\overset{0}{\mathbf{g}}$ be a semi-Riemannan metric and $\mathbf{h}$ a ρ-dependent, 2-step nilpotent symmetric bilinear form. Then for the metric*

$$\widetilde{\mathbf{g}} = 2\mathrm{d}(\rho t)\mathrm{d}t + t^2(\overset{0}{\mathbf{g}} + \mathbf{h}) \tag{3.12}$$

the possibly non-vanishing components of the Ricci tensor are given by

$$\overset{0}{g}{}^{kl} \overset{0}{\nabla}_k \dot{h}_{il} \quad \text{and} \quad \rho \ddot{h}_{ij} - \left(\frac{n}{2} - 1 \right) \dot{h}_{ij} + R_{ij}. \tag{3.13}$$

Here the dots denote the ρ derivatives of the h_{ij}'s and R_{ij} are the components of the Ricci tensor of $\mathbf{g} = \overset{0}{\mathbf{g}} + \mathbf{h}$.

Proof. The components of the Ricci tensor of $\widetilde{\mathbf{g}}$ are given by the left-hand sides of the Fefferman-Graham equations (2.3, 2.4, 2.5). Lemma 3.1 shows that the term in the third Fefferman-Graham equation (2.5) is zero.

In order to analyse the term in the second Fefferman-Graham equation (2.4), we use formula (2.6) for expressing ∇ in terms of $\overset{0}{\nabla}$ and the tensor $C^k_{ij} = C^k_{ji}$, i.e.,

(3.14)
$$\begin{aligned} g^{kl}\left(\nabla_k \dot{g}_{il} - \nabla_i \dot{g}_{kl}\right) &= (\overset{0}{g}{}^{kl} - h^{kl})(\overset{0}{\nabla}_k \dot{h}_{il} - \overset{0}{\nabla}_i \dot{h}_{kl} + C^p{}_{kl}\dot{h}_{ip} - C^p{}_{il}\dot{h}_{pk}) \\ &= \overset{0}{g}{}^{kl}(\overset{0}{\nabla}_k \dot{h}_{il} + C^p{}_{kl}\dot{h}_{ip} - C^p{}_{il}\dot{h}_{pk}) - h^{kl}\overset{0}{\nabla}_k \dot{h}_{il} - h^{kl}C^p{}_{kl}\dot{h}_{ip}, \end{aligned}$$

because $\mathbf{h}$ is trace free and because of Lemma 3.1. For $C^k{}_{ij}$, the formula (2.7) reduces to

$$C^k{}_{ij} = \frac{1}{2}(\overset{0}{g}{}^{kl} - h^{kl})(\overset{0}{\nabla}_l h_{ij} - \overset{0}{\nabla}_i h_{jl} - \overset{0}{\nabla}_j h_{il}), \tag{3.15}$$

again by Lemma 3.1. Hence

$$\dot{h}_{kp}C^p{}_{ij} = \frac{1}{2}\dot{h}_k{}^l(\overset{0}{\nabla}_l h_{ij} - \overset{0}{\nabla}_i h_{jl} - \overset{0}{\nabla}_j h_{il}).$$

Therefore the last term in (3.14) becomes

$$\begin{aligned} 2h^{kl}\dot{h}_{ip}C^p{}_{kl} &= h^{kl}\dot{h}_i{}^p(\overset{0}{\nabla}_p h_{kl} - \overset{0}{\nabla}_k h_{pl} - \overset{0}{\nabla}_l h_{pk}) \\ &= -h^{kl}(h_{kl}\overset{0}{\nabla}_p \dot{h}_i{}^p - h_{pl}\overset{0}{\nabla}_k \dot{h}_i{}^p - h_{pk}\overset{0}{\nabla}_l \dot{h}_i{}^p) = 0, \end{aligned}$$

because of (3.9) in Lemma 3.1. Similarly, the remaining term in (3.14) is

$$\overset{0}{g}{}^{kl}(C^p{}_{kl}\dot{h}_{ip} - C^p{}_{il}\dot{h}_{pk}) - h^{kl}\overset{0}{\nabla}_k \dot{h}_{il} = -\dot{h}_i{}^l\overset{0}{\nabla}_k h_l{}^k - h^{kl}\overset{0}{\nabla}_k \dot{h}_{il} + \tfrac{1}{2}\dot{h}_{kl}\overset{0}{\nabla}_i h^{kl} = 0.$$

This verifies the formula for the terms in the second Fefferman-Graham equation. The term in the first Fefferman-Graham equation (2.3) is seen to be equal to the second term in (3.13) by using Lemma 3.1. □

This proposition shows that, apart from the Ricci tensor of $\mathbf{g}$, the Fefferman-Graham equations contain only terms that are linear in $\mathbf{h}$. Thus, we now determine the Ricci tensor of a metric $\mathbf{g} = \overset{0}{\mathbf{g}} + \mathbf{h}$ in terms of the Ricci tensor of $\overset{0}{\mathbf{g}}$ and of $\mathbf{h}$ using formula (2.8) and apply this to a metric $\widetilde{\mathbf{g}} = 2\mathrm{d}(\rho t)\mathrm{d}t + t^2(\overset{0}{\mathbf{g}} + \mathbf{h})$. For this we note that for a metric as in (3.10) with inverse (3.11) the formula (2.8) for the Ricci tensor of $\mathbf{g}$ contains terms up to fourth order in $\mathbf{h}$. Hence we observe:

Proposition 3.2. *Let* $\overset{0}{\mathbf{g}}$ *be a semi-Riemannian metric and* $\mathbf{h}$ *be a 2-step nilpotent symmetric bilinear form. The Ricci tensor* R_{ij} *of* $\mathbf{g} = \overset{0}{\mathbf{g}} + \mathbf{h}$ *is given by*

$$R_{ij} = \overset{0}{R}_{ij} + \overset{0}{\nabla}{}^k \overset{0}{\nabla}_{(i} h_{j)k} - \tfrac{1}{2} \overset{0}{\nabla}{}^k \overset{0}{\nabla}_k h_{ij} + Q^{(2)}_{ij}(\mathbf{h}) + Q^{(3)}_{ij}(\mathbf{h}) + Q^{(4)}_{ij}(\mathbf{h}), \tag{3.16}$$

in which we raise the indices with $\overset{0}{g}_{ij}$ *and where the* $Q^{(r)}_{ij}(\mathbf{h})$ *are symmetric tensors that are of order* $r = 2, 3, 4$ *in* h_{ij}*, and which are given explicitly in* (3.20)*,* (3.19) *and* (3.18) *below.*

Now we are going to compute the $Q^{(k)}_{ij}(\mathbf{h})$'s by using equation (2.8) for the Ricci tensor of $\mathbf{g} = \overset{0}{\mathbf{g}} + \mathbf{h}$. First we note that the formula (3.15) for $C^k{}_{ij}$ and Lemma 3.1 implies

$$C^k{}_{ki} = -\tfrac{1}{2}(\overset{0}{g}{}^{kl} - h^{kl}) \overset{0}{\nabla}_i h_{kl} = 0.$$

Hence (2.8) simplifies to

$$R_{ij} = \overset{0}{R}_{ij} - \overset{0}{\nabla}_k C^k{}_{ij} - C^p{}_{jk} C^k{}_{ip}. \tag{3.17}$$

We start with the terms of fourth order in $\mathbf{h}$: by (3.9) in Lemma 3.1 we get

(3.18)

$$\begin{aligned} Q^{(4)}_{ij}(\mathbf{h}) &= -\tfrac{1}{4} h^{pq} h^{kl} (\overset{0}{\nabla}_q h_{jk} - \overset{0}{\nabla}_j h_{kq} - \overset{0}{\nabla}_k h_{jq})(\overset{0}{\nabla}_l h_{ip} - \overset{0}{\nabla}_i h_{lp} - \overset{0}{\nabla}_p h_{il}) \\ &= -\tfrac{1}{4} h^{pq} h^{kl} (\overset{0}{\nabla}_q h_{jk} - \overset{0}{\nabla}_k h_{jq})(\overset{0}{\nabla}_l h_{ip} - \overset{0}{\nabla}_p h_{il}) \\ &= \tfrac{1}{4} h^{ab} h^{cd} (\overset{0}{\nabla}_c h_{jb} - \overset{0}{\nabla}_b h_{jc})(\overset{0}{\nabla}_{\mathrm{d}} h_{ia} - \overset{0}{\nabla}_a h_{id}) \\ &= \tfrac{1}{4} h^{ab} h^{cd} (\mathbf{h}(\mathbf{e}_j, [\mathbf{e}_c, \mathbf{e}_b]))(\mathbf{h}(\mathbf{e}_i, [\mathbf{e}_d, \mathbf{e}_a])), \end{aligned}$$

where, for the last equality, we have written the summation in terms of the frame field $\mathbf{e}_i$ and used that $h_{ia} = 0$. Note that $Q^{(4)}_{ij}(\mathbf{h}) = 0$ if $[\mathbf{e}_a, \mathbf{e}_b] \in \mathcal{K}$.

Now we compute the third order terms and because of (3.7) in Lemma 3.1 we obtain

(3.19)

$$
\begin{aligned}
Q^{(3)}_{ij}(\mathbf{h}) &= -\tfrac{1}{2} h^{kl} \overset{0}{g}{}^{pq} \left(\overset{0}{\nabla}_i h_{kp} \overset{0}{\nabla}_j h_{lq} - (\overset{0}{\nabla}_p h_{ik} - \overset{0}{\nabla}_k h_{ip})(\overset{0}{\nabla}_q h_{jl} - \overset{0}{\nabla}_l h_{jq}) \right) \\
&= -\tfrac{1}{2} h^{ab} \overset{0}{g}{}^{\bar{c}d} \left(\overset{0}{\nabla}_i h_{a\bar{c}} \overset{0}{\nabla}_j h_{bd} - (\overset{0}{\nabla}_{\bar{c}} h_{ia} - \overset{0}{\nabla}_a h_{i\bar{c}})(\overset{0}{\nabla}_{\mathrm{d}} h_{jb} - \overset{0}{\nabla}_b h_{jd}) \right) \\
&\quad - \tfrac{1}{2} h^{ab} \overset{0}{g}{}^{CD} \left(\overset{0}{\nabla}_i h_{aC} \overset{0}{\nabla}_j h_{bD} - (\overset{0}{\nabla}_C h_{ia} - \overset{0}{\nabla}_a h_{iC})(\overset{0}{\nabla}_D h_{jb} - \overset{0}{\nabla}_b h_{jD}) \right) \\
&= \tfrac{1}{2} h^{ab} \overset{0}{g}{}^{\bar{c}d} (\overset{0}{\nabla}_{\bar{c}} h_{ia} - \overset{0}{\nabla}_a h_{i\bar{c}})(\overset{0}{\nabla}_{\mathrm{d}} h_{jb} - \overset{0}{\nabla}_b h_{jd}) \\
&\quad + \tfrac{1}{2} h^{ab} \overset{0}{g}{}^{CD} \left((\overset{0}{\nabla}_C h_{ia} - \overset{0}{\nabla}_a h_{iC})(\overset{0}{\nabla}_D h_{jb} - \overset{0}{\nabla}_b h_{jD}) \right) \\
&= \tfrac{1}{2} h^{ab} \left(\overset{0}{g}{}^{\bar{c}d} (\overset{0}{\nabla}_{\bar{c}} h_{ia} - \overset{0}{\nabla}_a h_{i\bar{c}}) \mathbf{h}(\mathbf{e}_j, [\mathbf{e}_d, \mathbf{e}_b]) \right) \\
&\quad + \tfrac{1}{2} h^{ab} \left(\overset{0}{g}{}^{CD} ((\overset{0}{\nabla}_C h_{ia} - \overset{0}{\nabla}_a h_{iC})(\mathbf{h}(\mathbf{e}_j, [\mathbf{e}_b, \mathbf{e}_D]) \right).
\end{aligned}
$$

Clearly, this vanishes if $[\mathbf{e}_a, \mathbf{e}_b] \in \mathcal{K}$ and $[\mathbf{e}_a, \mathbf{e}_B] \in \mathcal{K}$, and in particular if $\mathcal{K}$ is involutive.

Finally, we turn to the second order terms. They are given as

(3.20)

$$
\begin{aligned}
Q^{(2)}_{ij}(\mathbf{h}) &= \overset{0}{\nabla}_k h^{kl} \left(\tfrac{1}{2} \overset{0}{\nabla}_l h_{ij} - \overset{0}{\nabla}_{(i} h_{j)l} \right) + h^{kl} \left(\tfrac{1}{2} \overset{0}{\nabla}_k \overset{0}{\nabla}_l h_{ij} - \overset{0}{\nabla}_k \overset{0}{\nabla}_{(i} h_{j)l} \right) \\
&\quad - \tfrac{1}{4} \overset{0}{\nabla}_i h^{kl} \overset{0}{\nabla}_j h_{kl} - \tfrac{1}{4} \left(\overset{0}{\nabla}{}^k h_i{}^l - \overset{0}{\nabla}{}^l h_i{}^k \right) \left(\overset{0}{\nabla}_l h_{jk} - \overset{0}{\nabla}_k h_{jl} \right).
\end{aligned}
$$

First we rewrite the last term as

$$
\tfrac{1}{4} \left(\overset{0}{\nabla}{}^k h_i{}^l - \overset{0}{\nabla}{}^l h_i{}^k \right) \left(\overset{0}{\nabla}_l h_{jk} - \overset{0}{\nabla}_k h_{jl} \right) = \overset{0}{\nabla}_{[k} h_{l]i} \overset{0}{\nabla}{}^k h_j{}^l = \overset{0}{\nabla}_{[k} h_{l]j} \overset{0}{\nabla}{}^k h_i{}^l.
$$

Next, we analyse the term $h^{kl} \overset{0}{\nabla}_k \overset{0}{\nabla}_{(i} h_{j)l}$ using the divergence of $\mathbf{h}$, Lemma 3.1, the curvature, and the fact that $\mathbf{h}$ is 2-step nilpotent:

$$
\begin{aligned}
& h^{kl} \overset{0}{\nabla}_k \overset{0}{\nabla}_i h_{jl} \\
&\quad = -h_{jl} \overset{0}{\nabla}_k \overset{0}{\nabla}_i h^{kl} - \overset{0}{\nabla}_k h_{lj} \overset{0}{\nabla}_i h^{kl} - \overset{0}{\nabla}_k h^{kl} \overset{0}{\nabla}_i h_{jl} \\
&\quad = -h_{jl} \left(\overset{0}{\nabla}_i \overset{0}{\nabla}_k h^{kl} + h^{pl} \overset{0}{R}_{ki}{}^k{}_p + h^{kp} \overset{0}{R}_{ki}{}^l{}_p \right) - \overset{0}{\nabla}_k h_{lj} \overset{0}{\nabla}_i h^{kl} - \overset{0}{\nabla}_k h^{kl} \overset{0}{\nabla}_i h_{jl} \\
&\quad = -h_{jl} \overset{0}{\nabla}_i \overset{0}{\nabla}_k h^{kl} - h_j{}^l h^{kp} \overset{0}{R}_{kilp} - \overset{0}{\nabla}_k h_{lj} \overset{0}{\nabla}_i h^{kl} - \overset{0}{\nabla}_k h^{kl} \overset{0}{\nabla}_i h_{jl}.
\end{aligned}
$$

Hence, we obtain

$$Q^{(2)}_{ij}(\mathbf{h}) = \tfrac{1}{2}\overset{0}{\nabla}_k h^{kl}\overset{0}{\nabla}_l h_{ij} + h_{l(i}\overset{0}{\nabla}_{j)}\overset{0}{\nabla}_k h^{kl} + \tfrac{1}{2}h^{kl}\overset{0}{\nabla}_k\overset{0}{\nabla}_l h_{ij} - h^{kp}h^l{}_{(i}\overset{0}{R}_{j)klp} \\ + \overset{0}{\nabla}_k h_{l(i}\overset{0}{\nabla}_{j)}h^{kl} - \tfrac{1}{4}\overset{0}{\nabla}_i h^{kl}\overset{0}{\nabla}_j h_{kl} - \overset{0}{\nabla}_{[k}h_{l]i}\overset{0}{\nabla}^k h_j{}^l.$$

Therefore, if $\mathbf{h}$ is divergence free, i.e. $\nabla_k h^{kl} = 0$, we get formula (3.21) for $Q^{(2)}_{ij}(\mathbf{h})$.

Proposition 3.3. *Let* $\overset{0}{\mathbf{g}}$ *be a semi-Riemannian metric and* $\mathbf{h}$ *be a* 2*-step nilpotent symmetric bilinear form such that there is a totally null distribution* $\mathcal{N}$ *with* $\mathrm{Im}(\mathbf{h}^\sharp) \subset \mathcal{N}$ *and* $\mathcal{K} = \mathcal{N}^\perp$ *involutive. Then the Ricci tensor* R_{ij} *of* $\mathbf{g} = \overset{0}{\mathbf{g}} + \mathbf{h}$ *is at most quadratic in* $\mathbf{h}$*, i.e., the terms* $Q^{(3)}_{ij}(\mathbf{h})$ *and* $Q^{(4)}_{ij}(\mathbf{h})$ *in* (3.16) *vanish. If we assume in addition that* $\mathbf{h}$ *is divergence free, then*

$$Q^{(2)}_{ij}(\mathbf{h}) = \tfrac{1}{2}h^{kl}\overset{0}{\nabla}_k\overset{0}{\nabla}_l h_{ij} - h^{kp}h^l{}_{(i}\overset{0}{R}_{j)klp} + \overset{0}{\nabla}_k h_{l(i}\overset{0}{\nabla}_{j)}h^{kl} \\ - \tfrac{1}{4}\overset{0}{\nabla}_i h^{kl}\overset{0}{\nabla}_j h_{kl} - \overset{0}{\nabla}_{[k}h_{l]i}\overset{0}{\nabla}^k h_j{}^l. \tag{3.21}$$

We can apply these results to the metric $\widetilde{\mathbf{g}} = 2\mathrm{d}(\rho t)\mathrm{d}t + t^2\mathbf{g}$ as defined in (3.12): Under the assumption that $\mathcal{K}$ is involutive and that $\mathbf{h}$ is divergence free we can apply Proposition 3.1. Since $\dot{\mathbf{h}}$ is divergence free if $\mathbf{h}$ is divergence free, it implies that $\widetilde{\mathbf{g}}$ is Ricci-flat if and only if

$$\rho\ddot{h}_{ij} - \left(\frac{n}{2} - 1\right)\dot{h}_{ij} + \overset{0}{\nabla}^k\overset{0}{\nabla}_{(i}h_{j)k} - \frac{1}{2}\overset{0}{\nabla}^k\overset{0}{\nabla}_k h_{ij} + \overset{0}{R}_{ij} + Q^{(2)}_{ij}(\mathbf{h}) = 0, \tag{3.22}$$

where $Q^{(2)}_{ij}(\mathbf{h})$ is given as in (3.21). Moreover, that $\mathbf{h}$ is divergence free also allows us to simplify the term $\overset{0}{\nabla}^k\overset{0}{\nabla}_{(i}h_{j)k}$. In fact, if $\overset{0}{\nabla}^k h_{ik} = 0$ we get

$$\overset{0}{\nabla}^k\overset{0}{\nabla}_i h_{jk} = \overset{0}{R}^k{}_{ij}{}^l h_{kl} + \overset{0}{R}^k{}_{ik}{}^l h_{jl} + \overset{0}{\nabla}_i\overset{0}{\nabla}^k h_{jk} = \overset{0}{R}^k{}_{ij}{}^l h_{kl} + \overset{0}{R}_i{}^l h_{jl}. \tag{3.23}$$

This shows that we can eliminate all $\overset{0}{\nabla}_i$ derivatives from this term to obtain

Corollary 3.1. *Let* $\overset{0}{\mathbf{g}}$ *be a semi-Riemannian metric and* $\mathbf{h}$ *be a* 2*-step nilpotent symmetric bilinear form such that there is an involutive distribution* $\mathcal{K}$

such that $\mathrm{Im}(\mathbf{h}^\sharp) \subset \mathcal{N} = \mathcal{K}^\perp \subset \mathcal{K}$. *Then the metric*

$$\widetilde{\mathbf{g}} = 2\mathrm{d}(\rho t)\mathrm{d}t + t^2(\overset{0}{\mathbf{g}} + \mathbf{h})$$

is Ricci-flat if the perturbation $\mathbf{h}$ *is divergence free and*

$$\rho \ddot{h}_{ij} - \tfrac{n-2}{2}\dot{h}_{ij} - \frac{1}{2}\overset{0}{\Box} h_{ij} + \overset{0}{R}{}^k{}_{ij}{}^l h_{kl} + \overset{0}{R}{}^k{}_{(i} h_{j)k} + \overset{0}{R}_{ij} + Q^{(2)}_{ij}(\mathbf{h}) = 0, \tag{3.24}$$

where $Q^{(2)}_{ij}(\mathbf{h})$ *is given in* (3.21) *and* $\overset{0}{\Box} h_{ij} = \overset{0}{\nabla}{}^k \overset{0}{\nabla}_k h_{ij}$.

Now we are looking for geometric conditions such that $Q^{(2)}_{ij}(\mathbf{h})$ simplifies further and perhaps vanishes. In fact we show:

Theorem 3.1. *Let* $\overset{0}{\mathbf{g}}$ *be a semi-Riemannian metric and* $\mathbf{h}$ *be a divergence free, 2-step nilpotent symmetric bilinear form. If there is an involutive distribution* $\mathcal{K}$ *with* $\mathrm{Im}(\mathbf{h}^\sharp) \subset \mathcal{N} = \mathcal{K}^\perp \subset \mathcal{K}$ *and*

$$\overset{0}{\nabla}_Z Y \in \mathcal{K}^\perp, \qquad \text{for all } Y, Z \in \mathcal{K}^\perp \tag{3.25}$$

$$\overset{0}{\nabla}_X Y \in \mathcal{K}, \qquad \text{for all } X \in TM, Y \in \mathcal{K}^\perp, \tag{3.26}$$

then,

$$Q^{(2)}_{ij}(\mathbf{h}) = \tfrac{1}{2} h^{kl} \overset{0}{\nabla}_k \overset{0}{\nabla}_l h_{ij} - \overset{0}{\nabla}_{[k} h_{l]i} \overset{0}{\nabla}{}^k h_j{}^l. \tag{3.27}$$

Moreover, if in addition

$$\mathcal{L}_Y \mathbf{h} = 0, \qquad \text{for all } Y \in \mathcal{K}^\perp, \tag{3.28}$$

then $Q^{(2)}_{ij}$ *is zero, i.e., the Ricci tensor of* $\mathbf{g} = \overset{0}{\mathbf{g}} + \mathbf{h}$ *is linear in the perturbation* $\mathbf{h}$,

$$R_{ij} = \overset{0}{R}_{ij} + \overset{0}{\nabla}{}^k \overset{0}{\nabla}_{(i} h_{j)k} - \frac{1}{2} \overset{0}{\nabla}{}^k \overset{0}{\nabla}_k h_{ij}. \tag{3.29}$$

Proof. We work in a basis $(\mathbf{e}_a, \mathbf{e}_A, \mathbf{e}_{\bar{a}})$ and use the conventions as in Section 3.1. First note that assumption (3.26) implies that terms of the form $\overset{0}{\nabla}_k h_{al}$ or $\overset{0}{\nabla}_k h_{AB}$ are zero (where we use our index convention). This implies that in formula (3.21) for $Q^{(2)}_{ij}(\mathbf{h})$ the terms $\overset{0}{\nabla}_k h_{li} \overset{0}{\nabla}_j h^{kl}$ and $\overset{0}{\nabla}_i h^{kl} \overset{0}{\nabla}_j h_{kl}$ vanish.

Next we look at the curvature term in formula (3.21) for $Q^{(2)}_{ij}(\mathbf{h})$. Again by assumption (3.26) we have

$$\overset{0}{R}(\mathbf{e}_i,\mathbf{e}_a,\mathbf{e}_b,\mathbf{e}_c) = -\overset{0}{g}(\overset{0}{\nabla}_{\mathbf{e}_a}\mathbf{e}_b,\overset{0}{\nabla}_{\mathbf{e}_i}\mathbf{e}_c) + \overset{0}{g}(\overset{0}{\nabla}_{\mathbf{e}_i}\mathbf{e}_b,\overset{0}{\nabla}_{\mathbf{e}_a}\mathbf{e}_c),$$

which vanishes because of (3.25) and (3.26). This proves the first statement.

To prove the second point, assumption (3.26) gives

(3.30)
$$\overset{0}{\nabla}_{[k}h_{l]i}\overset{0}{\nabla}^k h_j{}^l = -\tfrac{1}{2}\overset{0}{g}{}^{\bar{a}b}\overset{0}{g}{}^{\bar{c}d}\overset{0}{\nabla}_{\mathrm{d}}h_{\bar{a}i}\overset{0}{\nabla}_b h_{\bar{c}j} + \tfrac{1}{2}\overset{0}{g}{}^{AB}\overset{0}{g}{}^{CD}(\mathbf{h}([\mathbf{e}_A,\mathbf{e}_C],\mathbf{e}_i))\overset{0}{\nabla}_B h_{Dj}.$$

Note that the last term in this formula is zero since $\mathcal{K}$ is involutive. On the other hand, we observe that for $Y \in \mathcal{K}^\perp$

$$\overset{0}{\nabla}_Y\mathbf{h} = \mathcal{L}_Y\mathbf{h},$$

because of (3.26). This also shows that in our situation $\mathcal{L}_Y\mathbf{h}$ is tensorial in $Y \in \mathcal{K}^\perp$. If we now assume that $\mathcal{L}_Y\mathbf{h} = 0$ for all $Y \in \mathcal{K}^\perp$, then $\nabla_Y\mathbf{h} = 0$ for all $Y \in \mathcal{K}^\perp$ and thus the remaining term in (3.30) vanishes, as well as the term $h^{kl}\overset{0}{\nabla}_k\overset{0}{\nabla}_l h_{ij}$. Consequently, $Q^{(2)}_{ij}(\mathbf{h})$ is zero. □

Theorem 3.1 gives another corollary.

Corollary 3.2. *Let* $\overset{0}{\mathbf{g}}$ *be a semi-Riemannian metric and* $\mathbf{h}$ *be a 2-step nilpotent symmetric bilinear form. If there is a totally null distribution* $\mathcal{N}$ *such that* $\mathrm{Im}(\mathbf{h}^\sharp) \subset \mathcal{N}$, $\mathcal{K} = \mathcal{N}^\perp$ *is involutive and conditions* (3.25) *and* (3.26) *of Theorem 3.1 are satisfied, then the metric* $\widetilde{\mathbf{g}} = 2\mathrm{d}(\rho t)\mathrm{d}t + t^2(\overset{0}{\mathbf{g}} + \mathbf{h})$ *is Ricci-flat if the following system of linear PDEs on* $\mathbf{h} = (h_{ij})$ *is satisified:*

$$\mathrm{div}(\mathbf{h}) = 0, \qquad (3.31)$$

$$\mathcal{L}_Y\mathbf{h} = 0,\ \forall\, Y \in \mathcal{K}^\perp, \qquad (3.32)$$

$$\rho\ddot{h}_{ij} - \tfrac{n-2}{2}\dot{h}_{ij} - \tfrac{1}{2}\overset{0}{\Box}h_{ij} + \overset{0}{R}{}^k{}_{ij}{}^l h_{kl} + \overset{0}{R}{}^k{}_{(i}h_{j)k} + \overset{0}{R}_{ij} = 0. \qquad (3.33)$$

The examples of conformal structures in [3, 26] satisfy the assumptions of Theorem 3.1 and the corollary, which enabled us to use the ansatz to find Ricci-flat ambient metrics.

Note that the assumptions of Theorem 3.1 imply that $\overset{0}{\nabla}_X\mathbf{e}_a \in \mathcal{K}$ but not that $\mathcal{K}$ or $\mathcal{K}^\perp = \mathrm{span}(\mathbf{e}_1,\ldots,\mathbf{e}_p)$ are parallel distributions. Indeed, the terms

$$2\overset{0}{\mathbf{g}}(\overset{0}{\nabla}_i\mathbf{e}_a, \mathbf{e}_A) = \overset{0}{g}([\mathbf{e}_i, \mathbf{e}_a], \mathbf{e}_A) + \overset{0}{g}([\mathbf{e}_A, \mathbf{e}_a], \mathbf{e}_i) + \overset{0}{g}([\mathbf{e}_A, \mathbf{e}_i], \mathbf{e}_a)$$

might be non-zero for $i = B$ or $i = \bar{c}$.

4. Ambient metrics for null Ricci Walker metrics

In this section we apply the results of the previous section to conformal classes given by a null Ricci Walker metric $\mathbf{g}$ as defined in the introduction. First we review some results about Walker metrics, then focus on the Ricci tensor, and derive a condition for being null Ricci Walker. Recall that we defined a *null Ricci Walker manifold* as a semi-Riemannian manifold that admits a vector distribution $\mathcal{N} \subset TM$ of rank $p > 0$ such that $\mathcal{N}$ is totally null, invariant under parallel transport with respect to the Levi-Civita connection, and contains the image of the Schouten tensor P, or equivalently of the Ricci tensor, when considered as endomorphisms. In regards to the ambient metric, given that of $\dot{h}|_{\rho=0} = 2\mathsf{P}$, our ansatz for h in the previous section was to assume that the image of h is also contained in $\mathcal{N}$. We will show here that for null Ricci Walker metrics this ansatz is in fact necessary, at least up to the critical order when n is even and hence proving Theorem 1.2 from the introduction. Finally we will draw the conclusions from the previous sections about the ambient metric of null Ricci Walker-manifolds.

Note that in Section 4.1 we drop the suffix 0 on $\overset{0}{\mathbf{g}}$ for brevity, and use it again in Section 4.2 when we need to distinguish between $\overset{0}{\mathbf{g}}$ and the ρ-dependent family $\mathbf{g}$. Moreover, in this section we will use the same index conventions as in (3.2).

4.1. Walker manifolds

A semi-Riemannian manifold $(M, \mathbf{g})$ is a *Walker manifold* if there is a vector distribution $\mathcal{N} \subset TM$ of rank $p > 0$ that is a totally null with respect to $\mathbf{g}$ and invariant under parallel transport with respect to the Levi-Civita connection of $\mathbf{g}$. The most comprehensive study of Walker manifolds can be found in [8]. In the following we will derive a description that is useful for our purpose and allows us to construct examples.

Proposition 4.1. *Let $(M, \mathbf{g})$ be a semi-Riemannian manifold of dimension n. Then the following conditions are equivalent*

(1) $(M, \mathbf{g})$ *is a* Walker manifold *with parallel null distribution $\mathcal{N}$.*

(2) *There exists local coordinates* $(x^1, \ldots, x^n)$*, so-called* Walker coordinates*, such that*

$$\mathbf{g} = 2\mathrm{d}x^{\bar{a}}(\delta_{\bar{a}b}\mathrm{d}x^b + F_{\bar{a}B}\mathrm{d}x^B + H_{\bar{a}\bar{b}}\mathrm{d}x^{\bar{b}}) + G_{AB}\mathrm{d}x^A\mathrm{d}x^B, \tag{4.1}$$

where the $F_{\bar{a}B}$ *and* G_{AB} *independent of the* x^a*'s. Here we use the same index conventions as in (3.2) as well as* $\delta_{\bar{a}b} = 1$ *if* $\bar{a} = n - p + b$ *and zero otherwise. In these coordinates, the parallel null distribution* $\mathcal{N}$ *is given by the span of the* $\partial_a = \frac{\partial}{\partial x^a}$*'s.*

(3) *There is a frame* $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$ *with dual frame* $(\Theta^1, \ldots, \Theta^n)$ *such that*

$$\mathbf{g} = 2g_{a\bar{c}}\Theta^a \circ \Theta^{\bar{c}} + g_{AB}\Theta^A \circ \Theta^B, \tag{4.2}$$

with constants $g_{a\bar{c}}$ *and* g_{AB} *and such that*

$$\begin{aligned} &\mathcal{K} = \mathrm{span}(\mathbf{e}_1, \ldots \mathbf{e}_{n-p}) \textit{ is involutive,} \\ &[\mathbf{e}_a, \mathbf{e}_b] = [\mathbf{e}_a, \mathbf{e}_B] = 0, \\ &[\mathbf{e}_a, \mathbf{e}_{\bar{c}}] \in \mathcal{K}^\perp, \quad [\mathbf{e}_B, \mathbf{e}_{\bar{c}}] \in \mathcal{K}, \quad \textit{and} \quad [\mathbf{e}_{\bar{a}}, \mathbf{e}_{\bar{c}}] \in \mathcal{K}^\perp. \end{aligned} \tag{4.3}$$

In this frame $\mathcal{N} = \mathcal{K}^\perp = \mathrm{span}(\mathbf{e}_1, \ldots, \mathbf{e}_p)$*.*

Proof. The equivalence of items (4.1) and (4.1) is due to Walker [36]. In order to show that (4.1) implies (4.1), we fix some Walker coordinates $(x^1, \ldots, x^n)$ such that

$$\mathbf{g} = 2\mathrm{d}x^{\bar{a}}(\delta_{\bar{a}b}\mathrm{d}x^b + F_{\bar{a}B}\mathrm{d}x^B + H_{\bar{a}\bar{b}}\mathrm{d}x^{\bar{b}}) + G_{AB}\mathrm{d}x^A\mathrm{d}x^B,$$

with $F_{\bar{a}B}$ and G_{AB} independent of the x^a's. Then we set

$$\mathbf{e}_a := \partial_a, \qquad \mathbf{e}_A := C_A^{\ B}\left(\partial_B - F_{\bar{a}B}\delta^{\bar{a}b}\partial_b\right), \qquad \mathbf{e}_{\bar{c}} := \partial_{\bar{c}} - H_{\bar{a}\bar{c}}\delta^{\bar{a}b}\partial_b,$$

where $C_A^{\ B}$ is a matrix such that $C_A^{\ B}G_{BE}C_D^{\ E} = \epsilon_A\delta_{AD}$, with $\epsilon_A = \pm 1$. Note that, since G_{AB} does not depend on the x^a's, also $C_A^{\ B}$ does not depend on the x^a's. We claim that this frame satisfies all the conditions (4.3). Clearly, the metric in this frame has the correct form and $[\mathbf{e}_a, \mathbf{e}_b] = 0$. But also the other commutator relations are satisfied:

$$\begin{aligned} [\mathbf{e}_a, \mathbf{e}_{\bar{c}}] &= \left[\partial_a, \partial_{\bar{c}} - H_{\bar{b}\bar{c}}\delta^{\bar{b}e}\partial_e\right] = -\mathrm{d}H_{\bar{b}\bar{c}}(\partial_a)\delta^{\bar{b}e}\partial_e \in \mathcal{K}^\perp, \\ [\mathbf{e}_a, \mathbf{e}_A] &= \left[\partial_a, C_A^{\ B}\left(\partial_B - F_{\bar{c}B}\delta^{\bar{c}d}\partial_d\right)\right] = 0, \\ [\mathbf{e}_{\bar{c}}, \mathbf{e}_A] &= \left[\partial_{\bar{c}} - H_{\bar{a}\bar{c}}\delta^{\bar{a}b}\partial_b, C_A^{\ B}\left(\partial_B - F_{\bar{e}B}\delta^{\bar{e}d}\partial_d\right)\right] \in \mathcal{K}. \end{aligned}$$

This shows that all the conditions (4.3) are satisfied.

Conversely, we have to show that the bracket relations (4.3) imply that $\nabla_X \mathbf{e}_a \in \mathcal{N} = \mathcal{K}^\perp$. For this we use the Koszul formula

$$2\mathbf{g}(\nabla_{\mathbf{e}_i}\mathbf{e}_a, \mathbf{e}_j) = \mathbf{g}([\mathbf{e}_i, \mathbf{e}_a], \mathbf{e}_j) + \mathbf{g}([\mathbf{e}_j, \mathbf{e}_a], \mathbf{e}_i) + \mathbf{g}([\mathbf{e}_j, \mathbf{e}_i], \mathbf{e}_a).$$

From (4.3) is follows that this is zero for all $j = a$ and $j = B$. Hence, $\nabla_X \mathbf{e}_a \in \mathcal{N} = \mathcal{K}^\perp = \operatorname{span}(\mathbf{e}_1, \dots, \mathbf{e}_p)$. □

Next we record formulas for the curvature of a Walker metric.

Lemma 4.1. *Let* $(M, \mathbf{g})$ *be a Walker manifold and let* $(\mathbf{e}_1, \dots, \mathbf{e}_n)$ *be a frame as in* (4.1) *of Proposition 4.1 such that* $\mathbf{g}$ *is given as in* (4.2)*.*

(1) *Let* ${\Gamma^k}_{ij}$ *the connection components with respect to the frame* $(\mathbf{e}_1, \dots, \mathbf{e}_n)$*, i.e., defined by* $\nabla_i \mathbf{e}_j = {\Gamma^k}_{ij}\mathbf{e}_k$*. Then*

$$\begin{aligned} {\Gamma^k}_{ab} = {\Gamma^k}_{ba} = {\Gamma^k}_{Ab} = {\Gamma^k}_{bA} &= 0, \\ {\Gamma^B}_{ai} = {\Gamma^B}_{ia} &= 0, \\ {\Gamma^{\bar{c}}}_{ai} = {\Gamma^{\bar{c}}}_{ia} = {\Gamma^{\bar{c}}}_{Ai} = {\Gamma^{\bar{c}}}_{iA} &= 0. \end{aligned} \tag{4.4}$$

(2) *The curvature tensor and and the Ricci tensor of* $\mathbf{g}$ *satisfy*

$$R_{ijab} = R_{ijaB} = 0, \tag{4.5}$$

and

$$R_{ab} = R_{aB} = 0, \tag{4.6}$$

for all $a, b = 1, \dots, p$*,* $B = p+1, \dots n-p$ *and* $i, j = 1, \dots n$*.*

Proof. The properties of the connection components are a direct consequence of $\mathcal{K}$ and $\mathcal{K}^\perp$ being parallel distributions and of the Koszul formula

$${\Gamma^k}_{ij} = \tfrac{1}{2} g^{kl} \left(\mathbf{g}([\mathbf{e}_i, \mathbf{e}_j], \mathbf{e}_l) + \mathbf{g}([\mathbf{e}_l, \mathbf{e}_j], \mathbf{e}_i) + \mathbf{g}([\mathbf{e}_l, \mathbf{e}_i], \mathbf{e}_j) \right).$$

As $\mathcal{K}$ and $\mathcal{K}^\perp$ are parallel distributions, in the given frame, the curvature tensor of a Walker manifold satisfies equations (4.5). Indeed, we have for example

$$R_{biAd} = \mathbf{g}(\overset{0}{R}(\mathbf{e}_b, \mathbf{e}_i)\mathbf{e}_A, \mathbf{e}_d) = 0,$$

since $\mathcal{K}$ is parallel and thus $R(\mathbf{e}_b, \mathbf{e}_i)\mathbf{e}_A \in \mathcal{K}$. This implies that the components of the Ricci tensor

$$R_{ai} = g^{b\bar{c}}(R_{bai\bar{c}} + R_{\bar{c}aib}) + g^{AB}R_{AaiB} = g^{b\bar{c}}R_{\bar{c}aib}$$

are zero unless $i = \bar{d}$. □

This shows that the terms of the Ricci tensor that could prevent a Walker metric from being null Ricci Walker are the following

$$\begin{aligned} R_{\bar{a}b} &= g^{\bar{c}d}R_{d\bar{a}b\bar{c}}, \\ R_{AB} &= g^{CD}R_{CABD}, \\ R_{\bar{a}B} &= g^{\bar{c}d}R_{\bar{c}B\bar{a}d} + g^{AC}R_{AB\bar{a}C}. \end{aligned} \tag{4.7}$$

We will now give conditions for these terms to vanish. The following results will also provide a method of constructing examples of null Ricci Walker metrics in Section 5, in particular for the examples of Lie groups with left-invariant metric.

Proposition 4.2. *Let* $\mathbf{g}$ *be a metric as in* (4.2) *and assume that the frame* $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$ *satisfies the following bracket relations*

$$[\mathbf{e}_i, \mathbf{e}_j] = r_{ij}^k \mathbf{e}_k$$

with smooth functions r_{ij}^k *satisfying the relations*

$$r_{ab}^k = r_{aB}^k = r_{AB}^{\bar{c}} = r_{a\bar{c}}^{\bar{b}} = r_{a\bar{c}}^B = r_{B\bar{c}}^{\bar{a}} = r_{\bar{a}\bar{c}}^{\bar{b}} = r_{\bar{a}\bar{c}}^B = 0 \tag{4.8}$$

(these are just the conditions in Proposition 4.1). If we assume in addition that

$$r_{AB}^C = 0, \tag{4.9}$$

and

$$\mathrm{d}r_{b\bar{c}}^d(\mathbf{e}_A) = 0, \tag{4.10}$$

$$\mathrm{d}r_{BC}^d(\mathbf{e}_A) = \mathrm{d}r_{B\bar{c}}^D(\mathbf{e}_A) = 0, \tag{4.11}$$

then $\mathbf{g}$ *is a Walker metric whose curvature satisfies in addition*

$$R_{ABCi} = R_{\bar{a}bD\bar{c}} = 0, \qquad R_{Ai} = 0,$$

and

$$R_{\bar{a}b\bar{c}d} = g_{f(\bar{a}} dr^{f}_{\bar{c})d}(\mathbf{e}_b).$$

Moreover, $\mathbf{g}$ *is null Ricci Walker, if and only if*

$$R_{b\bar{c}} = \tfrac{1}{2}\left(g_{f\bar{c}} g^{\bar{a}d} \mathrm{d}r^{f}_{\bar{a}d}(\mathbf{e}_b) + \mathrm{d}r^{d}_{\bar{c}d}(\mathbf{e}_b)\right) = 0. \tag{4.12}$$

Proof. First we compute the curvature components R_{bijd}. Because of the previous lemma we only have to compute $R_{b\bar{a}\bar{c}d}$ as all other are zero. In terms of the r^k_{ij}'s the connection coefficients $\Gamma^k{}_{ij}$ write as

$$\Gamma^k_{ij} = \frac{1}{2} r^k_{ij} + g^{kl} r^m_{l(i} g_{j)m} = \frac{1}{2} r^k_{ij} - g^{kl} g_{m(i} r^m_{j)l}. \tag{4.13}$$

After imposing the condition on the frame to define a Walker metric, i.e., after imposing equations (4.8), Lemma 4.1 leaves us with the only possibly non-vanishing connection coefficients $\Gamma^b{}_{a\bar{c}}$, $\Gamma^b{}_{AB}$, $\Gamma^C{}_{AB}$, $\Gamma^b{}_{A\bar{c}}$, $\Gamma^B{}_{A\bar{c}}$ and $\Gamma^k_{\bar{a}\bar{c}}$. Imposing the additional condition (4.9), $r^C_{AB} = 0$, implies

$$\Gamma^C{}_{AB} = -g^{CD} g_{E(A} r^E_{B)D} = 0,$$

This together with $\Gamma^{\bar{c}}_{AB} = 0$, implies that $\nabla_{\mathbf{e}_A} \mathbf{e}_B \in \mathcal{K}^\perp$ and hence, with $\mathcal{K}^\perp$ being parallel, that

$$R_{ABCD} = 0,$$

and therefore by (4.7) that

$$R_{AB} = 0.$$

Next, we look at the curvature terms in $R_{\bar{c}B} = g^{\bar{a}d} R_{\bar{a}B\bar{c}d} + g^{AC} R_{AB\bar{c}C}$ and compute

$$R_{\bar{a}B\bar{c}d} = -g_{\bar{b}d} \mathrm{d}\Gamma^{\bar{b}}{}_{\bar{a}\bar{c}}(\mathbf{e}_B) = \tfrac{1}{2} g_{b(\bar{a}} dr^b_{\bar{c})d}(\mathbf{e}_B).$$

This vanishes because of condition (4.10). Moreover,

$$\begin{aligned} R_{ABD\bar{c}} &= g_{b\bar{c}}\left(\mathrm{d}\Gamma^b{}_{BD}(\mathbf{e}_A) - \mathrm{d}\Gamma^b{}_{AD}(\mathbf{e}_B)\right) \\ &= -g_{b\bar{c}} \mathrm{d}r^b_{D(A}(\mathbf{e}_{B)}) - \mathrm{d}r^E_{D\bar{c}}(\mathbf{e}_{[A}) g_{B]E} + g_{ED} \mathrm{d}r^E_{\bar{c}[A}(\mathbf{e}_{B]}), \end{aligned}$$

vanishes because of condition (4.11). Hence we have

$$R_{ABD\bar{c}} = R_{\bar{a}B\bar{c}d} = 0,$$

and therefore $R_{Ai} = 0$. Furthermore, because of ${\Gamma^k}_{ab} = 0$ and $[\mathbf{e}_a, \mathbf{e}_{\bar{c}}] = r^b_{a\bar{c}}\mathbf{e}_b$ we obtain

$$\begin{aligned} R_{b\bar{a}d\bar{c}} &= \mathbf{g}(\nabla_b \nabla_{\bar{a}} \mathbf{e}_d, \mathbf{e}_{\bar{c}}) = \left(\mathrm{d}{\Gamma^f}_{\bar{a}d}(\mathbf{e}_b) + {\Gamma^k}_{\bar{a}d}{\Gamma^f}_{bk} \right) g_{f\bar{c}} = \mathrm{d}{\Gamma^f}_{\bar{a}d}(\mathbf{e}_b) g_{f\bar{c}} \\ &= g_{f(\bar{a}}\mathrm{d}r^f_{\bar{c})d}(\mathbf{e}_b), \end{aligned}$$

which implies the formula (4.12) for the Ricci components $R_{b\bar{c}}$. The metric is null Ricci Walker if and only if these components vanish. This proves the statement. □

Remark 4.1. Of course, when constructing examples, the r^k_{ij}'s in this proposition cannot be chosen freely as they have to obey Jacobi's identity. However in some situations, such as $\mathcal{N} = \mathcal{N}^\perp$, i.e., $n = 2p$, or when constructing examples of left-invariant metrics, i.e., when the r^k_{ij}'s are constant, the conditions (4.9), (4.10) and (4.11) can be imposed without yielding a contradiction.

Remark 4.2. In view of the examples we will construct in Section 5, note that in general the remaining Ricci components do not vanish, even if all the r^k_{ij}'s are constant:

$$\begin{aligned} R_{\bar{a}\bar{c}} &= 2g^{b\bar{d}} R_{b(\bar{a}\bar{c})\bar{d}} + g^{BD} R_{B(\bar{a}\bar{c})D} \\ &= 2\Big(\mathrm{d}{\Gamma^b}_{(\bar{a}\bar{c})}(\mathbf{e}_b) - \mathrm{d}{\Gamma^b}_{b(\bar{a}}(\mathbf{e}_{\bar{c})}) + {\Gamma^{\mathrm{d}}}_{b(\bar{a}} r^b_{\bar{c})d} - {\Gamma^{\mathrm{d}}}_{b(\bar{a}}{\Gamma^b}_{\bar{c})d} + {\Gamma^b}_{b\bar{d}}{\Gamma^{\bar{d}}}_{(\bar{a}\bar{c})} \\ &\quad + \mathrm{d}{\Gamma^A}_{(\bar{a}\bar{c})}(\mathbf{e}_A) - \mathrm{d}{\Gamma^A}_{A(\bar{a}}(\mathbf{e}_{\bar{c})}) + {\Gamma^A}_{B(\bar{a}} r^B_{\bar{c})A} + {\Gamma^A}_{B(\bar{a}}{\Gamma^B}_{\bar{c})A} + {\Gamma^{\bar{d}}}_{(\bar{a}\bar{c})}{\Gamma^A}_{A\bar{d}} \Big). \end{aligned}$$

4.2. Necessary conditions for the ambient metric of null Ricci Walker metrics

In this section we will derive conditions on the bilinear form h of the ambient metric for a conformal class that contains a null Ricci Walker metric $\overset{0}{g}$. This will show that, for null Ricci Walker metrics, our ansatz for h to have its image contained in $\mathcal{N}$ is in fact necessary, at least up to the critical order when n is even. The following theorem will imply Theorem 1.2 from the introduction.

Theorem 4.1. *Let $(M, \overset{0}{\mathbf{g}})$ be a null Ricci Walker metric of dimension $n > 2$ with Schouten tensor P whose image is contained in a $\overset{0}{\nabla}$-parallel totally null distribution $\mathcal{N}$. Let $\widetilde{\mathbf{g}} = 2\mathrm{d}t\, \mathrm{d}(\rho t) + t^2\mathbf{g}$ with $\mathbf{g} = \mathbf{g}(x^i, \rho)$ be an ambient metric for $\overset{0}{\mathbf{g}}$ in the sense of Definition 2.1. Then for*

$$\mathbf{h} = \mathbf{g} - \overset{0}{\mathbf{g}} = \sum_{m\geq 1} \frac{1}{m!} \overset{m}{h} \rho^m$$

with $\overset{m}{h} = \overset{m}{h}(x^i)$ the following holds:

(1) *If n is odd, then, for all $m \geq 1$,*

$$\mathrm{Im}\, \overset{m}{h}{}^{\sharp} \subset \mathcal{N}, \tag{4.14}$$

$$\overset{0}{\nabla}_k \overset{m}{h}_i{}^k = 0. \tag{4.15}$$

(2) *If n is even, then (4.14) and (4.15) must hold for $m \leq \frac{n}{2} - 1$ and the obstruction tensor satisfies*

$$\mathrm{Im}(\mathcal{O}^{\sharp}) \subset \mathcal{N}.$$

Moreover, one can choose an ambient metric such that the corresponding $\overset{m}{h}$ satisfy (4.14) and (4.15) for all $m \geq 1$.

Remark 4.3. The statement about the obstruction tensor in the case n even can also be obtained from results in [24].

Remark 4.4. Note that (4.14) is equivalent to $\overset{m}{h}_{ij} = 0$ unless $i, j \in \{n - p + 1, \ldots, n\}$. Moreover, we use the following convention: g^{kl} refers to the inverse of $g_{kl} = g_{kl}(x^i, \rho)$. However, whenever a raised index appears on a coefficient $\overset{m}{h} = \overset{m}{h}(x^i)$, the index is raised w.r.t. $\overset{0}{g}$, i.e. $\overset{m}{h}{}^i{}_j := \overset{0}{g}{}^{ik} \overset{m}{h}_{kj}$.

Proof. The proof is carried out by induction over m, where we assume $m \leq \frac{n}{2} - 1$ when n is even. When n is odd, we have that $Ric(\widetilde{g}) = O(\rho^{\infty})$ and when n is even that $Ric(\widetilde{g}) = O(\rho^{\frac{n}{2}-1})$. We will work in a coordinate frame as in (4.1) in Proposition 4.1.

Step 1: For $m = 1$, the statement follows from the assumption on P as well as the contracted version of the second Bianchi identity and $\mathsf{P}^i{}_i = 0$.

Assuming the induction hypothesis that the statement holds for $\overset{b}{h}$ with $1 \leq b \leq m - 1$, we show that the statement also holds for $\overset{m}{h}$. As a preparation, note that as a consequence of the induction hypothesis and parallelity of $\mathcal{N}$ we have

$$\overset{u}{h}{}^{ki}\overset{v}{h}_{kj} = 0, \quad \overset{u}{h}{}^{ki}\overset{0}{\nabla}_j\overset{v}{h}_{kl} = 0, \qquad \text{for all } 1 \leq u, v \leq m-1. \tag{4.16}$$

Moreover, for the inverse g^{ij} of g_{ij} the induction hypothesis implies that

$$g^{ij} = \overset{0}{g}{}^{ij} - \sum_{p=1}^{m-1} \frac{1}{p!}\overset{p}{h}{}^{ij}\rho^p + O(\rho^m). \tag{4.17}$$

Indeed, it is

$$g_{ik}\Big(\overset{0}{g}{}^{kj} - \sum_{p=1}^{m-1}\frac{1}{p!}\overset{p}{h}{}^{kj}\rho^p\Big) \quad = \quad \delta_i{}^j + \sum_{p,q=1}^{m-1}\frac{1}{q!p!}\overset{q}{h}_{ik}\overset{p}{h}{}^{kj}\rho^{p+q} + O(\rho^m),$$

so that the first equation in (4.16) verifies (4.17). Moreover, equations (4.16) and (4.17) then imply that

$$\partial^u_\rho (C^k{}_{ij})_{\rho=0} = -\frac{1}{2}\overset{0}{g}{}^{kl}\overset{0}{\nabla}_i\overset{u}{h}_{jl} = -\frac{1}{2}\overset{0}{\nabla}_i\overset{u}{h}_j{}^k, \tag{4.18}$$
$$\text{for } i \in \{1, \ldots, n-p\} \text{ and } u \leq m-1,$$

where the $C^k{}_{ij}$ were defined in Section 2.4.

Step 2: Here we show that the induction hypothesis implies that

$$\partial^a_\rho R_{ij}|_{\rho=0} = 0, \qquad \text{for } a \leq m-1 \text{ and } i \in \{1, \ldots, n-p\}, \tag{4.19}$$

where R_{ij} is the Ricci tensor of $g_{ij}(\rho)$. To this end, we rewrite this using (2.8) at $\rho = 0$

$$\partial^a_\rho R_{ij} = \partial^a_\rho\left(\overset{0}{\nabla}_i C^k{}_{kj} - \overset{0}{\nabla}_k C^k{}_{ij} + C^q{}_{ij}C^k{}_{kq} - C^q{}_{jk}C^k{}_{iq}\right). \tag{4.20}$$

Everywhere, not only at $\rho = 0$, we have $C^k{}_{kj} = -\frac{1}{2}\overset{0}{g}{}^{kl}\overset{0}{\nabla}_j g_{kl}$. Expanding the g-s in terms of the $\overset{u}{h}$ using the induction hypothesis as well as (4.16) and (4.17) reveals that $C^k{}_{kj} = O(\rho^{a+1})$. Thus, the first and third term in (4.20) vanish at $\rho = 0$. The fourth term is treated as follows:

Expanding $\partial_\rho^{m-1}\left(C^q{}_{jk}C^k{}_{iq}\right)$ at $\rho = 0$ gives a sum of certain coefficients times summands of the form $(\partial^u_\rho C^q{}_{jk})(\partial^v_\rho C^k{}_{iq})$ with $u + v \leq m - 1$. Assuming $i \in \{1, \dots, n-p\}$ and applying (4.18) to this yields

$$(\partial^u_\rho C^q{}_{jk})(\partial^v_\rho C^k{}_{iq}) = \frac{1}{4}\overset{0}{g}{}^{pq}\overset{0}{g}{}^{kl}\overset{0}{\nabla}_j\overset{u}{h}_{kp}\overset{0}{\nabla}_i\overset{v}{h}_{ql} = 0,$$

since u and v are $\leq m-1$ by the induction hypothesis and the fact that $\mathcal{N}$ is parallel. Thus, the fourth term in (4.20) vanishes at $\rho = 0$.

Finally, we show that the second term in (4.20) vanishes at $\rho = 0$: Assuming $i \in \{1, \dots, n-p\}$ and using (4.18) again, this term is given as

$$\frac{1}{2}\overset{0}{\nabla}_k\overset{0}{\nabla}_i\overset{a}{h}{}^k{}_j. \tag{4.21}$$

By the induction hypothesis, we must necessarily have that $k \in \{1, \dots, p\}$. As $\mathcal{N}$ is $\overset{0}{\nabla}$-invariant, it follows for the curvature of $\overset{0}{g}$ that

$$\overset{0}{R}_{ikhl} = 0 \text{ for all } i \in \{1, \dots, n-p\}, k \in \{1, \dots, p\}, \tag{4.22}$$

see also Lemma 4.1. This shows that the covariant derivatives in (4.21) commute and one obtains $\overset{0}{\nabla}_i$ applied to the divergence of $\overset{a}{h}$, which vanishes by the induction hypothesis. Thus, (4.19) is established.

Now we are going to differentiate the Fefferman-Graham equations (2.3, 2.4, 2.5) with respect to ρ and use that

$$\partial^k_\rho Ric(\widetilde{\mathbf{g}}) = 0, \quad \text{for all } k \text{ if } n \text{ is odd, and for } k \leq \frac{n}{2} - 2 \text{ if } n \text{ is even.}$$

Step 3: Applying ∂^{m-2}_ρ to the third Fefferman-Graham equation (2.5), where ∂_ρ always denotes the Lie derivative of a tensor in ρ-direction, and then evaluating at $\rho = 0$ yields using (4.16) that

$$\overset{0}{g}{}^{kl}\overset{m}{h}_{kl} = 0, \quad \text{for all } m \text{ if } n \text{ is odd and for } m \leq \frac{n}{2} \text{ if } n \text{ is even.} \tag{4.23}$$

Step 4: We apply ∂^{m-1}_ρ, for $m \leq \frac{n}{2} - 1$ if n is even, to the second Fefferman-Graham equation (2.4) and evaluate at $\rho = 0$. Using (4.23) and rewriting ∇ in terms of $\overset{0}{\nabla}$ and C, the result is

$$
\begin{aligned}
(4.24)\quad 0 &= \overset{0}{g}{}^{kl}\,\overset{0}{\nabla}_k\overset{m}{h}_{il}\\
&+ c_{u,v,w}\,\overset{u}{h}{}^{kl}\left(\partial_\rho^v(C^h{}_{ki})\overset{w}{h}_{hl} + \partial_\rho^v(C^h{}_{kl})\overset{w}{h}_{ih} - \partial_\rho^v(C^h{}_{ik})\overset{w}{h}_{hl} - \partial_\rho^v(C^h{}_{il})\overset{w}{h}_{kh}\right)_{\rho=0}
\end{aligned}
$$

for certain integer coefficients $c_{u,v,w}$, where $u+v+w=m$ and $1 \le w \le m-1$. Using $C^k{}_{ij} = C^k{}_{ji}$ as well as (4.16), the bracket reduces to

$$
\overset{u}{h}{}^{kl}\left(\partial_\rho^v(C^h{}_{kl})\overset{w}{h}_{ih}\right)_{\rho=0} - \left(\partial_\rho^v(C^h{}_{il})\right)_{\rho=0}\overset{w}{h}{}^l{}_h. \tag{4.25}
$$

In order for the second term in (4.25) to be nonzero, we must necessarily have that $l \in \{1,\dots,p\}$. In this situation, we can insert (4.18) for the C-term and it follows using (4.16) immediately that the resulting term vanishes. It remains to analyze the first term in (4.25). Unwinding the definitions, it is given by

$$
\overset{u}{h}{}^{kl}\,\partial_\rho^v\left(g^{hj}\left(\overset{0}{\nabla}_j g_{kl} - \overset{0}{\nabla}_k g_{jl} - \overset{0}{\nabla}_l g_{kj}\right)\right)_{\rho=0}\overset{w}{h}_{ih}. \tag{4.26}
$$

If the ρ-derivative falls on g^{hj}, then the resulting contraction with $\overset{w}{h}_{ih}$ is zero by (4.16). Thus g^{hj} in (4.26) can be replaced by $\overset{0}{g}{}^{hj}$. But then (4.26) involves a factor $\overset{w}{h}{}^j{}_i$, which can only be nonzero if $j \in \{1,\dots,p\}$, and (4.26) then reduces to

$$
\overset{u}{h}{}^{kl}\,\overset{0}{\nabla}_j\overset{v}{h}_{kl}\overset{w}{h}{}^j_i = 0. \tag{4.27}
$$

Thus, every term in (4.24) except for the first one vanishes and we obtain $\overset{0}{\nabla}_k\overset{m}{h}{}^k{}_i = 0$, which establishes (4.15).

Step 5: In order to prove (4.14), we apply ∂_ρ^{m-1} to the first Fefferman-Graham equation (2.3), assume that $i \in \{1,\dots,n-p\}$ and evaluate at $\rho = 0$. Using the induction hypothesis and (4.16) applied to the first-fifth term in the Fefferman-Graham equation (2.3), (4.23) applied to the fifth term, as well as (4.19), we obtain that at $\rho = 0$ and for $i \in \{1,\dots,n-p\}$ that

$$
\left(m - \frac{n}{2}\right)\overset{m}{h}_{ij} + \partial_\rho^{m-1}R_{ij}|_{\rho=0} = \left(m - \frac{n}{2}\right)\overset{m}{h}_{ij} = \partial_\rho^{m-1}(Ric_{ij}(\widetilde{\mathbf{g}})|_{\rho=0}. \tag{4.28}
$$

If n is odd, $\partial_\rho^{m-1}(Ric_{ij}(\widetilde{\mathbf{g}})|_{\rho=0} = 0$ for all m and hence equation (4.28) shows that $\overset{m}{h}_{ij} = 0$ for $i = 1, \ldots n-p$ completing the induction and establishing (4.14) for all m.

If n is even, $\partial_\rho^{m-1}(Ric_{ij}(\widetilde{\mathbf{g}})|_{\rho=0} = 0$ for all $m \leq \frac{n}{2} - 1$, and hence equation (4.28) shows that $\overset{m}{h}_{ij} = 0$ for $i = 1, \ldots n-p$ for all $m \leq \frac{n}{2} - 1$. But by taking $m = \frac{n}{2}$, it also gives a formula for the obstruction tensor $\mathcal{O}_{ij}$, in which c_n is a non-zero constant:

$$\mathcal{O}_{ij} = c_n \partial_\rho^{\frac{n}{2}-1} Ric_{ij}(\widetilde{\mathbf{g}})|_{\rho=0} = c_n \partial_\rho^{\frac{n}{2}-1} R_{ij}|_{\rho=0} = 0,$$

if $i \in \{1, \ldots n-p\}$ by (4.19). This verifies the statement about the obstruction tensor.

Finally, in the case that n is even, the terms $\overset{m}{h}_{ij}$, for $m \geq \frac{n}{2}$, in an ambient metric are not subject to any equation and we can choose them to be divergence free and with image in $\mathcal{N}$. This completes the proof of the Theorem. □

4.3. The Fefferman-Graham equations for null Ricci Walker metrics

Here we apply our results of Theorems 3.1 and 4.1 to null Ricci Walker metrics. The following theorem will imply Theorem 1.3 and consequently Theorem 1.1 from the introduction.

Theorem 4.2. *Let* $(M, \overset{0}{\mathbf{g}})$ *be a null Ricci Walker-manifold with parallel totally null distribution* $\mathcal{N}$ *such that* $\mathrm{Im}(\mathsf{P}^\sharp) \subset \mathcal{N}$. *Then an ambient metric* $\widetilde{\mathbf{g}} = 2\,\mathrm{d}t\,\mathrm{d}(\rho t) + t^2 \mathbf{g}(\rho)$ *for* $[\overset{0}{\mathbf{g}}]$ *in the sense of Definition 2.1 is given by* $\mathbf{g} = \overset{0}{\mathbf{g}} + \mathbf{h}$, *where* $\mathbf{h} = \mathbf{h}(\rho)$ *is divergence free bilinear form with* $\mathrm{Im}(\mathbf{h}^\sharp) \subset \mathcal{N}$ *that satisfies the the PDE*

(4.29)

$$\rho \ddot{h}_{ij} - \tfrac{n-2}{2}\dot{h}_{ij} - \tfrac{1}{2}\overset{0}{\Box} h_{ij} + \overset{0}{R}_{kijl} h^{kl} + \overset{0}{R}_{ij} + \tfrac{1}{2}\left(h^{kl} \overset{0}{\nabla}_k \overset{0}{\nabla}_l h_{ij} + \overset{0}{\nabla}_k h_{li} \overset{0}{\nabla}^l h^k{}_j \right) = O(\rho^m),$$

for $m = \infty$ *if* n *is odd and* $m = \frac{n-2}{2}$ *when* n *is even. Here* $\mathbf{h} = (h_{ij})$, $\overset{0}{R}_{ijkl}$ *denotes the curvature tensor,* $\overset{0}{R}_{ij}$ *the Ricci tensor and* $\overset{0}{\Box} h_{ij} = \overset{0}{\nabla}^k \overset{0}{\nabla}_k h_{ij}$, *all with respect to* $\overset{0}{\mathbf{g}}$.

Proof. Let $\widetilde{\mathbf{g}} = 2\,\mathrm{d}t\,\mathrm{d}(\rho t) + t^2\mathbf{g}(\rho)$ be an ambient metric for the conformal class of $\overset{0}{\mathbf{g}}$ in the sense of Definition 2.1. Then, from Theorem 4.1 we know that there is a $\mathbf{h} = \mathbf{g} - \overset{0}{\mathbf{g}}$ that is divergence free and its image is contained in $\mathcal{N}$. Then $\mathbf{h}$ and $\mathcal{K} = \mathcal{N}^{\perp}$ satisfy the assumptions of Corollary 3.1. Hence, the term quadratic in $\mathbf{h}$ in the Ricci tensor of $\mathbf{g} = \overset{0}{\mathbf{g}} + \mathbf{h}$ is given by equation (3.27). Note that, since $\mathcal{K}$ is parallel, the second term in (3.27) simplifies to

$$\overset{0}{\nabla}_{[k}h_{l]i}\overset{0}{\nabla}{}^{k}h_{j}{}^{l} = -\tfrac{1}{2}\overset{0}{\nabla}_{k}h_{li}\overset{0}{\nabla}{}^{l}h^{k}{}_{j}.$$

Moreover, since $\mathrm{Im}(\mathsf{P}^{\sharp}) \subset \mathcal{N}$ and $\mathrm{Im}(\mathbf{h}^{\sharp}) \subset \mathcal{N}$, in (3.24) the product of $\mathbf{h}$ with the Ricci tensor of $\overset{0}{g}$ vanishes,

$$\overset{0}{R}{}^{k}{}_{i}h_{jk} = \tfrac{1}{n-2}\mathsf{P}^{k}{}_{i}h_{jk} = 0.$$

This proves the statement. □

This theorem shows for a null Ricci Walker metric, that the terms in the Fefferman-Graham equations that are non-linear in $\mathbf{h}$ vanish whenever the components $h_{\bar{b}\bar{d}}$ of $\mathbf{h}$ do not depend on the coordinates x^a in Proposition 4.1 corresponding to the total null plane, i.e., if

$$\mathcal{L}_{\partial_a}\mathbf{h}_{\bar{b}\bar{d}} = \partial_a(h_{\bar{b}\bar{d}}) = 0.$$

In the following we will present two situations in which this assumption is satisfied.

4.4. Null Ricci Walker metrics with linear Fefferman-Graham equations

We have seen that the condition (1.7), i.e, that

$$\mathcal{L}_X\mathbf{h} = 0, \qquad \text{for all } X \in \mathcal{N},$$

is crucial for the Fefferman-Graham equations to linearise. We will now see special classes of null Ricci Walker metrics for which this is the case. It turns out that the relation between property (1.7) and the curvature when applied to $\mathcal{N}$ is crucial. First we observe:

Lemma 4.2. *Let* $\overset{0}{\mathbf{g}}$ *be a null Ricci Walker metric with parallel null distribution* $\mathcal{N}$ *and Schouten tensor* P. *Assume furthermore that*

$$X \lrcorner \overset{0}{R} = 0, \qquad \text{for all } X \in \mathcal{N}, \tag{4.30}$$

where $\overset{0}{R}$ *is the curvature tensor of* $\overset{0}{\mathbf{g}}$. *Then* $\mathcal{L}_X \mathsf{P} = 0$ *for all* $X \in \mathcal{N}$.

Proof. For a Walker manifold, the differential Bianchi identity ensures that condition (4.30) also implies that $\mathcal{N} \lrcorner \overset{0}{\nabla} \overset{0}{R} = 0$. This on the other hand implies that $\overset{0}{\nabla}_a \mathsf{P}_{ij} = 0$, which for a null Ricci Walker metrics this is equivalent to $\mathcal{L}_{\mathbf{e}_a} \mathsf{P} = 0$. □

Next we prove a result that strengthens Theorem 1.2 for this class:

Proposition 4.3. *Let* $\overset{0}{\mathbf{g}}$ *be a null Ricci Walker metric with parallel null distribution* $\mathcal{N}$ *and Schouten tensor* P *satisfying condition (4.30) for its curvature.*

Then an ambient metric $\widetilde{\mathbf{g}} = 2\,\mathrm{d}t\,\mathrm{d}(\rho t) + t^2 \mathbf{g}(\rho)$ *for* $[\overset{0}{\mathbf{g}}]$ *in the sense of Definition 2.1 is given by* $\mathbf{g} = \overset{0}{\mathbf{g}} + \mathbf{h}$, *where* $\mathbf{h} = \mathbf{h}(\rho)$ *satisfies* $\mathrm{Im}(\mathbf{h}^\sharp) \subset \mathcal{N}$, $\mathcal{L}_X \mathbf{h} = 0$ *for all* $X \in \mathcal{N}$ *and solves the linear PDE*

$$\rho \ddot{h}_{ij} - \tfrac{n-2}{2} \dot{h}_{ij} - \tfrac{1}{2} \overset{0}{\Box} h_{ij} + \overset{0}{R}_{ij} = O(\rho^m), \tag{4.31}$$

for all m *if* n *is odd and for* $m \leq \frac{n}{2} - 1$ *if* n *is even. When* n *is even, the obstruction tensor is given by*

$$\mathcal{O}_{ij} = c_n \, \overset{0}{\Box}{}^m \overset{0}{R}_{ij},$$

where c_n *is a non-zero constant depending on* n *and* $\overset{0}{\Box}{}^m$ *is the* m*-th power of the tensor Laplacian of* $\overset{0}{\mathbf{g}}$. *In particular,*

$$\mathrm{Im}(\mathcal{O}^\sharp) \subset \mathcal{N}, \qquad \mathcal{L}_X \mathcal{O} = 0, \quad \text{for all } X \in \mathcal{N}.$$

Proof. From Theorem 4.1 we know that $\mathbf{h}$ in the ambient metric satisfies (or, if n is even, can be chosen such) that $\mathrm{Im}(\mathbf{h}^\sharp) \subset \mathcal{N}$. The remaining properties of $\mathbf{h} = \sum_{m \geq 1} \frac{1}{m!} \overset{m}{h} \rho^m$ are proved in a similar way by induction over m as in the proof of Theorem 4.1. But now the computations are simplified, as we can use equations (4.29) in Theorem 4.2, which are equivalent to the Fefferman-Graham equations:

Applying $\overset{0}{\nabla}_a$ to equation (4.29), differentiating it $(m-1)$ times with respect to ρ, for $m \leq \frac{n}{2} - 1$ when n is even, and using the induction hypothesis yields

$$0 = \left(m - \frac{n}{2}\right)\overset{0}{\nabla}_a \overset{m}{h}_{ij} - \frac{1}{2}\overset{0}{g}{}^{kl}\overset{0}{\nabla}_a\overset{0}{\nabla}_k\overset{0}{\nabla}_l \overset{m-1}{h}{}_{ij} + \overset{0}{\nabla}_a\overset{0}{R}_{kijl}\overset{m-1}{h}{}^{kl} = \left(m - \frac{n}{2}\right)\overset{0}{\nabla}_a\overset{m}{h}_{ij}.$$

Here we use the Bianchi identity and that (4.30) allows to commute $\overset{0}{\nabla}_a$ with $\overset{0}{\nabla}_k$. This equation shows $\overset{0}{\nabla}_a h_{ij} = \mathcal{O}(\rho^m)$ for all m, when n is odd, and for $m = \frac{n}{2} - 1$, when n is even. Moreover, when n is even, the terms $\overset{m}{h}$ for $m \geq \frac{n}{2}$ are not determined by the Fefferman-Graham equations. So we can choose them in a way that $\mathcal{L}_{\mathbf{e}_a}\overset{m}{h}_{ij} = \partial_a(h_{ij}) = 0$, which is equivalent to $\overset{0}{\nabla}_a h_{ij}$. With this and the assumption $\overset{0}{R}_{aijk} = 0$, equation (4.29) reduces to equation (4.31). Note also that such a $\mathbf{h}$ is divergence free.

In order to obtain the formula for the obstruction tenser when n is even, we write equation (4.31) in terms of the $\overset{m}{h}_{ij}$ and obtain

$$m\overset{1}{h}_{ij} = \overset{0}{R}_{ij}, \qquad 2(k-m)\overset{k+1}{h}{}_{ij} = \overset{0}{\Box}\overset{k}{h}_{ij}, \qquad \text{for } k = 1, \dots, m-1.$$

This shows that the term of order ρ in (4.31), which is the obstruction tensor, is equal to $c_n\, \overset{0}{\Box}{}^m R_{ij}$ with a nonzero constant c_n. $\Box$

Note that for $\mathbf{h} = h_{\bar{a}\bar{c}}\Theta^{\bar{a}}\Theta^{\bar{c}}$ with $\overset{0}{\nabla}_a h_{ij} = 0$ the term $\overset{0}{\Box}h_{ij}$, i.e. the wave operator of $\overset{0}{\mathbf{g}}$ applied to the tensor $\mathbf{h}$ in (4.31), simplifies to

$$\overset{0}{\Delta}(h_{\bar{b}\bar{d}}) = \overset{0}{g}{}^{AC}\overset{0}{\nabla}_A\overset{0}{\nabla}_C(h_{\bar{b}\bar{d}}),$$

which is the wave operator for the metric $g_{AC}\Theta^A\Theta^C$ in $n-2p$ dimensions applied to the component functions $h_{\bar{a}\bar{c}}$ of $\mathbf{h}$. Finally, the vanishing of the curvature terms $\overset{0}{R}_{aijk}$ implies that the system (4.34), in addition to becoming linear, decouples to $\frac{p+1}{2}$ single equations on the $\frac{p+1}{2}$ components $h_{\bar{b}\bar{d}}$. These equations only differ in their inhomogeneity:

Corollary 4.1. *Let* $(M, \overset{0}{\mathbf{g}})$ *be a null Ricci Walker-manifold with parallel totally null distribution* $\mathcal{N}$ *and* $\mathrm{Im}(\mathsf{P}^\sharp) \subset \mathcal{N}$ *and such that that*

$$X \lrcorner \overset{0}{R} = 0, \qquad \textit{for all } X \in \mathcal{N},$$

where $\overset{0}{R}$ *is the curvature tensor of* $\overset{0}{\mathbf{g}}$*. Then, the an ambient metric metric* $\widetilde{\mathbf{g}} = 2\mathrm{d}(\rho t)\mathrm{d}t + t^2(\overset{0}{\mathbf{g}} + \mathbf{h})$ *for* $[\overset{0}{\mathbf{g}}]$ *is given by* $\mathbf{h}$ *whose components* $h_{\bar{b}\bar{d}}$ *of* $\mathbf{h}$

in a basis as in Proposition 4.1 satisfy the following inhomogeneous linear PDE

$$\Delta_-(h_{\bar{b}\bar{c}}) + 2\overset{0}{R}_{\bar{b}\bar{d}} \;=\; O(\rho^m), \tag{4.32}$$

where $m=\infty$ *when* n *is odd and* $m=\frac{n-2}{2}$ *when* n *is even and where* Δ_- *is the linear second order differential operator defined by*

$$\Delta_-(f) = 2\rho\ddot{f} + (2-n)\dot{f} - \overset{0}{\Delta}(f) \tag{4.33}$$

for the function $f=f(x^{p+1},\ldots,x^n,\rho)$ *and with* $\overset{0}{\Delta}(f) = \overset{0}{g}{}^{AC}\overset{0}{\nabla}_A\overset{0}{\nabla}_C(f) = \overset{0}{g}{}^{AC}\mathbf{e}_A(\mathbf{e}_C(f))$.

A special case of this situation is when the parallel null distribution has rank one, i.e. $p=1$ and $\mathcal{N}=\mathbb{R}\cdot\mathbf{e}_1$. Here the property $\mathcal{L}_{\mathbf{e}_1}\mathbf{h}=0$ follows directly from $\mathbf{h}=h(\Theta^n)^2$ being divergence free. Indeed, we have

$$\mathrm{div}(\mathbf{h}) = \overset{0}{\nabla}_k h^k{}_i = \mathcal{L}_{\mathbf{e}_1}\mathbf{h} = \partial_1(h).$$

Moreover, if the rank of $\mathcal{N}$ is one, also the curvature terms $\overset{0}{R}_{iklj}$ that occur in equation (4.29) have to vanish:

Lemma 4.3. *If* $\mathbf{g}$ *is a null Ricci Walker metric and if the null parallel distribution* $\mathcal{N}$ *has rank one, then* $R_{\bar{a}bd\bar{c}}=0$.

Proof. This is an immediate consequence of equations (4.7):

$$0 = R_{a\bar{c}} = g^{\bar{b}d}(R_{\bar{b}a\bar{c}d} + R_{da\bar{c}\bar{b}}) + g^{AB}R_{Aa\bar{c}B} = g^{\bar{b}d}R_{\bar{b}a\bar{c}d},$$

because of equation (4.5). □

Hence, we obtain:

Corollary 4.2. *Let* $(M,\overset{0}{\mathbf{g}})$ *be a null Ricci Walker manifold with a parallel null line* $\mathcal{N}=\mathbb{R}\cdot\mathbf{e}_1$, *a frame* $\mathbf{e}_1=\partial_1,\mathbf{e}_B,\mathbf{e}_n$ *with a dual frame* $\Theta^1,\Theta^B,\Theta^n$ *as in Proposition 4.1, and such that* $\mathrm{Im}(\mathsf{P}^\sharp)\subset\mathcal{N}$, *i.e.,* $\mathrm{Ric}=f(\Theta^n)^2$, *for a function* f *with* $\partial_1(f)=0$. *Then an ambient metric* $\widetilde{\mathbf{g}} = 2\,\mathrm{d}t\,\mathrm{d}(\rho t) + t^2\mathbf{g}(\rho)$ *for* $[\overset{0}{\mathbf{g}}]$ *in the sense of Definition 2.1 is given by* $\mathbf{g}=\overset{0}{\mathbf{g}}+\mathbf{h}$, *where* $\mathbf{h}=$

$h(\rho, x^i)(\Theta^n)^2$ and h satisfies $\partial_1(h) = 0$ and the following linear PDE

$$\Delta_-(h) + 2f = O(\rho^m), \tag{4.34}$$

where $m = \infty$ when n is odd and $m = \frac{n-2}{2}$ when n is even, and where Δ_- was defined in (4.33).

This corollary and Proposition 4.3 imply the statements in Corollary 1.1. Note that for null Ricci Walker metrics we have that $\mathcal{L}_X \mathcal{O} = \nabla_X \mathcal{O} = 0$ for all $X \in \mathcal{N}$. A construction method for metrics satisfying the assumptions is provided by Proposition 4.2. Explicit examples will be constructed in the next section.

Finally we show an example for which the condition (4.30) is not satisfied and analyse its Fefferman-Graham equations. It turns out that they are not linear in $\mathbf{h}$.

Example 4.1. We consider the following Walker metric in signature $(2,2)$ on $M = \mathbb{R}^4 \ni (x^1, x^2, y^1, y^2)$:

$$\begin{aligned}\mathbf{g} &= 2\mathrm{d}x^1\mathrm{d}y^1 + 2\mathrm{d}x^2\mathrm{d}y^2 + 2(x^1\mathrm{d}y^1)^2 + 2(x^2\mathrm{d}y^2)^2 - 4x^1x^2\mathrm{d}y^1\mathrm{d}y^2 \\ &= 2\left(\Theta^1\Theta^{\bar{1}} + \Theta^2\Theta^{\bar{2}}\right),\end{aligned}$$

where in our notation above we have a co-frame and its dual frame given by

$$\begin{aligned}
\Theta^1 &= \mathrm{d}x^1 + (x^1)^2\mathrm{d}y^1 - 2x^1x^2\mathrm{d}y^2, & \mathbf{e}_1 &= \tfrac{\partial}{\partial x^1},\\
\Theta^2 &= \mathrm{d}x^2 + (x^2)^2\mathrm{d}y^2 - 2x^1x^2\mathrm{d}y^1, & \mathbf{e}_2 &= \tfrac{\partial}{\partial x^2},\\
\Theta^{\bar{1}} &= \mathrm{d}y^1, & \mathbf{e}_{\bar{1}} &= \tfrac{\partial}{\partial y^1} - (x^1)^2\tfrac{\partial}{\partial x^1} + 2x^1x^2\tfrac{\partial}{\partial x^2},\\
\Theta^{\bar{2}} &= \mathrm{d}y^2, & \mathbf{e}_{\bar{2}} &= \tfrac{\partial}{\partial y^2} - (x^2)^2\tfrac{\partial}{\partial x^2} + 2x^1x^2\tfrac{\partial}{\partial x^1}.
\end{aligned}$$

This is a Walker metric with parallel null distribution $\mathcal{K} = \mathcal{K}^\perp = \mathrm{span}(\mathbf{e}_1, \mathbf{e}_2)$. Indeed, we have

$$\nabla \mathbf{e}_1 = 2(x^1\mathrm{d}y^1 - x^2\mathrm{d}y^2) \otimes \mathbf{e}_1 - 2x^2dy^1 \otimes \mathbf{e}_2,$$

and

$$\nabla \mathbf{e}_2 = -2(x^1\mathrm{d}y^1 - x^2\mathrm{d}y^2) \otimes \mathbf{e}_2 - 2x^1dy^2 \otimes \mathbf{e}_1.$$

Then by direct computation or using Proposition 4.2 we see that the Ricci tensor of $\mathbf{g}$ is given by

$$Ric = -12\left((x^1\Theta^{\bar{1}})^2 - 4x^1x^2\Theta^{\bar{1}}\Theta^{\bar{2}} + (x^2\Theta^{\bar{2}})^2\right),$$

and hence g is null Ricci Walker. The curvature tensor has the following non-vanishing terms

$$R_{1\bar{1}\bar{1}1} = -R_{1\bar{1}\bar{2}2} = R_{2\bar{2}\bar{2}2} = 2.$$

Moreover, the Bach tensor, which in dimension 4 is the obstruction tensor, does not vanish,

$$\mathcal{O} = -144\left((x^1\Theta^{\bar{1}})^2 - 4x^1x^2\Theta^{\bar{1}}\Theta^{\bar{2}} + (x^2\Theta^{\bar{2}})^2\right).$$

Hence, there is no smooth Ricci-flat ambient metric and we can only find an ambient metric whose Ricci tensor is of first order in ρ. From Theorem 4.1 we know that the ambient metric is of the form $\widetilde{\mathbf{g}} = 2\mathrm{d}t\,\mathrm{d}(\rho t) + t^2(\mathbf{g}+\mathbf{h})$, where $\mathbf{h} = \mathbf{h}(x^1, x^2, y^1, y^2, \rho)$ is of the form

$$\begin{aligned}\mathbf{h} = {} & A(x^1, x^2, y^1, y^2, \rho)(\Theta^{\bar{1}})^2 - 2B(x^1, x^2, y^1, y^2, \rho)\Theta^{\bar{1}}\Theta^{\bar{2}}) \\ & + C(x^1, x^2, y^1, y^2, \rho)(\Theta^{\bar{1}\bar{2}})^2,\end{aligned}$$

with $A_{\rho=0} = B_{\rho=0} = C_{\rho=0} = 0$ and

$$0 = \mathrm{div}(\mathbf{h}) = \left(\tfrac{\partial A}{\partial x^1} - \tfrac{\partial B}{\partial x^2}\right)\Theta^{\bar{1}} + \left(\tfrac{\partial B}{\partial x^1} - \tfrac{\partial C}{\partial x^2}\right)\Theta^{\bar{2}}.$$

A direct computation shows that the Fefferman-Graham equations for this example remain non-linear. For example, the $\bar{1}\bar{1}$-component of equation (4.29) is

$$\begin{aligned}&\rho\ddot{A} - \dot{A} - 2A - +\tfrac{1}{2}A\tfrac{\partial^2 A}{(\partial x^1)^2} - B\tfrac{\partial^2 A}{\partial x^2\partial x^1} + \tfrac{1}{2}C\tfrac{\partial^2 A}{(\partial x^2)^2} + \tfrac{1}{2}\left(\tfrac{\partial A}{\partial x^1}\right)^2 - \tfrac{\partial A}{\partial x^2}\tfrac{\partial B}{\partial x^1} \\ &\quad + \tfrac{1}{2}\left(\tfrac{\partial B}{\partial x^1}\right)^2 + (x^1)^2\tfrac{\partial^2 A}{(\partial x^1)^2} - 4x^2x^1\tfrac{\partial^2 A}{\partial x^2\partial x^1} - \tfrac{\partial^2 A}{\partial y^1\partial x^1} + 4x^2\tfrac{\partial A}{\partial x^2} - 4x^1\tfrac{\partial A}{\partial x^1} \\ &\quad + (x^2)^2\tfrac{\partial^2 A}{(\partial x^2)^2} - \tfrac{\partial^2 A}{\partial y^2\partial x^2} - 4x^1\tfrac{\partial B}{\partial x^2} - 12(x^1)^2.\end{aligned}$$

In this example our ansatz (1.7), i.e., that

$$\mathcal{L}_{\mathbf{e}_1}\mathbf{h} = \mathcal{L}_{\mathbf{e}_2}\mathbf{h} = 0, \tag{4.35}$$

does not yield a solution to the Fefferman-Graham equations, i.e., to $Ric(\tilde{g}) = O(\rho)$. Indeed, the ansatz (4.35) is equivalent to the components A, B and C being independent of x^1 and x^2, and hence the Ricci tensor of

$\tilde{g}$ has the components

$$\begin{aligned}&\left(\rho\ddot{A}(y^1,y^2,\rho)-\dot{A}(y^1,y^2,\rho)-2A(y^1,y^2,\rho)-12(x^1)^2\right)(\Theta^{\bar{1}})^2\\&+2\left(\rho\ddot{B}(y^1,y^2,\rho)-\dot{B}(y^1,y^2,\rho)-2B(y^1,y^2,\rho)-12x^1x^2\right)\Theta^{\bar{1}}\Theta^{\bar{2}}\\&+\left(\rho\ddot{C}(y^1,y^2,\rho)-\dot{C}(y^1,y^2,\rho)-2C(y^1,y^2,\rho)-12(x^2)^2\right)(\Theta^{\bar{2}})^2,\end{aligned}$$

which cannot be of the form ρQ for Q a tensor on M. Instead, a solution is for example given by

$$\mathbf{h}=-12\rho\left((x^1\Theta^{\bar{1}})^2-4x^1x^2\Theta^{\bar{1}}\Theta^{\bar{2}}+(x^2\Theta^{\bar{2}})^2\right),$$

which is divergence free but does not satisfy the ansatz (1.7). With this $\mathbf{h}$ the ambient metric $\widetilde{\mathbf{g}}=2\mathrm{d}(\rho t)\mathrm{d}t+t^2(\overset{\circ}{\mathbf{g}}+\mathbf{h})$ has Ricci tensor

$$Ric(\widetilde{\mathbf{g}})=-144\rho(3\rho-1)\left((x^1\Theta^{\bar{1}})^2-4x^1x^2\Theta^{\bar{1}}\Theta^{\bar{2}})+(x^2\Theta^{\bar{2}})^2\right)=\rho(3\rho-1)\mathcal{O}.$$

5. Examples with explicit ambient metrics

In this section we will provide examples of conformal classes of null Ricci Walker metrics for which we find explicit solutions to equation (4.32) obtaining explicit examples of Ricci-flat ambient metrics.

5.1. Solving the homogeneous equation

Equation (4.32) is a linear, inhomogeneous PDE for each of the functions $h_{\bar{a}\bar{c}}$ given by the linear differential operator

$$\Delta_-=2\rho\partial_\rho^2+(2-n)\partial_\rho-\overset{\circ}{\Delta}.$$

In the section we will find metrics for which we get an explicit solution of (4.32). Before this, we start by providing the solution to the homogeneous equation.

Lemma 5.1. *Let M be a smooth manifold of dimension n and $\mathcal{D}$ some linear differential operator on M. For a function $F\in C^\infty(M)$ we define the* *functions* $F_\pm \in C^\infty(M \times (-\epsilon, \epsilon))$ *as*

$$F_\pm := \sum_{k=1}^{\infty} \frac{\mathcal{D}^k(F)}{k! \prod_{i=1}^{k}(2i \pm n)} \rho^k,$$

where F_- *is only defined when* n *is odd or* $\mathcal{D}^{\frac{n}{2}}(F) = 0$*. Moreover, define the following linear differential operators on* $C^\infty(M \times (-\epsilon, \epsilon))$

$$\mathcal{D}_\pm := 2\rho\partial_\rho^2 + (2 \pm n)\partial_\rho - \mathcal{D}.$$

Then, for any $F \in C^\infty(M)$ *and* $f \in C^\infty(M \times (-\epsilon, \epsilon))$ *we have*

$$\mathcal{D}_\pm(F_\pm) = \mathcal{D}(F), \tag{5.1}$$

$$\mathcal{D}_-(\rho^{\frac{n}{2}} f) = \rho^{\frac{n}{2}} \mathcal{D}_+(f), \tag{5.2}$$

$$\mathcal{D}_-(\rho^{\frac{n}{2}} F_+) = \rho^{\frac{n}{2}} \mathcal{D}_+(F_+) = \rho^{\frac{n}{2}} \mathcal{D}(F), \tag{5.3}$$

$$\mathcal{D}_-(\rho^{\frac{n}{2}}(F + F_+)) = 0. \tag{5.4}$$

In particular, for each $F \in C^\infty(M)$*, the function* $f = \rho^{\frac{n}{2}}(F + F_+)$ *is a solution to the homogeneous equation* $\mathcal{D}_-(f) = 0$*.*

Proof. To verify equations (5.1) and (5.2) is a straightforward computation. Both together imply (5.3) which yields (5.4). □

5.2. Extensions of nilpotent Lie algebras

Let $\mathfrak{k}$ be a two-step nilpotent Lie algebra of dimension q and let $\mathfrak{z}$ be its centre of dimension $p < q$. We fix a complement $\mathfrak{m}$ of $\mathfrak{z}$,

$$\mathfrak{k} = \mathfrak{z} \oplus \mathfrak{m}$$

Then $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{z}$ and we can fix a basis $(\mathbf{e}_a)_{a=1,\dots,p}$ of $\mathfrak{z}$ and $(\mathbf{e}_A)_{A=p+1,\dots,q}$ of $\mathfrak{m}$ such that

$$[\mathbf{e}_a, \mathbf{e}_b] = 0, \qquad [\mathbf{e}_a, \mathbf{e}_B] = 0, \qquad [\mathbf{e}_A, \mathbf{e}_B] = r^c_{AB}\mathbf{e}_c,$$

where r^c_{AB} denote the structure constants of $\mathfrak{k}$. Note that there are no further conditions on these numbers other than $r^c_{AB} = -r^c_{BA}$. Denote by $\mathfrak{der}(\mathfrak{k})$ the derivations of $\mathfrak{k}$ which comes with a canonical Lie algebra structure induced from $\mathfrak{gl}(\mathfrak{k})$. Note that derivations leave the centre invariant.

Furthermore, let H be a Lie group with Lie algebra $\mathfrak{h}$ and of dimension $p = \dim(\mathfrak{z})$ and $\phi : \mathfrak{h} \to \mathfrak{Der}(\mathfrak{k})$ a Lie algebra homomorphism from $\mathfrak{h}$ to the derivations of $\mathfrak{k}$. By fixing a basis $(\mathbf{e}_{\bar{a}})_{\bar{a}=q+1,\dots,p+q}$ of $\mathfrak{h}$, we can write ϕ as

$$\phi(\mathbf{e}_{\bar{a}})\mathbf{e}_b = r^d_{b\bar{a}}\mathbf{e}_d, \qquad \phi(\mathbf{e}_{\bar{a}})\mathbf{e}_B = r^d_{B\bar{a}}\mathbf{e}_d + r^E_{B\bar{a}}\mathbf{e}_E,$$

with some constants $r^d_{b\bar{a}}$, $r^d_{B\bar{a}}$ and $r^E_{B\bar{a}}$. Finally, with respect to this basis denote the structure constants of $\mathfrak{h}$ by $r^{\bar{c}}_{\bar{a}\bar{b}}$, i.e.,

$$[\mathbf{e}_{\bar{a}}, \mathbf{e}_{\bar{b}}] = r^{\bar{c}}_{\bar{a}\bar{b}}\mathbf{e}_{\bar{c}}.$$

Now we define the Lie algebra $\mathfrak{g}$ to be semi-direct sum $\mathfrak{g} = \mathfrak{h} \ltimes_\phi \mathfrak{k}$ of $\mathfrak{h}$ and $\mathfrak{k}$ with respect to ϕ of dimension $n = p + q$. Clearly, the structure constants of $\mathfrak{g}$ are given by the numbers

$$r^c_{AB}, r^d_{b\bar{a}}, r^d_{B\bar{a}}, r^E_{B\bar{a}}, r^{\bar{c}}_{\bar{a}\bar{b}},$$

which are subject to the conditions $r^k_{ij} = -r^k_{ji}$ and

$$r^e_{AB} r^d_{e\bar{c}} = -2 r^C_{\bar{c}[A} r^d_{B]C},$$

i.e., that $\phi(\mathbf{e}_{\bar{c}})$ is a derivation, as well as

$$r^{\bar{c}}_{\bar{a}\bar{b}} r^e_{d\bar{c}} = 2 r^c_{d[\bar{a}} r^e_{\bar{b}]c}, \qquad r^{\bar{c}}_{\bar{a}\bar{b}} r^d_{A\bar{c}} = 2 r^c_{A[\bar{a}} r^d_{\bar{b}]c} + 2 r^B_{A[\bar{a}} r^d_{\bar{b}]B}, \qquad r^{\bar{c}}_{\bar{a}\bar{b}} r^B_{A\bar{c}} = 2 r^C_{A[\bar{a}} r^B_{\bar{b}]C},$$

which ensure that $\phi : \mathfrak{k} \to \mathfrak{Der}(\mathfrak{h})$ is a Lie algebra homomorphism. The frame $\mathbf{e}_1, \dots, \mathbf{e}_n$ on the Lie group G corresponding to $\mathfrak{g}$ satisfies the bracket relations of Proposition 4.2 with the parallel distribution $\mathcal{K}$ given by $\mathfrak{k}$. Now we define a left invariant metric by formula (3.4)

$$\mathbf{g} = g_{a\bar{c}}(\Theta^a \otimes \Theta^{\bar{c}} + \Theta^{\bar{c}} \otimes \Theta^a) + \overset{0}{g}_{AB}\Theta^A \circ \Theta^B,$$

where the Θ^i's are again the algebraic duals of the $\mathbf{e}_i$'s, the g_{ij} are constants with $g_{a\bar{c}}$ and $\overset{0}{g}_{AB}$ non degenerate and $\overset{0}{g}_{AB}$ symmetric. If the signature of $\overset{0}{g}_{AB}$ is (s, t), then the signature of the metric $\mathbf{g}$ is $(p + s, p + t)$ or $(p + t, p + s)$, so for example if $\overset{0}{g}_{AB}$ is definite, the signature of $\mathbf{g}$ is (p, q) or (q, p). The distribution $\mathcal{K}^\perp$ is given by $\mathfrak{z}$. Then Proposition 4.2 implies that $(G, \mathbf{g})$ is a null Ricci Walker manifold of dimension n, which, in general is not Ricci-flat. Its possibly non vanishing components are given by constants $R_{\bar{a}\bar{c}}$.

In order to determine the ambient metric for the conformal class given by $\mathbf{g}$ on G, we have to solve equations (4.32) in this setting, i.e., find a functions $h \in C^\infty((-\varepsilon, \varepsilon) \times G)$, such that

$$2\rho\ddot{h} + (2-n)\dot{h} - \Delta(h) + C = 0, \quad \text{with initial condition } h|_{\rho=0} \equiv 0, \tag{5.5}$$

with $\Delta(h) = g^{AB}\nabla_A\nabla_B h$, and for constants C that are given by the components of the Ricci tensor $R_{\bar{a}\bar{c}}$. Equation (5.5), when taken along $\rho = 0$ implies

$$\dot{h}|_{\rho=0} \equiv \frac{C}{n-2}.$$

Clearly, the problem (5.5) has a linear solution

$$h = \frac{C}{n-2}\rho,$$

but Lemma 5.1 shows that there are more solutions. From Corollary 4.1 we obtain Theorem 1.4 from the introduction. More precisely, we get the following.

Theorem 5.1. *Let $\mathfrak{k}$ be a two-step nilpotent Lie algebra of dimension q with centre $\mathfrak{z}$ of dimension $p \leq q$, and let H be a Lie group of dimension p and with Lie algebra $\mathfrak{h}$. Let $\phi : \mathfrak{h} \to \mathfrak{der}(\mathfrak{k})$ a Lie algebra homomorphism into the derivations of $\mathfrak{k}$ and G be the $n = q + p$-dimensional Lie group corresponding to the Lie algebra $\mathfrak{g}$ that is given as the semi-direct sum*

$$\mathfrak{g} = \mathfrak{h} \ltimes_\phi \mathfrak{k},$$

of $\mathfrak{h}$ and $\mathfrak{k}$ by ϕ. Fix a basis $(\mathbf{e}_{\bar{a}})_{\bar{a}=1,\ldots,p}$ of $\mathfrak{h}$, a basis $(\mathbf{e}_a)_{a=1,\ldots,p}$ of $\mathfrak{z}$ and complement it with $(\mathbf{e}_A)_{A=1,\ldots,q-p}$ to a basis of $\mathfrak{k}$. Let $(\Theta^i)_{i=1,\ldots,n}$ is the dual basis to $(\mathbf{e}_i)_{i=1,\ldots,n}$ and

$$\mathbf{g} = 2\, g_{a\bar{c}}\, \Theta^a \circ \Theta^{\bar{c}} + g_{AB}\Theta^A \circ \Theta^B$$

be the left-invariant semi-Riemannian metric $\mathbf{g}$ on G defined by real numbers $g_{a\bar{c}}$ and g_{AB}. Then the conformal class of $\mathbf{g}$ on G admits Ricci-flat ambient metrics given by

$$\widetilde{\mathbf{g}} = 2\mathrm{d}(\rho t)\mathrm{d}t + \\ + t^2\Big(\mathbf{g} + \Big(\frac{2\rho}{n-2}R_{\bar{a}\bar{c}} + \rho^{\frac{n}{2}}\Big(F_{\bar{a}\bar{c}} + \sum_{k=1}^{\infty}\frac{\overset{0}{\Delta}{}^{k}(F_{\bar{a}\bar{c}})}{k!\prod_{i=1}^{k}(2i+n)}\rho^k\Big)\Big)\Theta^{\bar{a}}\Theta^{\bar{c}}\Big),$$

where $R_{\bar{a}\bar{c}} = Ric^{\mathbf{g}}(\mathbf{e}_{\bar{a}}, \mathbf{e}_{\bar{c}})$ are the components of the Ricci tensor of $\mathbf{g}$ and $F_{\bar{a}\bar{c}} = F_{\bar{c}\bar{a}}$ are functions on G with $dF_{\bar{a}\bar{c}}(\mathbf{e}_a) = 0$. In particular, when n is *odd,* $F_{\bar{a}\bar{c}} \equiv 0$ *gives the unique analytic Ricci-flat Fefferman-Graham ambient metric.*

Note that in general the metrics $\mathbf{g}$ as in the theorem are neither Ricci-flat nor do they admit parallel null vector fields (see also Remark 4.2).

5.3. Generalised pp-waves

Another class of examples to which our Corollary 4.1 applies are the Lorentzian pp-waves for which we have determined the analytic ambient metric in [25]. The acronym "pp" stands for *plane fronted with parallel rays.* A Lorentzian pp-wave metric in dimension n is locally given by

$$\mathbf{g} = 2\mathrm{d}u\mathrm{d}v + H\mathrm{d}u^2 + \sum_{i=1}^{n-2}(\mathrm{d}x^i)^2,$$

where $H = H(x^i, u)$ is a function that does not depend on v.

Here we generalise this class and the results in [3, 25] to higher signature and, more importantly, determine all solutions to the Fefferman-Graham equations including the non-analytic ones, and determine the obstruction tensor in the case of Lorentzian pp-waves.

We will use the same index conventions as in the previous sections ($a = 1, \ldots p,\ B = p+1, \ldots n-p,\ \bar{c} = n-p+1, \ldots, n$), and define a modified Kronecker delta as

$$\delta_{\bar{a}b} = \begin{cases} 1, & \text{if } \bar{a} = b + n - p, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 5.1. Let $\mathcal{U} \subset \mathbb{R}^n \ni (x^1, \ldots, x^n)$ be an open set, and $H_{\bar{a}\bar{c}}$ and G_{AB} smooth functions on $\mathcal{U}$ satisfying $\det(G_{AB}) \neq 0$ and $\partial_a(H_{\bar{a}\bar{c}}) = 0$ and $\partial_a(G_{AB}) = \partial_{\bar{c}}(G_{AB}) = 0$. Then the semi-Riemannian metric

$$\mathbf{g} = 2\delta_{\bar{a}b}\mathrm{d}x^{\bar{a}}\mathrm{d}x^b + H_{\bar{a}\bar{b}}\mathrm{d}x^{\bar{a}}\mathrm{d}x^{\bar{b}} + G_{AB}\mathrm{d}x^A\mathrm{d}x^B, \tag{5.6}$$

is called a *generalised pp-wave*, or for short, a *gpp-wave.*

If all the G_{AB}'s are constants, we call $\mathbf{g}$ *plane fronted wave with parallel rays*, or for short, *pp-wave.*

To obtain Lorentzian gpp-waves, one sets $p = 1$ and G_{AB} positive definite. For all p, gpp-waves admit p parallel vector fields ∂_a and hence are

Walker metrics, however in general not null Ricci Walker metrics. As in Proposition 4.1, for gpp-waves we have the frame and dual co-frame

$$\mathbf{e}_a := \partial_a, \quad \mathbf{e}_B := E_B{}^A \partial_A, \quad \mathbf{e}_{\bar{c}} := \partial_{\bar{c}} - H_{\bar{a}\bar{c}} \delta^{\bar{a}\bar{b}} \partial_b,$$
$$\Theta^a = \mathrm{d}x^a + H_{\bar{a}\bar{c}} \mathrm{d}x^{\bar{c}}, \quad \Theta^B = F^B_A \mathrm{d}x^A, \quad \Theta^{\bar{c}} = \mathrm{d}x^{\bar{c}},$$

where $E_A{}^B$ is a matrix such that $E_A{}^B G_{BC} E_D{}^C = \delta_{AD}$ and F^B_A is the inverse of $E_A{}^B$. Note that, since G_{AB} does not depend on the x^a's or the $x^{\bar{c}}$'s neither does $E_A{}^B$. The gpp-wave metric in this frame is

$$\mathbf{g} = \delta_{a\bar{b}} \Theta^a \Theta^{\bar{b}} + g_{AB} \Theta^A \Theta^B,$$

with $g_{AB} = \epsilon_A \delta_{AB}$. The only non vanishing brackets for this frame are

$$\begin{aligned} [\mathbf{e}_A, \mathbf{e}_B] &= -E_{[A}{}^C E_{B]}{}^D dF_D{}^E(\partial_C)\mathbf{e}_E, \\ [\mathbf{e}_A, \mathbf{e}_{\bar{b}}] &= -dH_{\bar{b}\bar{c}}(\mathbf{e}_A)\delta^{\bar{c}\bar{d}}\mathbf{e}_d, \\ [\mathbf{e}_{\bar{a}}, \mathbf{e}_{\bar{b}}] &= 2\mathrm{d}H_{\bar{c}[\bar{a}}(\partial_{\bar{b}]})\delta^{\bar{c}\bar{d}}\mathbf{e}_d, \end{aligned}$$

Hence, the assumptions of Proposition 4.2 are satisfied whenever the G_{AB}'s are constant, i.e., whenever $\mathbf{g}$ is a pp-wave.

The Levi-Civita connection ∇ of a gpp-wave $\mathbf{g}$ is given by

$$\begin{aligned} \nabla_A \mathbf{e}_B &= \nabla^{\mathbf{G}}_A \mathbf{e}_B, \\ \nabla_{\bar{a}} \mathbf{e}_B &= dH_{\bar{a}\bar{c}}(\mathbf{e}_B)\delta^{\bar{c}\bar{b}}\mathbf{e}_b, \\ \nabla_{\bar{a}} \mathbf{e}_{\bar{b}} &= -2\mathrm{d}H_{\bar{a}[\bar{b}}(\partial_{\bar{c}]})\delta^{\bar{c}\bar{d}}\mathbf{e}_d - \mathrm{grad}^{\mathbf{G}}(H_{\bar{a}\bar{b}}), \end{aligned}$$

in which $\nabla^{\mathbf{G}}$ is the Levi-Civita connection of the metric $\mathbf{G} = G_{AB}\mathrm{d}x^A\mathrm{d}x^B$ and $\mathrm{grad}^{\mathbf{G}}$ the corresponding gradient. This allows us to compute the curvature, which satisfies $R_{aijk} = 0$, and the Ricci-curvature, whose only possibly non-vanishing terms are given as

$$\begin{aligned} R_{AB} &= R^{\mathbf{G}}_{AB}, \\ R_{\bar{a}\bar{c}} &= -\tfrac{1}{2} g^{BD} \mathbf{g}(\nabla_B(\mathrm{grad}(H_{\bar{a}\bar{c}}), \mathbf{e}_D) = -\tfrac{1}{2} g^{BD} \nabla^{\mathbf{G}}_B \nabla^{\mathbf{G}}_D (H_{\bar{a}\bar{c}}) = -\tfrac{1}{2}\Delta_{\mathbf{G}}(H_{\bar{a}\bar{c}}). \end{aligned}$$

Lemma 5.2. *The defined gpp-waves satisfy $R_{aijk} = 0$ and they are null Ricci Walker metrics if the metric $\mathbf{G}$ is Ricci-flat. In particular, pp-waves are null Ricci Walker metrics.*

Remark 5.1. If we drop the assumption on a pp-wave that the $\mathbf{e}_a$'s are parallel, i.e., that $\partial_a H \neq 0$, then the Ricci tensor is no longer two-step nilpotent.

For example in the Lorentzian case, i.e., when $p = 1$ and $\epsilon_i = 1$, if $\partial_1 H \neq 0$ we get that

$$Ric\,(\partial_1, \partial_n) = \partial_1^2(H), \qquad Ric\,(\partial_A, \partial_n) = \partial_A \partial_1(H),$$

which shows that Ric cannot be two-step nilpotent (see also [23]).

Remark 5.2. Using the necessary conditions that were derived in [20] for conformal Einstein metrics, a straightforward computation of the Weyl, Cotton and Bach tensors as in [25] shows that in general gpp-waves are not conformally Einstein. In fact, in [25] we gave explicit examples of Bach flat pp-waves that are not conformally Einstein.

When determining the ambient metric for a gpp-wave for which the metric $\mathbf{G}$ is Ricci-flat, we can apply Theorem 4.2 and Proposition 4.3. Moreover, since all the $\mathbf{e}_a = \partial_a$ are parallel, the curvature terms R_{aijk} vanish, but also the $\Theta^{\bar{a}}$'s are parallel. We obtain

Corollary 5.1. *Let $\mathbf{G} = G_{AB}\mathrm{d}x^A\mathrm{d}x^B$ be a Ricci-flat metric on $\mathbb{R}^{n-2p}$ and $H_{\bar{a}\bar{b}}$ functions of $(n-p)$ variables $(x^A, x^{\bar{b}})$ that define the gpp-wave*

$$\mathbf{g} = 2\delta_{\bar{a}b}\mathrm{d}x^{\bar{a}}\mathrm{d}x^b + H_{\bar{a}\bar{b}}\mathrm{d}x^{\bar{a}}\mathrm{d}x^{\bar{b}} + G_{AB}\mathrm{d}x^A\mathrm{d}x^B$$

on $\mathbb{R}^n$. Then an ambient metric for $[\mathbf{g}]$ is given by $\widetilde{\mathbf{g}} = 2\,\mathrm{d}t\,\mathrm{d}(\rho t) + t^2(\mathbf{g} + \mathbf{h}(\rho))$, where $\mathbf{h} = h_{\bar{b}\bar{d}}\mathrm{d}x^{\bar{b}}\mathrm{d}x^{\bar{d}}$ and whose components satisfy $\partial_a(h_{\bar{b}\bar{d}}) = 0$ and

$$2\rho\ddot{h}_{\bar{b}\bar{d}} + (2-n)\dot{h}h_{\bar{b}\bar{d}} - \Delta_{\mathbf{G}}(h_{\bar{b}\bar{d}}) - \Delta_{\mathbf{G}}(H_{\bar{b}\bar{d}}) = O(\rho^m), \tag{5.7}$$

with $m = \infty$ when n is odd and $m = \frac{n-2}{2}$ when n is even and where $\Delta_{\mathbf{G}}$ is the Laplacian of $\mathbf{G}$.

This corollary shows that in order to obtain Ricci-flat ambient metrics, for a function $H = H(x^{p+1}, \ldots, x^n)$ we have to solve the equation

$$2\rho\ddot{h} + (2-n)\dot{h} - \Delta_{\mathbf{G}}(h) - \Delta_{\mathbf{G}}(H) = 0, \tag{5.8}$$

for a function $h = h(\rho, x^{p+1}, \ldots, x^n)$. This can be solved by standard power series expansion, noticing that its indicial exponents are $s = 0$ and $s = n/2$. We extend our results in [3, 25], by the following more general existence statement for gpp-waves.

Theorem 5.2. *Let* $\mathbf{G}$ *be a semi-Riemannian metric on* $\mathbb{R}^{n-2p}$*. Then the following functions* $h = h(\rho, x^{p+1}, \dots, x^n)$ *are solutions to equation* (5.8) *with* $h(\rho) \to 0$ *when* $\rho \downarrow 0$*:*

When n *is odd:*

$$h = \sum_{k=1}^{\infty} \frac{\Delta_{\mathbf{G}}^k H}{k! \prod_{i=1}^k (2i-n)} \rho^k + \rho^{n/2} \Big(\alpha + \sum_{k=1}^{\infty} \frac{\Delta_{\mathbf{G}}^k \alpha}{k! \prod_{i=1}^k (2i+n)} \rho^k \Big), \tag{5.9}$$

where $\alpha = \alpha(x^{p+1}, \dots, x^n)$ *is an arbitrary function of its variables. In particular, if* $\alpha \equiv 0$ *this gives an analytic in* ρ *solution in a neighbourhood of* $\rho = 0$ *with* $h(0) = 0$*.*

When $n = 2s$ *is even:*

$$\begin{aligned} h = {} & \sum_{k=1}^{s-1} \frac{\Delta_{\mathbf{G}}^k H}{k! \prod_{i=1}^k (2i-n)} \rho^k + \rho^s \Big(\alpha + \sum_{k=1}^{\infty} \frac{\Delta_{\mathbf{G}}^k \alpha}{k! \prod_{i=1}^k (2i+n)} \rho^k \Big) \\ & + c_n \rho^s \left(\sum_{k=0}^{\infty} (\log(\rho) - q_k) \frac{\Delta_{\mathbf{G}}^{s+k} H}{k! \prod_{i=1}^k (2i+n)} \rho^k \right), \end{aligned} \tag{5.10}$$

where $\alpha = \alpha(x^{p+1}, \dots, x^n)$ *and* $q_0 = q_0(x^{n-p+1}, \dots, x^n)$ *and*

$$q_k(x^{n-p+1}, \dots, x^n) := q_0(x^{n-p+1}, \dots, x^n) + \sum_{i=1}^{k} \frac{n+4i}{i(n+2i)},$$

for $k = 1, 2, \dots$*, are arbitrary functions of their variables and the constant* c_n *is given as follows*

$$c_n := -\frac{1}{(s-1)! \prod_{i=0}^{s-1} (2i-n)}.$$

In particular, when $\Delta_{\mathbf{G}}^s H \equiv 0$ *there are solutions that are analytic in* ρ *in a neighbourhood of* $\rho = 0$ *and with* $h(0) = 0$*. These solutions are parametrized by the functions* α*.*

Proof. That the given function satisfy equation (5.8) can be checked directly. In the case n odd it follows from Lemma 5.1. For n even, the situation is a bit more subtle. We give the formulas for each term, ignoring the term

$(\rho^{\frac{n}{2}}(\alpha+\alpha_+))$, for which we have seen that it is in the kernel of $\mathcal{D}_-$:

$$\begin{aligned}
&\mathcal{D}_-\left(\sum_{k=1}^{s-1}\frac{\Delta^k_{\mathbf{G}}H}{k!\prod_{i=1}^k(2i-n)}\rho^k\right)\\
&\quad=\Delta_{\mathbf{G}}H-\frac{\Delta^s_{\mathbf{G}}H}{(s-1)!\Pi_{i=1}^{s-1}(2i-n)}\rho^{s-1},\\
&\mathcal{D}_-\left(\rho^s\Delta^s_{\mathbf{G}}\left(\log(\rho)(H+H_+)\right)\right)\\
&\quad=n\rho^{s-1}\Delta^sH+\frac{n+4}{n+2}\rho^s\Delta^{s+1}_{\mathbf{G}}H\\
&\qquad+\sum_{k=1}^{\infty}\frac{(n+4(k+1))}{(k+1)!\Pi_{i=1}^{k+1}(2i+n)}\Delta^{s+k+1}H\,\rho^{s+k}\\
&\mathcal{D}_-\left(\rho^s\Delta^s_{\mathbf{G}}\sum_{k=0}^{\infty}q_k\frac{\Delta^k_{\mathbf{G}}H}{k!\prod_{i=1}^k(2i-n)}\rho^k\right)\\
&\quad=(q_1-q_0)\rho^s\Delta^{s+1}_{\mathbf{G}}H\\
&\qquad+\sum_{k=1}^{\infty}\frac{(q_{k+1}-q_k)(n+2(k+1))}{k!\Pi_{i=1}^{k+1}(2i+n)}\Delta^{s+k+1}_{\mathbf{G}}H\,\rho^{s+k}.
\end{aligned}$$

Looking at the ρ^{s-1}-terms in these formulas we determine c_n as in the theorem by

$$-\frac{1}{(s-1)!\Pi_{i=1}^{s-1}(2i-n)}+nc_n=0.$$

Moreover, looking at the ρ^s-terms, we determine q_1 by

$$\frac{n+4}{n+2}-(q_1-q_0)=0$$

as given in the theorem, and finally the other q_k's by

$$n+4(k+1)-(q_{k+1}-q_k)(n+2(k+1))(k+1)=0.$$

This proves the theorem. □

Summarising, we obtain

Corollary 5.2. *Let*

$$\mathbf{g}=2\mathrm{d}x^{\bar{a}}(\delta_{\bar{a}b}\mathrm{d}x^b+H_{\bar{a}\bar{b}}\mathrm{d}x^{\bar{b}})+G_{AB}\mathrm{d}x^A\mathrm{d}x^B$$

be a gpp-wave with Ricci-flat metric $\mathbf{G} = G_{AB}\mathrm{d}x^A\mathrm{d}x^B$*. Then ambient metrics in the sense of Definition 2.1 for the conformal class* $[\mathbf{g}]$ *are*

$$\begin{aligned}\widetilde{\mathbf{g}} &= 2\mathrm{d}(\rho t)\mathrm{d}t + t^2\mathbf{g} \\ &+ t^2\Bigg(\Bigg(\sum_{k=1}^{m}\frac{\Delta^k_{\mathbf{G}}(H_{\bar a\bar b})}{k!\prod_{i=1}^k(2i-n)}\rho^k \\ &+ \rho^{n/2}\Bigg(F_{\bar a\bar b} + \sum_{k=1}^{\infty}\frac{\Delta^k_{\mathbf{G}}(F_{\bar a\bar b})}{k!\prod_{i=1}^k(2i+n)}\rho^k\Bigg)\Bigg)\mathrm{d}x^{\bar a}\mathrm{d}x^{\bar b}\Bigg)\end{aligned}$$

in which $m = \infty$ *when* n *is odd and* $m = \frac{n-2}{2}$ *when* n *is even, and* $F_{\bar a\bar c} = F_{\bar c\bar a}$ *are arbitrary functions on* M*, with* $\partial_a(F_{\bar a\bar c}) = 0$*. Moreover,*

(1) *When* n *is odd,* $F_{\bar a\bar c} \equiv 0$ *gives the unique analytic Ricci-flat Fefferman-Graham ambient metric.*

(2) *When* n *is even and* $\Delta^{\frac{n}{2}}_{\mathbf{G}}(H_{\bar a\bar c}) = 0$*, then the metric* $\widetilde{\mathbf{g}}$ *is Ricci-flat.*

(3) *When* n *is even and* $\Delta^{\frac{n}{2}}_{\mathbf{G}}(H_{\bar a\bar c}) \neq 0$*, then Ricci-flat but non analytic ambient metrics are given by formula* (5.10)*.*

5.4. Ambient metrics for Lorentzian pp-waves

Finally we consider Lorentzian pp-waves, i.e., gpp-waves with $p = 1$ and $G_{AB} = \delta_{AB}$. Since $p = 1$ we use a different convention as names for the variables: we replace coordinates x^1, x^A, $A = 2, \dots, n-2$, and x^n by $v := x^1$, $y^i = x^{i+1}$, $i = 1, \dots, n-2$, and $u = x^n$. We have seen solutions of equation (5.8) in Theorem 5.2. For Lorentzian pp-waves these are all of the solutions. Here $\Delta_{\mathbf{G}} = \Delta$ is just the flat Laplacian and we can use the Fourier transform to transform equation (5.8) into an ODE. In fact, in [3] we proved the following

Theorem 5.3 ([3]). *Let* Δ *be the flat Laplacian in* $(n-2)$ *dimensions.*

When n *is odd, the most general solutions* h *to equation (5.8) with* $h(\rho) \to 0$ *when* $\rho \downarrow 0$ *are given by formula (5.9) in Theorem 5.2 and parametrized by arbitrary functions* $\alpha = \alpha(x^1, \dots, x^{n-2}, u)$*. In particular, there is a unique solution that is analytic in* ρ *in a neighbourhood of* $\rho = 0$ *with* $h(0) = 0$*. This solution is given by* $\alpha \equiv 0$*.*

When $n = 2s$ is even, the most general solutions h to equation (5.8) with $h(\rho) \to 0$ when $\rho \downarrow 0$ are given by

(5.11)
$$h = \sum_{k=1}^{s-1} \frac{\Delta^k H}{k! \prod_{i=1}^{k}(2i-n)} \rho^k + \rho^s \Big(\alpha + \sum_{k=1}^{\infty} \frac{\Delta^k \alpha}{k! \prod_{i=1}^{k}(2i+n)} \rho^k \Big) + c_n \rho^s \sum_{k=0}^{\infty} \frac{1}{k! \prod_{i=1}^{k}(2i+n)} \Big((\log(\rho) - q_k)\, \Delta^{s+k} H + Q * \Delta^{s+k} H \Big) \rho^k,$$

where $\alpha = \alpha(y^i, u)$ and $Q = Q(x^i, u)$ are arbitrary functions of their variables, $*$ denotes the convolution of two functions with respect to the y^i-variables, c_n is the constant defined in Theorem 5.2, and the other constants are given as follows

$$q_0 := 0, \ q_k := \sum_{i=1}^{k} \frac{n+4i}{i(n+2i)}, \quad \textit{for } k = 1, 2, \dots .$$

In particular, only when $\Delta^s H \equiv 0$ there are solutions that are analytic in ρ in a neighbourhood of $\rho = 0$ and with $h(0) = 0$. These solutions are not unique but parametrized by the functions α.

With the results of Corollary 1.1, in particular with the formula for the obstruction tensor, for Lorentzian pp-waves we get the complete picture in Theorem 1.5:

Corollary 5.3. *Let*

$$\mathbf{g} = 2\mathrm{d}u\mathrm{d}v + H\,\mathrm{d}u^2 + \sum_{i=1}^{n-2} (\mathrm{d}y^i)^2 \tag{5.12}$$

be a Lorentzian pp-wave metric with $H = H(y^1, \dots, y^{n-2}, u)$ a function not depending on v. Let Δ be the flat Laplacian in $n-2$ dimensions.

(1) *If n is odd, the unique Ricci-flat ambient metric that is analytic in ρ is*

$$\widetilde{\mathbf{g}} = 2\mathrm{d}(\rho t)\mathrm{d}t + t^2 \mathbf{g} + +t^2 \left(\sum_{k=1}^{\infty} \frac{\Delta^k(H)}{k! \prod_{i=1}^{k}(2i-n)} \rho^k \right) \mathrm{d}u^2.$$

Moreover, all non-analytic solutions are parametrized by arbitrary functions $\alpha = \alpha(y^1, \dots, y^{n-2}, u)$ and given by formula (5.9) in Theorem 5.2, in which $\Delta_{\mathbf{G}}$ is replaced by the flat Laplacian.

(2) *If $n = 2s$ is even the obstruction tensor for $[\mathbf{g}]$ is a constant multiple of $\Delta^{n/2}(H)\mathrm{d}u^2$. If it vanishes, all Ricci-flat ambient metrics that are analytic in ρ are given by*

$$\widetilde{\mathbf{g}} = 2\mathrm{d}(\rho t)\mathrm{d}t + t^2\mathbf{g} + t^2 \left(\sum_{k=1}^{s-1} \frac{\Delta^k(H)}{k! \prod_{i=1}^{k}(2i-n)} \rho^k + \sum_{k=0}^{\infty} \frac{\Delta^k(\alpha)}{k! \prod_{i=1}^{k}(2i+n)} \rho^{\frac{n}{2}+k} \right) \mathrm{d}u^2,$$

where $\alpha = \alpha(y^1, \dots, y^{n-2}, u)$ is an arbitrary smooth function. Independently of the vanishing of the obstruction tensor, non-analytic ambient metrics can be obtained from formula (5.11) in Theorem 5.3.

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