Radiative B meson decays at the NNLO in QCD for the physical charm quark mass

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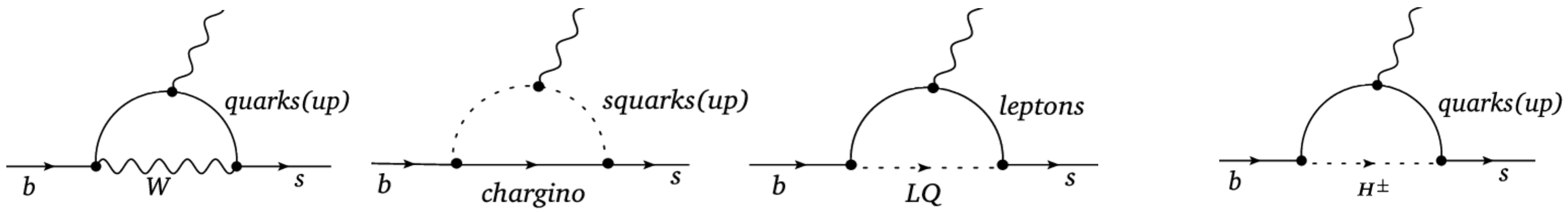
- **Introduction**
- **Charm-dependent $\mathcal{O}(\alpha_s^2)$ corrections at physical m_c**
- **Current SM predictions for $\mathcal{B}_{s\gamma}$**
- **Summary**

Motivation

FCNC process $\Rightarrow$ bounds on beyond-the-SM physics



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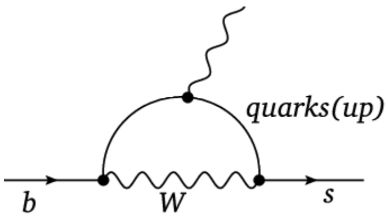

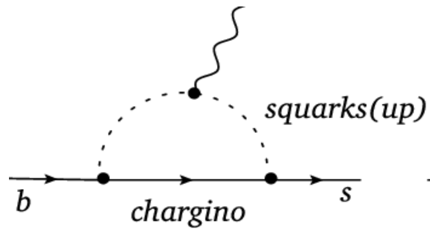

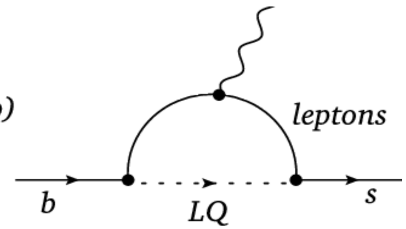

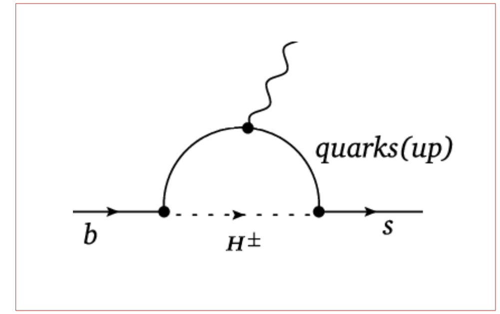

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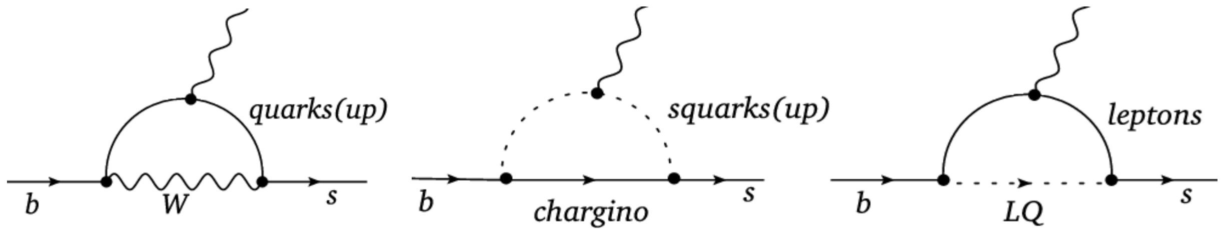
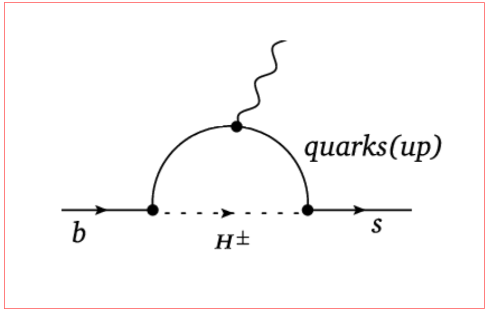




Measurements of $\mathcal{B}(\bar{B} \to X_s\gamma)$

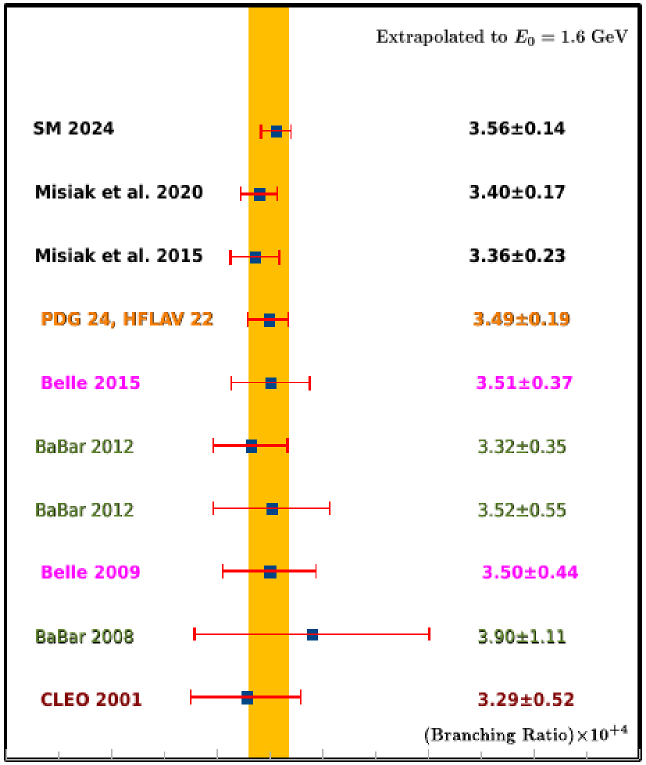
CLEO, BaBar and Belle measurements combined by PDG [2024] and HFLAV [arXiv:2206.07501].

$$\mathcal{B}^{\rm exp}_{s\gamma} = (3.49 \pm 0.19) \cdot 10^{-4} \qquad \text{(for } E_\gamma \gtrsim 1.6\,\text{GeV)}$$

(5.4%)

Background grows for smaller E_γ! Average CLEO, BELLE and BABAR with $E_0 \in [1.7, 2.0]$ GeV. Then, extrapolate to $E_0 = 1.6\,\text{GeV} \simeq m_b/3$.

THEORY: E_0 should be large ($\sim m_b/2$) but not too close to the endpoint ($m_b - 2E_0 \gg \Lambda_{\rm QCD}$), $E_0 = 1.6$ GeV is now conventional.



Belle-II: Better accuracy is expected, $\sim 2.6\%$ [arXiv:1808.10567]

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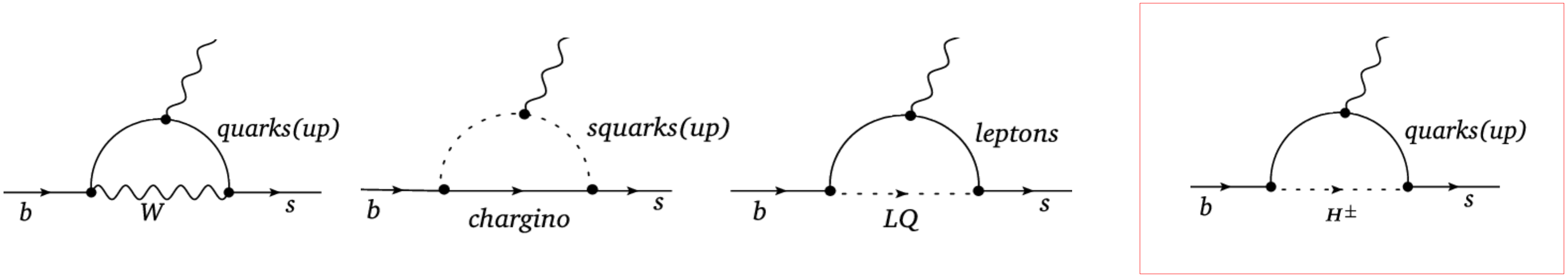

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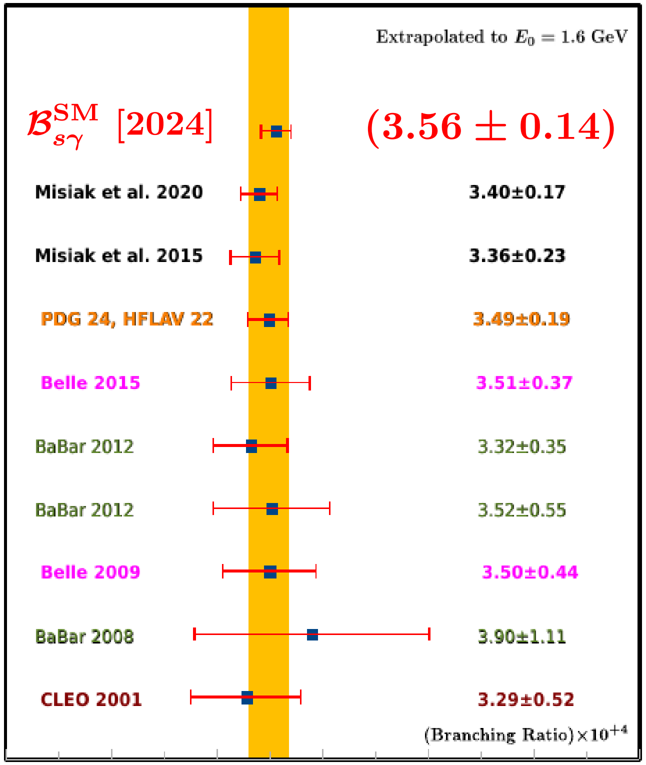

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SM Prediction 2024: [Preliminary]

$$\mathcal{B}_{s\gamma}^{\text{SM}} = (3.56 \pm 0.14) \cdot 10^{-4} \quad \longrightarrow \quad (4.0\%)$$

Theoretical framework

Radiative B meson decay is a flavor-changing process that takes place well below the electroweak scale and is conveniently described by the Low-energy Effective Field Theory that is derived from SM by decoupling the W-boson and other heavier SM particles (Z, t, H^0).

$$\Rightarrow \quad \textbf{effective Lagrangian:} \quad \rightarrow \quad \mathcal{L}_{\rm weak} = \frac{4G_F}{\sqrt{2}} V_{tb} V_{ts}^* \sum_{i=1}^{8} C_i(\mu_b) \mathcal{Q}_i$$

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$$\mathcal{Q}_1 = (\bar{s}_L \gamma_\mu T^a c_L)(\bar{c}_L \gamma^\mu T^a b_L) \qquad \mathcal{Q}_3 = (\bar{s}_L \gamma_\mu b_L) \sum\nolimits_q (\bar{q}\gamma^\mu q)$$

$$\mathcal{Q}_2 = (\bar{s}_L \gamma_\mu c_L)(\bar{c}_L \gamma^\mu b_L) \qquad \mathcal{Q}_4 = (\bar{s}_L \gamma_\mu T^a b_L) \sum\nolimits_q (\bar{q}\gamma^\mu T^a q)$$

$$\mathcal{Q}_7 = \frac{e}{16\pi^2} m_b \, (\bar{s}_L \sigma_{\mu\nu} b_R) \, F^{\mu\nu} \qquad \mathcal{Q}_5 = (\bar{s}_L \gamma_{\mu 1} \gamma_{\mu 2} \gamma_{\mu 3} b_L) \sum\nolimits_q (\bar{q}\gamma^{\mu 1} \gamma^{\mu 2} \gamma^{\mu 3} q)$$

$$\mathcal{Q}_8 = \frac{g}{16\pi^2} m_b \, (\bar{s}_L \sigma_{\mu\nu} T^a b_R) \, G^{a\mu\nu} \qquad \mathcal{Q}_6 = (\bar{s}_L \gamma_{\mu 1} \gamma_{\mu 2} \gamma_{\mu 3} T^a b_L) \sum\nolimits_q (\bar{q}\gamma^{\mu 1} \gamma^{\mu 2} \gamma^{\mu 3} T^a q)$$

Eight operators. Higher-order EW and/or CKM-suppressed effects ($|V_{ub}V_{us}^*/V_{tb}V_{ts}^*| < 0.02$) bring other operators.
$|C_7| : |C_{1,2}| : |C_8| \simeq 1 : 3 : 1/2$

current-current	photonic dipole	gluonic dipole	penguin
$Q_{1,2}$	Q_7	Q_8	$Q_{3,4,5,6}$

The most stringent constraints on C_7 Wilson coefficient comes from $\mathcal{B}_{s\gamma}$.

$\mathcal{B}(\bar{B} \to X_s\gamma)$ in the SM

The decay rate can be approximately evaluated in perturbation theory:

$$\Gamma(\bar{B} \to X_s\gamma)_{E_\gamma > E_0} = \boxed{\Gamma(b \to X_s^p\gamma) + (\text{non-perturbative effects})}_{E_\gamma > E_0}$$

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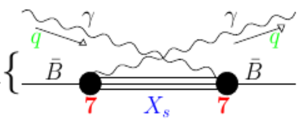
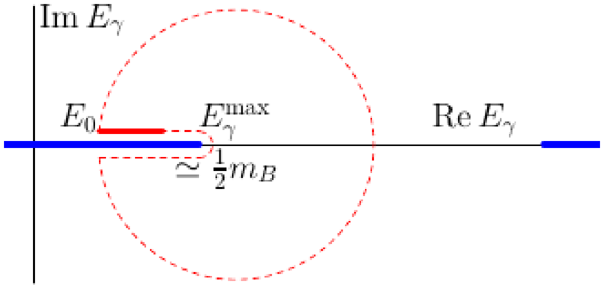
$$\Gamma(\bar{B} \to X_s\gamma)_{E_\gamma > E_0} = |C_7(\mu_b)|^2\, \Gamma_{77}(E_0) + (\text{others})$$

Optical theorem:

$$\frac{d\Gamma_{77}}{dE_\gamma} \sim \text{Im}\{\ \ldots\ \} \equiv \text{Im}\, A$$



[Integrate the amplitude A over E_γ]



OPE: non-perturbative corrections to $\Gamma_{77}(E_0)$ form a series in $\Lambda_{\rm QCD}/m_b$ and α_s that begins with:

$$\frac{\mu_\pi^2}{m_b^2},\ \frac{\mu_G^2}{m_b^2},\ \frac{\rho_D^3}{m_b^3},\ \frac{\rho_{LS}^3}{m_b^3}, \ldots;\ \frac{\alpha_s \mu_\pi^2}{(m_b - 2E_0)^2},\ \frac{\alpha_s \mu_G^2}{m_b(m_b - 2E_0)}; \ldots,$$

where μ_π, μ_G, ρ_D, $\rho_{LS} = \mathcal{O}(\Lambda)$, extracted from $\bar{B} \to X_c e \bar{\nu}$ decay spectra and the B-B^* mass difference.

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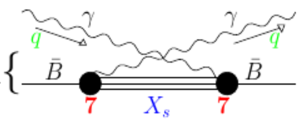

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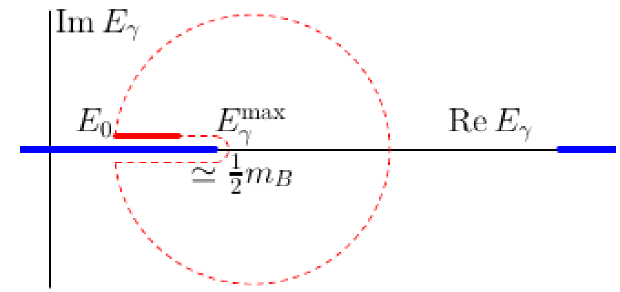

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Branching ratio formula for phenomenology:

$$\mathcal{B}(\bar{B} \to X_s\gamma)_{E_\gamma > E_0} = \mathcal{B}(\bar{B} \to X_c e\bar{\nu})_{\text{exp}} \left|\frac{V_{tb}V_{ts}^*}{V_{cb}}\right|^2 \frac{6\alpha_{\text{em}}}{\pi\, C}\left[\underbrace{P(E_0)}_{\sim 96\%} + \underbrace{N(E_0)}_{\sim 4\%}\right]$$

perturbative (P), non-perturbative (N)

$$\frac{\Gamma(b \to X_s^p\gamma)_{E_\gamma > E_0}}{|V_{cb}/V_{ub}|^2\,\Gamma(b \to X_u^p e\bar{\nu})} = \left|\frac{V_{tb}V_{ts}^*}{V_{cb}}\right|^2 \frac{6\alpha_{\text{em}}}{\pi} P(E_0)$$

semileptonic phase space factor

$$C = \left|\frac{V_{ub}}{V_{cb}}\right|^2 \Gamma(\bar{B} \to X_c e\bar{\nu})/\Gamma(\bar{B} \to X_u e\bar{\nu})$$

$$\Gamma(b \to X_s^p \gamma)_{E_\gamma > E_0} \simeq N \sum_{i,j=1}^{8} C_i(\mu_b) C_j(\mu_b)\, \hat{G}_{ij}(E_0, \mu_b)$$

$N = \frac{G_F^2 m_b^5}{32\pi^3}\left(\frac{\alpha_{em}}{\pi}\right)|V_{tb}V_{ts}^*|^2$
$\mu_b \sim m_b/2$
$\delta = 1 - 2E_0/m_b$
$\hat{G}_{ij} = \hat{G}_{ji}$
$z = m_c^2/m_b^2$

$G_{27}^{(2)}(z)$ is an interference of $\mathcal{Q}_2$ and $\mathcal{Q}_7$ operators.

$X_s^p = s,\ sg,\ sgg,\ sq\bar{q}\ :\rightarrow$ 2-, 3-, and 4-body contributions.

And,

$$\hat{G}_{ij} = \hat{G}_{ij}^{(0)} + \left(\frac{\alpha_s}{4\pi}\right)^1 \hat{G}_{ij}^{(1)} + \left(\frac{\alpha_s}{4\pi}\right)^2 \hat{G}_{ij}^{(2)} + \cdots$$

$$C_i = C_i^{(0)} + \left(\frac{\alpha_s}{4\pi}\right)^1 C_i^{(1)} + \left(\frac{\alpha_s}{4\pi}\right)^2 C_i^{(2)} + \cdots$$

The most important contributions: $\hat{G}_{77}^{(2)}$ (known); $\hat{G}_{17}^{(2)}$ and $\hat{G}_{27}^{(2)} \rightarrow$ interpolation (2015):

$F_1(z,\delta), F_2(z,\delta)$ interpolated in $\hat{G}_{17}^{(2)}(z)$ and $\hat{G}_{27}^{(2)}(z)$ } between $m_c \gg m_b$ and $m_c = 0$ limits, [arXiv: 1503.07916].

$\Longrightarrow$ $\pm 3\%$ uncertainty in $\mathcal{B}_{s\gamma}^{\rm SM}$.

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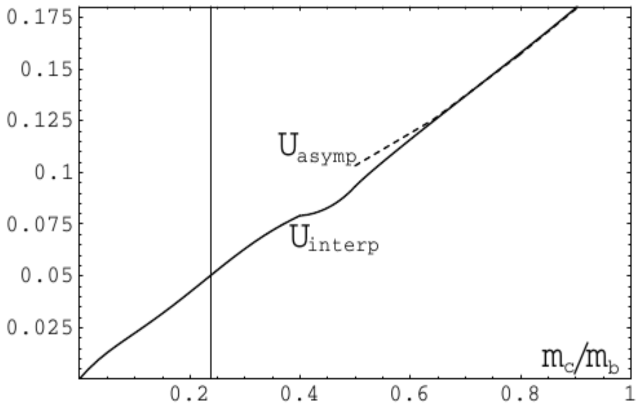
Let $\Delta\mathcal{B}_{s\gamma}$ denote the contribution from $F_{1,2}(z,\delta)$ to $\mathcal{B}_{s\gamma}$. The relative effect is given by:

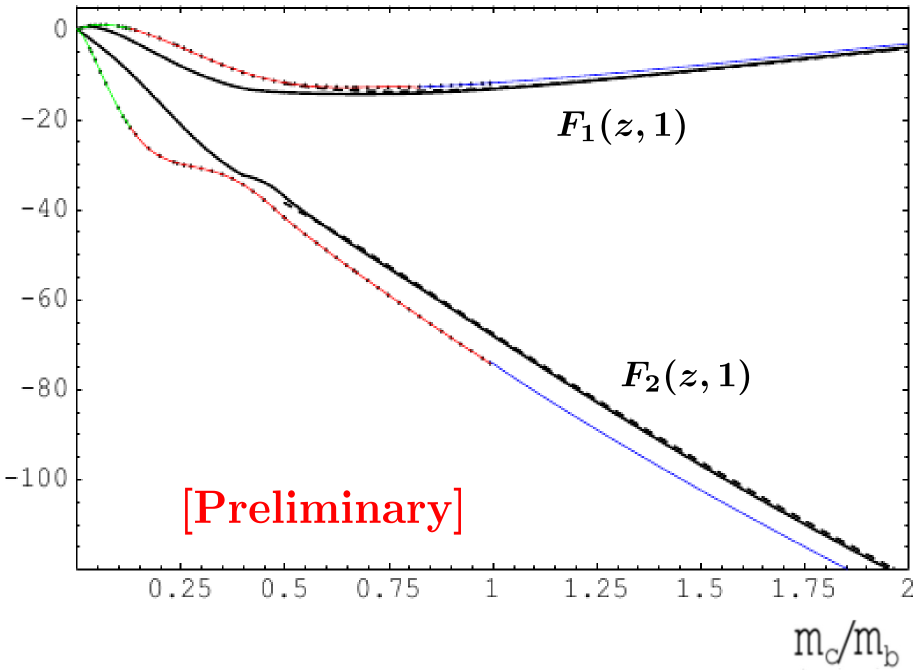


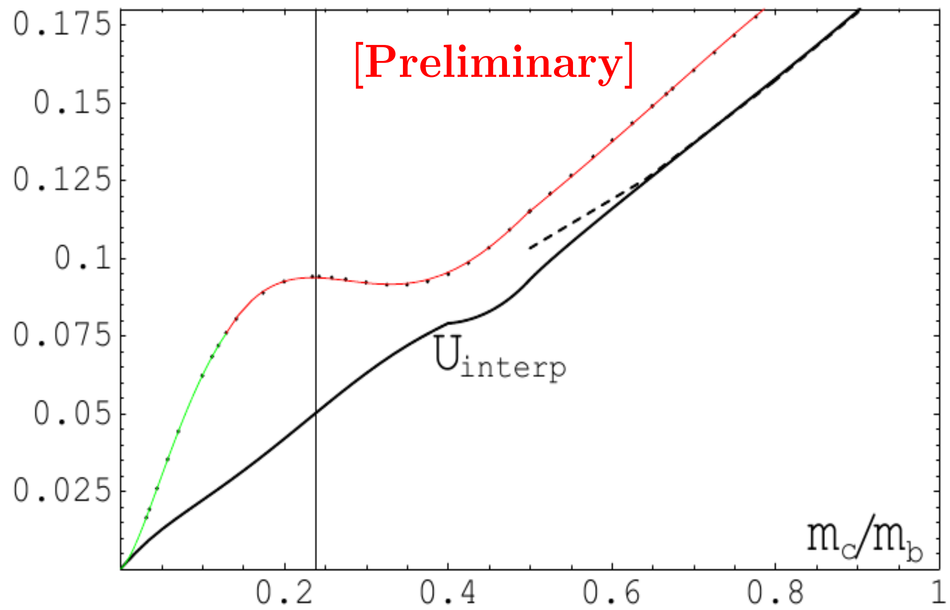
$$\frac{\Delta\mathcal{B}_{s\gamma}}{\mathcal{B}_{s\gamma}} \simeq U(z,\delta) \equiv \frac{\alpha^2(\mu_b)}{8\pi^2} \frac{C_1^{(0)}(\mu_b)F_1(z,\delta) + \left(C_2^{(0)}(\mu_b) - \frac{1}{6}C_1^{(0)}(\mu_b)\right)F_2(z,\delta)}{C_7^{(0)\rm eff}(\mu_b)}$$

$$C_7^{\rm eff} = C_7 + \sum_{i=1}^{6} y_i C_i \quad \text{with} \quad \vec{y} = \left\{0, 0, -\frac{1}{3}, -\frac{4}{9}, -\frac{20}{3}, -\frac{80}{9}\right\} \text{in } \overline{\rm MS} \text{ scheme.}$$

$\hat{G}_{17}^{(2)}$ and $\hat{G}_{27}^{(2)}$ are fully known for arbitrary m_c in 2024!

2024: $\hat{G}_{27}$ and $\hat{G}_{17}$ known at NNLO for arbitrary m_c





- The black dots are from numerical solution to differential equations.
- The green curves are the small-z expansions.
- The blue curves are the large-z expansions.
- The red curves are from fits to sample points.
- The black solid curves are from interpolation.
- The black dashed curves are the leading terms of the large-z expansions; $c_1 + c_2\,\mathrm{Log}[z] + c_3\,\mathrm{Log}[z]^2$.
- The solid vertical line corresponds to physical value of $z = 0.056$.
- Charm pair production threshold is at $\mathrm{m}_c/\mathrm{m}_b = 1/2$.

$\sqrt{z} = \mathrm{m}_c/\mathrm{m}_b$

Renormalization of $\hat{G}_{27}$ at NNLO for arbitrary m_c

$$
\begin{aligned}
\widetilde{\alpha}_s\, \hat{G}^{(1)}_{i7} + \widetilde{\alpha}_s^2\, \hat{G}^{(2)}_{i7} \;=\;& Z_b^{\rm OS}\, Z_m^{\rm OS}\, Z_{77} \Big\{ \widetilde{\alpha}_s^2\, s^{3\epsilon}\, \hat{G}^{(2)\rm bare}_{i7} + (Z_m^{\rm OS} - 1)\, s^{\epsilon} \left[Z_{i4}\, \hat{G}^{(0)m}_{47} + \widetilde{\alpha}_s\, s^{\epsilon}\, \hat{G}^{(1)m}_{i7} \right] \\
&+ \widetilde{\alpha}_s\, (Z_G^{\rm OS} - 1)\, s^{2\epsilon}\, \hat{G}^{(1)3P}_{i7} + Z_{i7}\, Z_m^{\rm OS} \left[\hat{G}^{(0)}_{77} + \widetilde{\alpha}_s\, s^{\epsilon}\, \hat{G}^{(1)\rm bare}_{77} \right] \\
&+ \widetilde{\alpha}_s\, Z_{i8}\, s^{\epsilon}\, \hat{G}^{(1)\rm bare}_{78} + \sum_{j=1,\ldots,6,11,12} Z_{ij}\, s^{\epsilon} \left[\hat{G}^{(0)}_{j7} + \widetilde{\alpha}_s\, s^{\epsilon}\, Z_g^2\, \hat{G}^{(1)\rm bare}_{j7} \right] \\
&+ 2\widetilde{\alpha}_s s^{2\epsilon} (Z_m - 1)\, z \frac{d}{dz} \hat{G}^{(1)\rm bare}_{i7} \Big\} + \mathcal{O}(\widetilde{\alpha}_s^3)
\end{aligned}
$$

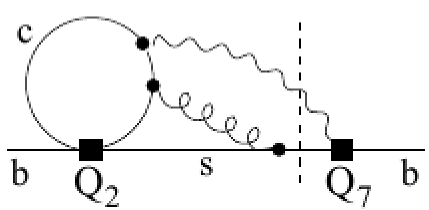
$\hat{G}^{(1)}_{27} = \hat{G}^{(1)2P}_{27} + \hat{G}^{(1)3P}_{27}$

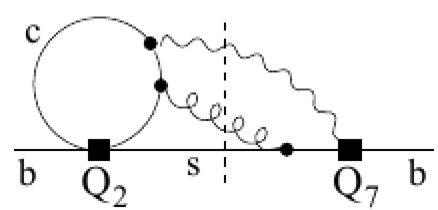
$\widetilde{\alpha}_s = \alpha_s/4\pi = g_s^2/16\pi^2$

$s = \frac{4\pi\mu_b^2}{m_b^2} e^{\gamma}$

$\hat{G}^{(0)}_{j7}$ vanish for $j = 1, 2, 11, 12$

Sample NNLO counterterms diagrams: M. Misiak, AR, M. Steinhauser, Phys. Lett. B770 (2017) 431 [arXiv:1702.07674].





$$\hat{G}^{(1)3P}_{27}(z) = g_0(z) + \epsilon\, g_1(z) + \mathcal{O}(\epsilon^2)$$

$$\hat{G}^{(1)3P}_{7(12)}(z) = 0 - \epsilon\,(20\, g_0(z)) + \mathcal{O}(\epsilon^2)$$

$$\hat{G}^{(1)m,3P}_{27}(z) = j_0(z) + \epsilon\, j_1(z) + \mathcal{O}(\epsilon^2)$$

$$\hat{G}^{(1)2P}_{27}(z) = -\frac{92}{81\epsilon} + f_0(z) + \epsilon\, f_1(z) + \mathcal{O}(\epsilon^2)$$

$$\hat{G}^{(1)2P}_{7(12)}(z) = \frac{2096}{81} + \epsilon\, e_1(z) + \mathcal{O}(\epsilon^2)$$

$$\hat{G}^{(1)m,2P}_{27}(z) = -\frac{1}{3\epsilon^2} + \frac{1}{\epsilon} r_{-1}(z) + r_0(z) + \epsilon\, r_1(z) + \mathcal{O}(\epsilon^2)$$

- Fully analytical solutions for 2-body in arXiv:2309.14706.
- Remaining 3-body functions recently calculated in fully analytical form but not yet published.

Sample analytical solutions from arXiv:2309.14706:

$f_0(z) =$

```
(4 * (((-2 * Pi^2 * w * (3 - 12 * Sqrt[w] + 30 * w - 36 * w^(3/2) + 52 * w^2 - 36 * w^(5/2) + 42 * w^3 -
12 * w^(7/2) + 9 * w^4)) / (9 * (1 + w)^6) - (13 - 284 * w - 738 * w^2 - 284 * w^3 + 13 * w^4) / (27 * (1
+ w)^4) - (4 * (1 + 4 * w + 18 * w^2 + 4 * w^3 + w^4) * H[-1, w]) / (9 * (1 + w)^4) + (4 * w * (9 + 24 * w
- 2 * w^2 - 4 * w^3 + w^4) * H[0, w]) / (9 * (1 + w)^5) - (2 * w^2 * (-1 + 2 * w + 3 * w^2 + 2 * w^3) *
H[0, w]^2) / (1 + w)^6 - I * Pi * ((-2 * (-1 + w) * (1 - 12 * w - 38 * w^2 - 12 * w^3 + w^4)) / (9 * (1 +
w)^5) - (8 * w^2 * (3 + 4 * w + 3 * w^2) * H[0, w]) / (3 * (1 + w)^6)) - (16 * w * (3 + 9 * w + 14 * w^2 + 9
* w^3 + 3 * w^4) * H[-1, -1, w]) / (3 * (1 + w)^6) + (8 * w * (3 + 9 * w + 14 * w^2 + 9 * w^3 + 3 * w^4) *
H[-1, 0, w]) / (3 * (1 + w)^6) + (8 * w * (1 + Sqrt[w] + w)^2 * (3 - 4 * Sqrt[w] + 8 * w - 4 * w^(3/2) + 3 *
w^2) * H[-1, 0, x]) / (3 * (1 + w)^6) + (8 * w^2 * (3 + 14 * w + 15 * w^2 + 6 * w^3) * H[0, -1, w]) / (3 * (1
+ w)^6) - (8 * w * (1 - Sqrt[w] + w)^2 * (3 + 4 * Sqrt[w] + 8 * w + 4 * w^(3/2) + 3 * w^2) * H[1, 0, x]) /
(3 * (1 + w)^6)) / 3 + (2 * (-(35 + 22 * w + 35 * w^2) / (4 * (1 + w)^2) - (2 * (1 + 4 * w + w^2) * H[-1, w])
/ (1 + w)^2 + (2 * w * (5 + w^2) * H[0, w]) / (1 + w)^3 - (w * (-1 + w + w^2) * H[0, w]^2) / (1 + w)^4 + ···
```

with $w = \frac{1-\sqrt{1-4\,z}}{1+\sqrt{1-4\,z}}$.

Exact counterterms improve the $\frac{1}{\epsilon}$ pole cancellation with better precision near the threshold.

Sample analytical solutions from arXiv:2309.14706:

$f_0(z) =$

```
(4 * (((-2 * Pi^2 * w * (3 - 12 * Sqrt[w] + 30 * w - 36 * w^(3/2) + 52 * w^2 - 36 * w^(5/2) + 42 * w^3 -
12 * w^(7/2) + 9 * w^4)) / (9 * (1 + w)^6) - (13 - 284 * w - 738 * w^2 - 284 * w^3 + 13 * w^4) / (27 * (1
+ w)^4) - (4 * (1 + 4 * w + 18 * w^2 + 4 * w^3 + w^4) * H[-1, w]) / (9 * (1 + w)^4) + (4 * w * (9 + 24 * w
- 2 * w^2 - 4 * w^3 + w^4) * H[0, w]) / (9 * (1 + w)^5) - (2 * w^2 * (-1 + 2 * w + 3 * w^2 + 2 * w^3) *
H[0, w]^2) / (1 + w)^6 - I * Pi * ((-2 * (-1 + w) * (1 - 12 * w - 38 * w^2 - 12 * w^3 + w^4)) / (9 * (1 +
w)^5) - (8 * w^2 * (3 + 4 * w + 3 * w^2) * H[0, w]) / (3 * (1 + w)^6)) - (16 * w * (3 + 9 * w + 14 * w^2 + 9
* w^3 + 3 * w^4) * H[-1, -1, w]) / (3 * (1 + w)^6) + (8 * w * (3 + 9 * w + 14 * w^2 + 9 * w^3 + 3 * w^4) *
H[-1, 0, w]) / (3 * (1 + w)^6) + (8 * w * (1 + Sqrt[w] + w)^2 * (3 - 4 * Sqrt[w] + 8 * w - 4 * w^(3/2) + 3 *
w^2) * H[-1, 0, x]) / (3 * (1 + w)^6) + (8 * w^2 * (3 + 14 * w + 15 * w^2 + 6 * w^3) * H[0, -1, w]) / (3 * (1
+ w)^6) - (8 * w * (1 - Sqrt[w] + w)^2 * (3 + 4 * Sqrt[w] + 8 * w + 4 * w^(3/2) + 3 * w^2) * H[1, 0, x]) /
(3 * (1 + w)^6)) / 3 + (2 * (-(35 + 22 * w + 35 * w^2) / (4 * (1 + w)^2) - (2 * (1 + 4 * w + w^2) * H[-1, w])
/ (1 + w)^2 + (2 * w * (5 + w^2) * H[0, w]) / (1 + w)^3 - (w * (-1 + w + w^2) * H[0, w]^2) / (1 + w)^4 + ···
```

with $w = \frac{1-\sqrt{1-4\,z}}{1+\sqrt{1-4\,z}}$.

Exact counterterms improve the $\frac{1}{\epsilon}$ pole cancellation with better precision near the threshold.

HPL functions: **Harmonic polylogarithms (HPL) are generalizations of the usual polylogarithms $\mathrm{Li}_a(z)$ and Nielsen polylogarithms $\mathrm{S}_{a,b}(z)$.**

- Defined as recursive integration of weight functions: $f(\pm1;t) = \frac{1}{1\mp t}$ and $f(0;t) = \frac{1}{t}$.
- The corresponding weight-one HPL are: $H(\pm1;w) = \int_0^w f(\pm1;t)dt = \mp\,\mathrm{Log}(1\mp w)$; $H(0;w) = \mathrm{Log}(w)$.
- HPLs of higher weigths are then given by: $H(0,0,\cdots,0_n;w) = \frac{1}{n!}\mathrm{Log}^n(w)$ and

$$H(a_1,a_2,...,a_n;w) = \int_0^w f(a_1;t)\,H(a_2,a_3,...,a_n;t)\,dt \quad \text{with } a_i = -1,\ 0,\ +1.$$

- Examples:

  weight 1: $H(0;i) = \frac{i\pi}{2}$ $\quad H(\pm1;i) = \frac{i\pi}{4} \mp \frac{\mathrm{Log}(2)}{2}$; ...

  weight 2: $H(-1,0;w) = \mathrm{Log}(1+w)\,\mathrm{Log}(w) + \mathrm{Li}_2(-w)$; ...

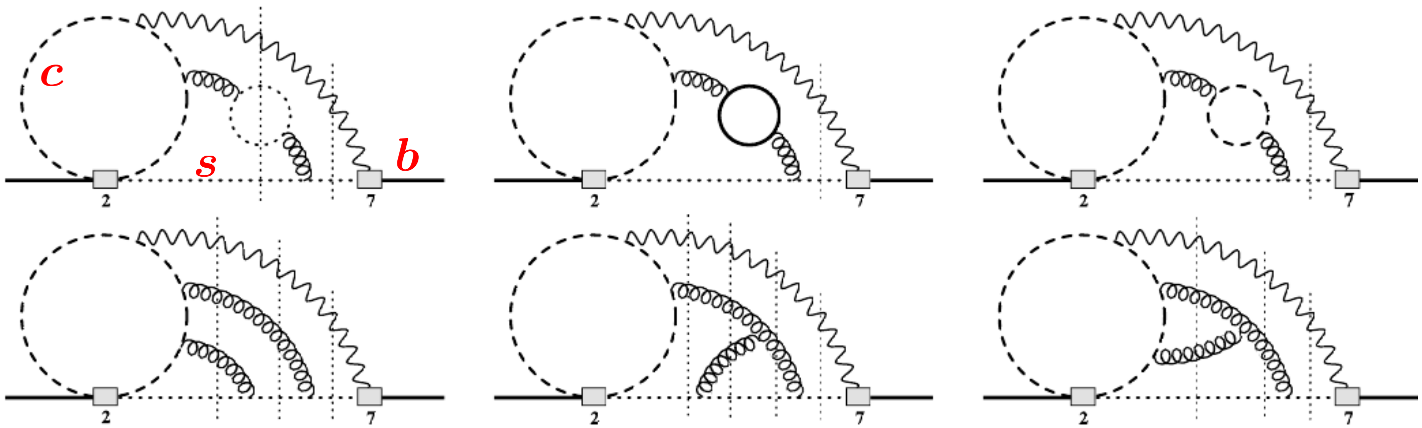
  $\vdots$

- HPLs have logarithmic divergences at 1, -1, 0 and branch cuts on $(-\infty,-1), (\infty,1), (-\infty,0)$, respectively.

HPL Mathematica implementation: D. Maître, [arXiv:hep-ph/0703052]

Evaluation of bare $\hat{G}_{27}$ at NNLO

Sample diagrams:



Dirac algebra $\Longrightarrow$ $\mathcal{O}(10^5)$ **four loop two scale scalar integrals with** $\mathcal{O}(500)$ **families having various unitarity cuts.**

- Scalar Feynman integrals that have the same structure of the integrand with various distributions of powers of propagators:

$$F(a_1, ..., a_2) = \int ... \int \frac{d^D k_1 ... d^D k_n}{D_1^{a_1} ...} \qquad D = 4 - 2\epsilon$$

- Find relations between them, IBP relations [Chetyrkin, Tkachov 81]:

$$\int ... \int d^D k_1 ... \frac{\partial}{\partial k_{\mu,i}} \left(p_{\mu,i} \frac{1}{D_1^{a_1} ...} \right) = 0$$

$$F(a) = \int d^D k \frac{1}{(-k^2 + m^2)^a}$$

$$\int d^D k \frac{\partial}{\partial k_\mu} \left[k_\mu \frac{1}{(-k^2 + m^2)^a} \right] = 0$$

$$(D - 2a)F(a) + 2am^2 F(a+1) = 0$$

$$F(a) = \frac{-(D - 2a + 2)F(a-1)}{(2a-2)m^2}$$

$$F(a) = \frac{-(D - 2a + 2)1^- F(a)}{(2a-2)m^2}$$

$$F(a) = 0; \quad a \leq 0$$

$F(a); a > 1$ express by $F(1)$

$F(1)$ is master integral

$$F(1) = -\Gamma(1 - \frac{D}{2})(m^2)^{D/2-1}$$

Reduction to master integrals $\mathcal{M}_n$ with the help of Integration By Parts (IBP) [KIRA, FIRE, LiteRed]. $\Longrightarrow$ $\mathcal{O}(1\,\text{TB})$RAM and weeks of CPU!

Extending the set of master integrals so that it closes under differentiation to construct a system of differential equations for masters w.r.t. z,

$$\frac{\mathrm{d}}{\mathrm{d}z}\mathcal{M}_n(z,\epsilon) = \sum_m R_{nm}(z,\epsilon)\mathcal{M}_m(z,\epsilon)$$

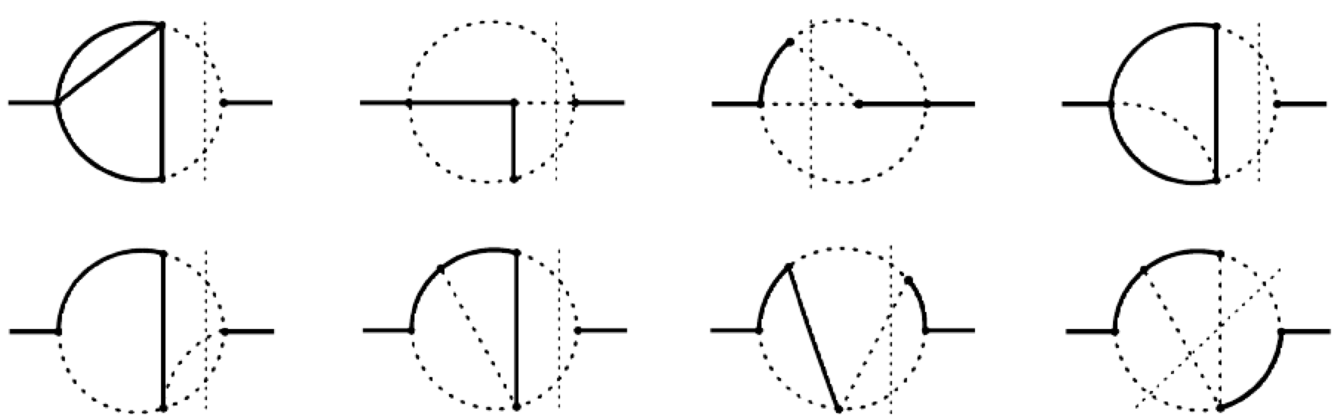
R_{nm}: are rational functions in z and ϵ.

Boundary conditions for masters at $m_c \gg m_b$ using automatized procedure. Involve the calculation of three-loops single-scale master integrals for their completion. Higher order terms in $u = 1/z$ using, power-log ansatz,

$$\mathcal{M}_i(u,\epsilon) = \sum_{n,m,k} c_{inmk}\epsilon^n u^m \mathrm{Log}^k(u)$$

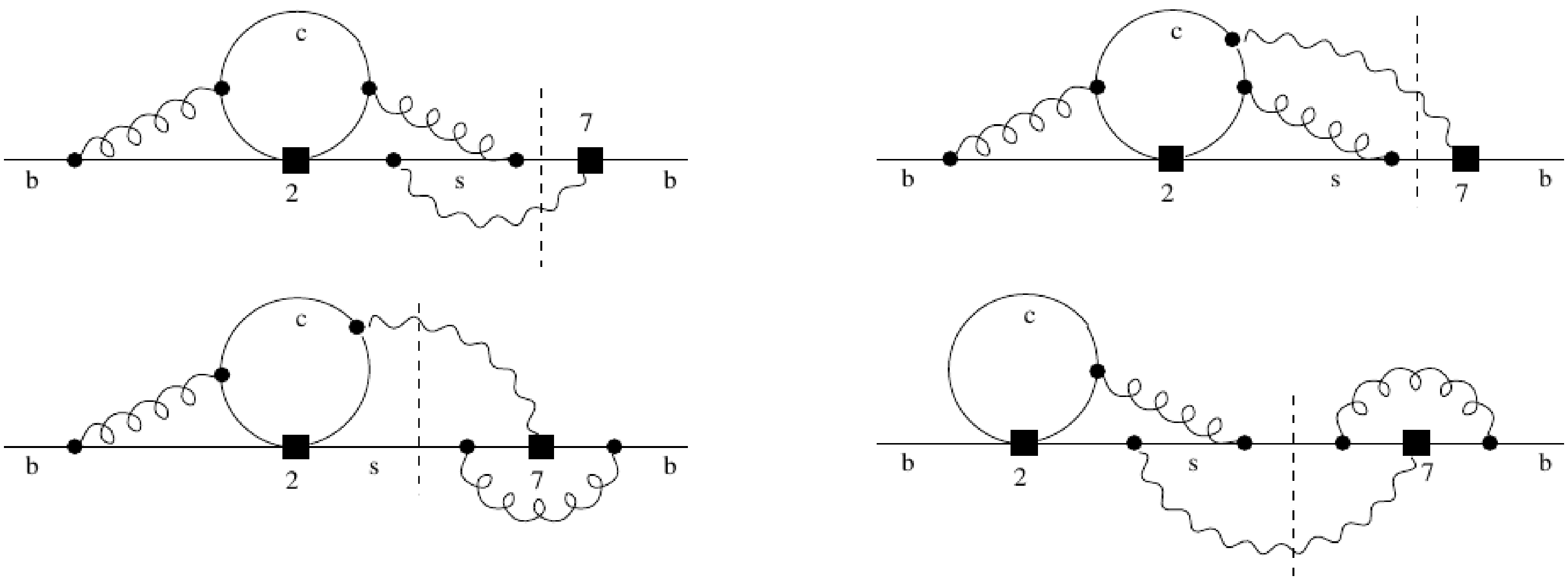
Solving the system of DEs numerically along an ellipse in the complex z-plane for the physical value of z.

Sample three-loop propagator type integrals appear in the large-z expansions of master integrals $\mathcal{M}_n$.



Bare 2-body contributions

arXiv:2309.14707: M. Czaja, M. Czakon, T. Huber, M. Misiak, M. Niggetiedt, AR, K. Schoenwald, M. Steinhauser
(Eur. Phys. J. C 83 (2023) 12, 1108)



: $\Delta_{30}\hat{G}_{27}^{(2)2P}$

: $\Delta_{21}\hat{G}_{27}^{(2)2P}$

- MIs evaluation at the physical value of m_c using AMFlow [arXiv:2201.11669].
- The 2-body contributions are not IR safe; both UV and IR divergences dimensionally regulated.
- Sample results:

$$\Delta_{21}\hat{G}_{27}^{(2)2P}(z) = \frac{368}{243\epsilon^3} + \frac{736-324f_0(z)}{243\epsilon^2} + \frac{1}{\epsilon}\left(\frac{1472}{243} + \frac{92}{729}\pi^2 - \frac{8f_0(z)+4f_1(z)}{3}\right) + p(z),$$

where $p(z=0.04) \simeq 144.959811.$

The large- z expansions of $p(z)$ reads:

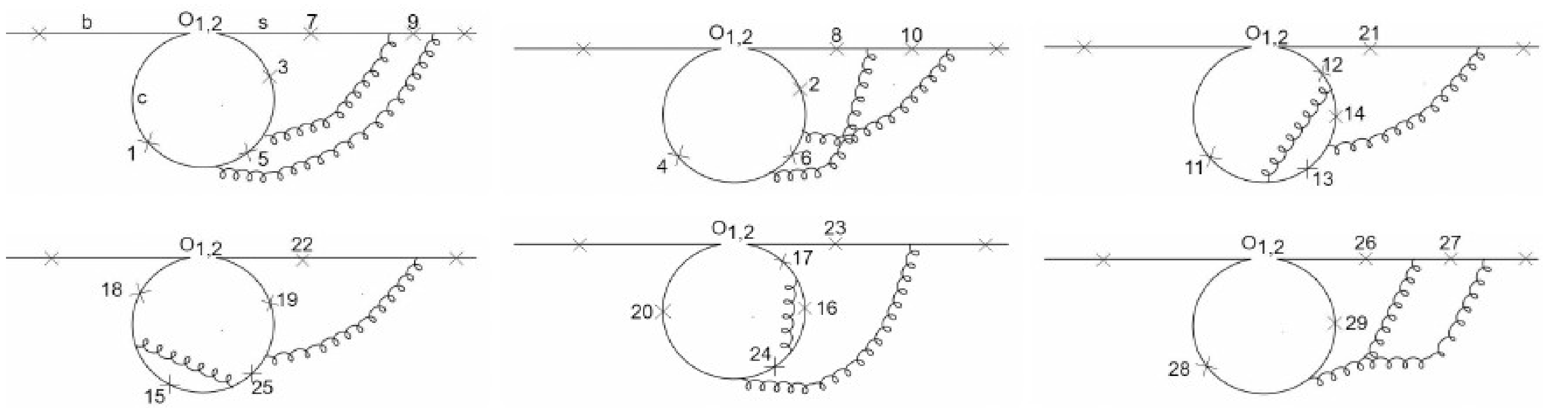
$$p(z) = \frac{138530}{6561} - \frac{3680}{729}\zeta(3) - \frac{6136}{243}L + \frac{5744}{729}L^2 - \frac{1808}{729}L^3 + \frac{1}{z}\left(-\frac{4222952}{1366875} - \frac{602852}{273375}L + \frac{34568}{18225}L^2 - \frac{532}{1215}L^3\right)$$

$$+ \frac{1}{z^2}\left(-\frac{333957725469}{26254935000} - \frac{111861263}{93767625}L + \frac{156358}{178605}L^2 - \frac{172}{1215}L^3\right) + \mathcal{O}\left(\frac{1}{z^3}\right), \quad \text{with } L = \log z.$$

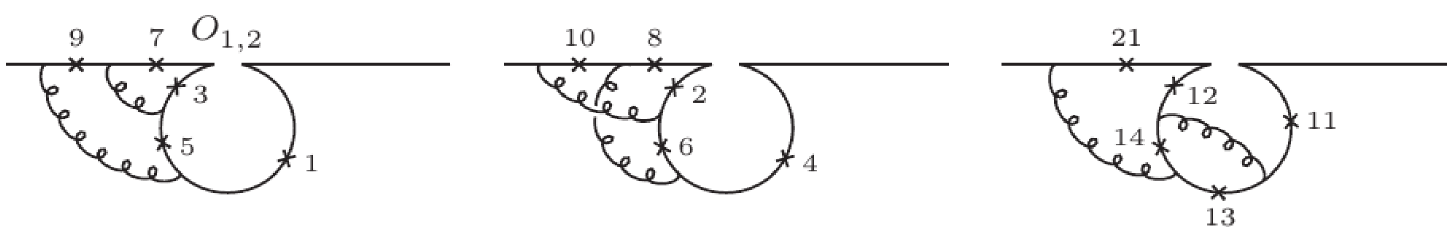
Vertex diagrams approach for bare 2-body contributions: $\Delta_{30}\hat{G}_{27}^{(2)2P}$

arXiv:2303.01714 [C. Greub, H.M. Asatrian, F. Saturnino, C. Wiegand]
arXiv:2309.14706 [M. Fael, F. Lange, K. Schoenwald, M. Steinhauser] $\rightarrow$ Fully completed $\Delta_{30}\hat{G}_{27}^{(2)2P}$
arXiv:2407.17270 [C. Greub, H.M. Asatrian, H.H. Asatryan, L. Born, J. Eicher]

In arXiv:2303.01714, only diagrams with no gluon-(b-quark) couplings.



In arXiv: 2407.17270, remaining diagrams with gluon-(b-quark) couplings.



- **Only amplitudes; no interferences.**
- **IBP as usual. Then either AMFlow or differential equations starting from $m_c \gg m_b$.**
- **Simplifying the differential equations and solving them analytically in many cases.**

The 3- and 4-body contributions Completed:

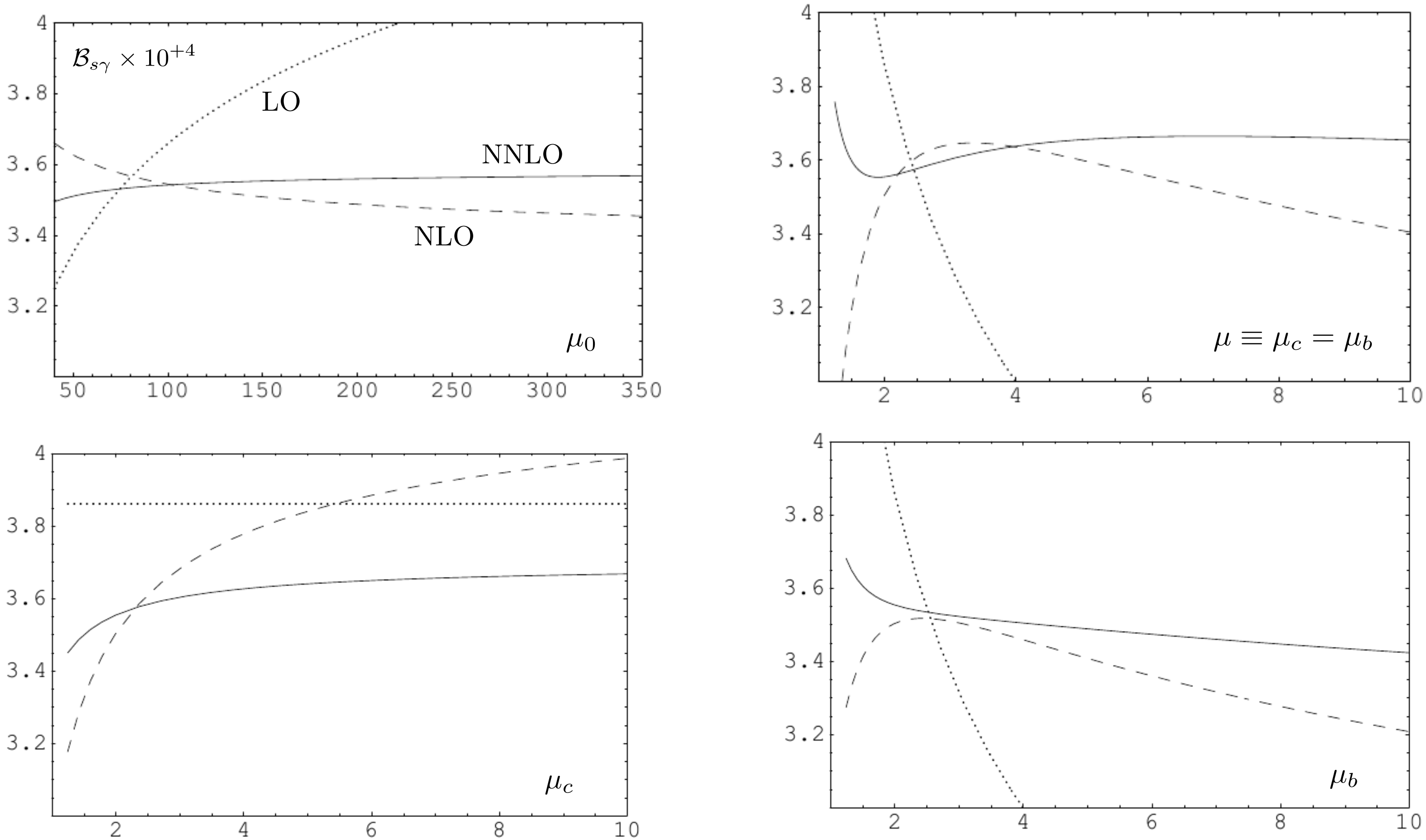
M. Czaja, M. Czakon, T. Huber, M. Misiak, M. Niggetiedt, AR, K. Schoenwald, M. Steinhauser [arXiv:24xx.xxxxx]

- **Cut-propagator approach is more convenient for higher-multiplicity final states.**

Uncertainties due to higher order $\mathcal{O}(\alpha_s^3)$ corrections

Renormalization scale dependence of $\mathcal{B}_{s\gamma}$:

- **The default values of μ_0, μ_b and μ_c are 160, 2, and 2, respectively.**



Estimate through perturbative series:

$$\pm 3\% \Leftrightarrow \boxed{40} \times \left(\frac{\alpha_s(\mu_b)}{\pi}\right)^3 \times \text{LO}$$

$\left(\frac{\alpha_s(\mu_b)}{\pi}\right)^1 \simeq 0.093$ $\left(\frac{\alpha_s(\mu_b)}{\pi}\right)^2 \simeq 0.0087$

$\left(\frac{\alpha_s(\mu_b)}{\pi}\right)^3 \simeq 0.00081$

Updated SM predictions for $\mathcal{B}_{s\gamma}$

arXiv:24xx.xxxxx
[Preliminary]

Mikolaj Misiak, Abdur Rehman, Matthias Steinhauser
JHEP 114 (2020) 175 [arXiv: 2002.01548]

$$\mathcal{B}_{s\gamma}^{\rm SM} = (3.56 \pm \overset{\pm 4.0\%}{0.14}) \times 10^{-4} \quad \longleftarrow \quad (3.40 \pm \overset{\pm 5.0\%}{0.17}) \times 10^{-4}$$

Uncertainties in $\mathcal{B}_{s\gamma}$ with comparison between 2024 and 2020:

$\mathcal{B}_{s\gamma}$	higher-order	interpolation	parametric	non-pert.	total
2024	$\pm 3\%$	$\pm 0\%$	$\simeq \pm 2.5\%$	incl. in para.	$\simeq \pm 4.0\%$
2020	$\pm 3\%$	$\pm 3\%$	$\simeq \pm 2.5\%$	incl. in para.	$\simeq \pm 5.0\%$

- Recall, expected experimental accuracy, 2.6%. Still room for the SM precision improvement!
- A breakdown of parametric uncertainties in the SM prediction:

$\mathcal{B}(B \to X_c e\bar{\nu})$	1.50%	$m_b^{\rm kin}$	0.25%
$\alpha_s(M_Z)$	0.90%	$m_c(\mu_c)$	0.53%
$m_t^{\rm pole}$	0.19%	m_b/m_q	0.18%
$\lambda = s_{12}$	0.02%	μ_G^2	0.72%
$A = s_{23}/s_{12}^2$	0.01%	μ_π^2	0.02%
$\bar{\rho}$	0.12%	ρ_D^3	0.75%
$\bar{\eta}$	0.01%	ρ_{LS}^3	0.04%

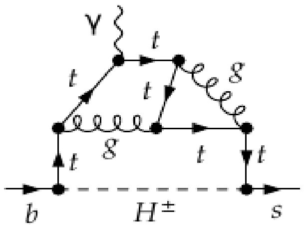
- Uncertainties from non-perturbative effects, $\delta\Gamma_c/\Gamma$, κ_V, and κ_{88}:

2024 and 2020 [Preliminary] $\to 0.74\% + 0.88\% + 0.92\% = 2.54\%$ (in quadrature, 1.48%) $\to \simeq 2.5\%$ (parametric) $\left(\sqrt{1.48^2 + 2^2}\right)\%$

Current bounds on the charged Higgs mass in 2HDM-II

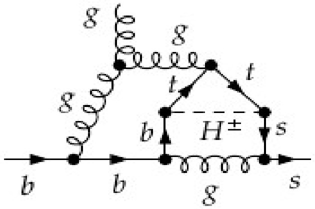
$$\mathcal{L}_{H^+} = (2\sqrt{2}G_F)^{1/2} \sum_{i,j=1}^{3} \overline{u}_i \left(A_u m_{u_i} V_{ij} P_L - A_d\, m_{d_j} V_{ij} P_R\right) d_j H^+ + h.c.$$

- The couplings A_d and A_u reads:



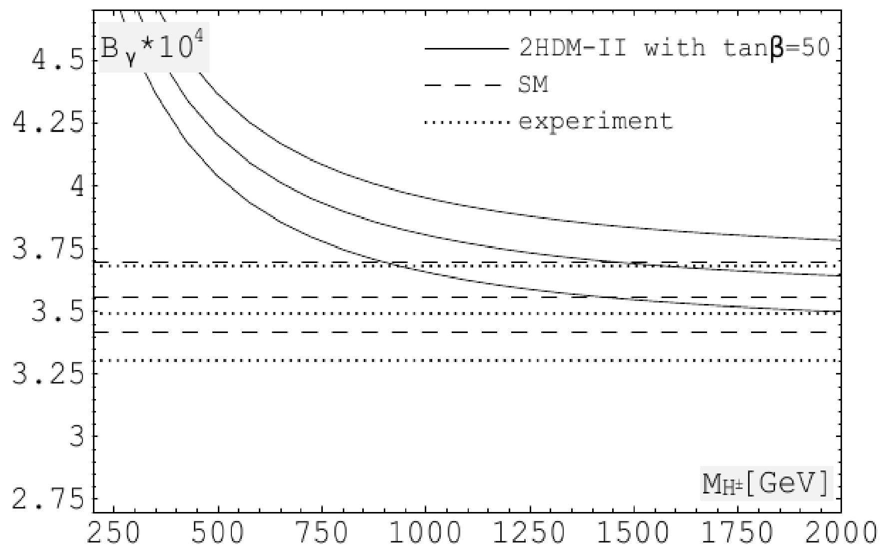
In 2HDM I $\rightarrow A_u = A_d = \frac{1}{\tan\beta}$

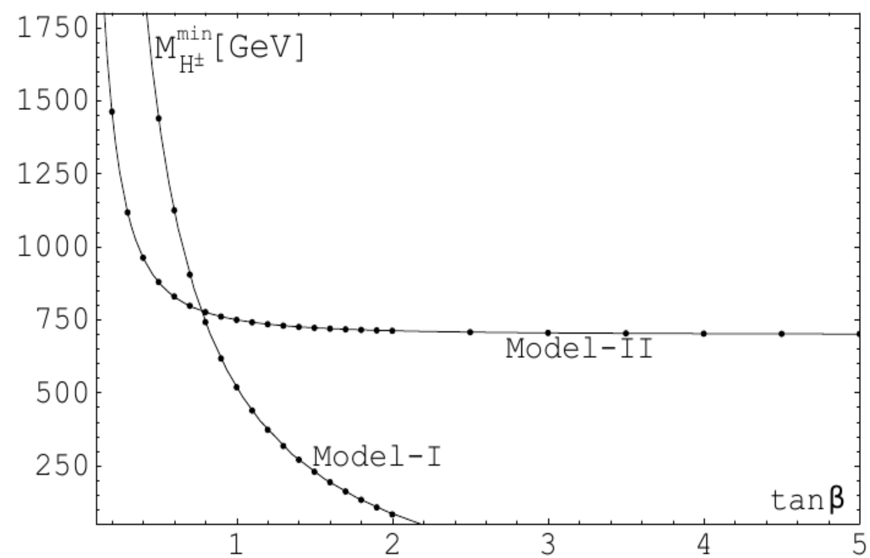
In 2HDM II $\rightarrow A_u = -\frac{1}{A_d} = \frac{1}{\tan\beta}$



- Terms involving $A_d^\star$ are suppressed by the strange-quark mass,

$$C_i^{t,2\mathrm{HDM}} = A_d A_u^\star\, C_{i,A_d A_u^\star}^{t,2\mathrm{HDM}} + A_u A_u^\star\, C_{i,A_u A_u^\star}^{t,2\mathrm{HDM}}$$





[Preliminary] **Current bounds on $M_{H^\pm}$ in 2HDM II is $M_{H^\pm} > 700\,\mathrm{GeV}$ at 95% C.L.**

Summary

- The $\bar{B} \to X_s \gamma$ process constrains extensions of the SM in a strong manner.
- The NNLO calculations of $\Gamma(b \to X_s^p \gamma)$ that matter for m_c-interpolation are finalized.
- The current (preliminary) SM prediction at $E_0 = 1.6\,\text{GeV}$ reads:

$$\mathcal{B}_{s\gamma}^{\text{SM}} = (3.56 \pm 0.14) \times 10^{-4}$$

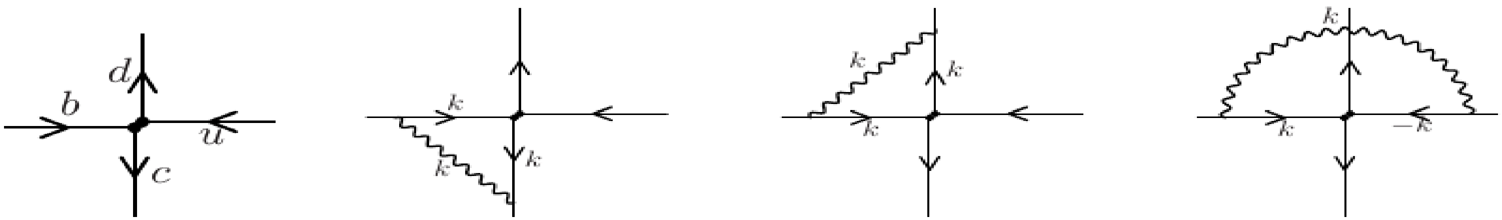
- The current bound on $M_{H^\pm}$ in 2HDM II is $M_{H^\pm} > 700\,\text{GeV}$ at 95% C.L.

BACKUP

Operator renormalization: A simple example

$$\langle \mathcal{Q}_{ib} \rangle = Z_q^2 Z_{ij}(\alpha_s(\mu)) \langle \mathcal{Q}_j(\mu) \rangle$$

$$Z_q = \left[1 - C_F \frac{\alpha_s}{4\pi\epsilon}(1-\xi) + \mathcal{O}(\alpha_s^2)\right]$$ → **quark field renormalization constant in $\overline{\text{MS}}$ scheme**



$\langle \mathcal{Q}_2 \rangle$ $\quad$ $\boxed{C_F \langle \mathcal{Q}_2 \rangle \frac{\alpha_s}{4\pi}\frac{1}{\epsilon}(1-\xi)}$ $\quad$ $T_F \left(\langle \mathcal{Q}_1 \rangle - \frac{\langle \mathcal{Q}_2 \rangle}{N_c} \right) \frac{\alpha_s}{4\pi}\frac{1}{\epsilon}(1-\xi)$ $\quad$ - $T_F \left(\langle \mathcal{Q}_1 \rangle - \frac{\langle \mathcal{Q}_2 \rangle}{N_c} \right) \frac{\alpha_s}{4\pi}\frac{1}{\epsilon}(4-\xi)$

Eliminated by quark field renormalization constant

What about other divergences? **Eliminated by operator renormalization.**

Sum diagrams with symmetric counter-parts:

$$\langle \mathcal{Q}_{1b} \rangle = \langle \mathcal{Q}_1 \rangle - 6T_F \frac{\alpha_s}{4\pi}\frac{1}{\epsilon}\left(\langle \mathcal{Q}_2 \rangle - \frac{\langle \mathcal{Q}_1 \rangle}{N_c} \right)$$
$$\langle \mathcal{Q}_{2b} \rangle = \langle \mathcal{Q}_2 \rangle - 6T_F \frac{\alpha_s}{4\pi}\frac{1}{\epsilon}\left(\langle \mathcal{Q}_1 \rangle - \frac{\langle \mathcal{Q}_2 \rangle}{N_c} \right)$$

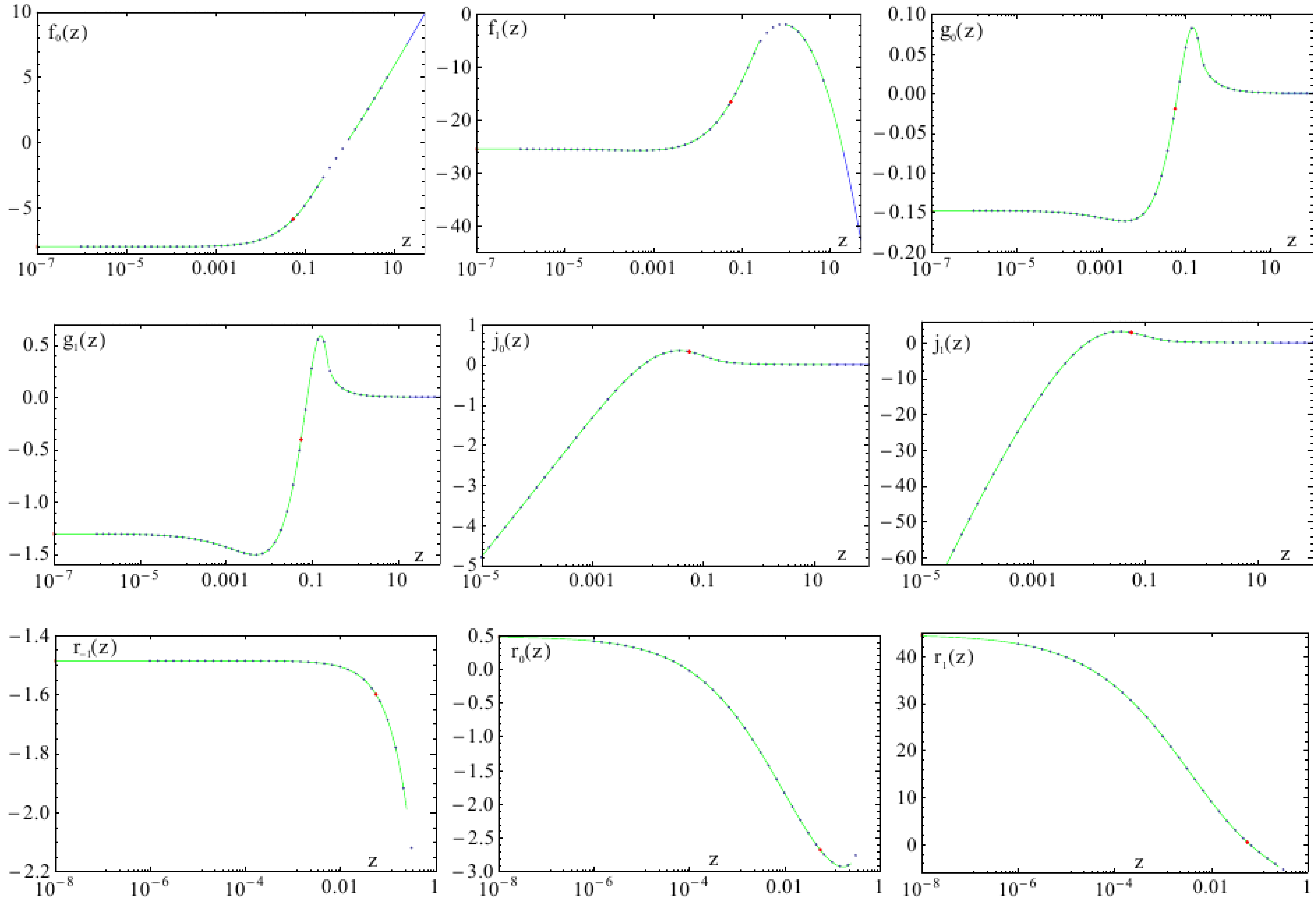
$$\Longrightarrow \quad \hat{Z} = 1 + 6T_F \frac{\alpha_s}{4\pi}\frac{1}{\epsilon} \begin{pmatrix} 1/N_c & -1 \\ -1 & 1/Nc \end{pmatrix}$$

Generally, in $\overline{\text{MS}}$ scheme:

$$\hat{Z}_i = 1 + \sum_{k=1}^{\infty} \frac{1}{\epsilon^k} Z_{i,k}(\alpha_s)$$

No explicit renormalization scale dependence and no mass dependence!

M. Misiak, AR, M. Steinhauser, Phys. Lett. B770 (2017) 431 [arXiv:1702.07674].



- **Fully analytical solutions for 2-body case is known in arXiv:2309.14706.**
- **For 3-body case, the analytical solutions are known but not yet published.**

Short overview on non-perturbative effects

A. Gluon-to-photon conversion in the QCD medium: Q_7-Q_8 interference

No hard photon but rather a hard gluon is emitted in the b-quark decay [Lee, Neubert and Paz in hep-ph/0609224]. Next, this gluon scatters on the remnants of the $\bar{B}$ meson (a diluted target whose size scales like Λ^{-1}); viz, valence quark or sea quark (u, d or s), which results in emission of a hard photon in this Compton-like scattering.

Dominant contribution to Isospin asymmetry, Δ_{0-} : $\rightarrow$ $\dfrac{\Gamma[\bar{B}^0 \to X_s\gamma] - \Gamma[B^- \to X_s\gamma]}{\Gamma[\bar{B}^0 \to X_s\gamma] + \Gamma[B^- \to X_s\gamma]}$
(measurment by Belle, arXiv: 1807.04236)

$$\delta\Gamma_c/\Gamma \simeq -\frac{1}{3}(-0.48 \pm 1.49 \pm 0.97 \pm 1.15)\% \times (1 \pm 0.3) = (0.16 \pm 0.74)\%$$

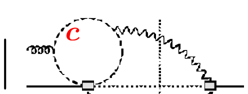
arXiv:0911.1651

$SU(3)_F$ violation; 30%
arXiv:1003.5012

B. Resolved photon contribution to Q_7-$Q_{1,2}$ interference

First pointed out in [M.B. Voloshin, hep-ph/9612483]; and established it amounts to 3.2% in the BR from [G. Buchalla, G. Isidori, G. Rey, hep-ph/9705253] (at LO in powers of $m_b\Lambda/m_c^2$).
M. Benzke, S.J. Lee, M. Neubert, G. Paz, arXiv:1003.5012; [A. Gunawardana, G. Paz, arXiv:1908.02812].

$$\delta N(E_0) = N_{127}\left[-\frac{\mu_G^2}{27m_c^2} + \frac{\Lambda_{17}}{m_b}\right] := N_{127}\left[-\kappa_V \frac{\mu_G^2}{27m_c^2}\right]$$

$\langle \bar{B} |$  $| \bar{B} \rangle$

$$N_{127} = (C_2 - \frac{1}{6}C_1)\, C_7$$

μ_G^2 HQET parameter

$$\Longrightarrow \quad \kappa_V = 1 - \frac{27m_c^2\Lambda_{17}}{m_b\mu_G^2} = 1.2 \pm 0.3$$

κ_V: uncorrelated!
$\Lambda_{17} \to [-24, 5]$ MeV for $m_c = 1.17$ GeV
Factor of 3 improvement wrt BLNP.

C. Contribution proportional to $|C_8(\mu_b)|^2$

A. Kapustin, Z. Ligeti H. D. Politzer [hep-ph/9507248]; A. Ferroglia & U. Haisch [arXiv:1009.2144].

Collinear logs $\ln\frac{m_b}{m_s}$ in the corresponding contributions to $P(E_0)$.

$\rightarrow$ fragmentation functions $\Longrightarrow$ effects below 1% in the BR.

In phenomenological analyses: $[\ln 10, \ln 50] \simeq \left[\ln\frac{m_B}{m_K}, \ln\frac{m_B}{m_\pi}\right]$ in the ballpark of frag. function estimates.
$\rightarrow$ light hadron masses serve as the physical collinear regulators.

M. Benzke, S.J. Lee, M. Neubert & G. Paz [arXiv:1003.5012]: non-perturbative effects that are unrelated to the collinear logs which affect BR by relative corrections in the range [-0.3, 1.9]% wrt the $\frac{m_b}{m_s} = 50$ case in $P(E_0)$, for $\mu_b = 1.5\,\text{GeV}$ and $E_0 = 1.6\,\text{GeV}$.

Numerically, we can reproduce [-0.3,1.9]% range by following replacement,

$$\ln\frac{m_b}{m_s} \rightarrow \kappa_{88} \ln 50 \quad \text{with} \quad \kappa_{88} = 1.7 \pm 1.1$$

in all the perturbative contributions proportional to $|C_8(\mu_b)|^2$.

T. Hurth, R. Szafron [arXiv: 2301.01739]: no numerical estimates available.

Not only $\delta\Gamma_c/\Gamma$, κ_V and κ_{88} but several other inputs parameterize non-perturbative effects namely, collinear requlators as well as the HQET parameters. $\delta\Gamma_c/\Gamma$, κ_V and κ_{88} are treated like all other parameters that $\mathcal{B}_{s\gamma}$ depends on.

Remarks on Non-perturbative corrections - I

Fixed-order corrections to the decay rate due to Q_7 alone give well-defined series in Λ/m_b and α_s.

Identified non-perturbative contributions begin with the following suppression factors.

$$\alpha_s \frac{\Lambda}{m_b}; \qquad \frac{\mu_\pi^2}{m_b^2}, \frac{\mu_G^2}{m_b^2}, \frac{\rho_D^3}{m_b^3}, \frac{\rho_{LS}^3}{m_b^3}, \ldots; \quad \frac{\mu_\pi^2}{(m_b - 2E_0)^2}, \frac{\mu_G^2}{m_b(m_b - 2E_0)}; \ldots,$$

where μ_π, μ_G, ρ_D, $\rho_{LS} = \mathcal{O}(\Lambda)$, extracted from $\bar{B} \to X_c e \bar{\nu}$ decay spectra and the B-B^* mass difference.

$\mathcal{O}(\alpha_s^n \frac{\Lambda}{m_b})_{n=0,1,2,\ldots} = 0,$

$\mathcal{O}(\frac{\Lambda^2}{m_b^2})$ Bigi, Blok, Shifman, Uraltsev, Vainstein, 1992.
Falk, Luke, Savage, 1993.

$\mathcal{O}(\frac{\Lambda^3}{m_b^3})$ Bauer, 1997.

$\mathcal{O}(\alpha_s \frac{\Lambda^2}{m_b^2})$ Ewerth, Gambino, Nandi, 2009.

They are well controlled for $E_0 = 1.6$ GeV.

Remarks on Non-perturbative corrections - II

Dominant non-perturbative uncertainty arises from the $\mathcal{Q}_7$ and $\mathcal{Q}_{1,2}$ interference.

Early analyses,

$$\frac{\Lambda^2}{m_c^2}\sum_{n=0}\left(\frac{\Lambda m_b}{m_c^2}\right)^n w_n$$

Voloshin, 1996; Khodjamirian, Rueckl, Stoll, Wyler, 1997;
Grant, Morgan, Nussinov, Peccei, 1997;
Ligeti, Randall, Wise, 1997; Buchalla, Isidori, Rey, 1997.

Enhance $\mathcal{B}(\bar{B}\to X_s\gamma)$ by around **3%**.

M. Benzke, S.J. Lee, M. Neubert and G. Paz, JHEP 1008 (2010) 099

Treating these effects as $\mathcal{O}(\Lambda/m_b)$ ones and estimated them using models of subleading shape functions, finding effects in the range $[-1.7\%, +4.0\%]$ of the $\mathcal{B}(\bar{B}\to X_s\gamma)$.

Combining them linearly with the other uncertainties, they estimated the overall non-perturbative uncertainty at the $\pm\,\mathbf{5}\%$ level.

This value has been adopted as it stands in the phen. update of 2015.