NEW NUMERICAL METHODS FOR FEYNMAN INTEGRALS AND THEIR APPLICATIONS TO B MESON DECAYS

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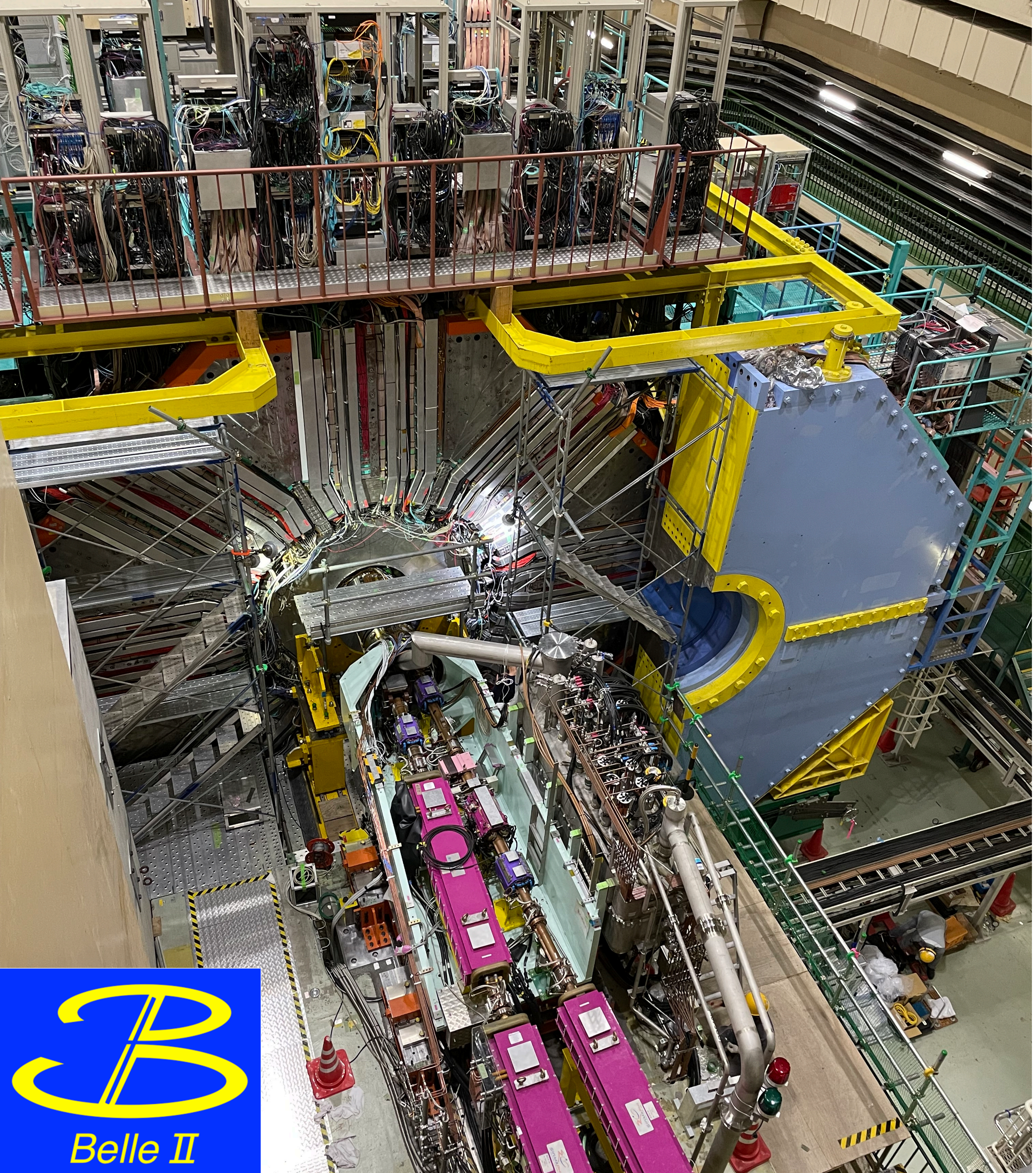
Seminar in Warsaw - March 14th 2024

MARIE CURIE ACTIONS

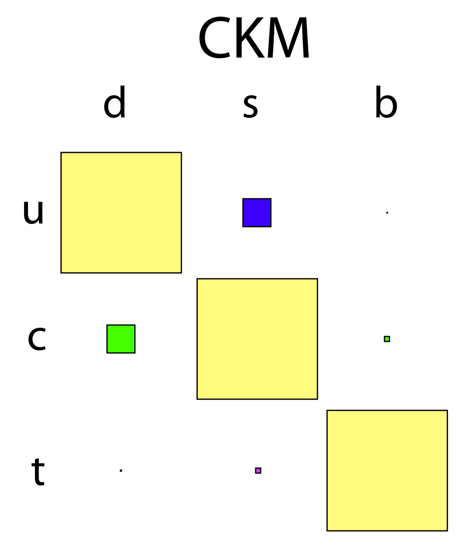
Funded by the European Union



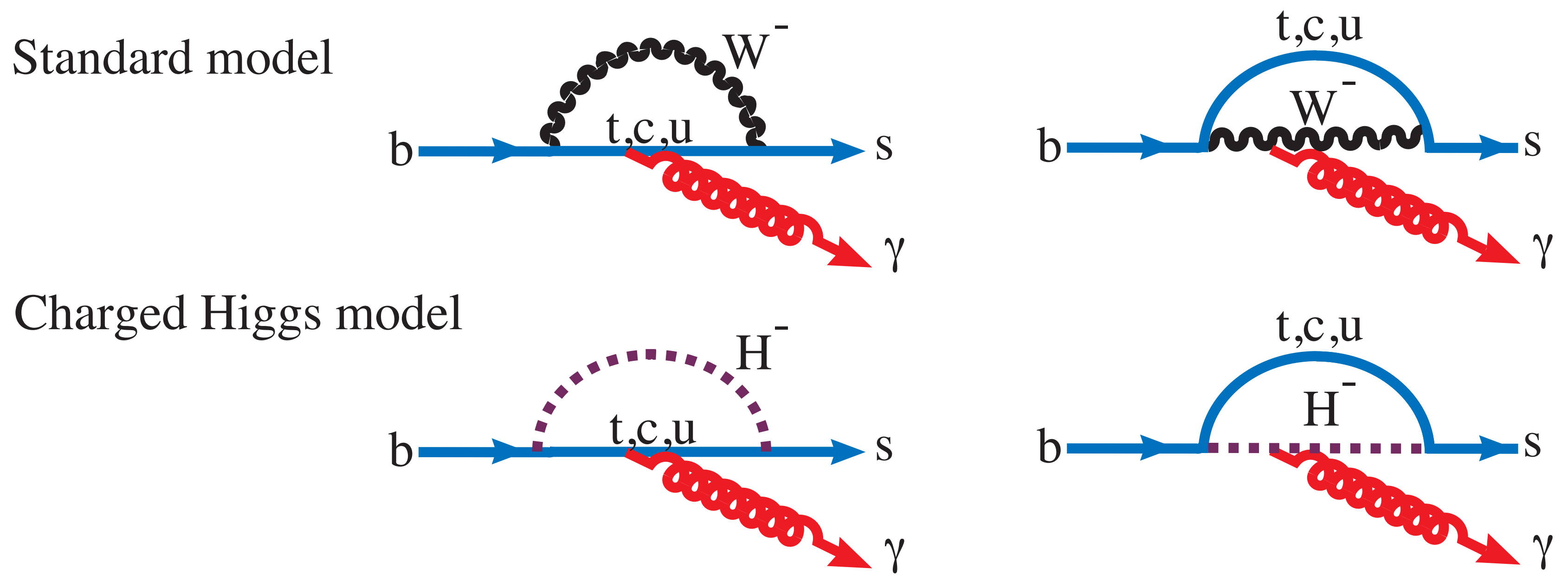
FLAVOUR PHYSICS

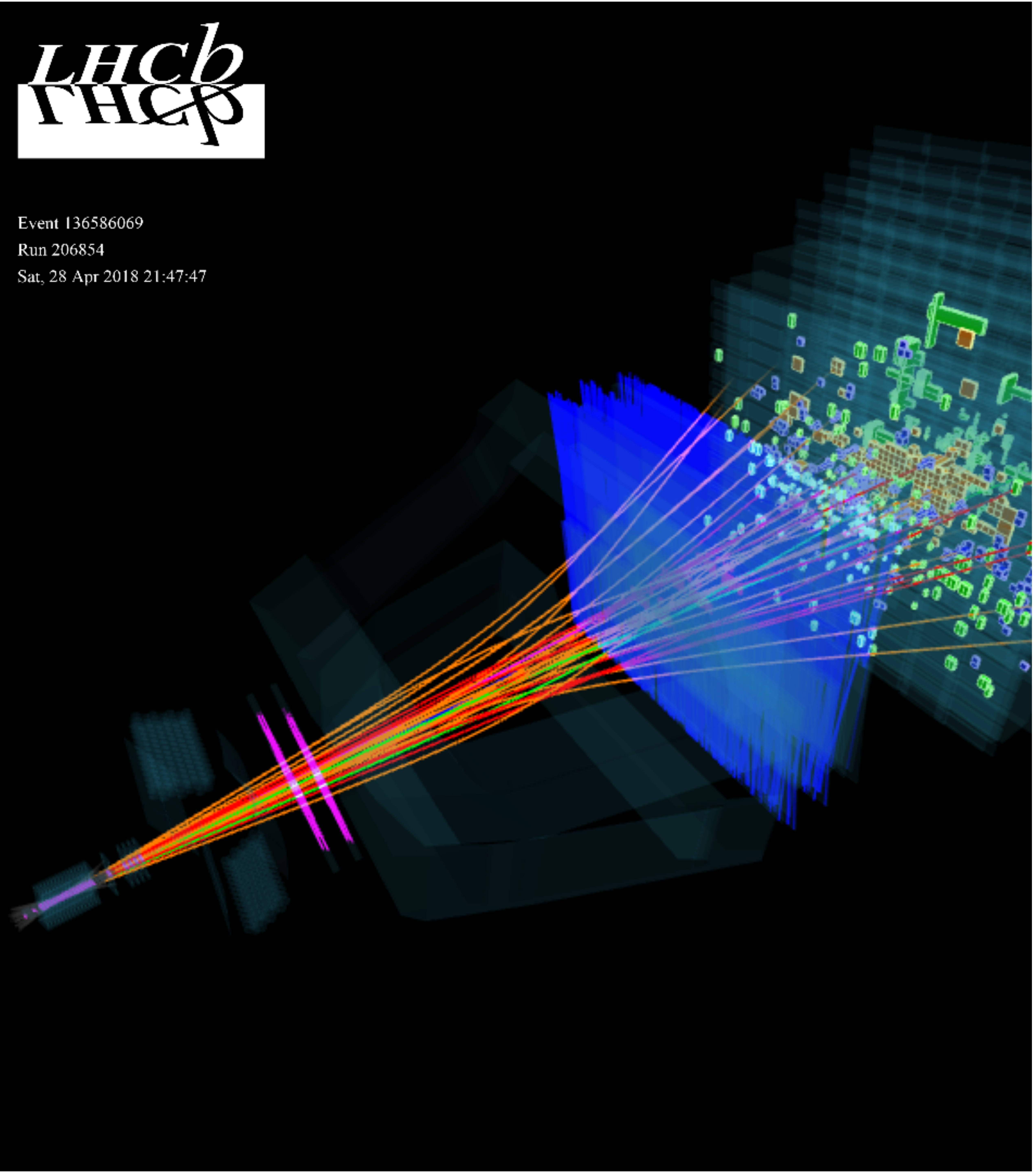
- Study of the properties of B mesons
- Measurements of CKM matrix elements

$V_{CKM} \simeq$



- CP violation
- Search for rare decays



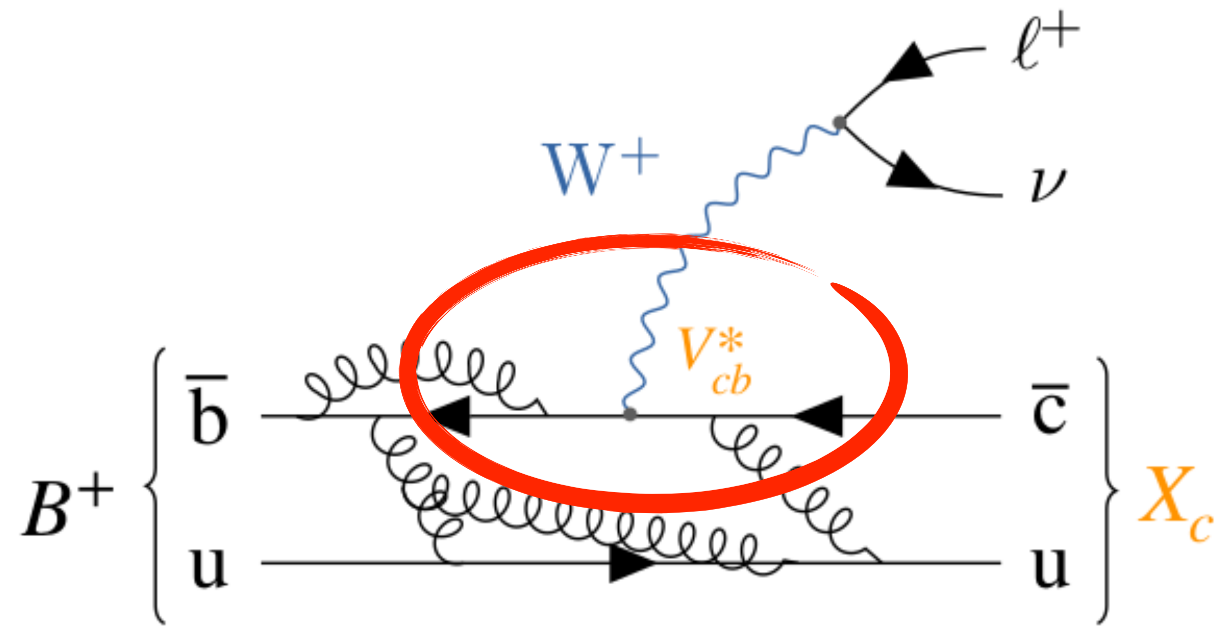


QUEST FOR PRECISION

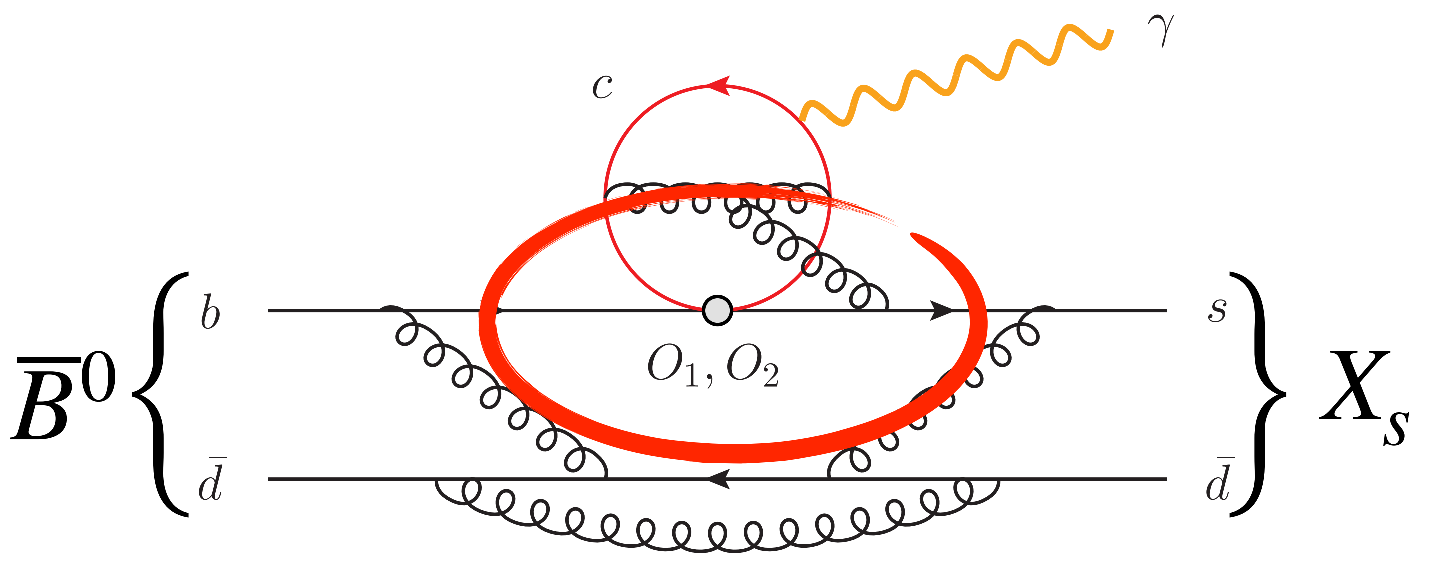
Accurate description

of fundamental interactions

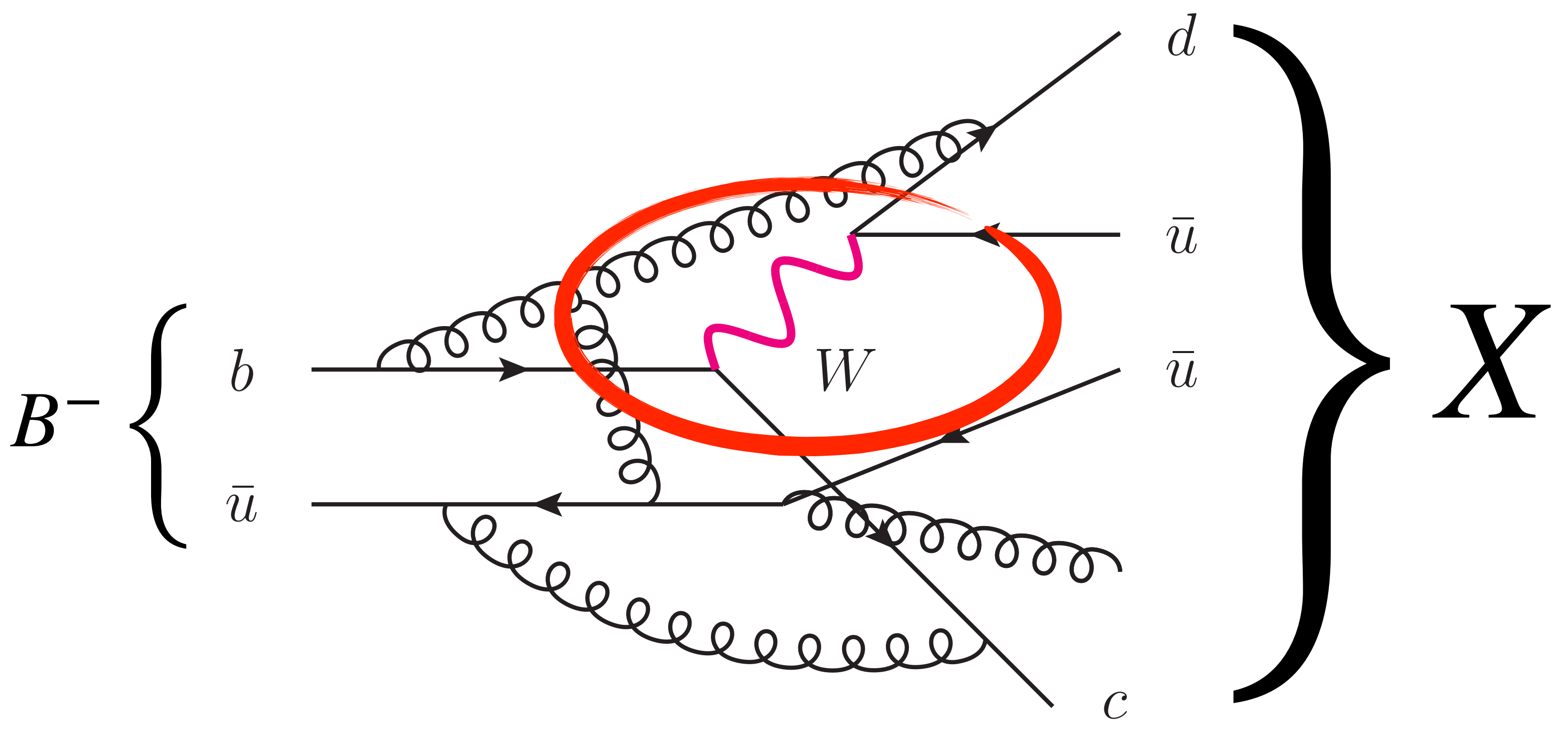
- Make predictions & suggest analyses
- Study implications of the data
- Develop new techniques to perform calculations



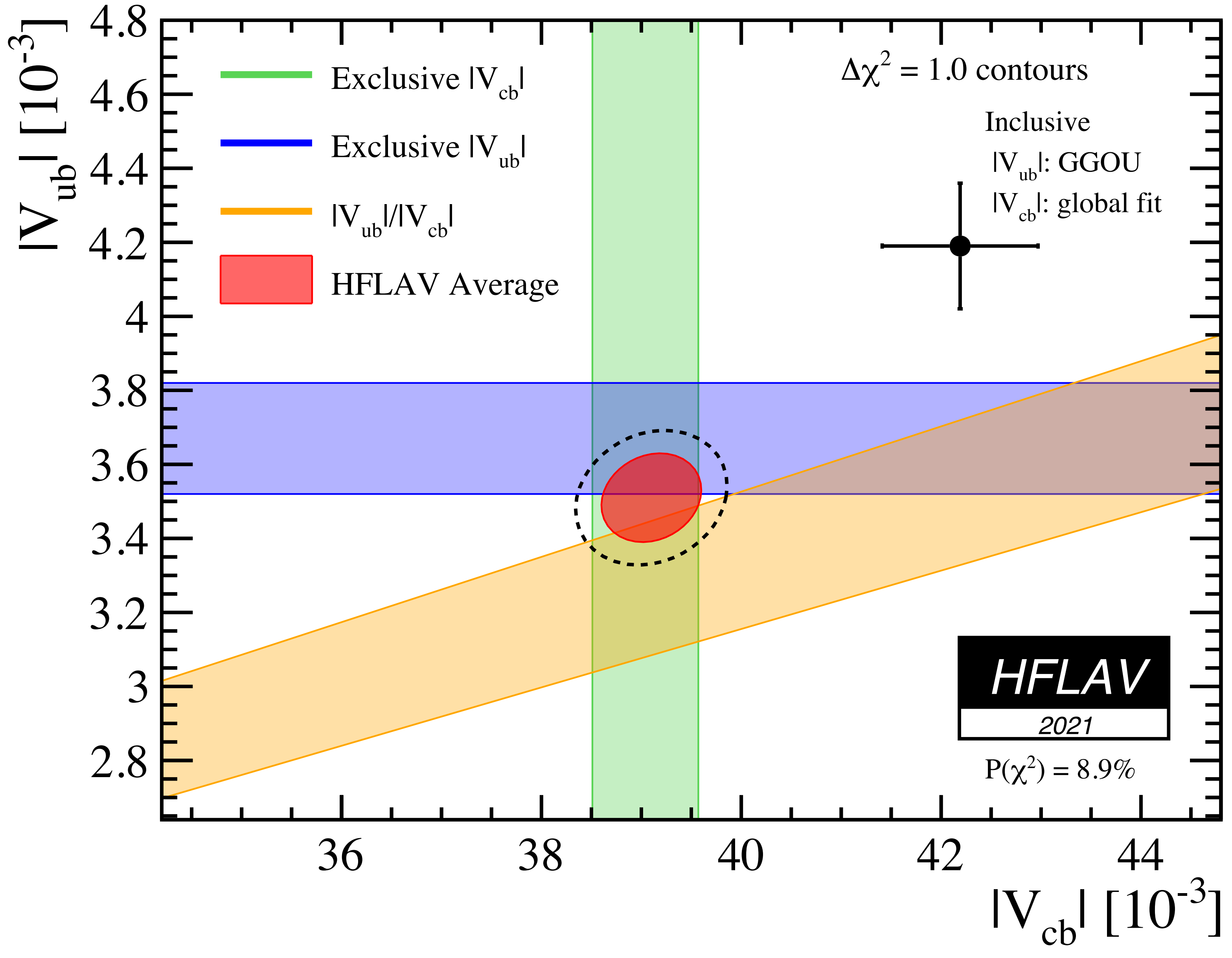
Semileptonic $B \to X_c l \bar{\nu}_l$



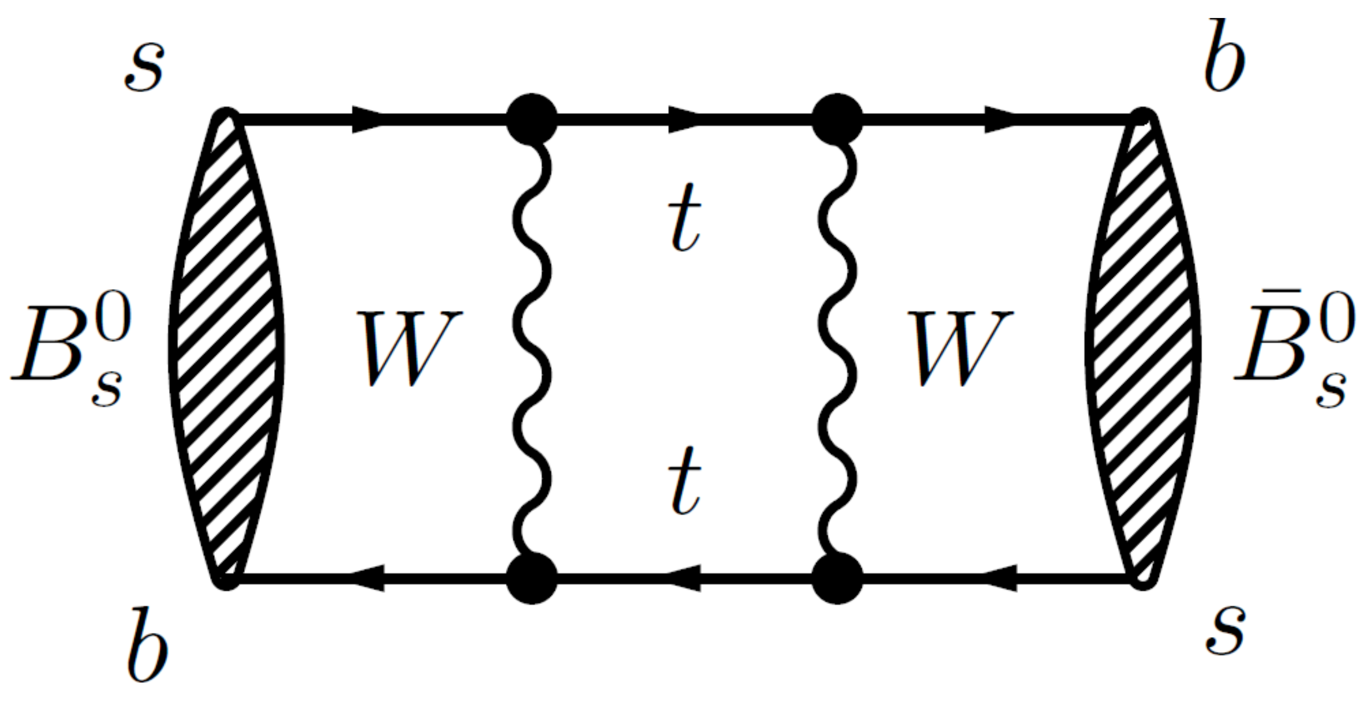
Rare decay $B \to X_s \gamma$



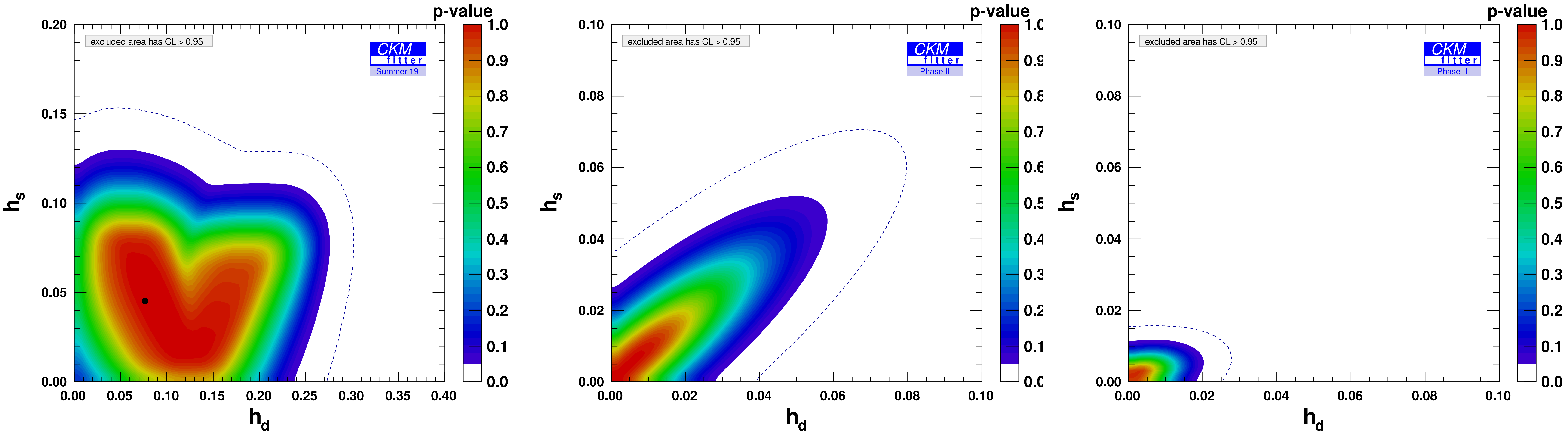
Lifetimes



WE NEED PRECISE PREDICTIONS IN THE SM, OFTEN AT THE 1% LEVEL!



$$M_{12} = (M_{12})_{SM}(1 + h_{d,s}e^{2i\sigma_{d,s}})$$



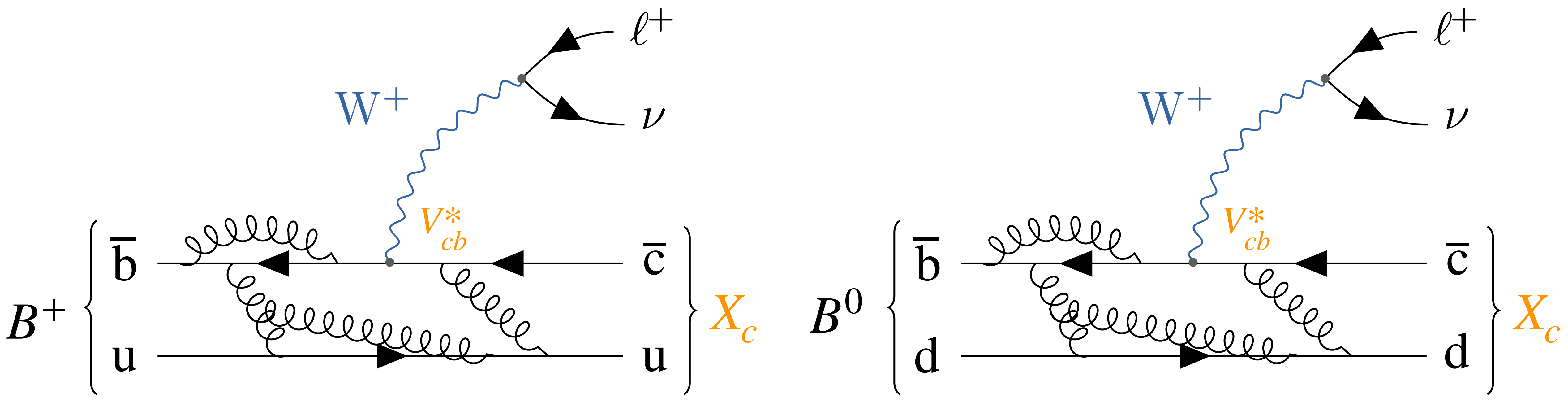
current status

updated LHCb + Belle II

without error on $|V_{cb}|$ and lattice

Charles, Descotes-Genon, Ligeti, Monteil, Papucci, Trabelsi, Vale Silva, 2006.04824

INCLUSIVE DECAYS



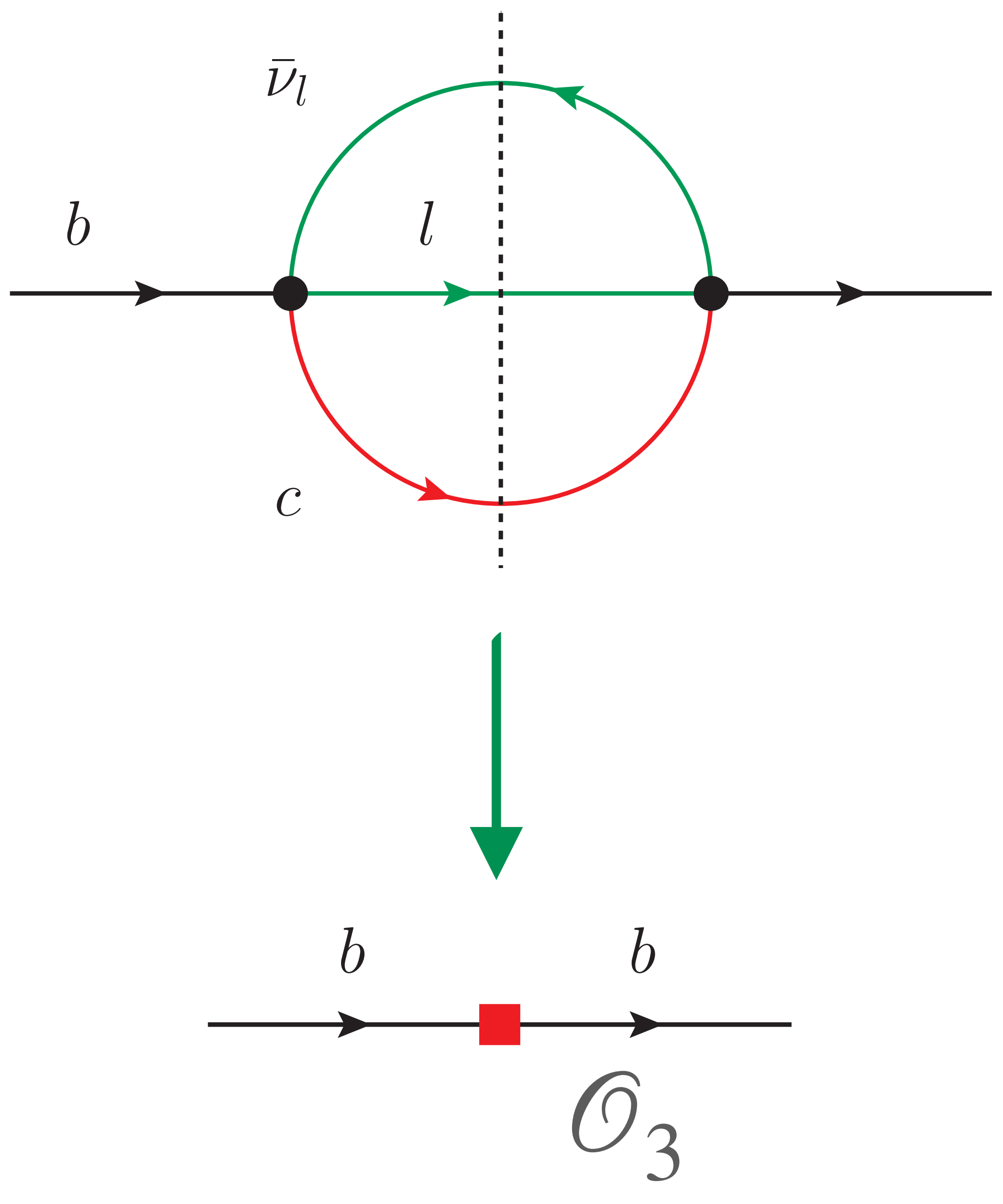
The rate can be expressed as double expansion in Λ_{QCD}/m_b and α_s

The Heavy Quark Expansion

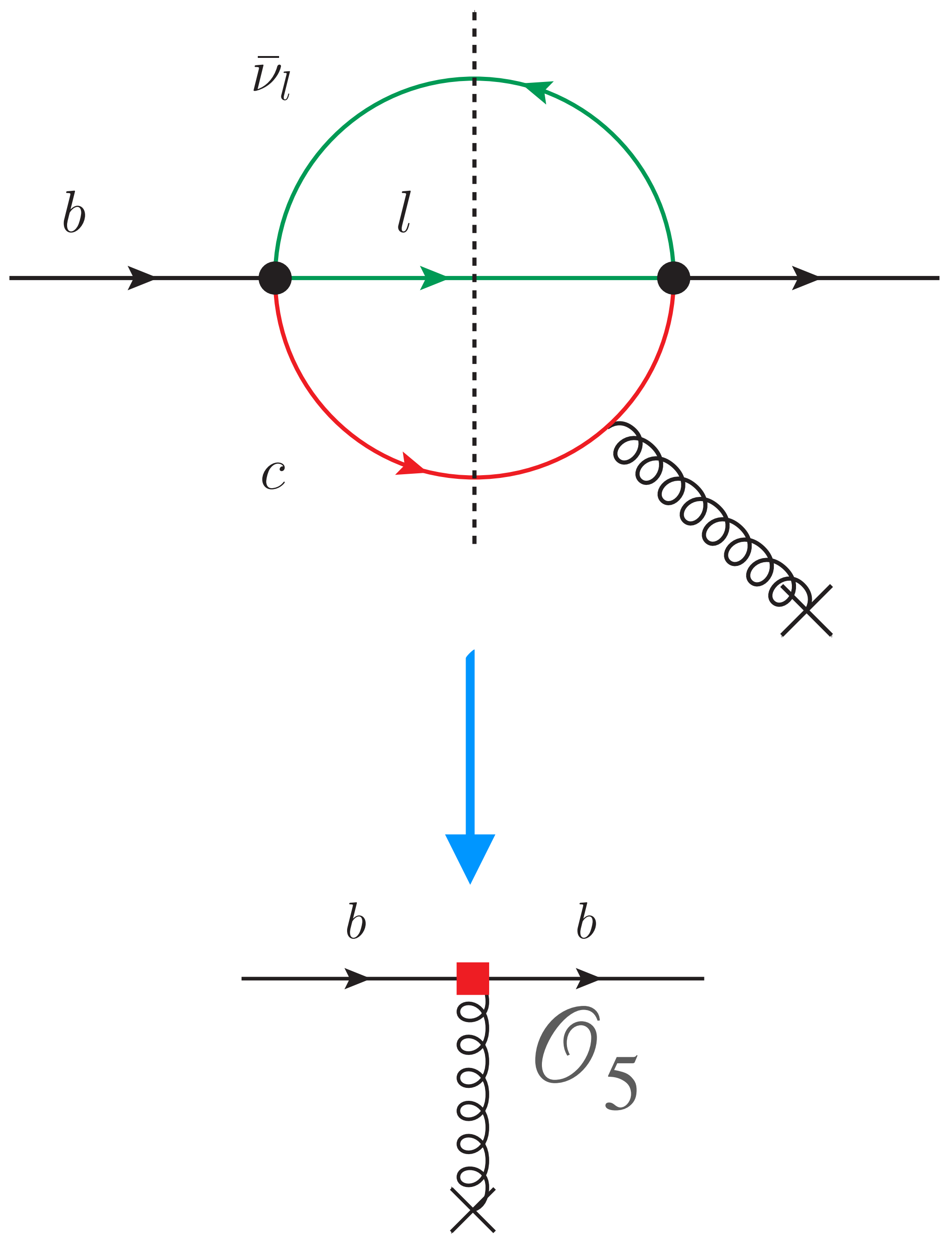
$$\Gamma = \Gamma_3 + \Gamma_5 \frac{\langle B | \mathcal{O}_5 | B \rangle}{m_b^2} + \Gamma_6 \frac{\langle B | \mathcal{O}_6 | B \rangle}{m_b^3} + \ldots + 16\pi^2 \left(\tilde{\Gamma}_6 \frac{\langle B | \mathcal{O}^{4q} | B \rangle}{m_b^3} + \ldots \right)$$

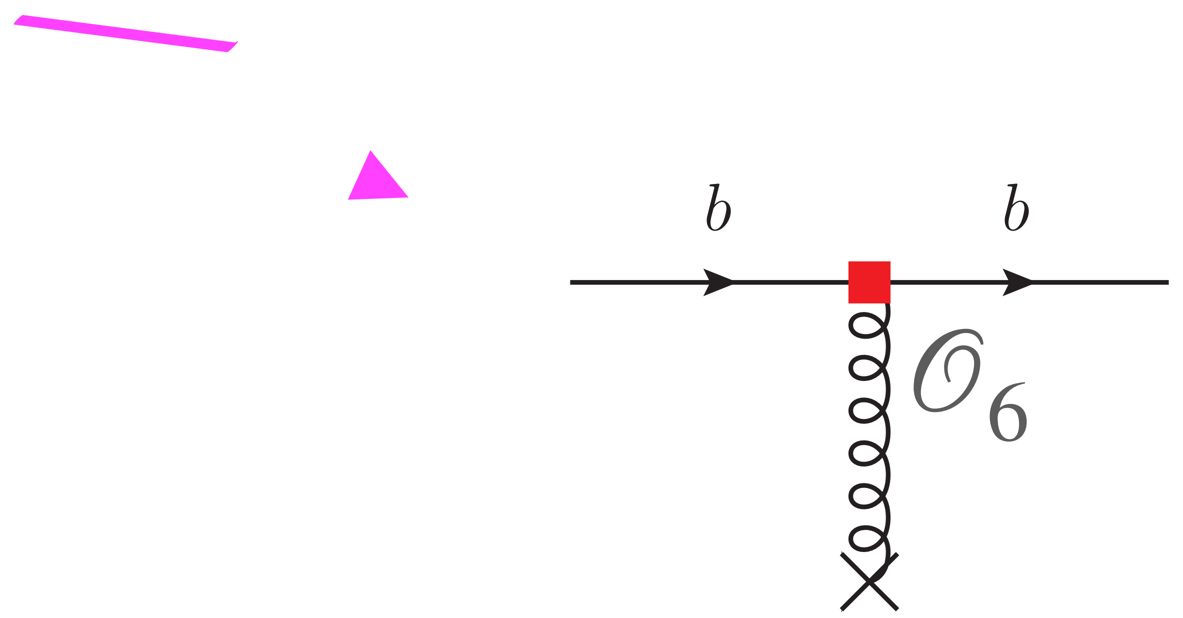
THE HEAVY QUARK EXPANSION

$$\Gamma = \Gamma_3 + \Gamma_5 \frac{\langle B | \mathcal{O}_5 | B \rangle}{m_b^2} + \Gamma_6 \frac{\langle B | \mathcal{O}_6 | B \rangle}{m_b^3} + \ldots + 16\pi^2 \left(\tilde{\Gamma}_6 \frac{\langle B | \mathcal{O}^{4q} | B \rangle}{m_b^3} + \ldots \right)$$



Free quark decay





Darwin term ρ_D^3

Spin-Orbit term ρ_{LS}^3

Kinetic term μ_π^2, chromomagnetic term μ_G^2

PERTURBATIVE VS NON-PERTURBATIVE

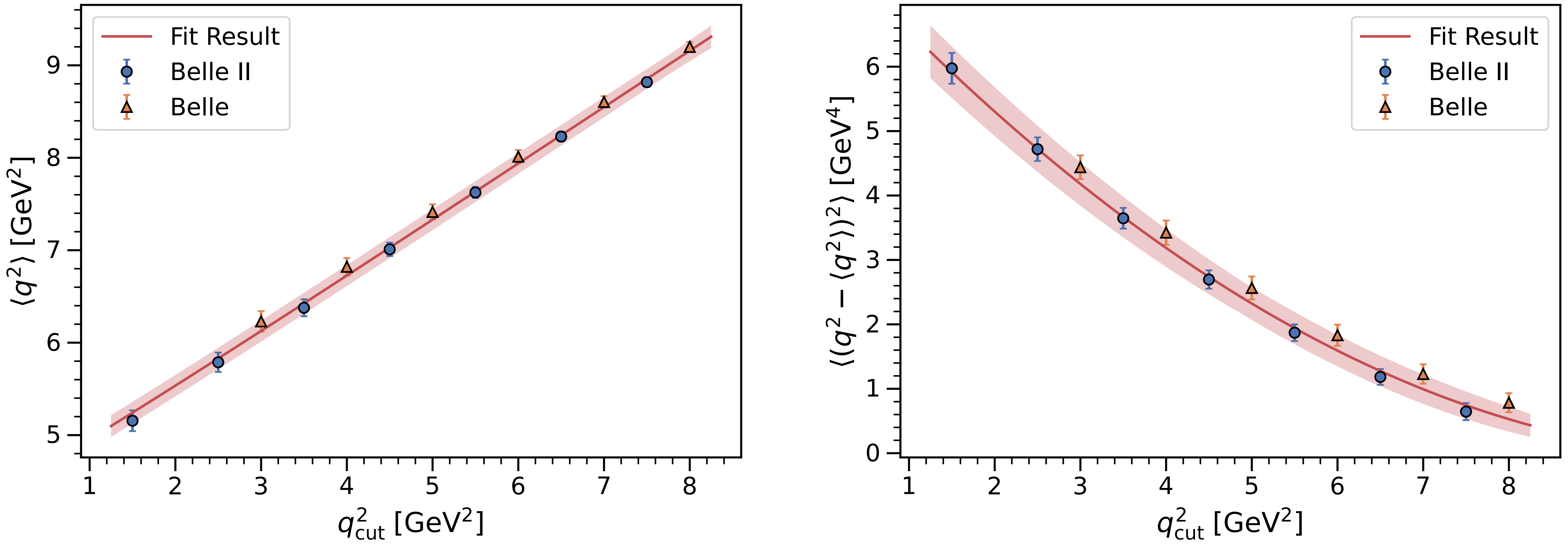
Perturbative

- Process specific
- Calculable in perturbative QCD
- Multi-loop techniques for Feynman integrals

$$\Gamma_i = \Gamma_i^{(0)} + \frac{\alpha_s}{\pi}\Gamma_i^{(1)} + \left(\frac{\alpha_s}{\pi}\right)^2 \Gamma_i^{(2)} + \dots$$

Non Perturbative

- Universal and process independent
- Extracted from data for $B \to X_c l \bar{\nu}_l$
- Preliminary studies in lattice QCD

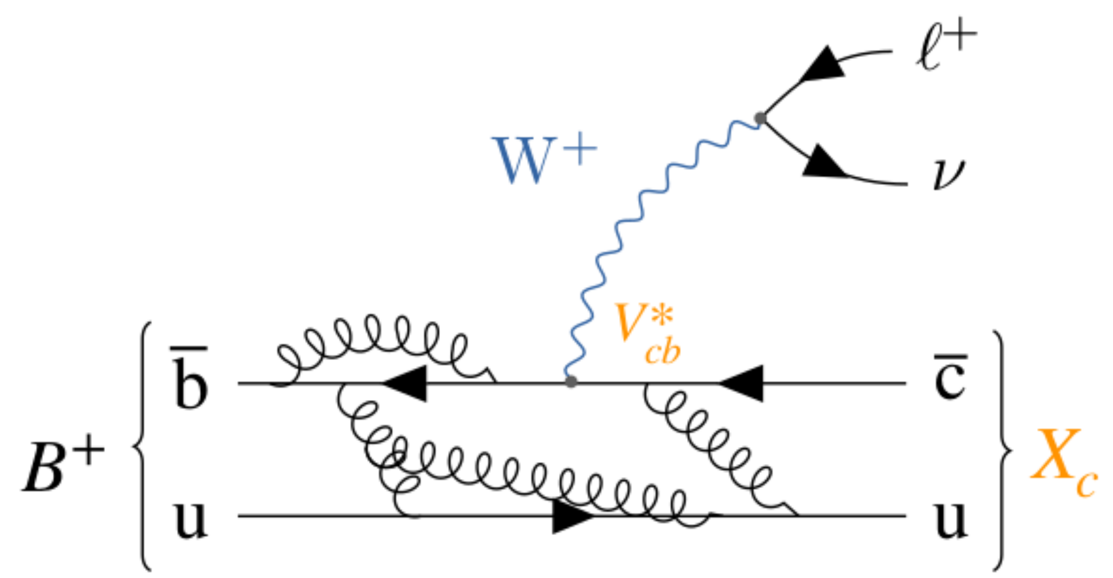


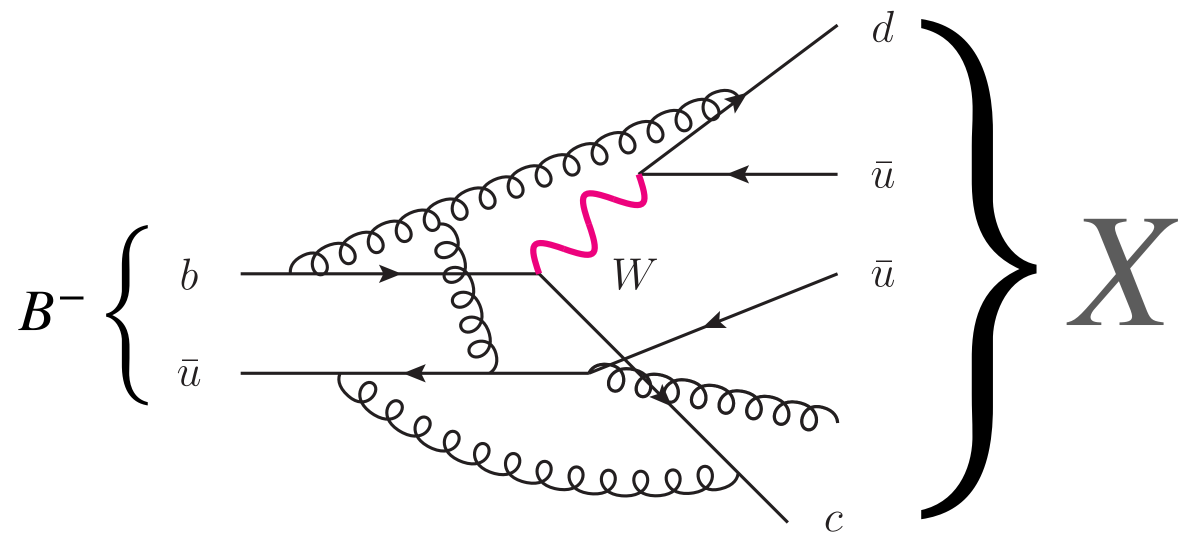
Bernlochner, MF, et al, 2205.10274 [hep-ph]

OUTLOOK

- Numerical methods for Feynman integrals
  - "Expand and match"
    MF, Lange, Schönwald, Steinhauser JHEP 09 (2021) 152
  - AMFLOW
    Xiao Liu, Yan-Qing Ma, Comput.Phys.Commun. 283 (2023) 108565

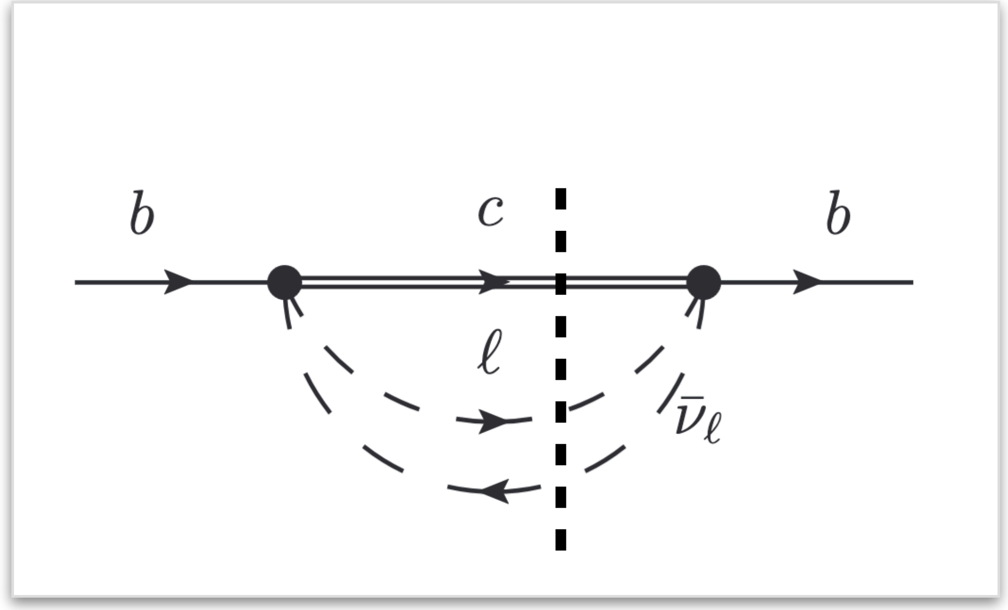
- Application to B-meson phenomenology
  - Lifetimes
    Egner, **MF**,, Schönwald, Steinhauser, in preparation
  - Third order corrections to $b \to u l \bar{\nu}_l$
    MF, Usovitsch, Phys.Rev.D 108 (2023) 114026
  - $O(\alpha_s^2)$ corrections to q^2 spectrum in $B \to X_c l \bar{\nu}_l$
    MF, Herren,, hep-ph/2403.03976



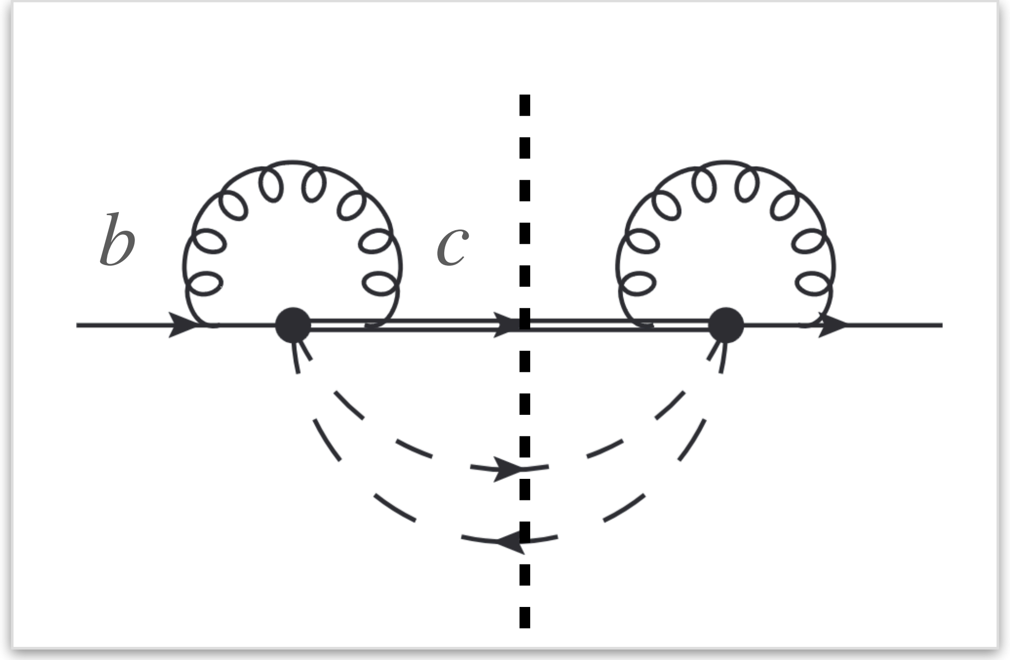


SCATTERING AMPLITUDES AND FEYNMAN INTEGRALS

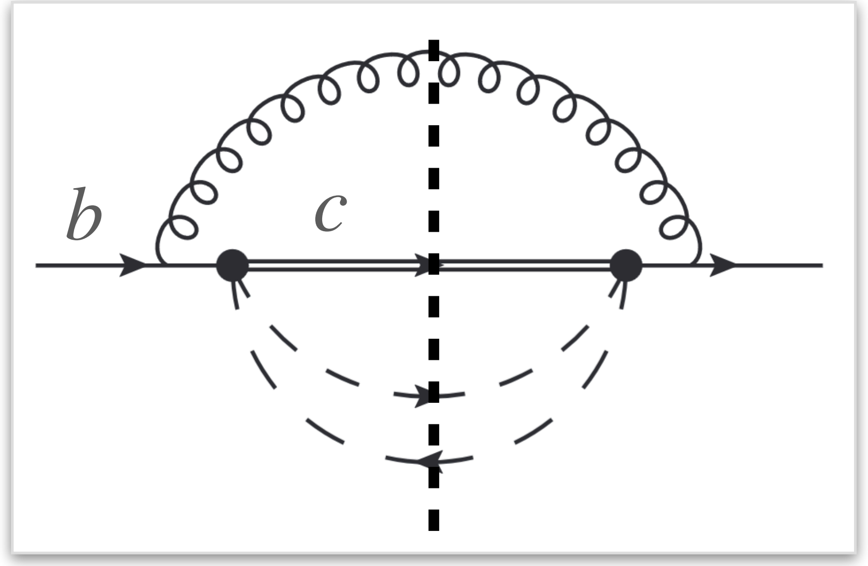
$$\Gamma_i = \Gamma_i^{(0)} + \frac{\alpha_s}{\pi}\Gamma_i^{(1)} + \left(\frac{\alpha_s}{\pi}\right)^2 \Gamma_i^{(2)} + \ldots$$



Leading order (LO)

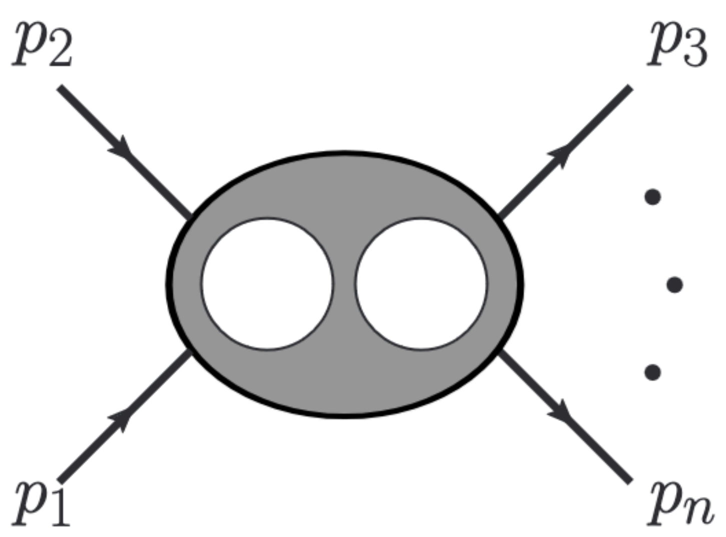


Next-to-next-to-leading order (NNLO)



Next-to-leading order (NLO)

$\mathcal{M} =$



$= \sum_i c_i I_i$

RATIONAL FUNCTIONS

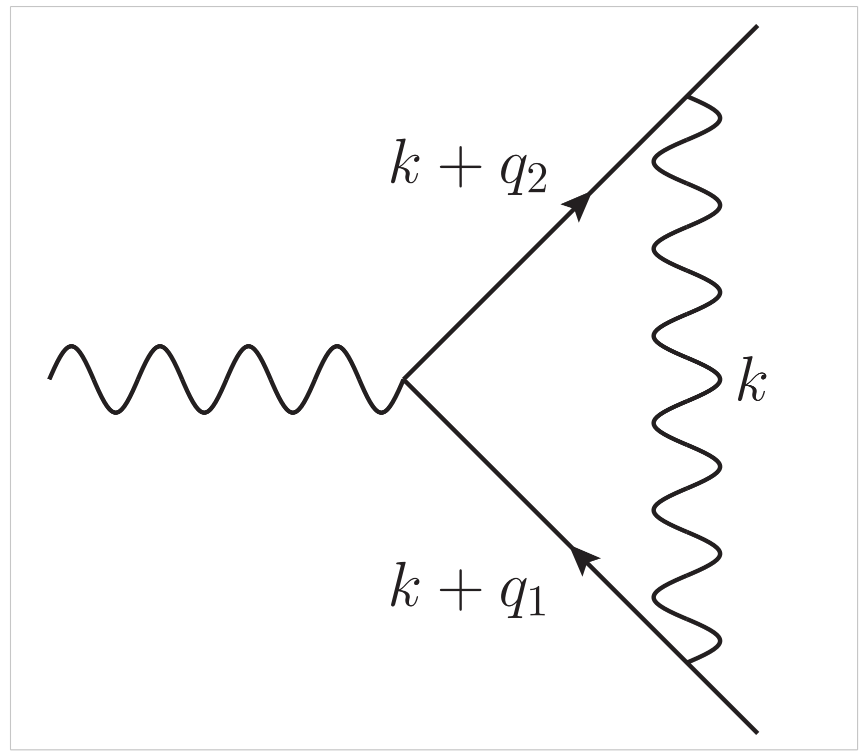
- Integration-by-part relations
- Analytic or numerical methods

FEYNMAN INTEGRALS

- Complicated loop integrations
- Polylogarithms and Elliptic functions
- Analytic/numerical method

Integral family

$$I(a_1, a_2, a_3) = \int d^d k \frac{1}{[k^2]^{a_1} [(k+q_1) - m^2]^{a_2} [(k+q_2) - m^2]^{a_3}}$$



with $s = (q_1 - q_2)^2$

Integration-by-part reduction

Master integrals

$$I(2,1,1) = \frac{(d-2)\left(4dm^2 + ds - 20m^2 - 4s\right)}{2(d-6)(d-5)m^4 s^2} I(0,0,1) + \frac{4(d-3)}{(d-6)s^2} I(0,1,1)$$

Differential Equations

Kotikov, Phys. Lett. B 254 (1991) 158;
Gehrmann, Remiddi, Nucl. Phys. B 580 (2000) 485

$$\frac{\mathrm{d}}{\mathrm{ds}} I(0,1,1) = \frac{\mathrm{d}}{\mathrm{ds}} \int d^d k \frac{1}{k^2[(k+q_1-q_2)^2 - m^2]}$$

$$= \frac{I(-1,2,1)}{s-4} - \frac{I(0,1,1)}{s-4} + \frac{2I(0,1,1)}{(s-4)s} - \frac{2I(0,2,0)}{(s-4)s}$$

$$\overset{IBP}{=} \frac{(d-2)}{s\left(4m^2-s\right)} I(0,0,1) + \frac{\left(-4dm^2+ds+12m^2-4s\right)}{2s\left(s-4m^2\right)} I(0,1,1)$$

Differential Equations

Kotikov, Phys. Lett. B 254 (1991) 158;
Gehrmann, Remiddi, Nucl. Phys. B 580 (2000) 485

$$\frac{d}{ds}\begin{pmatrix} I(0,0,1) \\ I(0,1,1) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \frac{d-2}{s\left(4m^2-s\right)} & \frac{-4dm^2+ds+12m^2-4s}{2s\left(s-4m^2\right)} \end{pmatrix}\begin{pmatrix} I(0,0,1) \\ I(0,1,1) \end{pmatrix}$$

Boundary conditions

$$I(0,0,1)\,|_{s=0} = (m^2)^{1-\epsilon}\Gamma(\epsilon-1)$$
$$I(0,1,1)\,|_{s=0} = \ldots$$

Analytic solution

- Solve in terms of known constants/functions
- Function properties well understood
- Known analytic structures and series expansions
- Fast and generic numerical evaluation tools

Numerical solution

- Oriented to phenomenological studies
- Applicable to larger class of problems
- Finite numerical accuracy

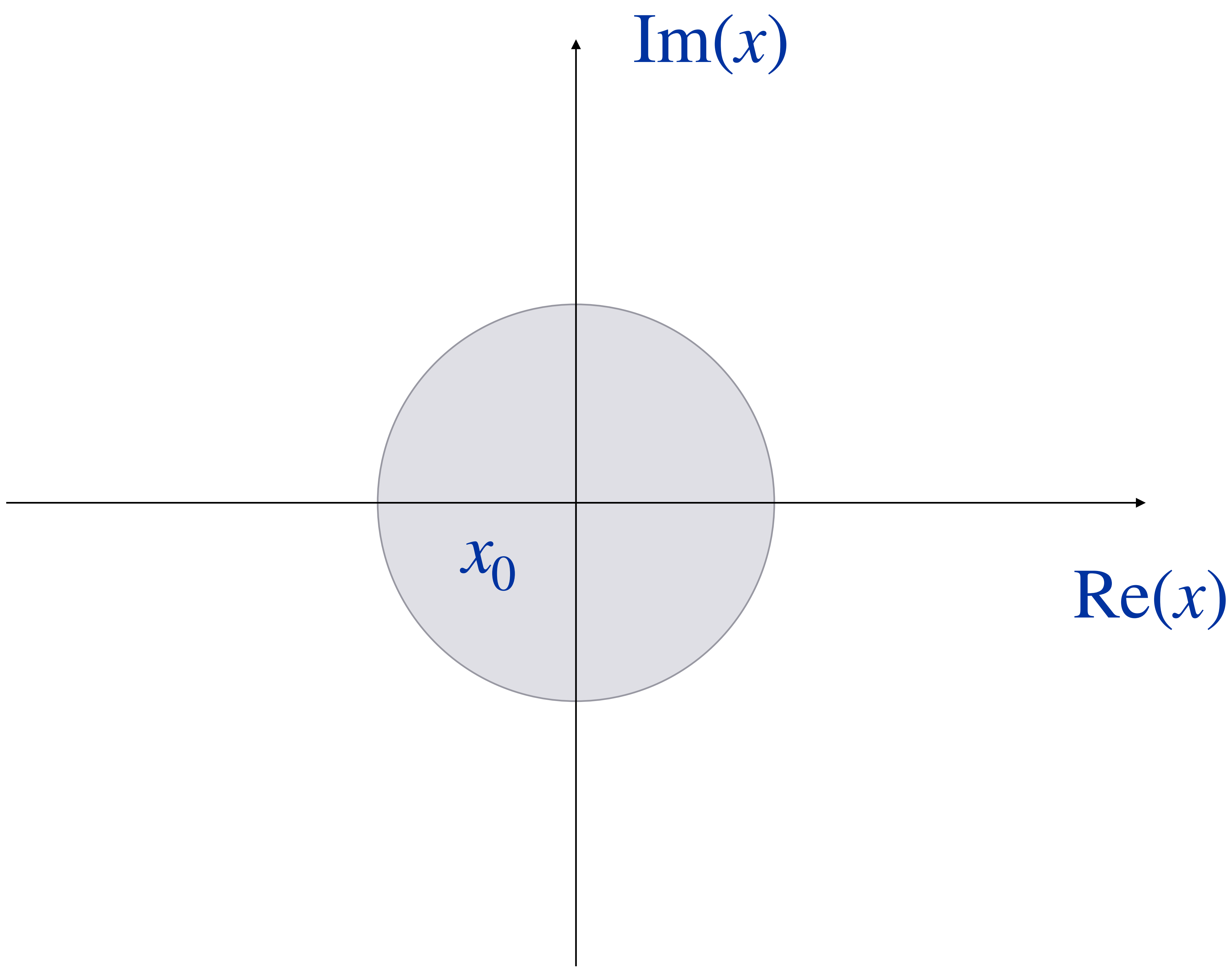
A SEMI-ANALYTIC METHOD

MF, Lange, Schönwald, Steinhauser JHEP 09 (2021) 152

$$I_a(x,\epsilon) = \sum_{m=m_{\min}}^{m_{\max}} \sum_{n=0}^{n_{\max}} c_{a,mn}\, \epsilon^m (x - x_0)^n$$

Construct a series expansion around some point x_0
[and $\epsilon = (d-4)/2$]

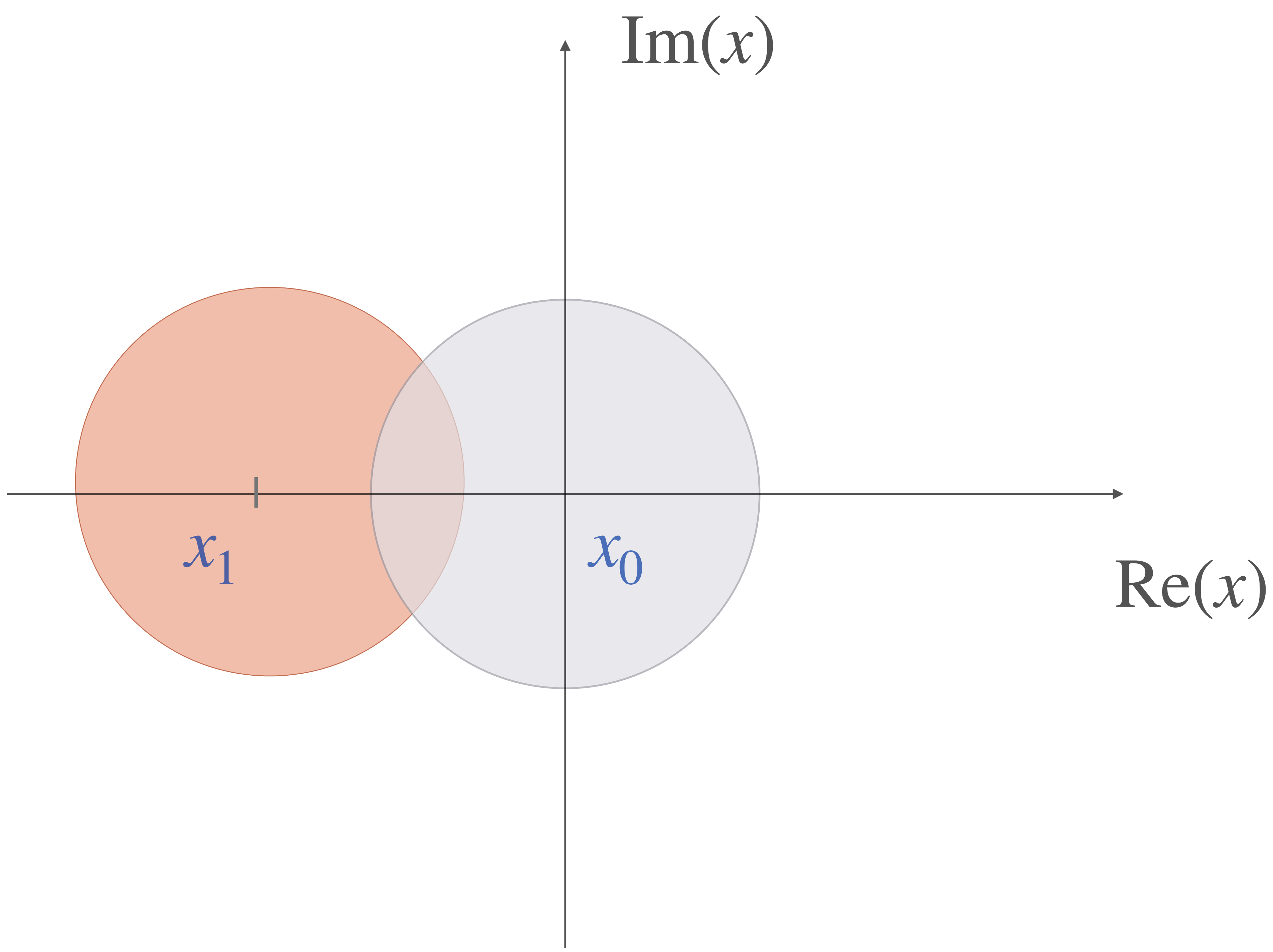
$$\frac{\partial \vec{I}}{\partial x} = M(x,\epsilon)\vec{I}$$



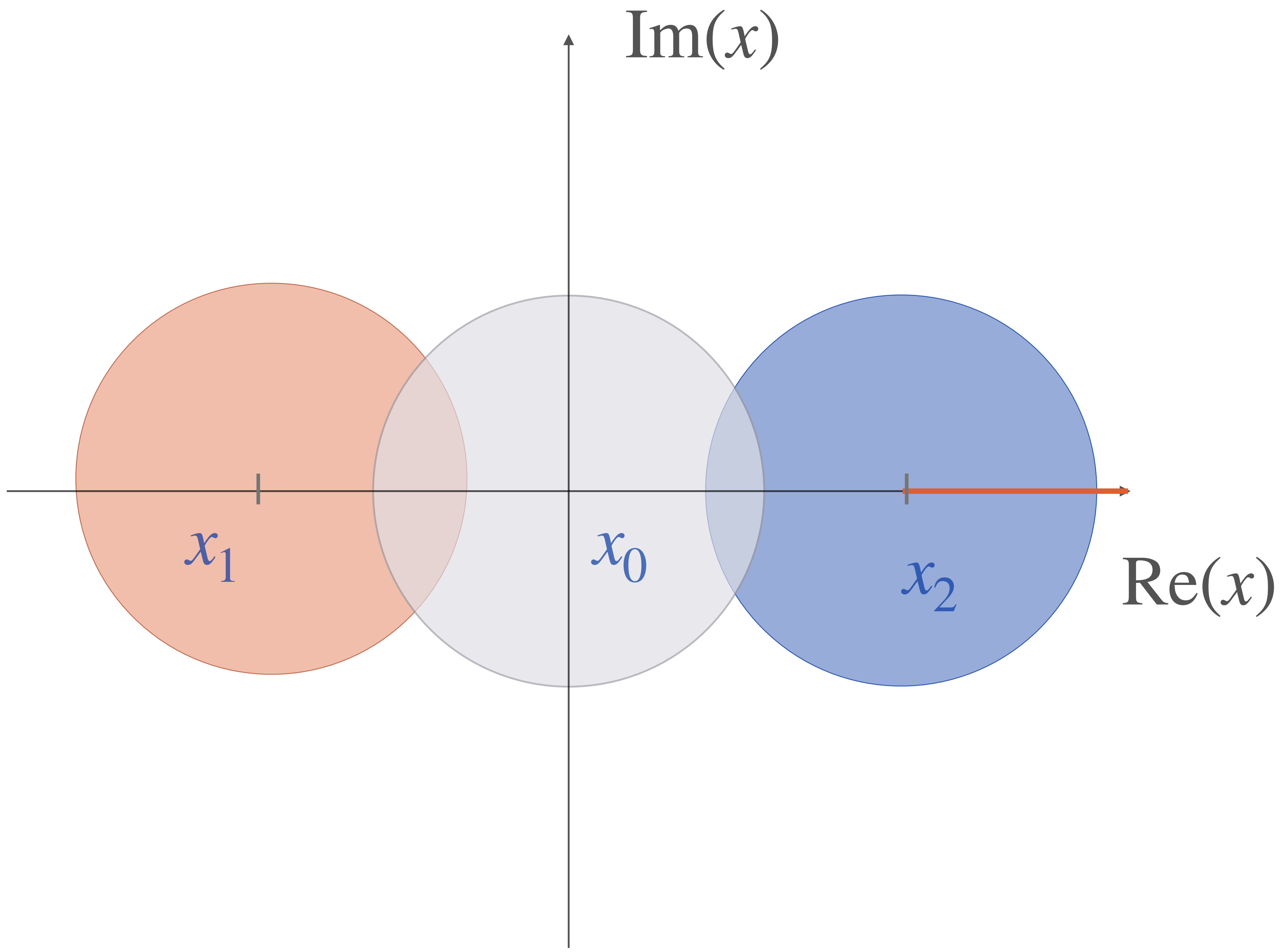
Compare order by order in x_0 and ϵ

$$\underbrace{\sum_m \sum_{n=1} n c_{a,mn} \epsilon^m (x - x_0)^{n-1}}_{\partial I_a / \partial x} = \sum_b M_{ab}(x, \epsilon) \underbrace{\sum_m \sum_{n=0} c_{b,mn} \epsilon^m (x - x_0)^n}_{I_b}$$

- Linear system of equations for the expansion coefficients $c_{k,mn}$
- Solve the liner system in term of a minimal set of coefficients
- The minimal set of undetermined coefficients are fixed from boundary conditions



- Proceeds with a new expansion around
- Match new expansion to the previous one (with finite accuracy)
- Iterate until all range of is covered



- Proceeds with a new expansion around
- Match new expansion to the previous one (with finite accuracy)
- Iterate until all range of is covered

- Power-log expansion around singular points (poles and thresholds)

$$I_a(x,\epsilon) = \sum_{m=m_{\min}}^{m_{\max}} \sum_{n=0}^{n_{\max}} \sum_{l\geq 0} c_{a,mnl}\, \epsilon^m\, (x-x_2)^{\alpha n-\beta} \log^l(x-x_2)$$

FEATURES

MF, Lange, Schönwald, Steinhauser JHEP 09 (2021) 152

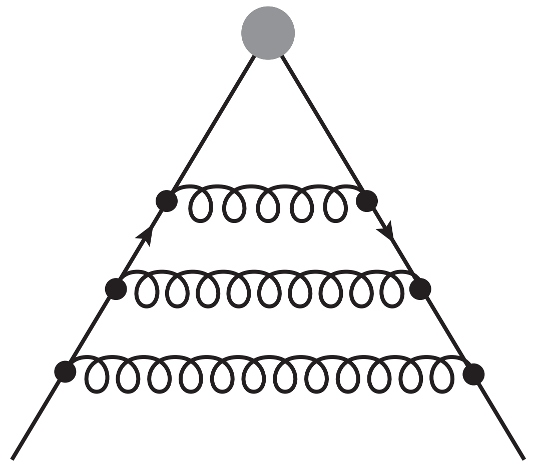
- GOAL: cover physical range of with several expansions
- No special form of the differential equations
- Well suited for fast numerical evaluation
- Precision systematic improvable:
  - more expansion points
  - deeper expansion
  - Möbius transformation
- Bottleneck
  - Problems with $O(10^2)$ masters integrals
  - Solve linear system with $O(10^6)$ equations
  - Match expansion in numerically stable way

Similar approaches

- DESS
  Lee, Smirnov, Smirnov, JHEP 03 (2018) 008
- DiffExp
  Hidding, Comput.Phys.Commun. 269 (2021) 108125
- SeaSide
  Armadillo, Bonciani, Devoto, Rana, Vicini, Comput.Phys.Commun. 282 (2023) 108545
- AMFlow
  Xiao Liu, Yan-Qing Ma, Comput.Phys.Commun. 283 (2023) 108565

Heavy quark form factors at $O(\alpha_s^3)$

MF, Lange, Schönwald, Steinhauser Phys.Rev.Lett. 128 (2022) 17; Phys.Rev.D 106 (2022) 3, 034029; Phys.Rev.D 107 (2023), 094017



AUXILIARY MASS METHOD

Xiao Liu, Yan-Qing Ma, Comput.Phys.Commun. 283 (2023) 108565

$$I(\vec{n}) = \int \prod_{i=1}^{L} d^D\ell_i \frac{1}{D_1^{n_1} \ldots D_N^{n_N}}$$

$$= \lim_{\eta \to i0^-} I_{\text{aux}}(\vec{n}, \eta)$$

Integrals with auxiliary mass parameter η

$$I_{\text{aux}}(\vec{n}, \eta) = \int \prod_{i=1}^{L} d^D\ell_i \frac{1}{(D_1 - \eta)^{n_1} \ldots (D_K - \eta)^{n_K} \ldots D_N^{n_N}}$$

Method of regions

$$\frac{1}{(\ell+p)^2-m^2-\eta}=\frac{1}{\ell^2-\eta}\sum_i\left(-\frac{2p\cdot\ell+p^2-m^2}{\ell^2-\eta}\right)^i$$

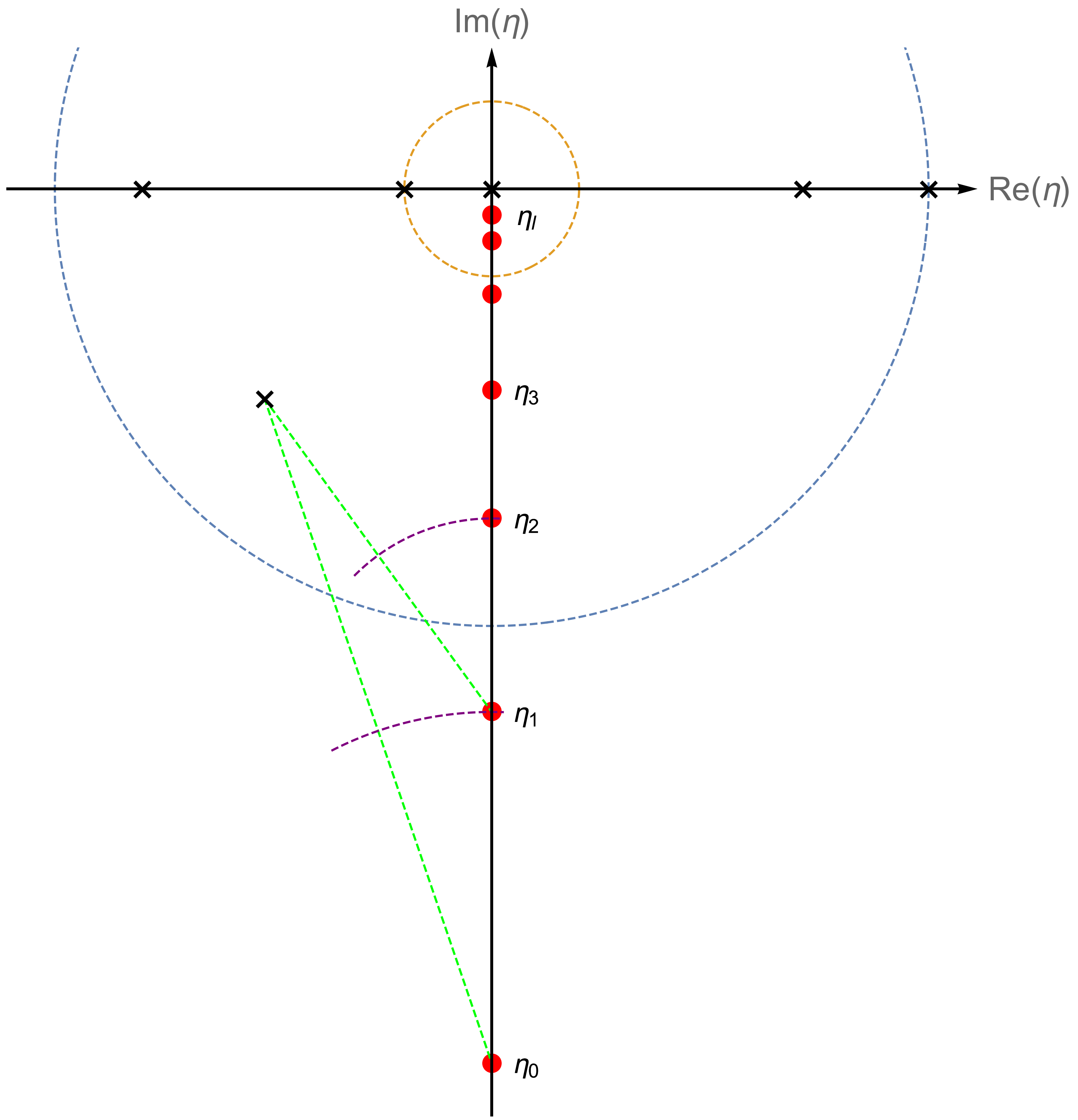
Differential equations

$$\frac{\partial I_{\text{aux}}(\eta)}{\partial\eta}=A(\eta)I_{\text{aux}}(\eta)$$

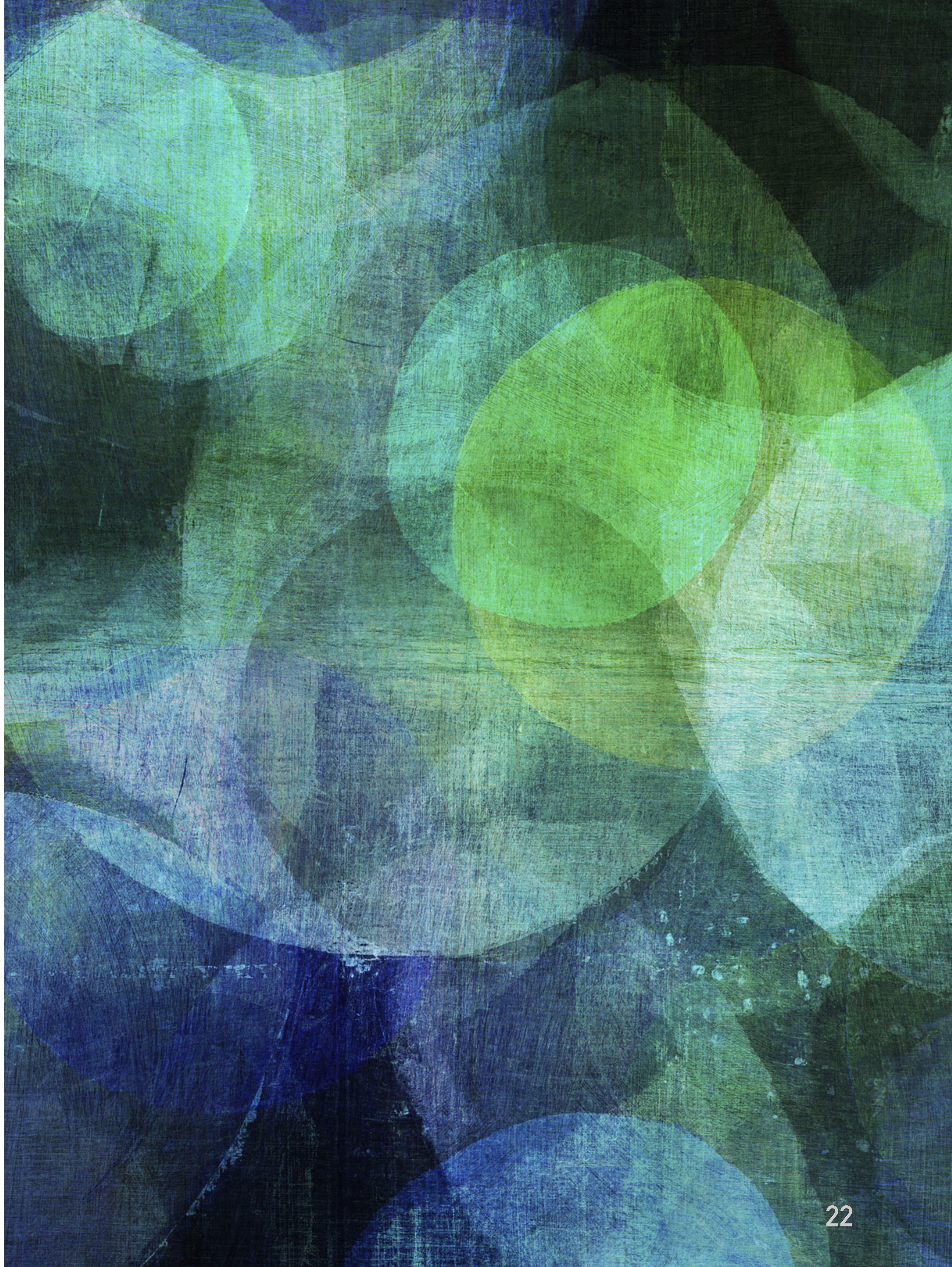
Boundary conditions at $\eta=i\infty$: Equal mass vacuum integrals

Davydychev and Tausk, Nucl, Phys. B, 1993, Broadhurst, Eur. Phys. J. C, 1999, Schroder and Vuorinen, JHEP, 2005, Kniehl, Pikelner and Veretin, JHEP, 2017, Luthe, phdthesis, 2015, Luthe, Maier, Marquard et al, JHEP, 2017

https://gitlab.com/multiloop-pku/amflow

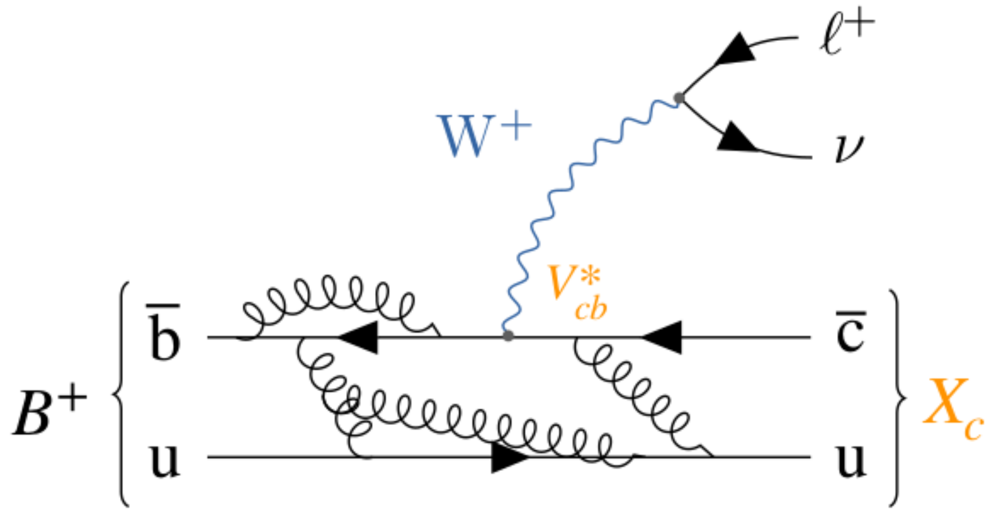


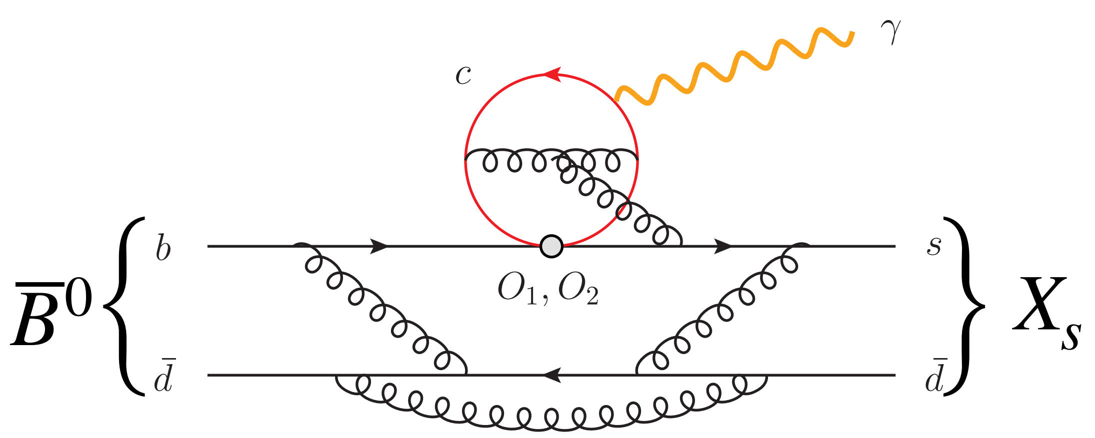
APPLICATIONS TO B PHYSICS



CHALLENGES WITH BOTTOM AND CHARM MASS

- Crucial sensitivity on $m_c/m_b \simeq 0.25$





- Needs for a short-distance mass scheme

$$m_b^{\text{OS}} : m_c^{\text{OS}} \qquad \Gamma(B \to X_c \ell \bar{\nu}_\ell) \sim 1 - 1.78\left(\frac{\alpha_s}{\pi}\right) - 13.1\left(\frac{\alpha_s}{\pi}\right)^2 - 163.3\left(\frac{\alpha_s}{\pi}\right)^3$$

$$m_b^{\text{kin}}(1\text{ GeV}) : \overline{m}_c(2\text{ GeV}) \qquad \Gamma(B \to X_c \ell \bar{\nu}_\ell) \sim 1 - 1.24\left(\frac{\alpha_s}{\pi}\right) - 3.65\left(\frac{\alpha_s}{\pi}\right)^2 - 1.0\left(\frac{\alpha_s}{\pi}\right)^3$$

$$m_b^{1\text{S}} : m_c \text{ via HQET} \qquad \Gamma(B \to X_c \ell \bar{\nu}_\ell) \sim 1 - 1.38\left(\frac{\alpha_s}{\pi}\right) - 6.32\left(\frac{\alpha_s}{\pi}\right)^2 - 33.1\left(\frac{\alpha_s}{\pi}\right)^3$$

- Estimate theoretical uncertainties

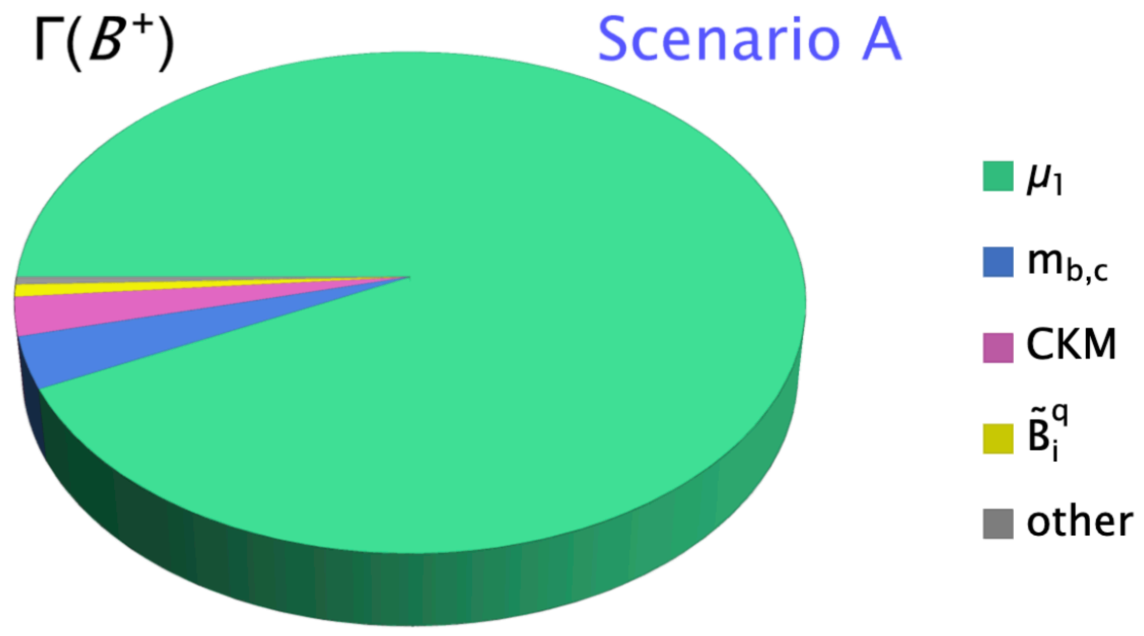
$$\overline{m}_c(\mu_c), \overline{m}_b(\mu_b), m_b^{\text{kin}}(\mu_{WC}), \ldots$$

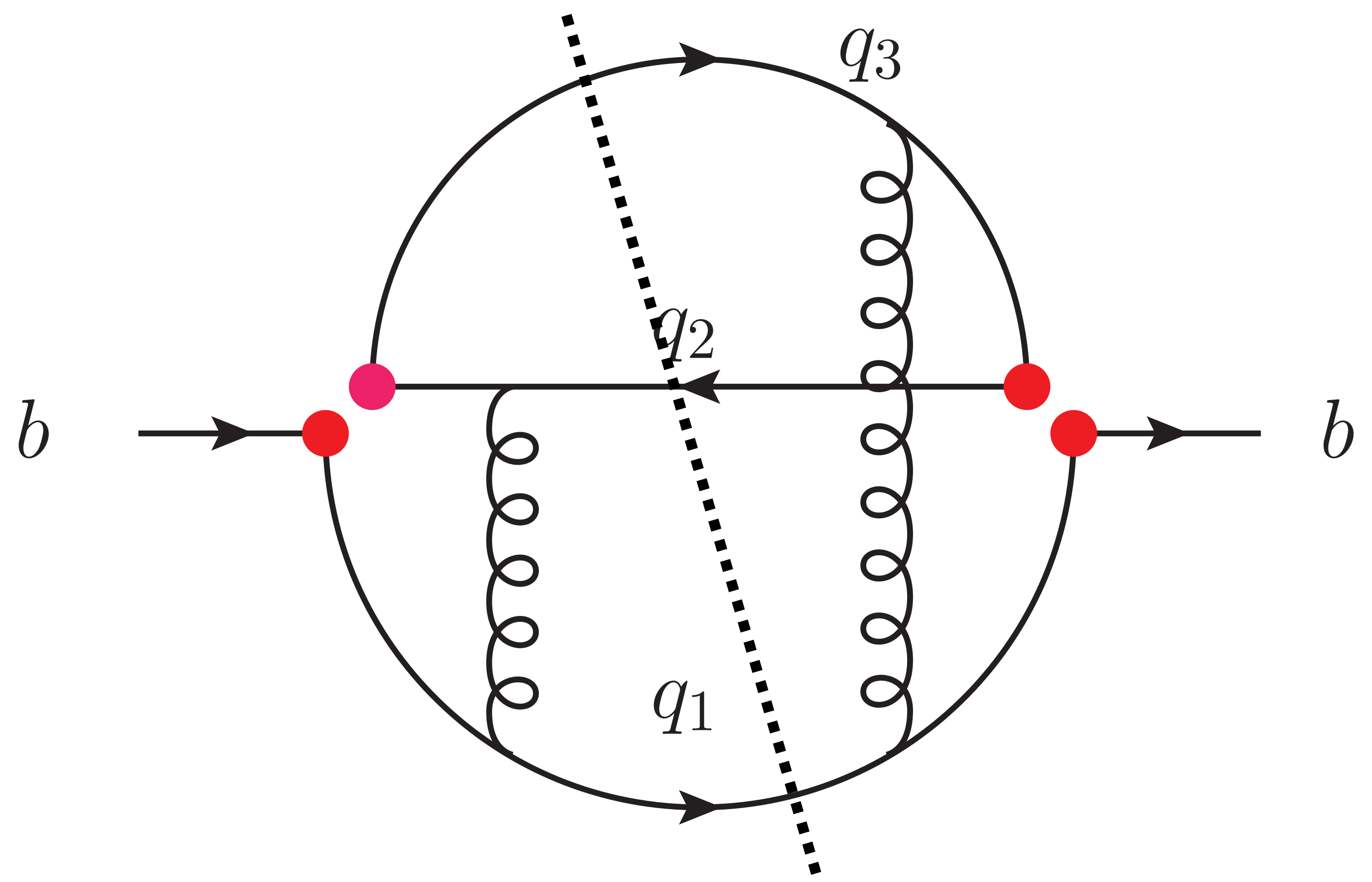
LIFETIMES OF B MESON

$$\frac{1}{\tau_B} = \Gamma(B_q) = \Gamma_3 + \Gamma_5 \frac{\langle B | \mathcal{O}_5 | B \rangle}{m_b^2} + \Gamma_6 \frac{\langle B | \mathcal{O}_6 | B \rangle}{m_b^3} + \ldots$$

$$\Gamma(B^+) = (0.59^{+0.11}_{-0.07})\ \mathrm{ps}^{-1}$$
$$\Gamma(B_d) = (0.63^{+0.11}_{-0.07})\ \mathrm{ps}^{-1}$$
$$\Gamma(B_s) = (0.63^{+0.11}_{-0.07})\ \mathrm{ps}^{-1}$$

Lenz, Piscopo, Rusov, JHEP 01 (2023) 004

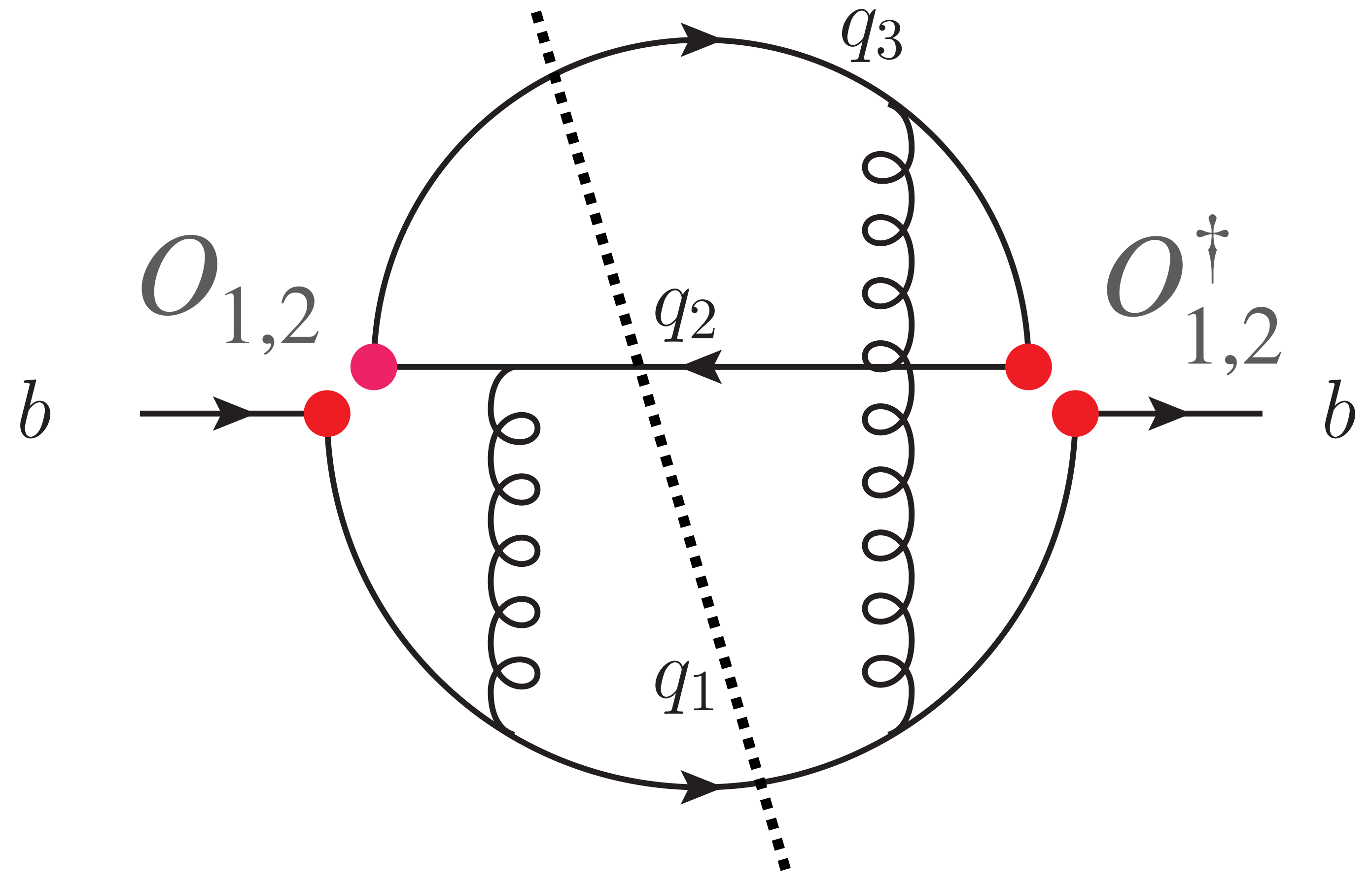




- $\Gamma_3 = \Gamma(b \to c l \bar{\nu}_l) + \Gamma(b \to c \bar{u} d) + \Gamma(b \to c \bar{c} s) + \ldots$
- NLO QCD corrections to nonleptonic decays

Bagan, Patricia Ball, Braun, Gosdzinsky, Nucl.Phys.B 432 (1994) 3;
Krinner, Lenz, Rauh, Nucl.Phys.B 876 (2013) 31

$$\mathcal{H}_{\text{eff}} = \frac{4G_F}{\sqrt{2}} \sum_{q_{1,3}=u,c} \sum_{q_2=d,s} \lambda_{q_1 q_2 q_2} \left(C_1(\mu_b) O_1^{q_1 q_2 q_3} + C_2(\mu_b) O_2^{q_1 q_2 q_3} \right) + \text{h.c.}$$



To use anti-commuting γ_5 we adopt the *Traditional* basis

Buras, Weisz, NPB 333 (1990) 66

$$O_1^{q_1 q_2 q_3} = (\bar{q}_1^\alpha \gamma^\mu P_L b^\beta)(\bar{q}_2^\beta \gamma_\mu P_L q_3^\alpha),$$

$$O_2^{q_1 q_2 q_3} = (\bar{q}_1^\alpha \gamma^\mu P_L b^\alpha)(\bar{q}_2^\beta \gamma_\mu P_L q_3^\beta),$$

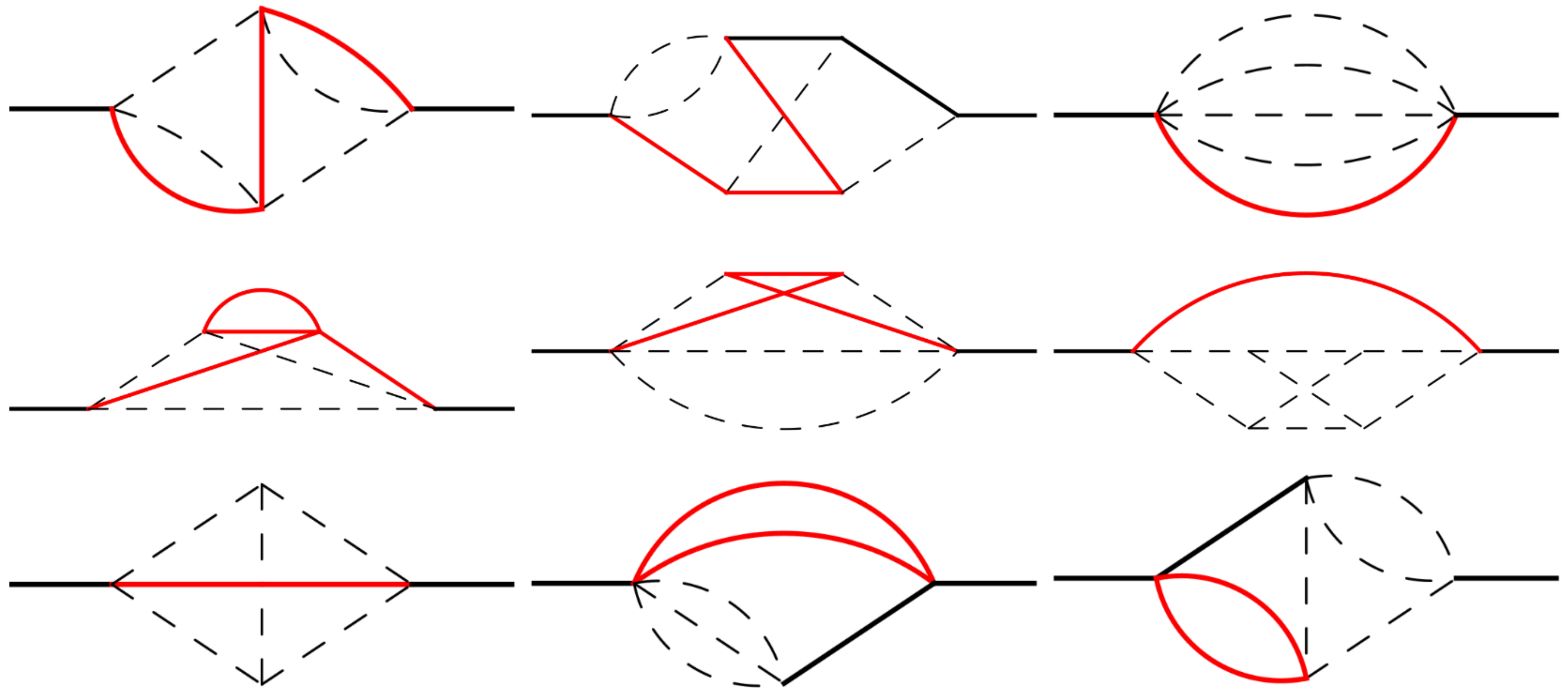
Matching conditions and ADMs known in the *CMM* basis

Chetyrkin, Misiak, Munz, hep-ph/9711280

$$O_1^{'q_1 q_2 q_3} = (\bar{q}_1 T^a \gamma^\mu P_L b)(\bar{q}_2 T^a \gamma_\mu P_L q_3),$$

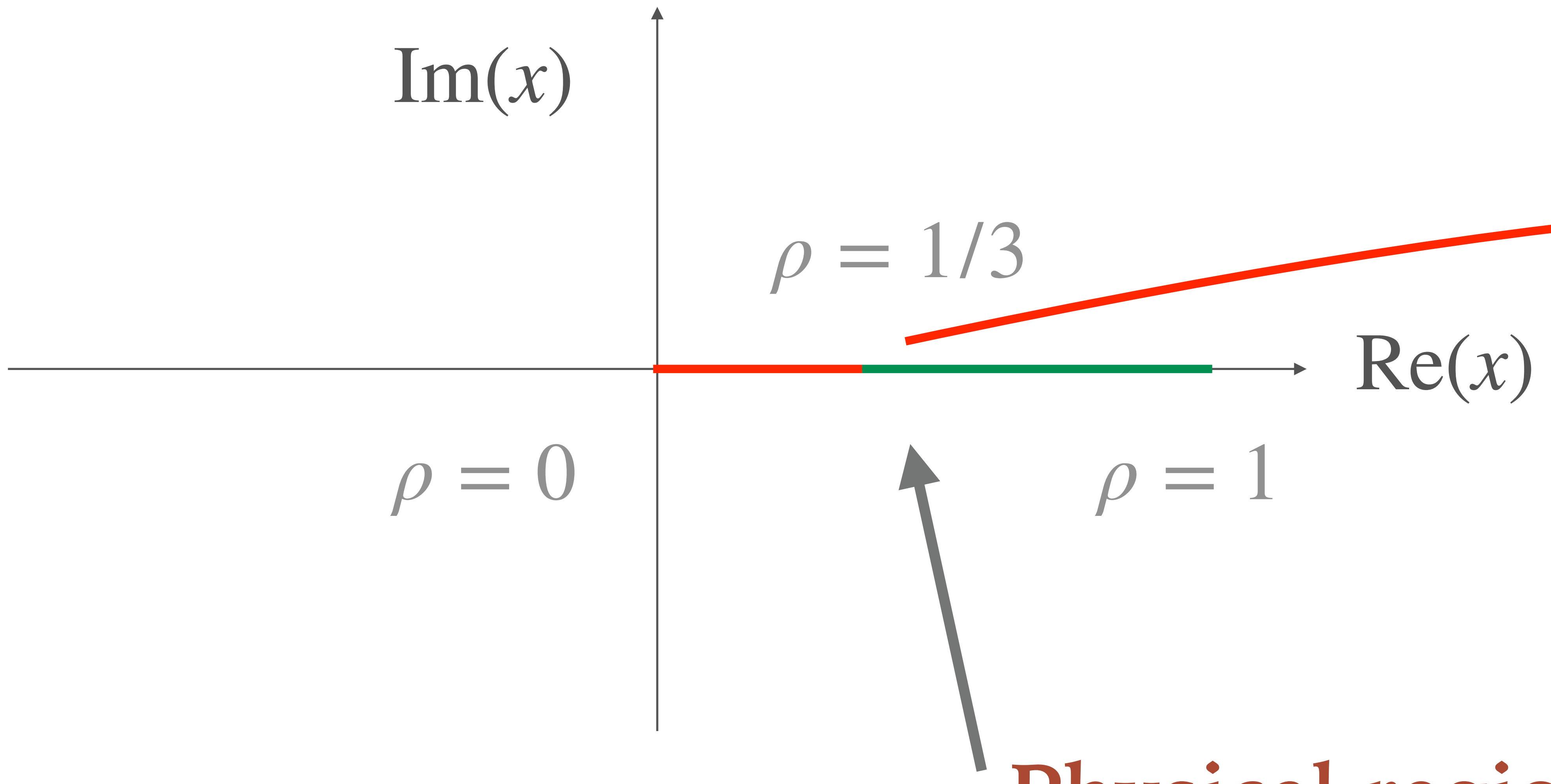
$$O_2^{'q_1 q_2 q_3} = (\bar{q}_1 \gamma^\mu P_L b)(\bar{q}_2 \gamma_\mu P_L q_3),$$

MASTER INTEGRALS



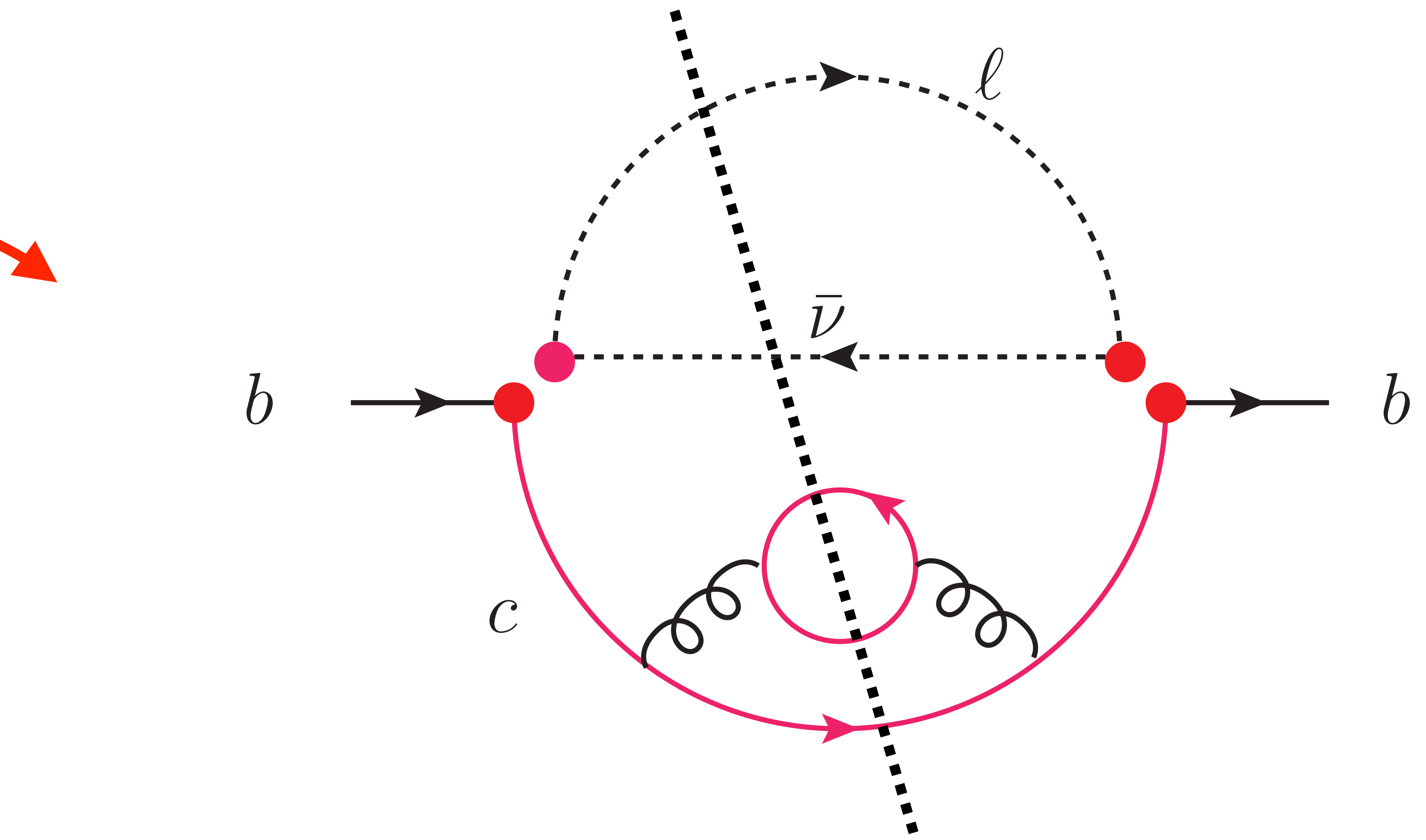
- Master integrals depend on $\rho = m_c/m_b$
- Use "expand and match" method

  Fael, Lange, Schönwald, Steinhauser JHEP 09 (2021) 152
- Boundary conditions at $\rho = 1/2$ with AMFlow



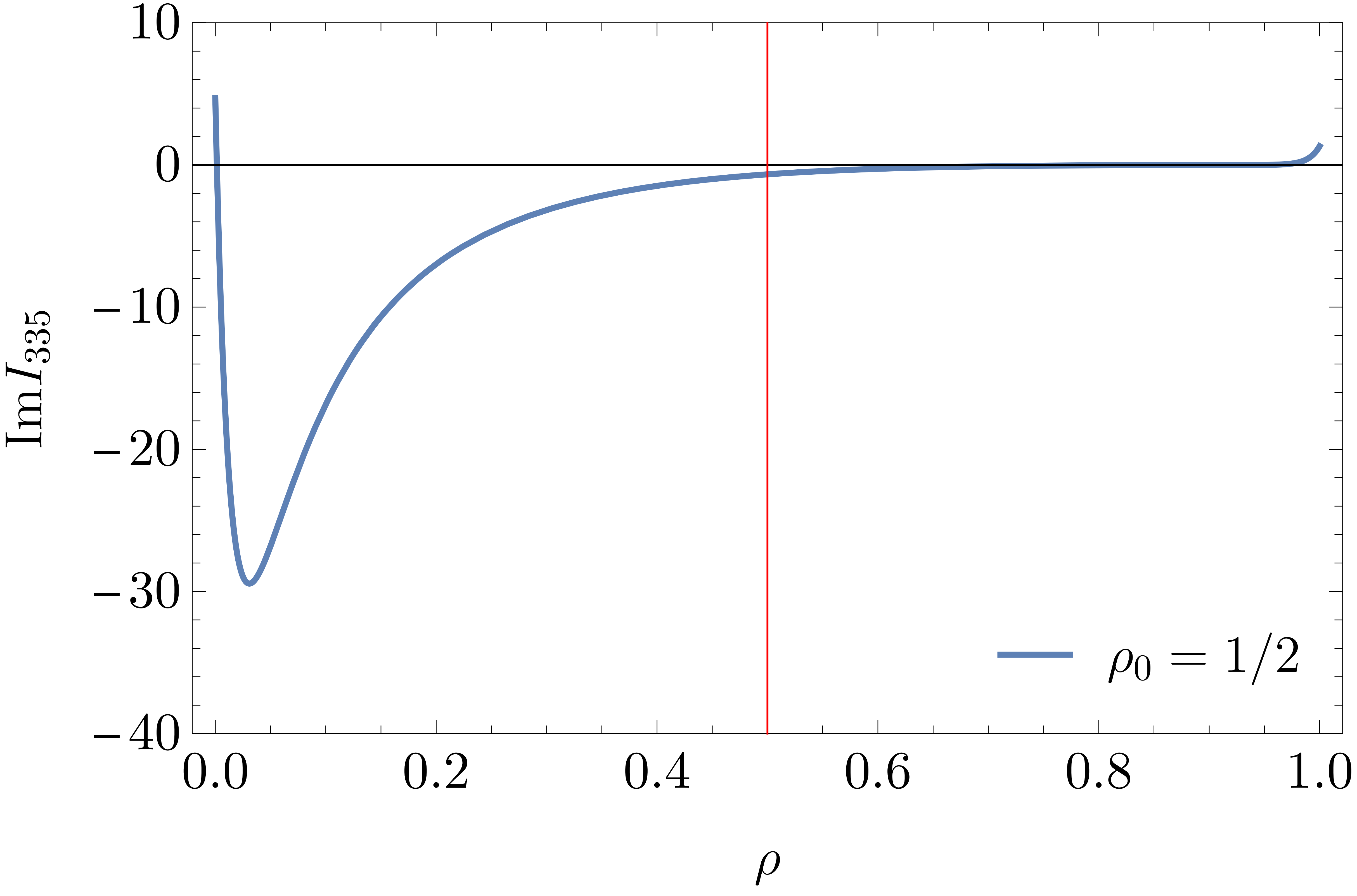
Physical region $0 \leq \rho < 1$

Threshold for 3 charm quarks

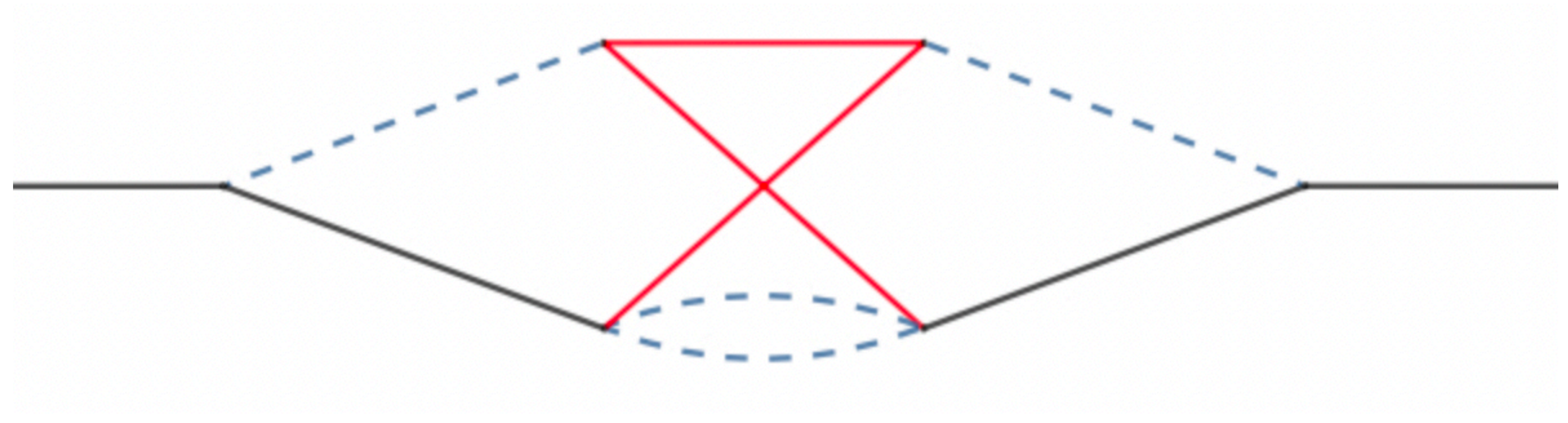


EXAMPLE

Plot of the imaginary at $O(\epsilon^0)$



$I_{335} =$



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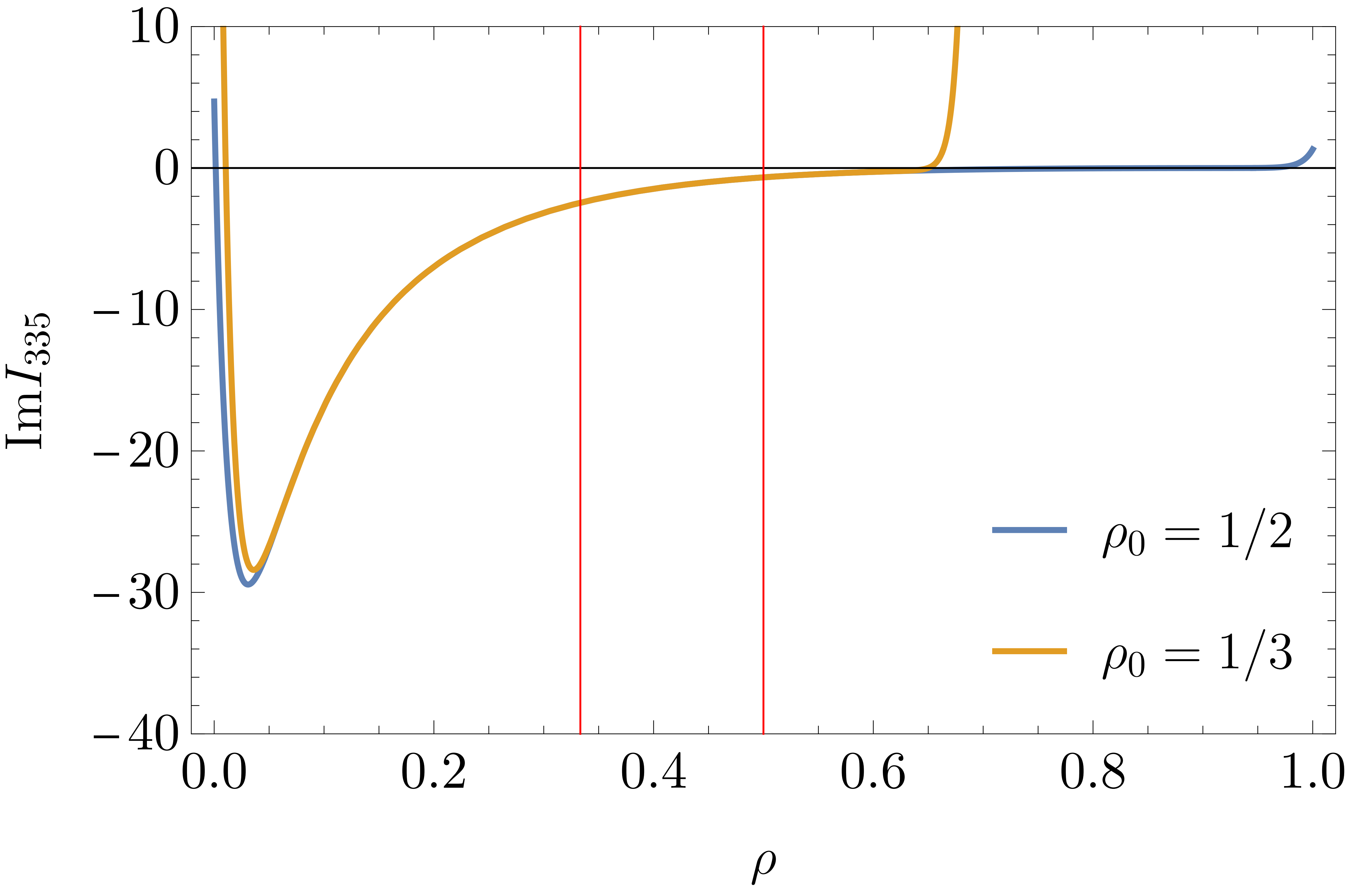
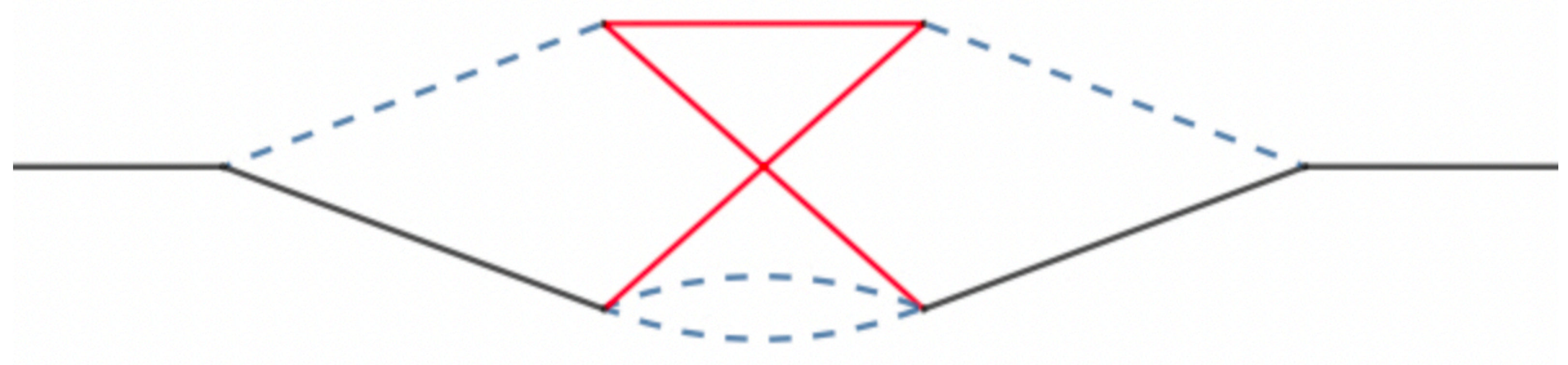


$I_{335} =$ 

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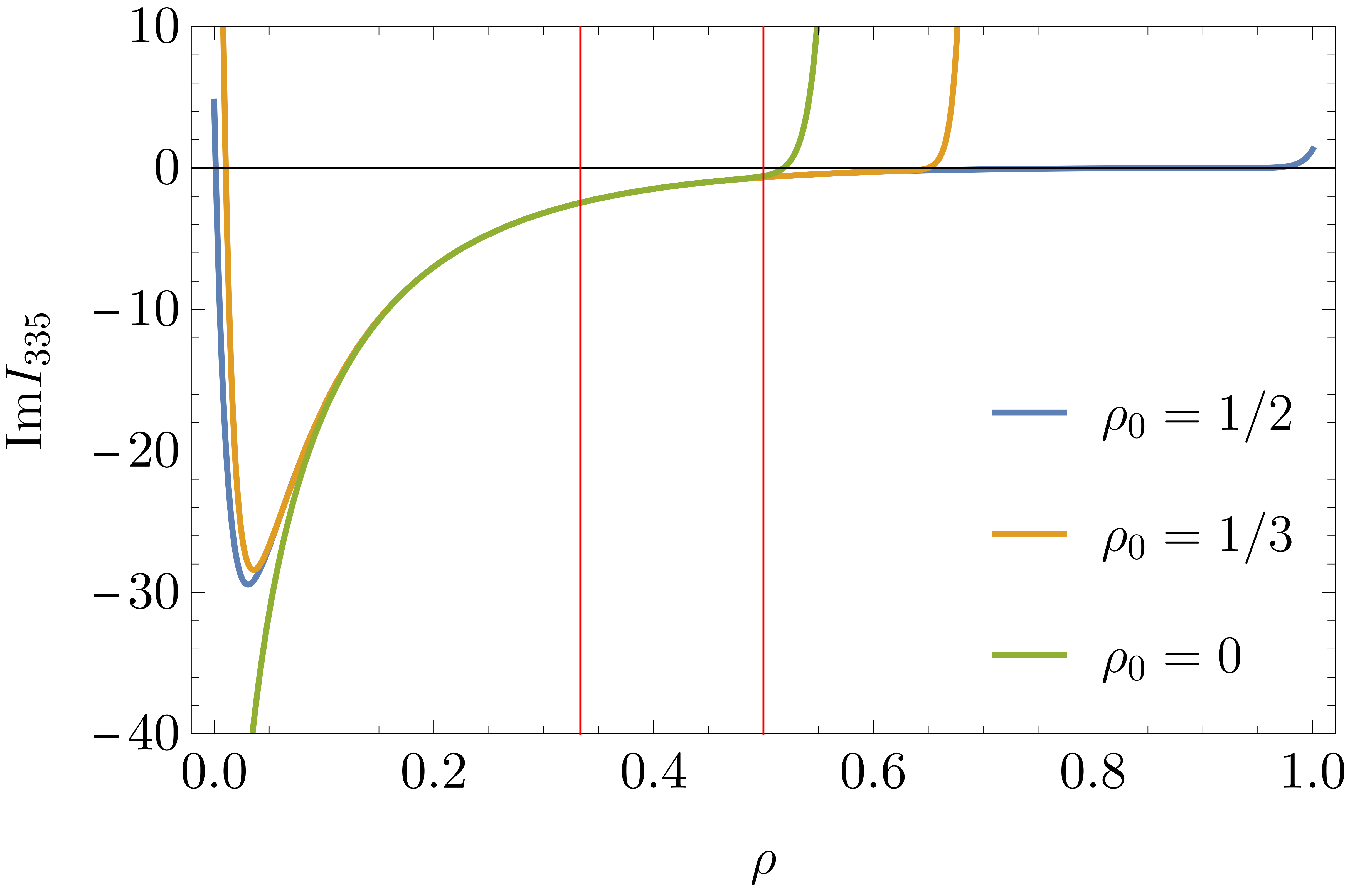
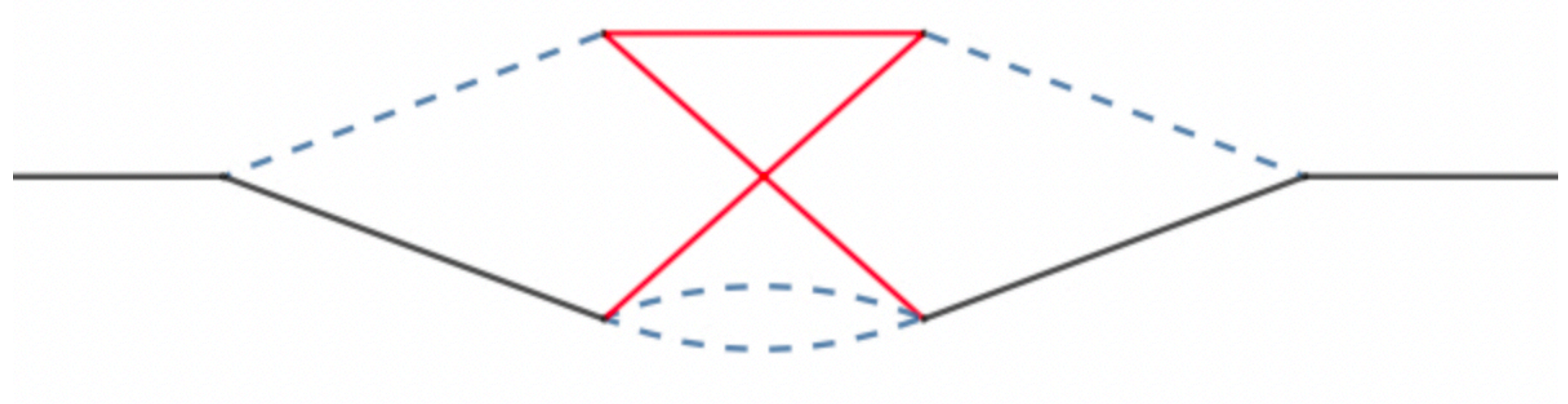


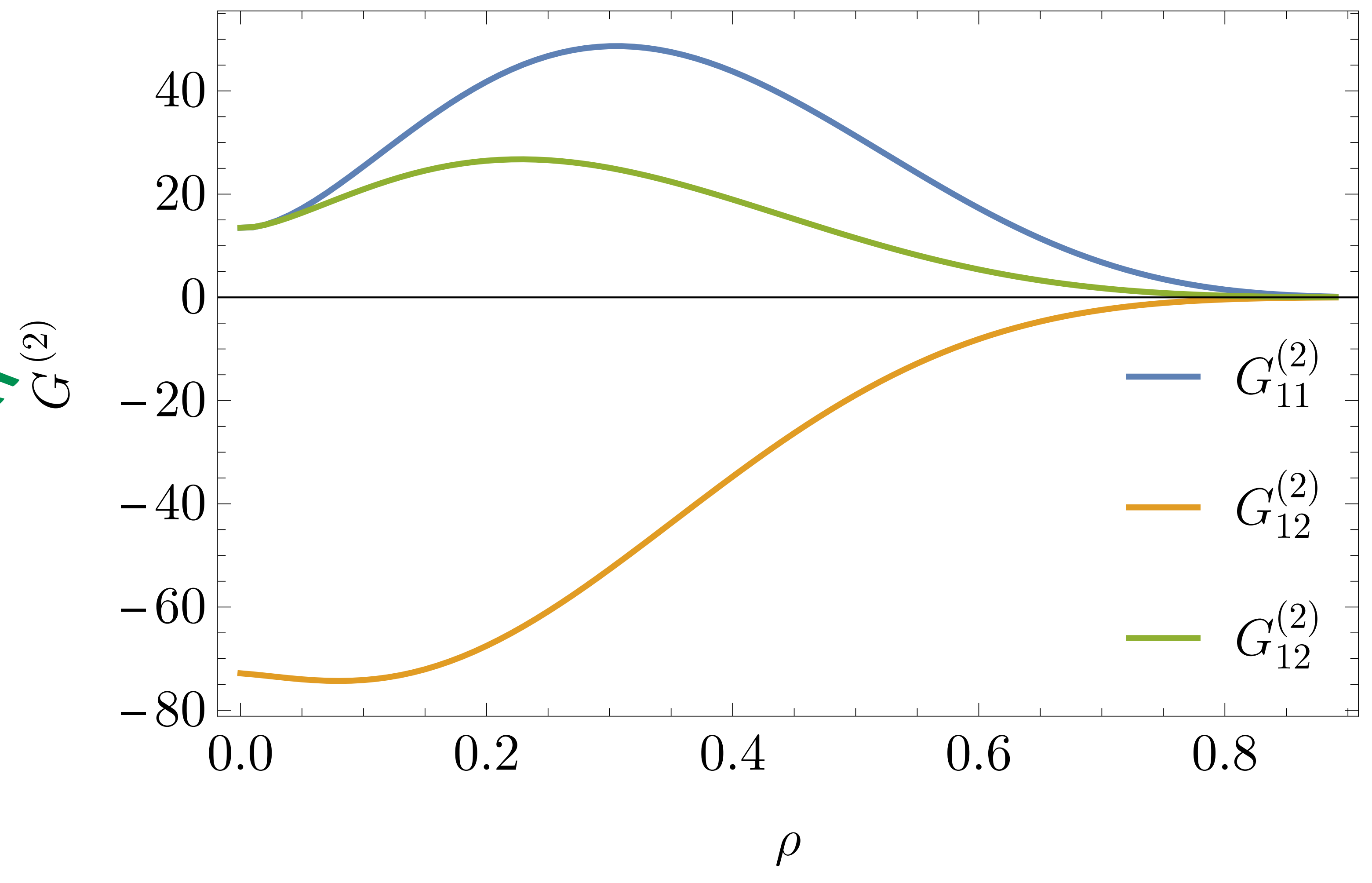
$I_{335} =$ 

RESULTS

Egner, **MF**,, Schönwald, Steinhauser, in preparation

$$\Gamma^{q_1q_2q_3} = \frac{G_F^2 m_b^5}{192\pi^3}|V_{CKM}|^2 \left[C_1^2(\mu_b)\, G_{11} + C_1(\mu_b)C_2(\mu_b)\, G_{12} + C_2^2(\mu_b)\, G_{22} \right]$$

NNLO interference terms



Total rate is scheme independent

$$\Gamma^{cd\bar{u}} = \Gamma_0 \left[1.89907 + 1.77538 \frac{\alpha_s}{\pi} + 14.1081 \left(\frac{\alpha_s}{\pi} \right)^2 \right]$$

with $m_b^{\rm OS} = 4.7\,{\rm GeV}, m_c^{\rm OS} = 1.3\,{\rm GeV}, \alpha_s = \alpha_s^{(5)}(m_b)$

NNLO Wilson coefficients

	$i=1$	$i=2$
$C_i^{(0)}(\mu_b)$	-0.2511	1.100
$C_i^{(1)}(\mu_b)$	4.382	-2.016
$C_i^{(2)}(\mu_b)$	36.63	-82.19

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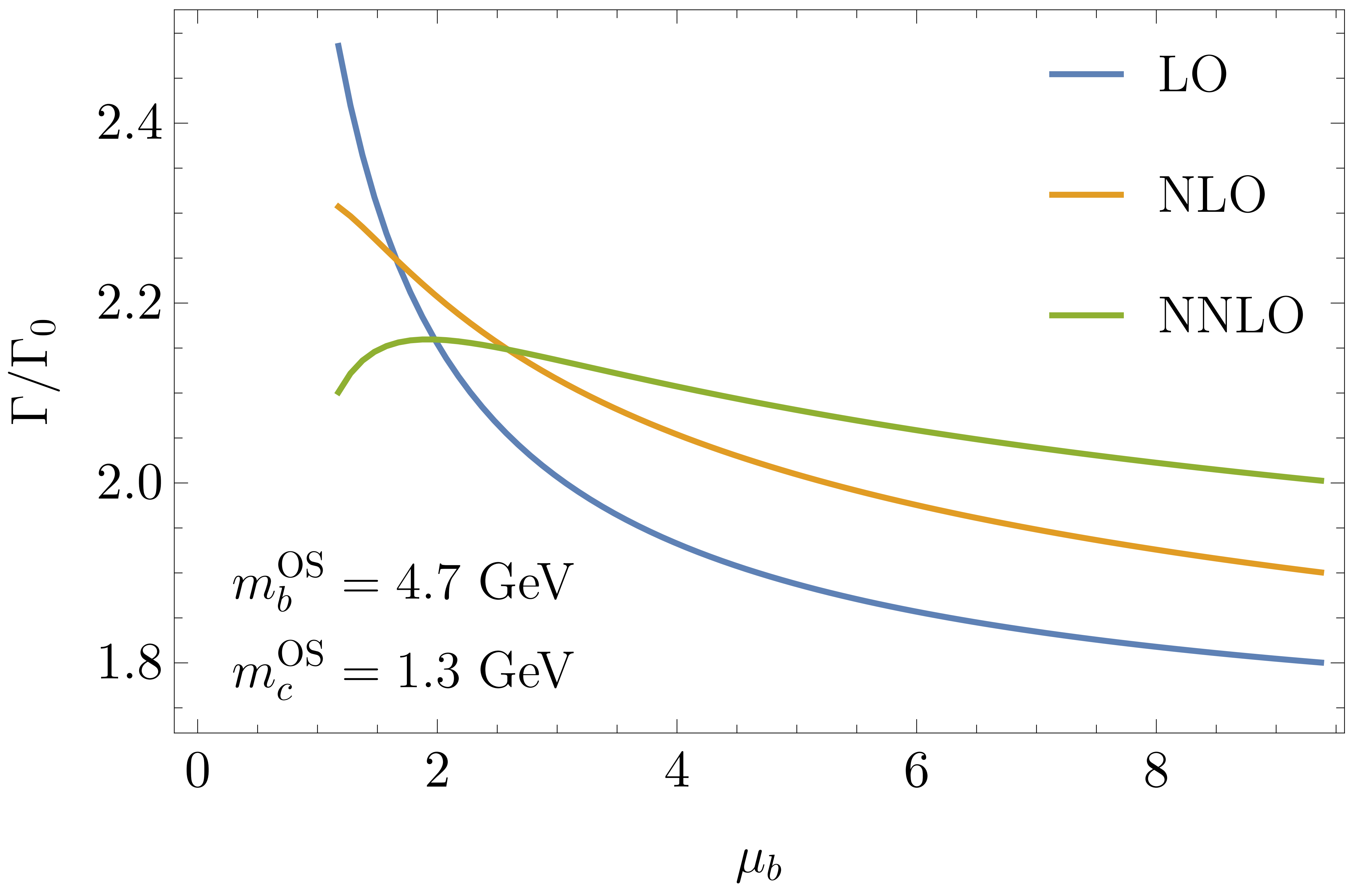
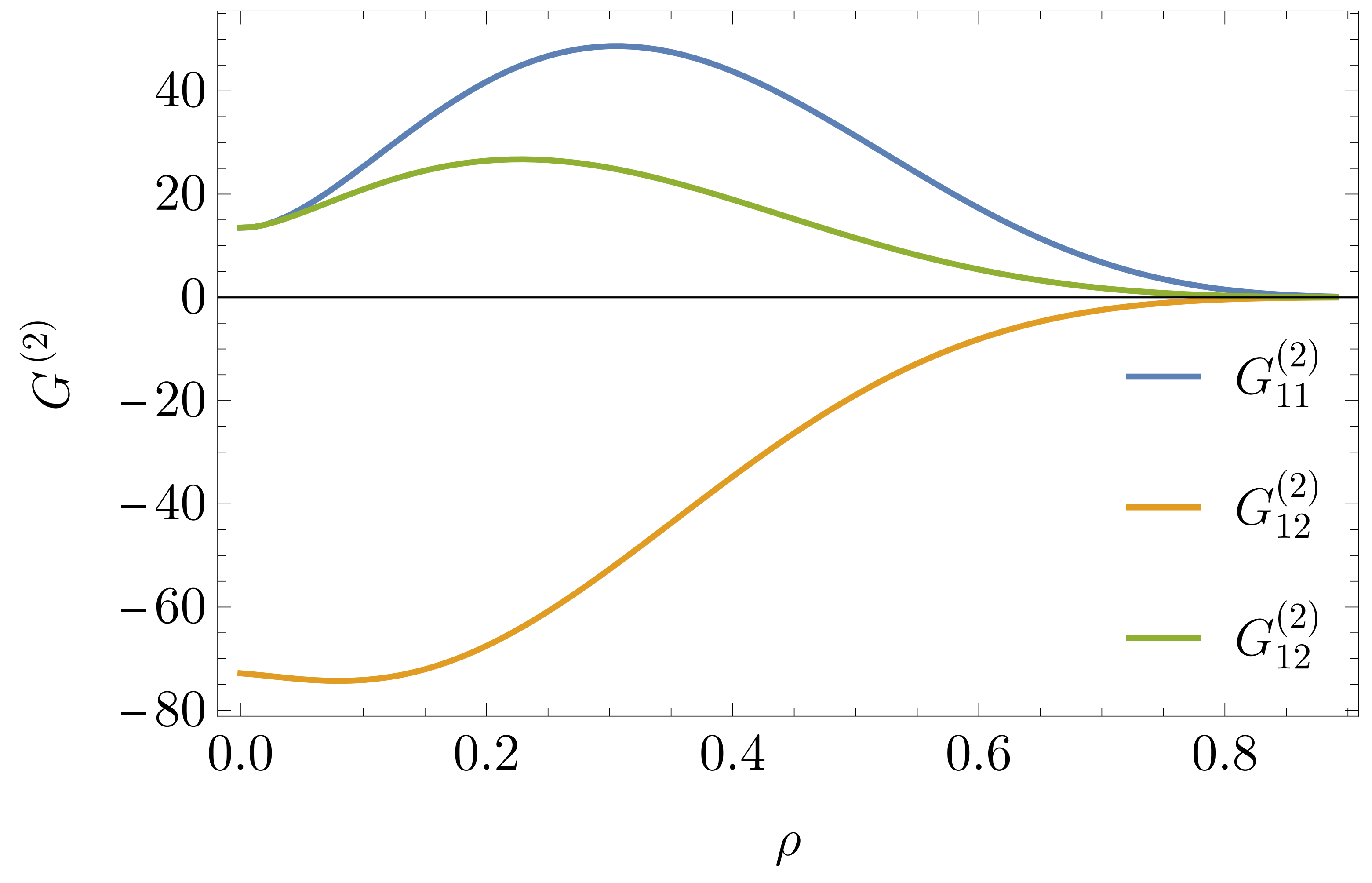


NNLO interference terms



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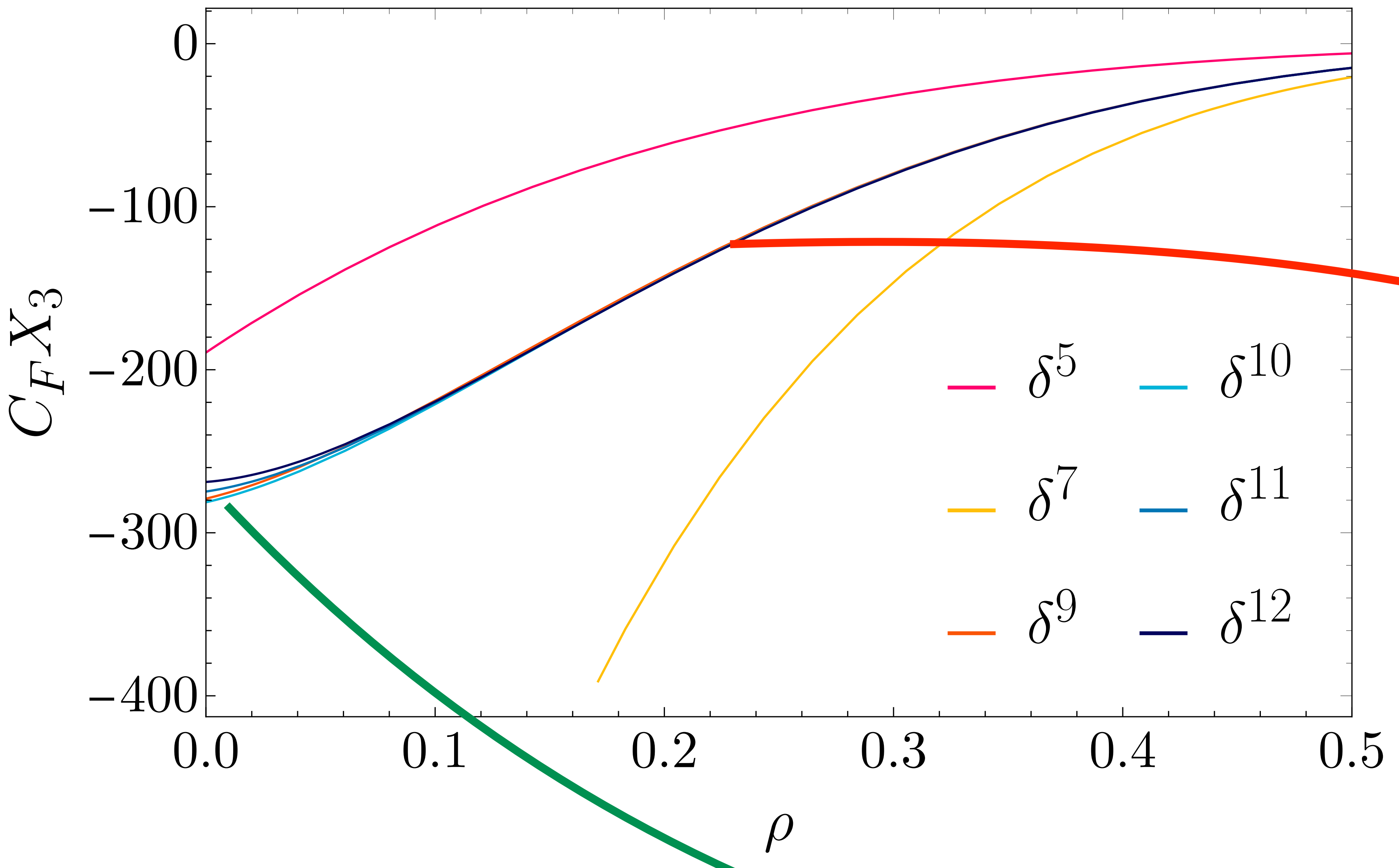
THIRD ORDER CORRECTIONS TO $b \to u l \bar{\nu}_l$ DECAY

MF, Usovitsch, hep-ph/2310.03685

Equal mass expansion $\delta = 1 - m_c/m_b \ll 1$



$$\Gamma_{\rm sl} = \frac{G_F^2 m_b^5 A_{\rm ew}}{192\pi^3} |V_{cb}|^2 \left(X_0(\rho) + C_F \sum_n \left(\frac{\alpha_s}{\pi}\right)^n X_n(\rho) \right)$$

with $\rho = m_c/m_b$

$$C_F X_3(\rho = 0.28) = -91.2 \pm 0.4\,(0.4\%)$$

$$C_F X_3(\rho = 0) = -269 \pm 27\,(10\%)$$

MF, Schönwald, Steinhauser, *Phys.Rev.D* 104 (2021) 016003, JHEP 08 (2022) 039.

$m_b^{\rm kin}(1\text{ GeV}) : \overline{m}_c(3\text{ GeV})$

$$\Gamma_{\rm sl} \simeq 1 - 0.019\,|_{\alpha_s} + 0.019\,|_{\alpha_s^2} + 0.032\,(9)\,|_{\alpha_s^3}$$

PHASE SPACE RATIO C

$$C = \left|\frac{V_{ub}}{V_{cb}}\right|^2 \frac{\Gamma(B \to X_c l \bar{\nu}_l)}{\Gamma(B \to X_u l \bar{\nu}_l)}$$

$$= 0.568 \pm 0.007 \pm 0.010\,(2.1\%)$$

Gambino, Misiak, hep-ph/0104034,
Gambino, Giordano, hep-ph/0805.0271,
Alberti, et al, hep-ph/1411.6560

Significant source of uncertainty

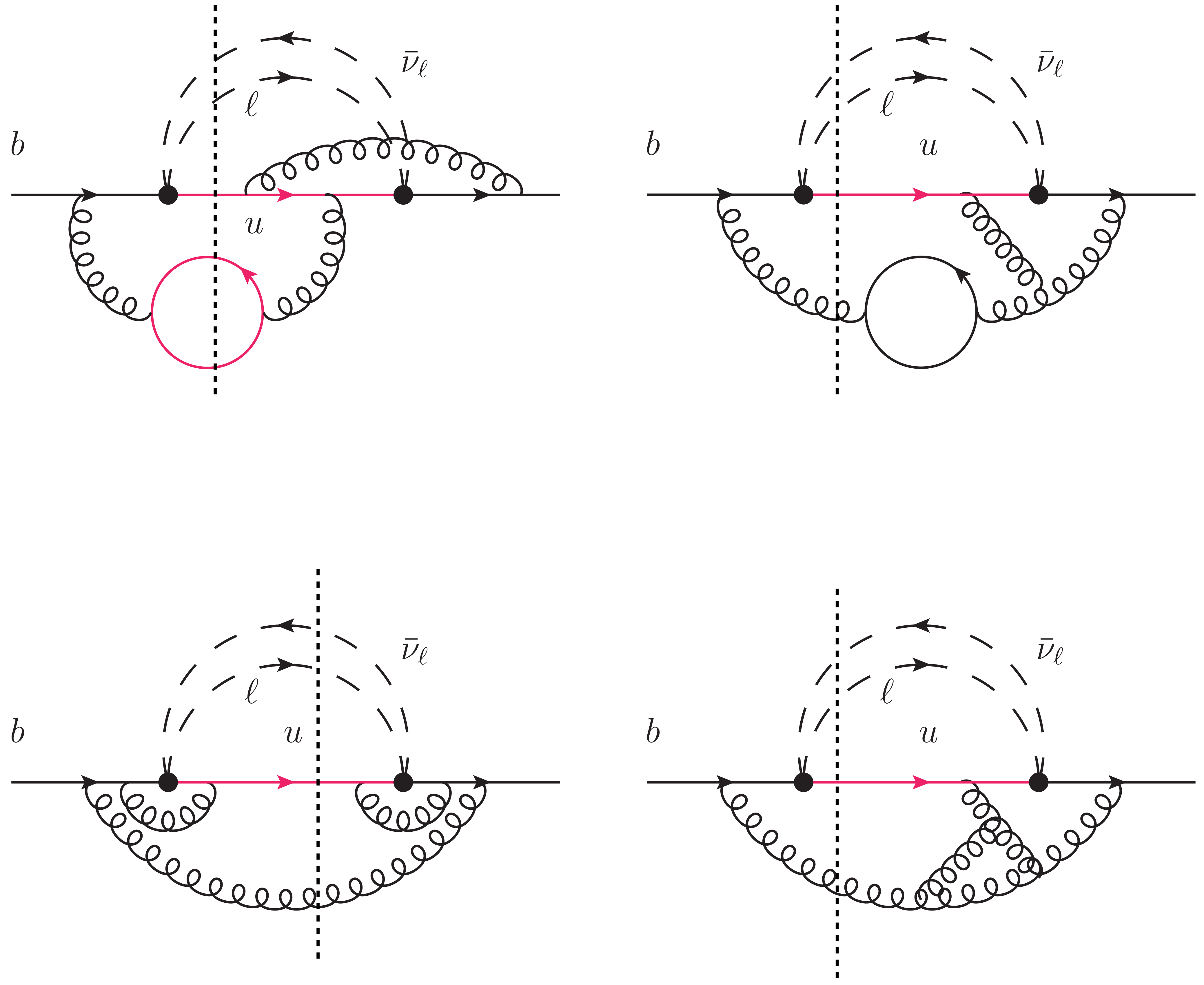
- $B \to X_s \gamma$
- $B \to X_s l \bar{l}$

Known up to NNLO

$$\mathrm{Br}(B \to X_s \gamma)_{E_\gamma > E_0} = \frac{\mathrm{Br}(B \to X_c l \bar{\nu}_l)}{C} \left|\frac{V_{ts}^{\star} V_{tb}}{V_{cb}}\right|^2 \frac{6\alpha_{\mathrm{em}}}{\pi} \left[1 + \delta_{NP}\right] P(E_0)$$

Normalisation factor: known up to N3LO?

$$|V_{ts}^{\star} V_{tb}|^2 = [1 + \lambda^2(2\bar{\rho} - 1) + O(\lambda^4)]\,|V_{cb}|^2$$



Fermionic corrections

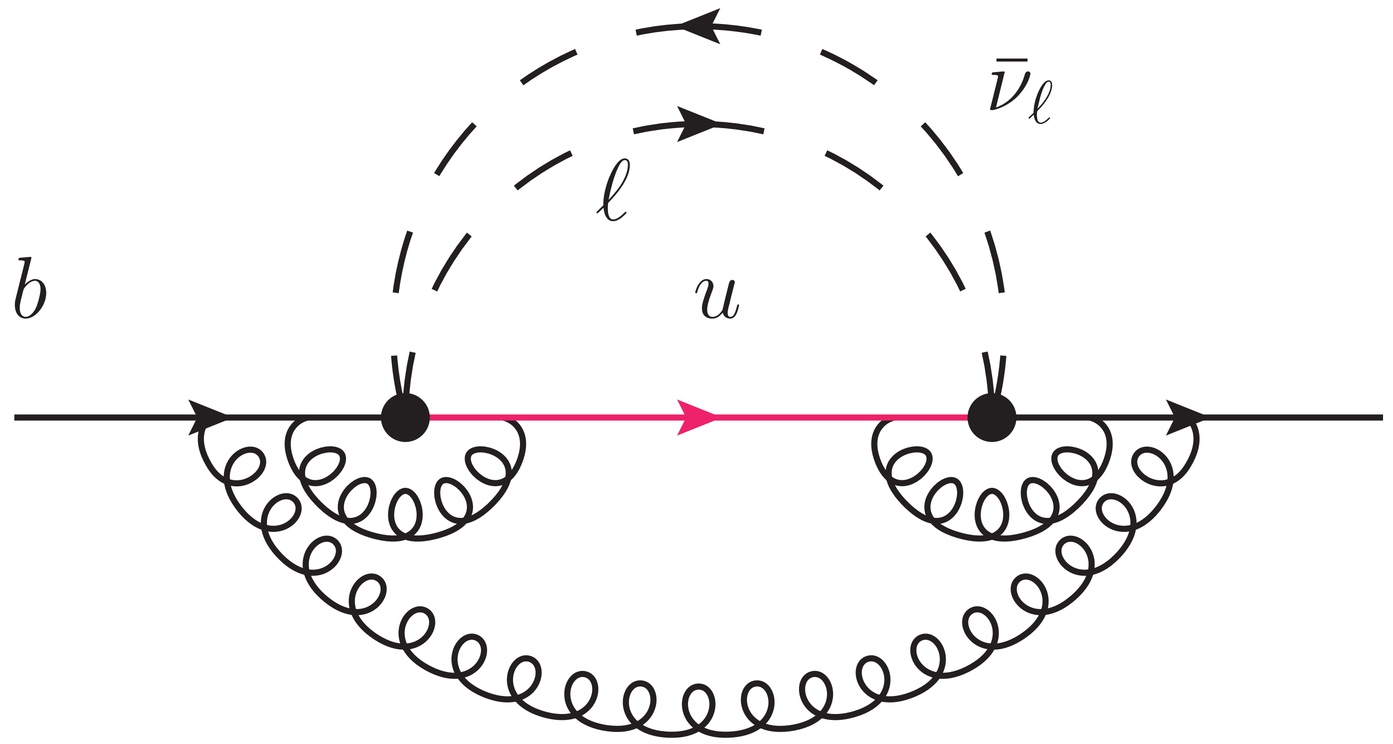
$$X_3 = N_L^2 T_F^2 X_{N_L^2} + N_H^2 T_F^2 X_{N_H^2} + N_H N_L T_F^2 X_{N_H N_L}$$
$$+ N_L T_F (C_F X_{N_L C_F} + C_A X_{N_L C_A})$$
$$+ N_H T_F (C_F X_{N_H C_F} + C_A X_{N_H C_A})$$
$$+ C_F^2 X_{C_F^2} + C_F C_A X_{C_F C_A} + C_A^2 X_{C_A^2}$$

Bosonic corrections

IBP REDUCTION AT 5 LOOPS

Challenging 5loop families:

12 propagators + 8 numerators



Trade electron-neutrino loop for a denominator raised to a symbolic power

$$\int d^dp \frac{p^{\mu_1}\ldots p^{\mu_N}}{(-p^2)[-(p-q)^2]} = \frac{i\pi^{2-\epsilon}}{(-q^2)^{\epsilon}} \sum_{i=0}^{[N/2]} f(\epsilon, i, N) \left(\frac{q^2}{2}\right)^i \{[g]^i[q]^{N-2i}\}^{\mu_1\ldots\mu_N}$$

Map 5-loop families into 4-loop ones

$$I_5(n_1, n_2, \ldots, n_{20}) \leftrightarrow \sum_{\vec{m}\in M} f_{\vec{m}}(\epsilon)\, J_{4\epsilon}(m_1, m_2, \ldots, m_{14})$$

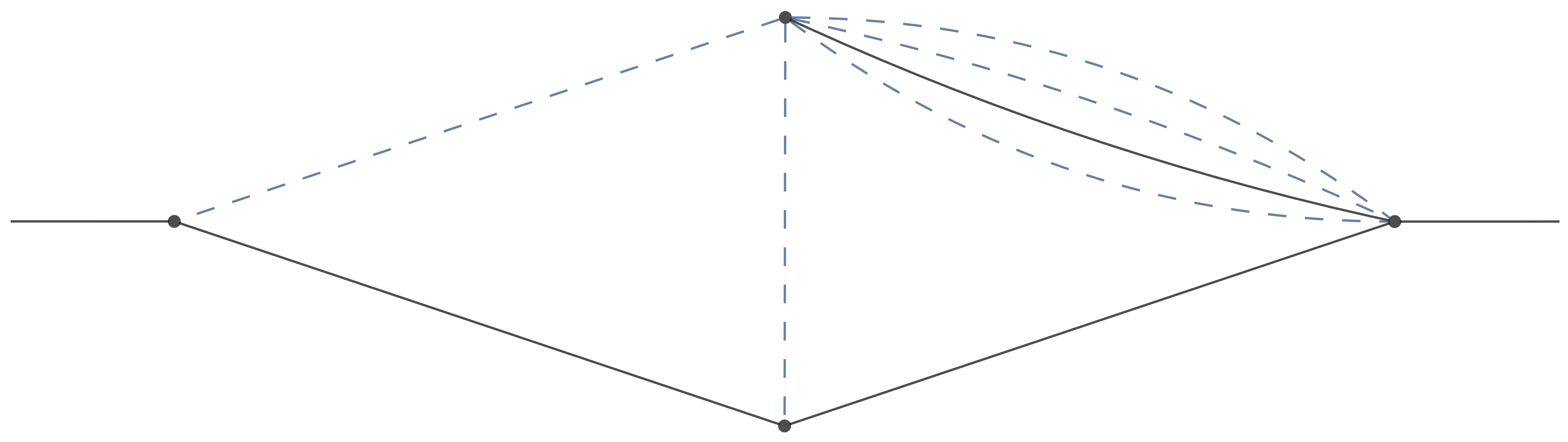
Use Kira with: **`symbolic_ibp: [1]`**

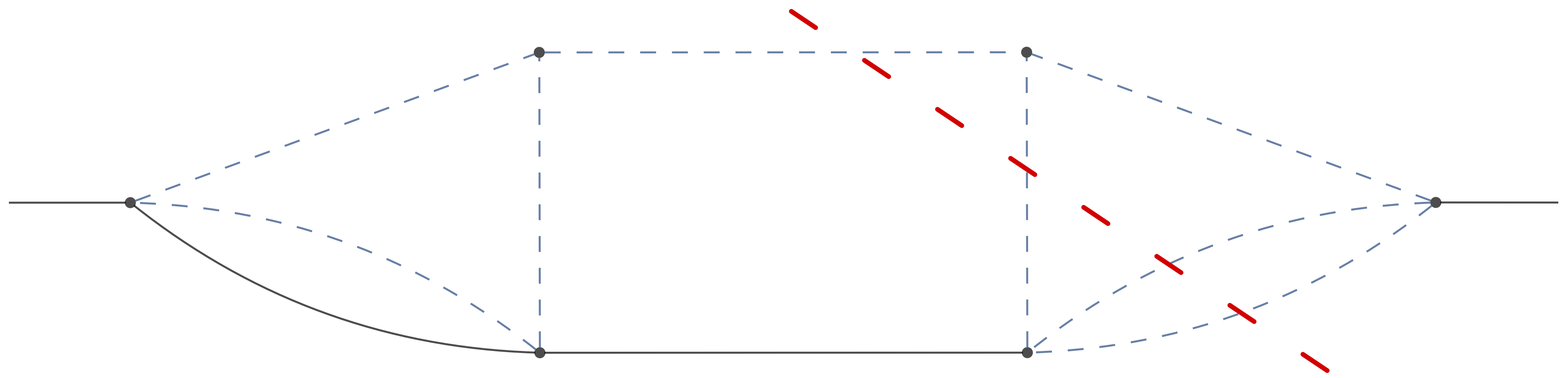
Klappert, Lange, Maierhöfer, Usovitsch, Comput. Phys. Commun. 266 (2021) 108024
Klappert, Lange, Comput.Phys.Commun. 247 (2020) 106951

ELIMINATE SECTORS WITHOUT CUTS

We eliminate up to 70% of the non-trivial sectors

- Identify non-trivial sectors
- For each family, identify the sectors with a physics cut



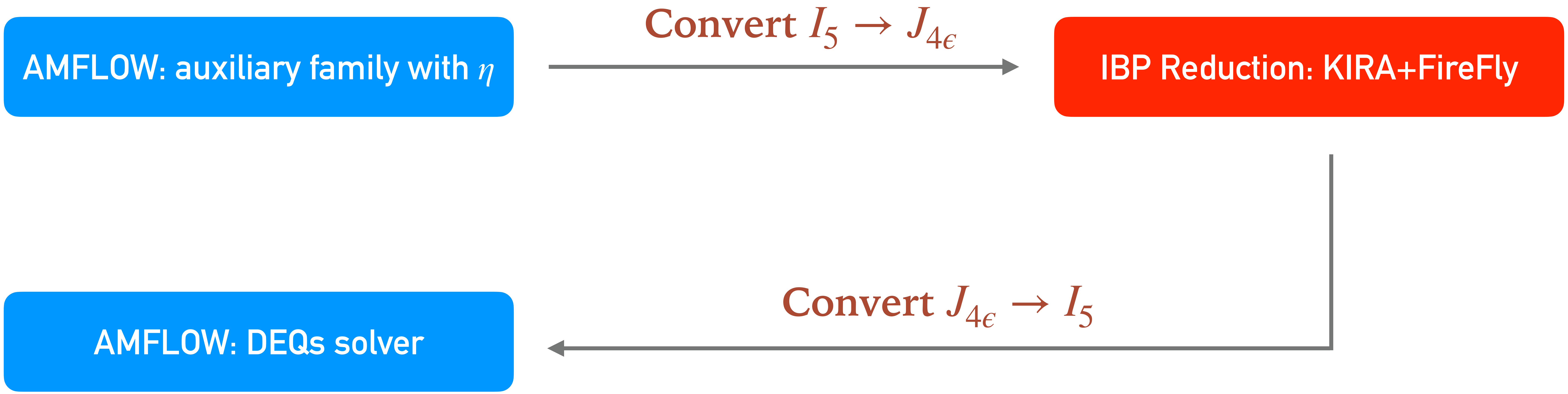


- Set to zero sectors without cuts: **zero_sectors: [1,2, ...]**
- Full reduction (up to 5 scalar products) with Kira+FireFly

$$I_5(n_1, n_2, \ldots, n_{20}) \leftrightarrow \sum_{\vec{m}\in M} f_{\vec{m}}(\epsilon)\, J_{4\epsilon}(m_1, m_2, \ldots, m_{14})$$

NUMERICAL EVALUATION WITH AMFLOW

- 48 families - 1369 master integrals



- All non-trivial sectors must be included
- Requires 40 digits of precision

RESULTS

	This work	Ref. [28]	Difference
$T_F^2 N_L^2$	-6.9195	-6.34 (42)	8.3%
$T_F^2 N_H^2$	-1.8768×10^{-2}	$-1.97\,(42) \times 10^{-2}$	5.0%
$T_F^2 N_H N_L$	-1.2881×10^{-2}	$-1.1\,(1.1) \times 10^{-2}$	12%
$C_F T_F N_L$	-7.1876	$-5.65\,(55)$	22%
$C_A T_F N_L$	42.717	39.7 (2.1)	7%
$C_F T_F N_H$	2.1098	2.056 (64)	2.5%
$C_A T_F N_H$	-0.45059	$-0.449\,(18)$	0.4%

$C_F X_3 = 280.2$ fermionic

-536.4 bosonic, large N_c

$-11.6\,(2.7)$ bosonic, subleading N_c

$= -267.8\,(2.7)$

MF, Usovitsch, hep-ph/2310.03685

- ➤ Parallel calculation large-N_c limit

Chen, Li, Li, Wang, Wand, Wu, hep-ph/2309.00762

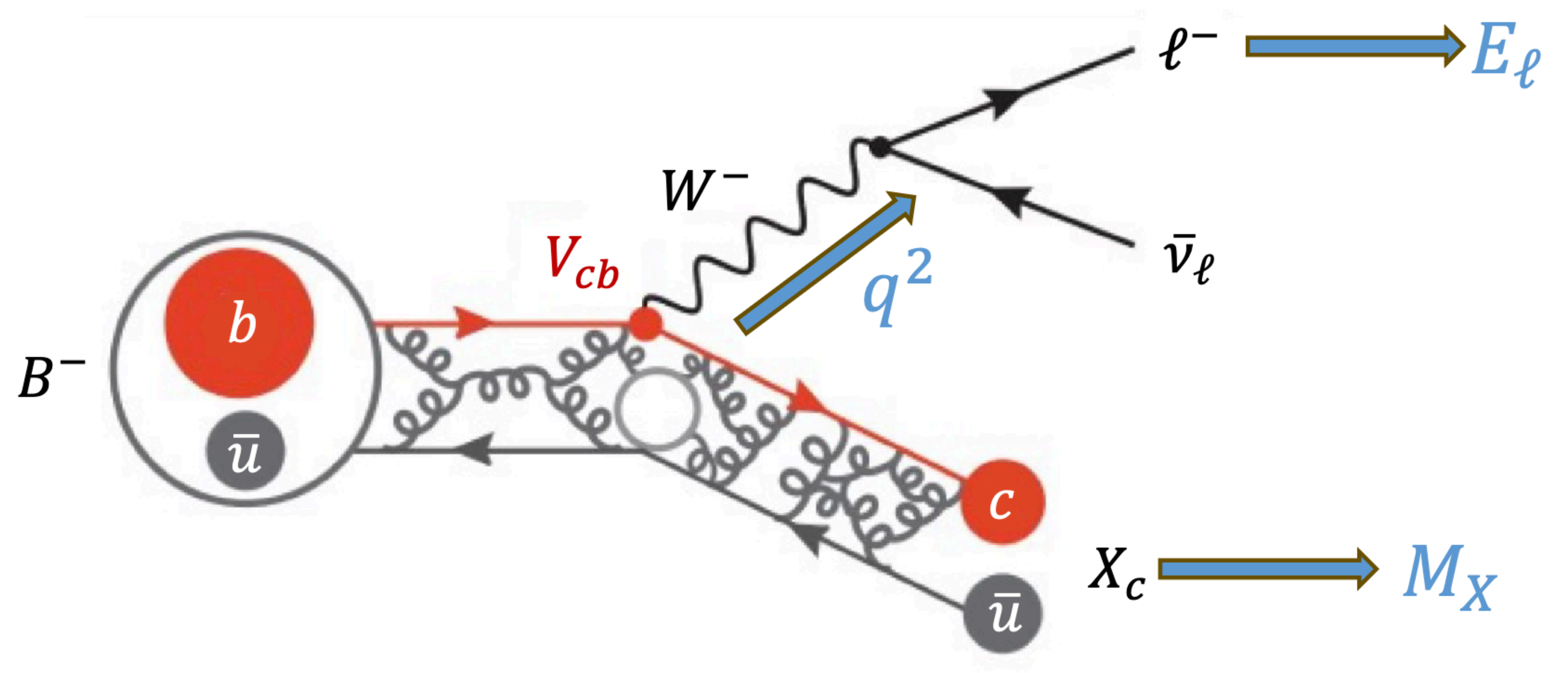
- ➤ Compatible with previous estimate

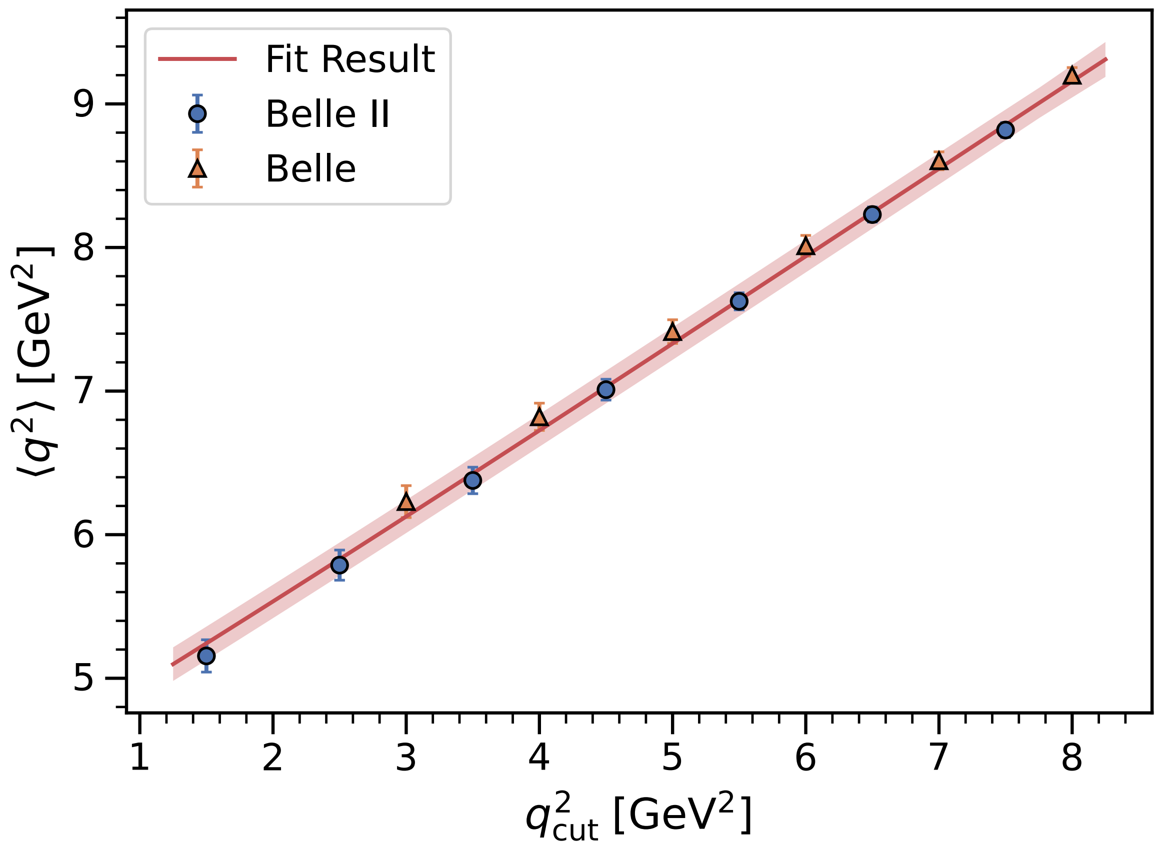
$$C_F X_3(\rho = 0) = -269 \pm 27\,(10\%)$$

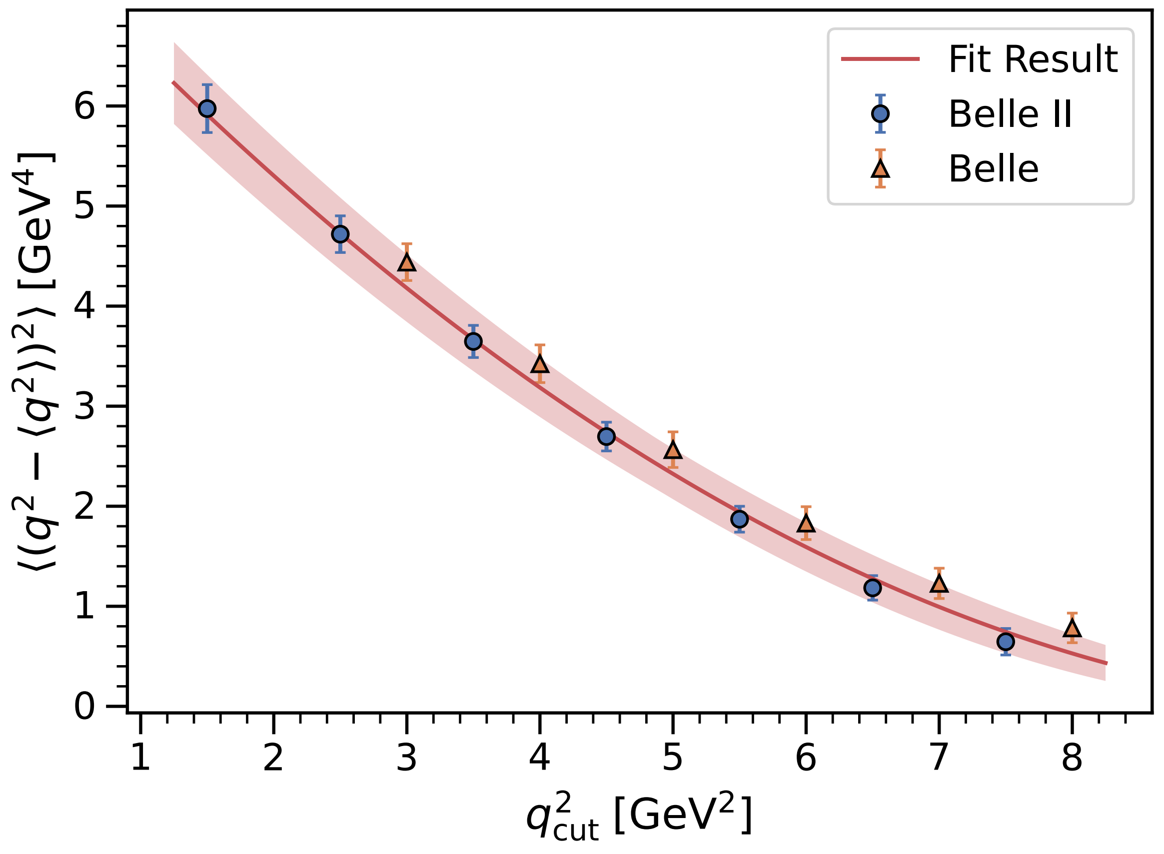
MF, Schönwald, Steinhauser, *Phys.Rev.D* 104 (2021) 016003, JHEP 08 (2022) 039.

NNLO CORRECTIONS TO q^2 SPECTRUM

MF, Herren,, hep-ph/2403.03976







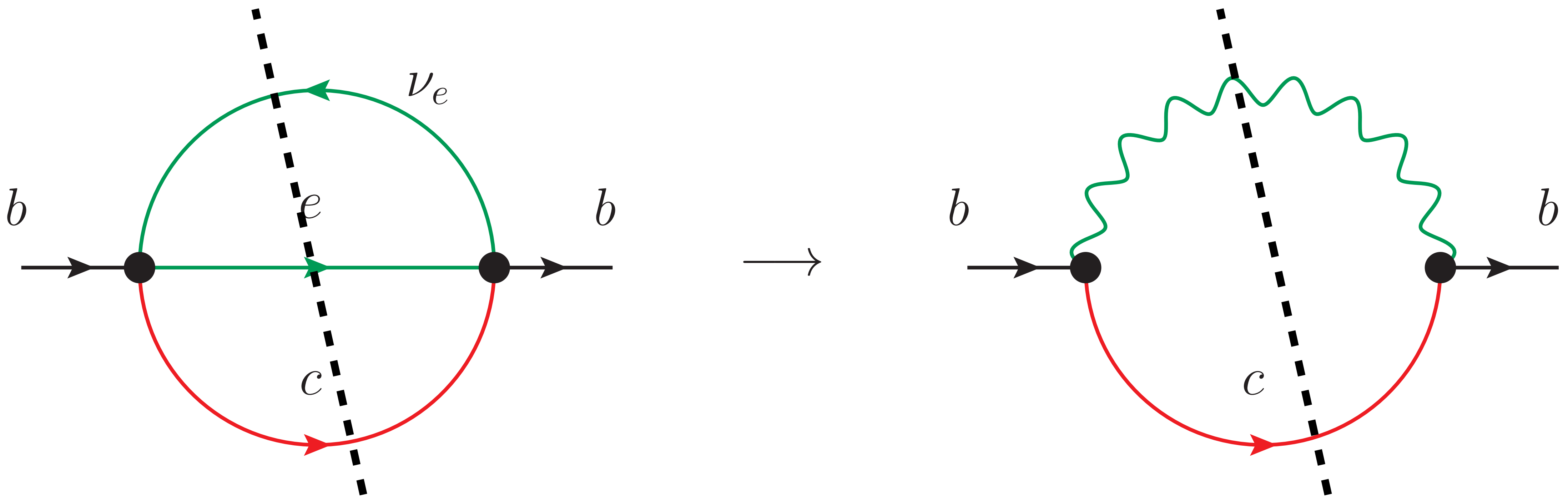
Bernlochner, MF, et al, 2205.10274 [hep-ph]
see also: **MF**, Mannel, Vos, JHEP 02 (2019) 177

$$\frac{d\Gamma}{d\hat{q}^2} = \frac{G_F^2 m_b^5}{192\pi^3} |V_{cb}|^2 \left[F_0(\rho, \hat{q}^2) + \frac{\alpha_s}{\pi} F_1(\rho, \hat{q}^2) + \left(\frac{\alpha_s}{\pi}\right)^2 F_2(\rho, \hat{q}^2) \right] + O\left(\frac{1}{m_b^2}\right)$$

Jezabeck, Kühn, Nucl. Phys. B 314 (1989) 1
Moreno, Mannel, Pivovarov, *Phys.Rev.D 105 (2022) 5, 054033*

Normalised moments $\langle (q^2)^n \rangle_{q^2_{\rm cut}} = \int_{q^2 > q^2_{\rm cut}} (q^2)^n \frac{d\Gamma}{dq^2} dq^2 \Big/ \int_{q^2 > q^2_{\rm cut}} \frac{d\Gamma}{dq^2} dq^2$

$$\delta(p_L^2 - q^2) \times$$



Jezabeck, Kühn, Nucl. Phys. B 314 (1989) 1
Moreno, Mannel, Pivovarov, *Phys.Rev.D 105 (2022) 5, 054033*

Integration w.r.t. neutrino-electron phase space

$$\mathscr{L}^{\mu\nu}(p_L) = \int L^{\mu\nu} d\Phi_2(p_L; p_l, p_\nu) = \frac{1}{384\pi^5}\left(1 - \frac{m_\ell^2}{p_L^2}\right)^2 \left[\left(1 + \frac{2m_\ell^2}{p_L^2}\right) p_L^\mu p_L^\nu - g^{\mu\nu} p_L^2 \left(1 + \frac{m_\ell^2}{2p_L^2}\right)\right]$$

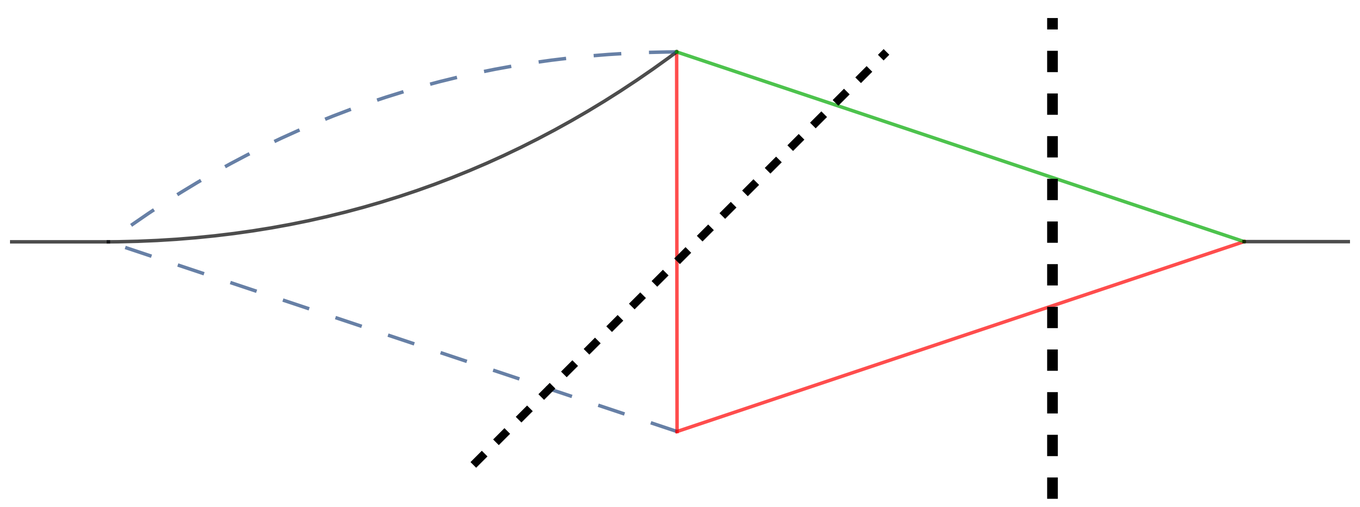
Inverse unitarity

$$\delta(p_L^2 - q^2) \to \frac{1}{2\pi i}\left[\frac{1}{p_L^2 - q^2 - i0} - \frac{1}{p_L^2 - q^2 + i0}\right]$$

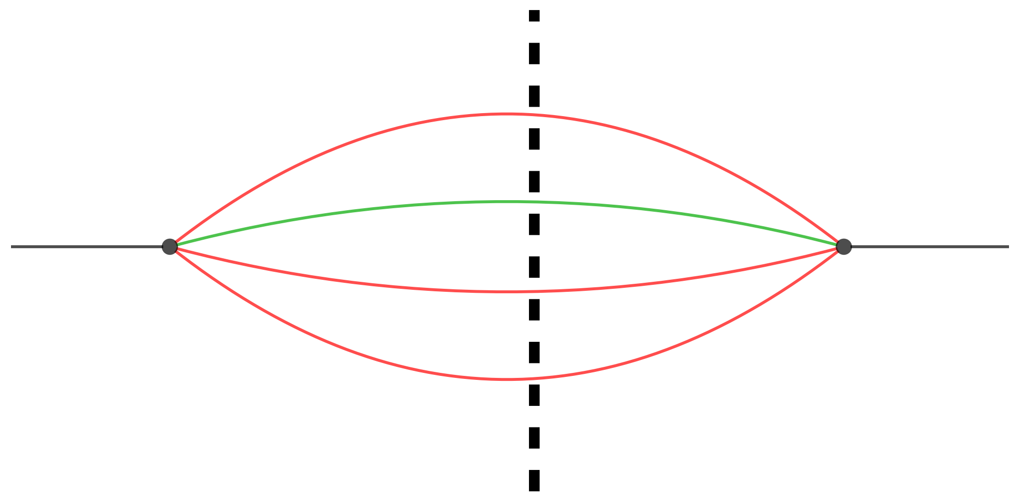
MASTER INTEGRALS

- 98 master integrals with cuts
- Ignore cuts through 3 charm quarks

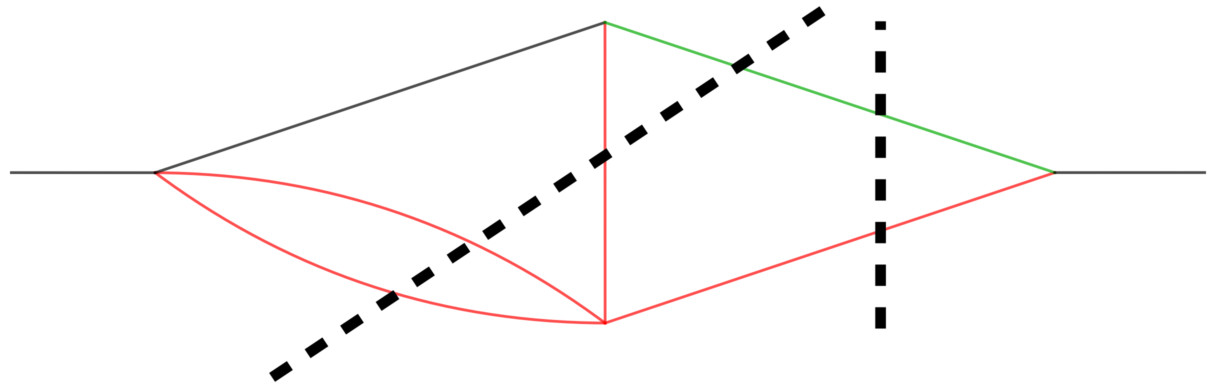
see also: Egner, **MF**, Schönwald, Steinhauser, HEP 09 (2023) 112



ONE CHARM CUTS

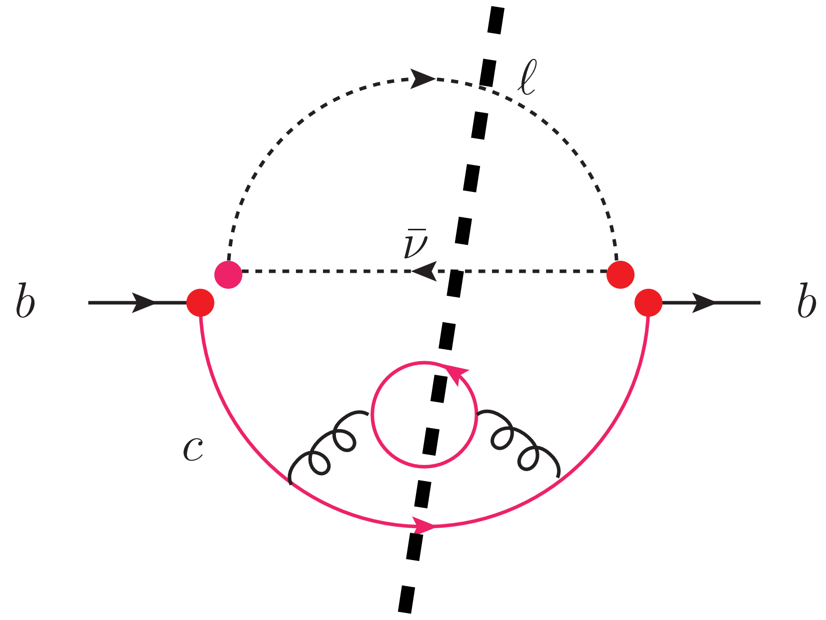


THREE CHARMS



THREE CHARMS

ONE CHARM CUTS



$$\mathrm{Br}(b \to c c \bar{c} l \bar{\nu}_l) \simeq 10^{-7}$$

CANONICAL FORM

Henn, Rev. Lett. 110 (2013) 251601

Find rational transformation $\mathbb{T}(u, \rho; \epsilon)$

Libra, R.N. Lee, Comput. Phys. Commun. 267 (2021) 108058

$$\frac{\partial \vec{I}}{\partial \rho} = \hat{M}_\rho(\hat{q}^2, \rho, \epsilon)\, \vec{I}(\hat{q}^2, \rho, \epsilon)$$

$$\frac{\partial \vec{I}}{\partial \hat{q}^2} = \hat{M}_{q^2}(\hat{q}^2, \rho, \epsilon)\, \vec{I}(\hat{q}^2, \rho, \epsilon)$$

$$\vec{I} = \mathbb{T}\vec{I}' \quad \longrightarrow$$

$$\frac{\partial \vec{I}'}{\partial \rho} = \epsilon \hat{M}_\rho(u, \rho)\, \vec{I}'(u, \rho, \epsilon)$$

$$\frac{\partial \vec{I}'}{\partial u} = \epsilon \hat{M}_{q^2}(u, \rho)\, \vec{I}'(u, \rho, \epsilon)$$

Analytic solution expressed via Generalised Polylogarithms

$$G(a_1, \ldots, a_n; z) = \int_0^z \frac{dt}{t - a_1} G(a_2, \ldots, a_n; t)$$

$$G(a_1, \ldots, a_n; z) = \int_0^z \frac{dt}{t - a_1} G(a_2, \ldots, a_n; t)$$

Examples

$$G(0; z) = \log(z) \qquad G(x, z) = \log\left(1 - \frac{z}{x}\right)$$

$$G(\underbrace{0,\ldots,0}_{n}; z) = \frac{\log^n(z)}{n!} \qquad G(\underbrace{0,\ldots,0}_{n}, x, z) = -\,\mathrm{Li}_n\left(\frac{z}{x}\right)$$

Fast numerical evaluation: GiNaC+PolyLogTools

http://www.ginac.de
Duhr, Dulat, JHEP 08 (2019) 135

$$G\left(x, \frac{1+x^2}{x}, x, \frac{1}{x}; z\right)\Bigg|_{x=1/2,\, z=1/3} = 0.00151860208899279\ldots$$

BOUNDARY CONDITIONS

AMFLOW with ~ 250 digits

GPLs with ~ 250 digits

PolyLogTools + GiNaC: Duhr, Dulat, JHEP 08 (2019) 135; Bauer, Frink, Kreckel, , J. Symb. Comput. 33 (2002) 1

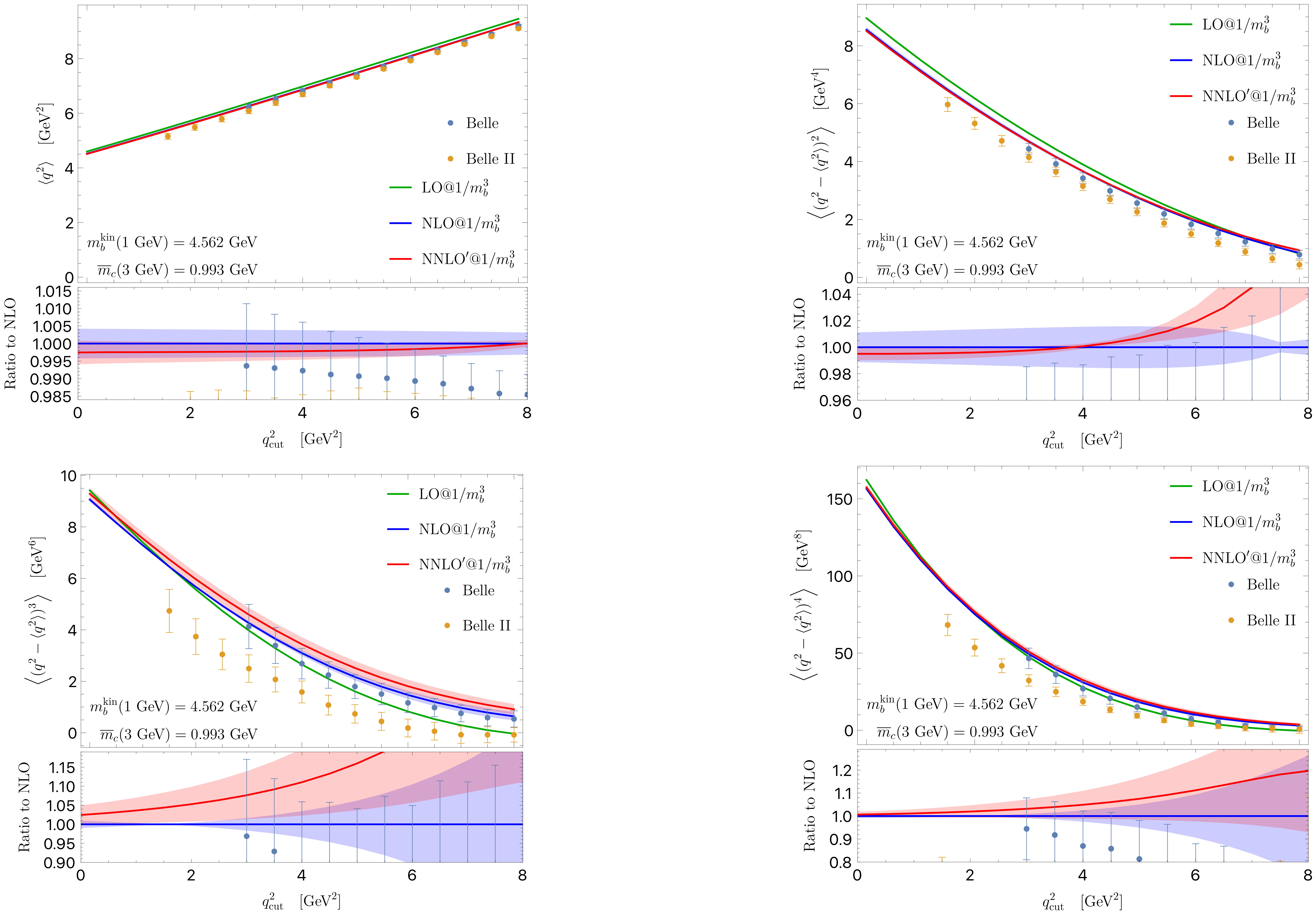
$$(\hat{q}^2, \rho^2) \in \{(1/5, 1/5), (4/25, 4/25)\}$$

PSLQ constant reconstruction

Ferguson, Bailey, Arno, Mathematics of Computation 68 (1999) 351.

Analytic boundary

$$2.18269754013877673 46\ldots = \frac{13\pi^2}{72} + \frac{\zeta_3}{3}$$



MF, Herren,, hep-ph/2403.03976

CONCLUSIONS

- New numerical and semi-analytic methods for Feynman integrals.
- Breakthrough for phenomenology!
- Several calculations are now possible, improved theoretical predictions.

- Predictions for B mesons affected by large $\alpha_s = 0.22$ and m_c/m_b
- NNLO corrections to q^2 moments allows for the better extraction of V_{cb}
- New independent evaluations of $\Gamma(b \to ul\bar{\nu}_l)$.
- Implications for the phase-space ratio C ongoing.
- Soon update for B-meson lifetimes including NNLO QCD corrections.