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PHYSICA VIVA

Contents

Introduction

The philosophy department of the Harvard University is located in a beautiful "red-brick" Emerson Hall. Emerson Hall was named in honor of Ralph Waldo Emerson, an American philosopher and theologian. The front of Emerson Hall has on its facade the inscription (installed when the R. W. Emerson was a dekan of the philosophy department): *What is man that you are mindful of him*?

The sentence was taken from the Psalm 8 of David:

> When I consider your heavens
> the work of your fingers
> the moon and the stars,
> which you have set in place
> what is man that you are mindful of him
> the son of man that you care for him
> Psalm 8:3

To measure the strength of religious belief in an Emerson time, the eminent researcher J. Leuba conducted a landmark survey in 1916. He found that 60 per cent of 1000 randomly selected scientists did not believe in a God and predicted that such disbelief would increase as education spread [2]. To test that prediction E. J. Larson and L. Witham replicated Leuba's survey in 1996 [1]. The result: about 40 per cent of scientists still believe in a personal God and an afterlife.

Ninety years ago Leuba asserted: "The essential problem facing organized Christianity is constituted by the widespread rejection of its two fundamental dogmas". Though a noted psychologist, Leuba misjudged either the human mind or the ability of science to satisfy all human needs.

In a book *Scientists as Theologians* [3] John Polkinghorne surveyed the thinking of three scientist-theologians, Ian Barbour, Artur Peacocke and John Polikinghorne. For each of them intellectual formation had lain in science and it was only later in life that they turned to theology.

Where theological understanding does come in to augment and complement scientific understanding is in relation to certain limit questions that arise out of scientific experience but which transcend science's own self limited range of enquiry. They revolve around two fundamental metaquestions.

(1) Why is the universe so deeply intelligible? Putting it more bluntly, why science is possible? Our ability to understand the physical world seems vastly to exceed anything that could plausibly be held to correspond to evolutionary necessity or to be a happy accidental spin-off from survival requirements. Science exploits the wonderful, rational transparency of the physical world but does not explain it. If the universe is the creation of the rational God, then it is possible to

understand its intelligibility as due to its being shot through with signs of the mind of its Creator signs that are accessible to the thoughts of creatures made in the image of the Creator.

(2) Why is the Universe so special? This question arises from the recognition enshrined in the Anthropic Principle that the laws of nature are fine-tuned to the high degree of specifity found to be necessary to make the evolution of carbon-based life a possibility. Positive response to the both metaquestions is my credo as the 65-year-old physicist and teacher of the science teachers. During my ten-year academic work at Science Teachers College of Warsaw University I lifted the taboo regarding theology and anthropic arguments in physics teacher education. My position is defended by several arguments: (a) Regardless of scholars position is this issue, students will continue to encounter endless teleological (anthropomorphic formulations in popular science movies and books. (b) A review of scientists attitude to religion shows that there is definitely no consensus on a universal rejection of theological formulation and explanations.

Theology is a complement to science and not an alternative. Cosmology and physics have now moved onto new stage. Attention focusses on the Anthropic Principle's recognition of the astonishing specifity that is required of the fundamental physical laws of a universe if it is to be capable of evolving carbon-based life and in the result the human beings.

The book is an elementary invitation to physics and reflects the taste of its author. I am of the opinion that *fides quaerers intellectum* (faith seeking understanding). The Physica Viva is a first step on this road.

Chapter 1

Motion in space

1.1 One-dimensional motion

In each moment of our life we are prodding numbers. More precisely, each of our gestures puts us into contact with three numbers: the real numbers that at least in this small region of the universe describe the location of every point in space. There is no escaping those numbers. Wherever you go, you live and breathe and *move* amid a swarm of constantly changing coordinates. They are your destiny.

It is not clear who first conceived of a world saturated with numerical addresses. The idea of identifying points by longitude, latitude goes back at least to *Archimedes*, but it was not formalized until 2000 years later when the seventeenth century French mathematicians *Pierre de Fermat* and *René Descartes* forged the link between geometry and algebra. Then at some points in the nineteenth century, mathematicians took an important leap of logic. If an ordered list of numbers describes a space perfectly, they reasoned, why not say that those lists of numbers are the space. As in that case why stop at three. They then boldly proceeded to define n-dimensional Euclidean space (n-space for short) for any positive integer n as the set of all n–tuples of real numbers $(x_1, \ldots, x_n)$. The symbol for such a space is $\mathbf{R}$ (for the real numbers) garnished with superscript n: $\mathbf{R}^n$. Nowadays the concept of n–dimensional Euclidean space permeates all branches of mathematics, physics and biology.

We might expect that as the number of dimensions gets larger and larger, space gets stranger and more interesting. And so it does, in the trivial sense that any space has all lower –dimensional spaces packed inside it. If planes (2–space) contain lines (1–space) and three dimensional space contains planes, then in a way, anything that can take place on a line also takes place in 3–space as well as in any higher dimensional space.[1]

1.1.1 Kinematics in one-dimension

To start with we shall describe the motion of the body in $\mathbf{R}^1$. In this case the *displacement* from some reference point $\mathbf{r}$ is equal $\mathbf{x}i$. The magnitude of the displacement x is the function of *time*, $x - x(t)$. The nature of time is complicated* and still debated by physicists and philosophers. Crudely speaking we can speak on two categories of

[1]From a deeper points of view, however every Euclidean space has its own character and as far as the number of dimensions is concerned, more is often less.

time:

Chronos – objective time which is measured by watches.

Tempus – psychological time.

Let us start with the displacement of a body:

$$x(t) = d + bt + ht^2 + pt^3, \tag{1.1}$$

where t is the *chronos*.

For a longer time, $t+\Delta t$, where Δ means "change" in the displacement equal:

$$x[t + \Delta t] = d + b(t + \Delta t) + h(t + \Delta t)^2 + p(t + \Delta t)^3,$$

and

$$\begin{aligned}\frac{\Delta x}{\Delta t} &= \frac{x[t + \Delta t] - x[t]}{\Delta t} \\ &= \frac{d - d + b(t + \Delta t - t)}{\Delta t} + \frac{h(t^2 + 2\Delta tt + (\Delta t)^2 - t^2)}{\Delta t} \\ &+ \frac{p(t^3 + 3t^2\Delta t + 3t(\Delta t)^2 + (\Delta t)^3 - t^3}{\Delta t}.\end{aligned} \tag{1.2}$$

We the *velocity* of the body

$$v = \frac{\Delta x}{\Delta t}, \quad \text{when} \quad \Delta t \to 0. \tag{1.3}$$

From formulae (1.2) and (1.3) we obtain

$$v = b + 2th + 3t^2 p, \tag{1.4}$$

and $\mathbf{v}$

$$\mathbf{v} = (b + 2th + 3t^2 p)\mathbf{i}.$$

In the same manner we define the acceleration of the body

$$a = \frac{\Delta v}{\Delta t}, \quad \text{when} \quad \Delta t \to 0. \tag{1.5}$$

From formulae (1.4) and (1.5) one obtains

$$\mathbf{a} = (2h + 6pt)\mathbf{i}. \tag{1.6}$$

Let us assume that body starts to move at the moment $t = t_0 = 0$ with velocity v_0. In that case we obtain from formula (1.4) and (1.6):

$$\begin{aligned} \mathbf{v} &= (v_0 + 2th + 3t^2p)\mathbf{i}, \\ \mathbf{a} &= (2h + 6pt)\mathbf{i}. \end{aligned} \tag{1.7}$$

On the Earth (and of course on the planets) body falls with constant $\vec{a}$. On the Earth

$$\mathbf{a} = g\mathbf{i}, \qquad g \cong 9.81\,\mathrm{m/s^2}. \tag{1.8}$$

(On the Mars $g = 3.7\,\mathrm{m/s^2}$).

Considering formula (1.8) we obtain $p = 0$

$$\boxed{\mathbf{a} = 2h\mathbf{i} = g\mathbf{i},} \tag{1.9}$$

$h = g/2$

$$\boxed{\mathbf{v} = (v_0 + gt)\mathbf{i},}$$

and

$$x(t) = d + v_0 t + \frac{gt^2}{2}. \tag{1.10}$$

For $t = t_0 = 0, \quad x(t_0) = x_0, \quad$ in that case, $d = x_0$

$$\boxed{x(t) = x_0 + v_0 t + \frac{gt^2}{2},} \tag{1.11}$$

and

$$\mathbf{x}(t) = \left(x_0 + v_0 t + \frac{g t^2}{2} \right) \mathbf{i}. \quad (1.12)$$

The limiting processes described by formulae (1.3) and (1.5) are the basis for *differential* calculus. The *derivative* of any function $f(t)$ is defined by:

$$\frac{df(x)}{dx} \equiv \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}. \quad (1.13)$$

Comparing formulae (1.3), (1.5) and (1.13) we define

> Velocity $\mathbf{v} = v\mathbf{i}$ is the first derivative of the displacement.
> Acceleration $\mathbf{a} = a\mathbf{i}$ is the first derivative of the velocity.

1.1.2 Dynamics in one-dimension, Newton Laws

In 1687 Newton published his *Principia* in which he put forth his three Laws of Motion. The First Law of Motion describes a body in the *absence* of the net force (The net force is the vector sum of all force acting on a body). Newton First Law of Motion states:

NEWTON'S FIRST LAW

> In the absence of a net force a body at rest remains at rest and body in motion continues motion along the same straight line and at constant speed.

We often call this the Law of Inertia and describe this characteristic of matter to remain in its particular state of motion as *inertia*.

Newton's Second Law explains what happens when external forces are present.

NEWTON'S SECOND LAW

> The net force on a body causes that body to accelerate. The acceleration is in the direction of the force, proportional to the force and inversely proportional to the mass of the body.

$$\mathbf{F} = F\mathbf{i}, \qquad \mathbf{a} = a\mathbf{i},$$

$$\mathbf{F} = m\mathbf{a}. \tag{1.14}$$

Equation (1.14) means that if we know the cause of the motion – that is the force **F** – then we know the change in the motion – that is, the acceleration.

We can also use Newton's Second Law as a definition of the mass of a body. Mass is a measure of the resistance to change in motion. If identical forces are applied to two bodies of masses m_1 and m_2, and two different accelerations are measured a_1 and a_2, then Eq. (1.14) can be used to relate the two masses by

$$\begin{aligned} F &= m_1 a_1 = m_2 a_2, \\ m_1 &= m_2 \frac{a_2}{a_1}. \end{aligned} \tag{1.15}$$

Newton's Third Law describes the forces involved when two bodies interact with each other:

NEWTON'S THIRD LAW

Let $\mathbf{F_{12}}$ be the force exerted by body 1 on body 2. There is also a force exerted by body 2 on body 1, $\mathbf{F_{21}}$. These forces lie along the same straight line and in opposite directions and are of identical magnitude. That is $\mathbf{F_{12}} = -\mathbf{F_{21}}$.

This has often been referred to as the principle of action and reaction.

1.1.3 Force as the function of position

Force $\mathbf{F}$ can be the function of time position and velocity, i.e.:

$$\mathbf{F} = \mathbf{F}(x, v, t). \tag{1.16}$$

Let us consider the very important example, the force as the function of the position

$$\mathbf{F} = \mathbf{F}(x), \qquad F = F(x) = ma. \tag{1.17}$$

We rewrite a as (Appendix A)

$$a = \frac{dv}{dt} = \frac{dv}{dx}v. \tag{1.18}$$

Therefore,

$$F(x) = mv\frac{dv}{dx}. \tag{1.19}$$

this can still be rewritten as

$$F(x) = \frac{1}{2}m\frac{d}{dx}(v^2), \tag{1.20}$$

or

$$F(x) = \frac{d}{dx}\left(\frac{1}{2}mv^2\right),$$

assuming the mass doesn't change. We define $\frac{1}{2}mv^2$ as the *kinetic energy* T; that is

$$T \equiv \frac{1}{2}mv^2. \tag{1.21}$$

Therefore,

$$F(x) = \frac{dT}{dx}. \tag{1.22}$$

We can now multiply by dx and integrate to get (Appendix A)

$$\int_{x_0}^{x} F(x)dx = \int_{T_0}^{T} dT = T - T_0. \tag{1.23}$$

The left-hand side represents the work done on the body by the force as it moves from x_0 to x. This work is equal to the change in the kinetic energy.

1.1.4 Total energy and conservation of energy

We can get more meaning out of the equation (1.23) if we define a potential $V = V(x)$ such, that

$$F(x) = -\frac{dV}{dx}. \tag{1.24}$$

Equation (1.24) can, of course, be turned around and written as

$$V(x) = -\int_{x_s}^{x} F(x)dx. \tag{1.25}$$

We can use the potential energy to the left-hand side of Eq. (1.23). Then

$$-V(x) + V(x_0) = T - T_0,$$

or

$$T_0 + V(x_0) = T + V(x). \tag{1.26}$$

This means that

$$\boxed{T(x) + V(x) = E = \text{constant.}}$$

The sum of the kinetic energy and the potential energy remains constant throughout the motion; this is called the *total energy.* We describe this by saying that the total energy is *conserved.* This is true only for forces definable by Eq. (1.24) or (1.25), which we call *conservative* forces.

1.1.5 The one dimensional harmonic oscillator

Harmonic oscillators are important for several reasons. First of all, they occur throughout nature. Indeed, any motion about a stable equilibrium is harmonic as long as it is small. To the first approximation the electrons in atoms, the nucleons in nuclei and even quarks (if they exist!) in nucleons move as the simple oscillators.

The simplest model of the harmonic motion is the mass attached to a spring in a frictionless environment. We shall assume that the spring obeys the very simple law (Hooke's Law)

$$\mathbf{F} = -k\mathbf{x}, \qquad F = -kx. \tag{1.27}$$

Thus,

$$F = ma = -kx,$$
$$m\frac{d^2x}{dt^2} = -kx. \tag{1.28}$$

The complete solution of the equation (1.28) is

$$x(t) = A\sin\omega t + B\cos\omega t, \tag{1.29}$$

where

$$\omega = \sqrt{\frac{k}{m}}.$$

From Eq. (1.29) we know that a simple harmonic oscillator like the mass and spring will move back and forth-oscillate sinusoidally – with an angular frequency $\omega = \sqrt{k/m}$. The frequency ω is the *natural* frequency for the simple oscillator and we will indicate this a subscript, $\omega_0 = \sqrt{k/m}$.

Consider now as a move realistic system an oscillator immersed in fluid. To the first approximation the force exerted on mass can be written as

$$F = m\frac{d^2x}{dt} = -kx - c\frac{dx}{dt}, \tag{1.30}$$

where $-c(dx)/(dt)$ is the friction force proportional to the velocity of the motion. We try the function $x(t) = Ae^{qt}$ as the solution of equation (1.30) and obtain simple quadratic equation for q's:

$$mq^2 + cq + k = 0, \tag{1.31}$$

with the elementary solution

$$q_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}. \tag{1.32}$$

We are happy if $c^2 - 4mk > 0$, then

$$\begin{aligned} q_1 &= \frac{-c + \sqrt{c^2 - 4mk}}{2m}, \\ q_2 &= \frac{-c - \sqrt{c^2 - 4mk}}{2m}. \end{aligned} \tag{1.33}$$

Both q's are negative, so we can define

$$\gamma_1 = -q_1, \qquad \gamma_2 = -q_2, \qquad \gamma_1,\, \gamma_2 > 0, \tag{1.34}$$

and write solution of Eq. (1.30) as

$$x(t) = A_1 e^{-\gamma_1 t} + A_2 e^{-\gamma_2 t}. \tag{1.35}$$

Both of the terms in Eq. (1.35) decay exponentially with time so the motion will be as shown in Fig.(). This case of the oscillator is called overdamped case.

For $c^2 = km$ the solution of the Eq. (1.30) has the form

$$x(t) = Ae^{-\gamma t} + Bte^{-\gamma t}, \qquad \gamma = \frac{c}{2m}. \tag{1.36}$$

This can be verified by simple substitution of Eq. (1.36) to Eq. (1.30).This case is called *critical* damping.

The most interesting case is when $c^2 < 4km$. But how to calculate square root of number smaller than zero? This problem has been solved by *Leonard Euler* in a letter to *John Bernoulli.* (October 18, 1740 Basel). To appreciate the reasoning of *L. Euler* let us come back to the free oscillator equation (1.28)

$$\frac{d^2x(t)}{dt^2} + \omega^2 x(t) = 0, \tag{1.37}$$

and assume that $v(0) = 0, \quad x(0) = 2$. In that case the solution of Eq. (1.37) has the form

$$x(t) = 2\cos\omega t. \tag{1.38}$$

But we know (Appendix) there exists the exponential function $x(t) = e^{\alpha t}$. Let us try the $x(t)$ as the solution of Eq. (1.37)

$$\alpha^2 + \omega^2 = 0, \tag{1.39}$$

then $\alpha = \pm i\omega$. The strange $i = \sqrt{-1}$ was introduced for the first time by ()* From Eq. (1.39) we deduce that

$$x(t) = e^{i\omega t} + e^{-i\omega t}. \tag{1.40}$$

By comparing Eq. (1.38) and (1.40) we obtain the relaxation

$$\cos\omega t = \frac{e^{i\omega t} + e^{-i\omega t}}{2}, \tag{1.41}$$

and we are on the safe side because

$$\sin\omega t = \frac{e^{i\omega t} - e^{-i\omega t}}{2i}, \tag{1.42}$$

$$\boxed{e^{\pm i\omega t} = \cos\omega t \pm i\sin\omega t.}$$

(Formula Eulera-Nahin)*

Let us now impress an external force upon oscillator. We can do this, for example, by putting an electron in a varying electromagnetic field. In fact we can simply write Eq. (1.30) as

$$m\frac{d^2x}{dt^2} = -kx - c\frac{dx}{dt} + F_{ex}, \tag{1.43}$$

where external force is equal

$$F_{ex} = F_0 \sin \omega t, \qquad F_0 \cos \omega t,$$

or generally

$$F_{ex} = F_0 e^{i(\omega t + \Theta)}. \tag{1.44}$$

We can try as the solution of Eq. (1.43)

$$x = A e^{i(\omega t + \delta)}. \tag{1.45}$$

After substitution Eq. (1.45) to Eq. (1.43) we obtain the quadratic equation for ω and A

$$(k - m\omega^2) + ic\omega = \frac{F_0}{A} \cos(\Theta - \delta) + i \frac{F_0}{A} \sin(\Theta - \delta). \tag{1.46}$$

A single complex equation, just like a vector equation is actually two equations

$$k - m\omega^2 = \frac{F_0}{a} \cos(\Theta - \delta),$$

and

$$c\omega = \frac{F_0}{A} \sin(\Theta - \delta). \tag{1.47}$$

We define $\varphi = \Theta - \delta$ and obtain from Eq. (1.47)

$$\tan \varphi = \frac{2\gamma\omega}{\omega_0^2 - \omega^2},$$

and

$$A = \frac{F_0}{m} \left[(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2 \right]^{-1/2} \tag{1.48}$$

where $\gamma = c/(2m)$ and $\omega_0^2 = k/m$. In Fig.A as the function ω is shown. Maximum amplitude occurs, then, when the external force has a frequency

$$\omega = \omega_r = \sqrt{\omega_0^2 - 2\gamma^2}. \tag{1.49}$$

This is known as the resonance frequency.

Amplitude $A(\omega)$ can be written as

$$A = \frac{F_0}{m}\left[((\omega_0 - \omega) + 2ij\omega)(\omega_0 - \omega) - 2ij\omega)\right]^{-1/2}. \tag{1.50}$$

From the Eq.(1.50) we conclude that the existence of complex numbers guarantees the stability of the bodies. For $\omega \to \omega_r$, $A(\omega)$ is finite when $\mathrm{Im}(2ij\omega) \neq 0$.

1.2 Three-dimensional motion

1.2.1 *Vectors*, *vector* algebra, *vector* analysis

In science and engineering we frequently encounter quantities which have magnitude and magnitude only: mass, time and temperature. These we label *scalar* quantities. In contrast, many interesting physical quantities have magnitude and, in addition, an associated direction. This second group includes *displacement*, *velocity*, *acceleration*, *force*, *momentum*, and *angular momentum*. Quantities with magnitude and direction are labelled *vector* quantities. Usually in elementary treatment, a *vector* is defined as a quantity having magnitude and direction. To distinguish *vectors* from scalars we identify *vector* quantities with boldface type, that is $\mathbf{V}$.

Vector may be conveniently represented by an arrow with length proportional to magnitude. The direction of the arrow gives the direction of the *vector*. In this representation *vector* addition

$$\mathbf{C} = \mathbf{A} + \mathbf{B} \tag{1.51}$$

consists in placing the rear end of the *vector* **B** at the point of *vector* **A**. *Vector* **C** is then represented by an arrow drawn from the rear of **A** to the point of **B**. This procedure, the triangle law of addition, assigns meaning of Eq (1.51) and is illustrated in Fig.By completing the parallelogram we see that

$$\mathbf{C} = \mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}, \tag{1.52}$$

as shown in Fig.

Note that the *vectors* are treated as geometrical objects which are independent of any *coordinate system*. Indeed we have not yet introduced a coordinate system. This concept of independence is developed in detail in the next section.

The representation of *vector* **A** by an arrow suggests a second possibility. Arrow **A**, Fig. starting from the origin terminates at point (x_1, y_1, z_1).

One particularly important *vector* quantity is the *displacement* from origin to the point (x_1, y_1, z_1) and is denoted by the special symbol **r**. We then have a choice of referring to the displacement either as the *vector* **r** or the collection of the three numbers (x_1, y_1, z_1).

Using **r** for the magnitude of *vector* **r**, Fig.shows that the end-point coordinates and the magnitude are related by

$$x_1 = r\cos\alpha,$$
$$y_1 = r\cos\beta,$$
$$z_1 = r\cos\gamma. \tag{1.53}$$

The $\cos\alpha, cos\beta$ and $\cos\gamma$ are called the direction cosines, α being the angle between the given *vector* and the positive x-axis and so on. One further bit of vocabulary: the quantities x_1, y_1, z_1 are the (cartesian) components of **r** or the projections of **r**.

Any *vector* **A** may be resolved into its components to yield

$$\begin{aligned} A_x &= A\cos\alpha, \\ A_y &= A\cos\beta, \\ A_z &= A\cos\gamma. \end{aligned} \tag{1.54}$$

At this stage it is convenient to introduce unit *vectors* along each of the coordinate axes. Let **i** be a *vector* of unit magnitude pointing in the positive x-direction, **j** a *vector* of unit magnitude in positive y-direction and **k** a *vector* of unit magnitude in the positive z-direction. Let $\mathbf{i}A_x$ be the *vector* with magnitude equal to A_x and in positive x-direction.

By *vector* addition

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}, \tag{1.55}$$

which states that a *vector* equals to the *vector* sum of its components. Note that if **A** vanishes all its components must vanish individually:

that is, if

$$\mathbf{A} = 0, \quad \text{then} \quad A_x = A_y = A_z = 0. \tag{1.56}$$

Finally, by the Pythagorean theorem the magnitude of *vector* **A** is

$$A = (A_x^2 + A_y^2 + A_z^2)^{1/2}. \tag{1.57}$$

Our naive approach is awkward to generalize for we encounter quantities such as elastic constants and index of refraction in anisotropic crystals which have magnitude and direction but which are not *vectors.*

In the subsequent we shall assume that *space* is isotropic: that is, there is no preferred direction or all direction are equivalent. Then the physical system being analyzed cannot and must not depend on our choice or *orientation* of our coordinate system.

We consider *vector* **r** as a geometric object independent of the coordinate system. Let us look at **r** in two different systems, one rotated in relation to the other.

For simplicity we consider the two dimensional case. The three dimensional coordinate systems are described in Appendix B. If the coordinates (x, y) are rotated counterclockwise through an angle φ, keeping **r** fixed, we get the following relations between the components resolved in the original system (unprimed) and those resolved in the new rotated system (primed) (Fig)

$$\begin{aligned} x' &= x\cos\varphi + y\sin\varphi, \\ y' &= -x\sin\varphi + y\cos\varphi. \end{aligned} \tag{1.58}$$

We know from paragraph ()that a *vector* could be represented by coordinates of a point; that is, the coordinates were proportional to the vector components. Hence the components of the *vector* must transform under rotation as coordinates of point (such as $\mathbf{r}$).

Therefore, whenever any pair of quantities (A_x, A_y) in (x, y) coordinate system is transformed into $(A_x^{'}, A_y^{'})$ by this rotation of the coordinate system with

$$\begin{aligned} A_x^{'} &= A_x \cos\varphi + A_y \sin\varphi, \\ A_y^{'} &= -A_x \sin\varphi + A_y \cos\varphi. \end{aligned} \tag{1.59}$$

We define A_x and A_y as the components of the *vector* $\mathbf{A}$. If A_x and A_y do not show this behavior when the coordinates are rotated they do not form a *vector*

Scalar and *vector* products

We define

$$\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z, \tag{1.60}$$

as the scalar (or dot) product of $\mathbf{A}$ and $\mathbf{B}$. We note that from definition $\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$.

If we reorient our axes and let A define a new x-axis, then

$$\begin{aligned} &A_x = A, \quad A_y = 0, \quad A_z = 0, \\ &B_x = B\cos\Theta, \quad B_y, \quad B_z. \end{aligned} \tag{1.61}$$

Then by Eq. (1.60)

$$\mathbf{AB} = A \cdot B\cos\Theta, \tag{1.62}$$

which may be taken as a second definition of scalar product.

A second form of *vector* multiplication employs the sine of the included angle instead of cosine. We define the *vector* product, or cross product as

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} \tag{1.63}$$

with

$$C = AB \sin \Theta.$$

C is now a vector, and we assign in a direction perpendicular to the plane of **A** and **B** such that $\mathbf{A}, \mathbf{B}, \mathbf{C}$ form a right-handed system. With this choice of direction we have

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}. \tag{1.64}$$

An alternate definition of the *vector* product $\mathbf{C} = \mathbf{A} \times \mathbf{B}$ consists in specifying the components of **C**

$$\begin{aligned} C_x &= A_y B_z - A_z B_y, \\ C_y &= A_z B_y - A_x B_z, \\ C_z &= A_x B_y - A_y B_x, \end{aligned} \tag{1.65}$$

or

$$C_i = A_j B_k - A_k B_j, \quad i, j, k \quad \text{different},$$

and with cyclic permutation of the indices i, j, k.

Let *vector* **A** be the function of position

$$\mathbf{A} = \mathbf{A}(x),$$

Table 1.1: Physical examples of scalar and *vector* products

Scalar product	Vector product
Work= force · displacement · cos Θ	**Angular momentum**= radius arm linear momentum = distance linear momentum sin Θ **Lorentz force**= charge velocity magnetic field sin Θ

then the first derivative of a *vector* **A** is defined as

$$\frac{d\mathbf{A}}{dx} = \lim_{\Delta x \longrightarrow 0} \frac{\mathbf{A}(x + \Delta x) - \mathbf{A}(x)}{\Delta x}. \tag{1.66}$$

Since we can write *vectors* in component form, we can also write derivatives of *vectors* in component form. Thus if we continue to focus our attention on the position *vector* **r**,

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}. \tag{1.67}$$

Then

$$\mathbf{v} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}. \tag{1.68}$$

We can likewise define a *vector* acceleration by

$$\mathbf{a} = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j} + \frac{d^2z}{dz^2}\mathbf{k}. \tag{1.69}$$

Differentiation of *vectors* follows rules similar to those we have already seen for differentiation of scalars. In particular we have:

$$\begin{aligned}\frac{d}{dt}[\mathbf{A}+\mathbf{B}] &= \frac{d\mathbf{A}}{dt}+\frac{d\mathbf{B}}{dt},\\ \frac{d}{dt}[\mathbf{A}\cdot\mathbf{B}] &= \frac{d\mathbf{A}}{dt}\cdot\mathbf{B}+\mathbf{A}\cdot\frac{d\mathbf{B}}{dt},\\ \frac{d}{dt}[\mathbf{A}\times\mathbf{B}] &= \frac{d\mathbf{A}}{dt}\times\mathbf{B}+\mathbf{A}\times\frac{d\mathbf{B}}{dt}.\end{aligned} \tag{1.70}$$

1.2.2 Reference frames

We have previously described a *vector* in terms of its components along the axes of a rectangular coordinate system. For many situations – for example, the motion of a Moon encircling Earth - there are easier ways of describing motion than using rectangular coordinates, e.g. plane polar coordinates.

Consider the point P in Fig.We know it is located by position *vector* $\mathbf{r}$, given by

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j}. \tag{1.71}$$

We can also locate it by giving its distance from the origin r and the polar angle Θ. The polar angle Θ is measured counterclockwise from the x-axis. We can then define two new perpendicular unit vectors, $\hat{\mathbf{r}}$ which points in the direction P, and $\hat{\boldsymbol{\Theta}}$ which points in the direction that P would move as the polar angle Θ increases. In terms of these unit *vectors*, we can write

$$\mathbf{r} = r\hat{\mathbf{r}}, \tag{1.72}$$

where

$$\hat{r} = \hat{r}(\Theta).$$

The *vectors* $\mathbf{r}$ and $\mathbf{\Theta}$ form a new coordinate system, the plane polar coordinate system, and are usually referred to simply as *polar coordinates*. If we need to switch back and forth from $\mathbf{i}$ and $\mathbf{j}$ to $\hat{r}$ and $\hat{\Theta}$ we use formula (1.58)

$$\begin{aligned} \hat{\mathbf{r}} &= \mathbf{i}\cos\Theta + \mathbf{j}\sin\Theta, \\ \hat{\mathbf{\Theta}} &= -\mathbf{i}\sin\Theta + \mathbf{j}\cos\Theta. \end{aligned} \tag{1.73}$$

To find the velocity expressed in polar coordinates, we obtain the time derivative of r from (1.72). Be careful, though, unlike $\mathbf{i}, \mathbf{j}, \mathbf{k}$ the radial unit *vector* is time dependent. We obtain

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\Theta}{dt}\frac{d\hat{\Theta}}{dt}. \tag{1.74}$$

Analogously we obtain for acceleration in polar coordinates

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \left(\frac{d^2r}{dt^2} - r\left(\frac{d\Theta}{dt}\right)^2\right)\mathbf{r} + \left(r\frac{d^2\Theta}{dt^2} + 2\frac{dr}{dt}\frac{d\Theta}{dt}\mathbf{\Theta}\right). \tag{1.75}$$

1.2.3 Separable forces

Earlier we solved the equation $F = m(d^2x)/(dt^2)$ for motion along a straight line. We now want to consider the general case of $F = m(d^2\mathbf{r})/(dt^2)$ where the force $\mathbf{F}$ and the motion it causes can have components in all three dimensions. The most general form of the force will depend on time, position and velocity $\mathbf{F}(\mathbf{r}, \mathbf{v}, t)$. Of course

r and **v** are *vectors* so we have three coupled equations

$$\begin{aligned}
m\frac{d^2x}{dt^2} &= F_x\left(t, x, y, z, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right),\\
m\frac{d^2y}{dt^2} &= F_y\left(t, x, y, z, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right),\\
m\frac{d^2z}{dt^2} &= F_z\left(t, x, y, z, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right).
\end{aligned} \tag{1.76}$$

Now we discuss elementary *solvable* Eq.(1.51). The earliest type of three-dimensional force to handle is a separable force where

$$\begin{aligned}
m\frac{d^2x}{dt^2} &= F_x\left(t, x, \frac{dx}{dt}\right),\\
m\frac{d^2y}{dt^2} &= F_y\left(t, y, \frac{dy}{dt}\right),\\
m\frac{d^2z}{dt^2} &= F_z\left(t, z, \frac{dz}{dt}\right).
\end{aligned} \tag{1.77}$$

That is, the force in a particular direction, depends on the component of position in that direction, the component of velocity in that direction and perhaps on time.

1.2.4 Three dimensional harmonic oscillator

A useful approximation of the motion of an atom in a crystal with cubic structure is a mass m held in place by three mutually perpendicular sets of springs (Fig).

If there is no damping, the oscillations are given by:

$$\begin{aligned}
m\frac{d^2x}{dt^2} &= -k_x x,\\
m\frac{d^2y}{dt^2} &= -k_y y,\\
m\frac{d^2z}{dt^2} &= -k_z z.
\end{aligned} \tag{1.78}$$

Each of these equations describes a simple harmonic oscillator so we can immediately write:

$$\begin{aligned} x &= A\cos(\omega_x t + \alpha), \\ y &= B\cos(\omega_y t + \beta), \\ z &= C\cos(\omega_z t + \gamma). \end{aligned} \tag{1.79}$$

where $\omega_x = \sqrt{k_x/m},\ \omega_y = \sqrt{k_y/m},\ \omega_z = \sqrt{k_z/m}$. If ω's are related by integers through

$$\frac{\omega_x}{n_x} = \frac{\omega_y}{n_y} = \frac{\omega_z}{n_z}, \tag{1.80}$$

they are said to be commensurable and the motion of the mass (electron, atom) either repeats itself or follows a closed path. If one of the amplitudes of Eq.(1.79) is also zero then the path taken is a closed path in a plane. The path taken is then called a *Lissajou figures*. A few such paths are sketched in Figfor the case of $z = 0$.

1.2.5 Motion in electromagnetic fields

Electric charges at rest produce forces on the body of charge **q** that can be described in terms of an electric field $\mathbf{E}(\mathbf{r})$ by

$$\mathbf{F}_{\mathrm{el}}(\mathbf{r}) = q\mathbf{E}(\mathbf{r}). \tag{1.81}$$

Moving charges or currents produce additional forces that can describe in terms of another field, magnetic induction $\mathbf{B}(\mathbf{r})$. These forces depend on the motion of the body of charge q – they are velocity-dependent forces given by

$$\mathbf{F}_m(\mathbf{r}) = q\mathbf{v} \times \mathbf{B}. \tag{1.82}$$

We can write the total force (the *Lorentz force*) on a charged, in the presence of both **E** and **B**, field as

$$\mathbf{F}(\mathbf{r}) = q\mathbf{E}(\mathbf{r}) + q\mathbf{v} \times \mathbf{B}. \tag{1.83}$$

When the electric field is unform and $\mathbf{B}(\mathbf{r}) = 0$, then

$$\mathbf{E} = E_0\mathbf{k}, \qquad \mathbf{B} = \mathbf{0} \tag{1.84}$$

(we choose the z–axis to lie parallel to the direction of **E**). Thus

$$\mathbf{F} = qE_0\mathbf{k}, \tag{1.85}$$

and with a constant force we can immediately write down the equation of motion. With the initial velocity

$$\begin{aligned} v_x(0) &= v_{x_0}, \\ v_y(0) &= v_{y_0}, \\ v_z(0) &= v_{z_0}, \end{aligned}$$

the equation of motion is

$$\begin{aligned} x &= x_0 + v_{x_0}t, \\ y &= y_0 + v_{y_0}t, \\ z &= z_0 + v_{z_0}t + \frac{1}{2}\frac{qE_0}{m}t^2. \end{aligned} \tag{1.86}$$

These have the same forum as the equation of motion discussed in paragraph As it must because the acceleration is constant, $a = (qE_0)/(m)$. When $\mathbf{E}(\mathbf{r}) = 0$ and $\mathbf{B} = B\mathbf{k}$, to solve the motion produced by Lorentz force:

$$\begin{aligned} m\frac{d^2x}{dt^2} &= v_yBq \\ m\frac{d^2y}{dt^2} &= -v_xBq, \\ m\frac{d^2z}{dt^2} &= 0. \end{aligned} \tag{1.87}$$

The solution of Eq.(1.53) for z–axis we find at once:

$$z = z_0 + v_{0z}t.$$

For x, y–axes we calculate the third derivative

$$\begin{aligned} \frac{d^3x}{dt^3} &= \frac{Bq}{m}\frac{d^2y}{dt^2}, \\ \frac{d^3y}{dt^3} &= -\frac{Bq}{m}\frac{d^2x}{dt^2}. \end{aligned} \tag{1.88}$$

Substituting the second equation to the first we obtain

$$\frac{d^3x}{dt^3} + \left(\frac{Bq}{m}\right)^2\frac{dx}{dt} = 0. \tag{1.89}$$

Putting in equation (1.89) $\eta = (dx)/(dt)$ we obtain for the motion in the x–axis

$$\frac{d^2\eta}{dt^2} + \omega^2\eta = 0, \tag{1.90}$$

where $\omega = (Bq)/(m)$. According to y–axis ($\zeta = (dy)/(dt)$) we obtain

$$\frac{d^2\zeta}{dt^2} + \omega^2\zeta = 0. \tag{1.91}$$

Both Eqs.(1.90) and (1.91) are the equations which describe the free harmonic oscillator. The solution of Eqs. (1.90,1.91) are

$$\begin{aligned} &\zeta = v_0\sin\omega t = v_y, \quad \zeta(0) = 0, \\ &\eta = v_0\cos\omega t = v_x, \quad \eta(0) = v_0, \end{aligned} \tag{1.92}$$

and consequently

$$y = \int \zeta dt = -\frac{v_0 \cos \omega t}{\omega} + y_0,$$
$$x = \int \eta dt = \frac{v_0 \sin \omega t}{\omega} + x_0. \tag{1.93}$$

From formulae (1.93) we obtain

$$(x - x_0)^2 + (y - y_0)^2 = \frac{v_0^2}{\omega}. \tag{1.94}$$

Eq. (1.94) describes the circular motion in the (x, y) plane, central about $x = x_0, y_0 = 0$ with radius $r = (v_0)/(\omega)$. Coupling the motion with constant velocity in the z direction, we see that the most general motion for a particle is travelling along a right circular helix. The position, pitch and diameter of the helix depend on the initial conditions This result has immense application for many particle accelerators, the cyclotron in particular – indeed, it is only because of this that they can operate at all.

1.2.6 Weighing of the electron

As early as about 1880 electrical discharges in gases were intriguing a number of experimental physicists in Europe. In 1881, at the Cavendish Laboratory at the University of Cambridge. *J. J. Thomson* began experimenting with gaseous discharges, and continued to do so for the next 50 years.

In the paper published in *Philosophical Magazine* in 1897 *Thomson* reported that "cathode rays" were charged particles, which he called "corpuscules". *Thomson*'s electron was the first *elementary particle* discovered and indeed the first evidence of the existence of an elementary particle

Before the discovery of the electron, streams of the electrons were referred to as “cathode rays”. The properties of these so-called cathode rays had been studied in later half of the nineteenth century, especially in the elegant demonstrations of *Crookes* and *Lenard*. Many minerals and glass fluoresce with a characteristic colour when placed in a beam of cathode rays. That these rays travel in straight lines normal to the cathode is shown by placing an object, often in the shape of a Maltese cross, in the path of the cathode rays. The shadow of the cross can be seen on the end of the tube. They also carry energy which can be converted into heat by directing them on to a thin platinum foil which quickly becomes red or even white hot. Cathode rays are also deflected by electric and magnetic fields in a way which clearly indicates that they carry a negative charge. All these can be concluded when the experimentalist understands and geniously applicates the Lorentz force described in paragraph.

In 1897 *J. J. Thomson* devised an experiment by which the ratio of charge to mass of electrons could be determined. Fig.shows a vacuum tube in which cathode rays streaming from the cathode C fall upon the anode A. In A there is a small hole, so that a “pencil” of cathode rays passes on to D containing a similar small hole. The narrow beam of cathode rays can be deflected in the vertical plane by an electric field $\mathbf{E}$, between the parallel plates, as shown. It can also be deflected in the same plane by a magnetic field B perpendicular to the paper. The point at which the cathode rays impinge upon the screen S is shown by a fluorescent spot of light, since the screen is coated internally with a fluorescent material such as zinc sulphide.

In our analysis of the *J. J. Thomson's* experiment we will assume that the electric and magnetic fields are confined to the space between the parallel plates. Suppose the magnitude of the electric field **E** and magnetic field are adjusted so that the beam of cathode rays is not deflected but falls on S. The *Lorentz force*, formula (**??**)must be equal zero, i.e.

$$Ee - qBv = 0, \tag{1.95}$$

where e is charged on the particle and v is its velocity. The velocity is then given by

$$v = \frac{E}{B}. \tag{1.96}$$

The velocity was found to be very high, up to 10^7 ms^{-1}. In the electric field alone (Fig.)the particle suffers deflection due to the acceleration it receives perpendicular to its direction of motion.

Using formula for displacement along y–axis one obtains

$$y = \frac{1}{2}\frac{d^2y}{dt^2}t^2, \tag{1.97}$$

for $(dy)/(dt) = v_y(0) = 0$, where t is the time electron takes to traverse the electric field. Using Newton's Second Law,

$$m\frac{d^2y}{dt^2} = eE. \tag{1.98}$$

and substituting formula (1.98) to formula (1.97), we obtain

$$\frac{e}{m} = \frac{2y}{Et^2} = \frac{2yv^2}{El^2}, \tag{1.99}$$

where $l = vt$. From similar triangles we find that

$$\frac{y}{\frac{1}{2}l} = \frac{y'}{L}, \tag{1.100}$$

hence

$$\frac{e}{m} = \frac{E_{y'}}{B^2 L l}. \tag{1.101}$$

Considering the results of the *J. J. Thomson* and *Dunmington* (1933) the value of e/m is equal

$$\frac{e}{m} = 1.7588\,10^{11}\frac{\mathrm{C}}{\mathrm{kg}}. \tag{1.102}$$

If we know the charge on the electron we can calculate, from formula (1.102) the mass m of the electron. The method for measurement of e was refined and developed by *R. Millikan* in 1911. *Millikan* early result was $e = 1.64\,10^{-19}$C. Refinements gave further results, viz.

$$e = 1.602\,10^{-19}\mathrm{C}. \tag{1.103}$$

Millikan's apparatus consisted of two circular brass plates 220 mm in diameter and 16 mm apart forming an air condenser. The upper plate had a minute hole at the center through which oil may drop, formed by a spray in an upper chamber. The oil drops were illuminated from the side by a carefully collimated beam of light. This light showed up the oil drops as bright specks against a dark background.The drops were charged by friction or X–ray ionization of the air as they were formed in the spray. They normally fall under the action of gravity, but could be made to rise again by applying an electric field in a suitable direction. The electric field is calculated from the potential difference ($V = 5$ kV) and the separation of the plates ($d = 16$ mm). From observations of the rate of rise and fall of the drop with and without the electric field, the electric charge on an oil drop was found.

When a drop of radius a falls under the action of gravity alone its weight P, is

$$P = \frac{4}{3}\pi a^3 \rho g, \tag{1.104}$$

where ρ is the density of the oil and g is the acceleration due to gravity. The upthrust due to the displaced air is, by *Archimedes'* principle

$$F_A = \frac{4}{3}\pi a^3 \rho_0 g, \tag{1.105}$$

where ρ_0 is the air density. The retarding force due to viscous drag as the drop moves through the air is given by *Stokes* law

$$F_{\text{drag}} = 6\pi\eta a v_0, \tag{1.106}$$

where η is the viscosity of air and v_9 is the velocity of the drop. When the velocity becomes uniform, the resultant force on the drop is zero (First Newton Law, paragraph so that for zero electric field we may write

$$P - F_A - F_{\text{drag}} = 0, \tag{1.107}$$

$$\frac{4}{3}\pi a^3(\rho - \rho_0)g = 6\pi\eta a v_0. \tag{1.108}$$

Suppose now that under the action of an electric field E the drop moves upward with a constant velocity v_E. Again the resultant force is zero, as the velocity is uniform, so that

$$P - F_A + F_{\text{drag}} - E_q = 0, \tag{1.109}$$

$$\frac{4\pi}{3}\pi a^3(\rho - \rho_0)g + 6\pi\eta a v_E = Eq, \tag{1.110}$$

where q is the charge on the drop. From equations (1.108) and (1.110) we calculate q

$$q = \frac{6\pi\eta a}{E}(v_0 + v_E). \tag{1.111}$$

Millikan found that values of q for different drops were always multiples of a common value e that is $q = ne$, where n is an integer.For e *Millikan* found

$$e = 1.591\,10^{-19}\,\text{C}. \tag{1.112}$$

Later experiments with the *Millikan*'s method gave $e = 1.602\,10^{-19}$ C.

Using the values of constants to four decimal places, the mass of the electron is given by

$$m = \frac{em}{e} = 9.1096\,10^{-31}\text{kg}. \tag{1.113}$$

This number only became significant when compared with mass M of a single hydrogen atom, which was obtained by dividing the relative atomic mass of hydrogen by N_A (*Avogadro's* number)

$$M = \frac{1.0078\,10^{-3}}{6.022\,10^{23}} = 1.6735\,10^{-27}\text{kg}. \tag{1.114}$$

Comparing the masses M and m we get

$$\frac{M}{m} = 1837. \tag{1.115}$$

Thus we see that the electron is a particle having a mass of just a little more than one of two-thousandth of that of proton (for a hydrogen atom is composed of one proton and one electron).

The *Millikan* fundamental result

$$q = ne, \qquad n = \text{integer} \tag{1.116}$$

is still under debate. Still the searching of the particles with fractional charge ($n =$ not integer) is the part of the elementary particle physics. Such a particle might have a charge of $0.1q$, $(2/3)q$ or even πq where $q = e$. *M. Perl* and his colleagues have been carrying out the general search for particles with fractional electric charge using a highly automated version of the *Millikan* oil drop experiment [2] In *Perl*'s experiment (1997!) a small liquid drop, 7 μm in diameter, falls through air. The air resistance and the small drop size cause the drops to rapidly attain a terminal velocity proportional to the force on the drop. A vertical electric field that periodically changes direction makes it possible to measure two terminal velocities, one for each electric direction. The difference between two terminal velocities is proportional to the charge on the drop. The sum of the two terminal velocities is proportional to the drop mass.

In esu units the 1920's the charge e of the electron was $4.774 \pm 0.009)10^{-10}$ esu. Its mass was 1/(1845) that of hydrogen atom. If we look at the most recent edition of the *Review of Particle Physics* we

[2] M. L. Perl and E. R. Lee, The search for elementary particles with fractional electric charge and the philosophy of speculative experiments, *Am. J. Phys*, **65** (1997) 698.

find that the charge of the electron is[3]

$$e = (4.8032068 \pm 0.0000015)10^{-10}\text{esu}$$

and its mass is

$$m = 9.1093897 \pm 0.0000054)10^{-31}\text{kg}, \tag{1.117}$$

approximately 1/(1837) the mass of the hydrogen atom.

Allowing for improvements in both the precision and accuracy of these measurements, it seems fairly to say that the properties of the electron have remained constant. It is still a negatively charged particle with a definite charge and mass. The electron, as an entity has remained constant even though the theories physicist use to describe it have evolved dramatically. *Thomson*'s early work used *Newton*'s and *Maxwell*'s theories. That was followed by *Bohr*'s quantum theory the nonrelativistic ($v << c, \quad c$ is the light velocity) *Schrödinger* quantum mechanics and relativistic ($v \to c$) *Dirac* theory.

As early as in 1909 *Bucherer* repeated the modified version of the *Thomson* experiments for the electrons with large velocity $v \to 0.7c$. Electrons from the β decay of radioactive nucleus enter a velocity selector, which determines the speed of those that emerge and then enter a uniform magnetic field, where the radius of their circular path R can be measured. In the region 2 magnetic field $\mathbf{B}$ is perpendicular to the velocity $\mathbf{B} \perp \mathbf{v}$ so the *Lorentz* force $F_L = evB$. The *Lorentz* force F_L equals the centripetal force

$$F_c = \frac{mv^2}{R}, \tag{1.118}$$

[3] $4.8032068(15) \times 10^{-10}\text{esu} = 1.602177733(49) \times 10^{-19}$ C

where the R is the radius of electron trajectory (circle), viz.:

$$\frac{mv^2}{R} = evB \tag{1.119}$$

and

$$R = \frac{mv}{eB} = \frac{m}{e}\frac{E}{B^2} \tag{1.120}$$

Bucherer's results are shown in Table 1.2

Table 1.2: *Bucherer*'s results

Measured $[v/c]$	Measured e/m $[10^{11}$ C/kg$]$
0.32	1.66
0.38	1.63
0.43	1.59
0.52	1.51
0.69	1.28

The first column gives the measured speeds in terms of the fraction of the speed of light. The second column gives the ratio e/m computed from the measured quantities Eq.(1.120). It is clear that the value of e/m varies with the speed of the electrons. When we ask why the e/m is smaller for high electron velocity we have two answers. We might have concluded, for example, that the charge on electron is the function of the velocity

$$e = e_0\sqrt{1-\left(\frac{v}{c}\right)^2}, \tag{1.121}$$

where e_0 is the charge on electron for $v = 0$. Actually we have implicitly assumed above that the charge on the electron is independent of its speed. This assumption is a direct consequence of the *Maxwell* electrodynamics. The experimental verification of the assumption that $e = e_0$ can be found in the paper: R. Kollath and D. Menzel, Measurement of the Charge on Moving Electron, *Z. Phys.* **134** (1953) 530. Another possibility is

$$m = \frac{m_0}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}, \tag{1.122}$$

i.e. mass of the electron is velocity dependent, and m_0 is the mass of the electron for $v = 0$, $m_0 = (9.1093897 \pm 0.0000054) \times 10^{-31}$ kg – formula (1.117). The formula (1.122) was also verified by Bertozzi (W. Bertozzi, *Speed and kinetic energy of relativistic electrons*, *Am. J. Phys.*, **32** (1964), 551). As in paper by Bertozzi we will call the particles (electrons) with $v \to c$ as the relativistic particles (electrons). As can be seen from formula (1.122) for $v \to c, \quad m \to \infty$. Because for the creation of the infinity mass we need the infinity energy we can suspect that c is the limiting velocity for all massive particles ($m_0 \neq 0$).

1.3 Newtonian World System

1.3.1 History of the Newtonian World System

In *Aristotle*'s day space was associated with the distribution of things directly observed. How to define motion by combining intervals of space and time was not at all clear, and motion was poorly distinguished from other forms of change. *Aristotle*'s law of motion may be expressed by the relation:

$$\text{applied force} = \text{resistance} \times \text{speed} \tag{1.123}$$

But really he had no general formula, and no precise way of measuring force, resistance and speed. He argued qualitatively, reasoning from the everyday experience that effort is needed to maintain that motion. It means that in the absence of all resistance, a body would move from place to place in no time at all, that is at infinite speed. *Aristotle*'s common-sense law of motion endured until finally eclipsed by the *Newton*'s law:

$$\text{applied force} = \text{mass} \times \text{acceleration} \tag{1.124}$$

In the fourteenth century scholars of Merton College, Oxford, such as *William Heytestbury*, showed that if v represents speed at any moment and a represents a constant positive acceleration, then in time interval t the speed attained is

$$v = at \tag{1.125}$$

and the distance S traveled is

$$S = \frac{at^2}{2}. \tag{1.126}$$

Their calculations by graphical methods anticipated the discovery of calculus by *I. Newton* and *G. Leibnitz* three hundred years later. *William Ockham* also of Merton College, argued that forces can act at a distance without any need for direct contact between bodies.

In the sixteenth century Polish astronomer *Mikołaj Kopernik* (Nicolaus Copernicus (1473–1543)) demonstrated the feasibility of a heliocentric universe. *Copernicus*'s great work, *Revolutions of the Celestial*

Spheres appeared in print in 1543 shortly before his death. *Johannes Kepler* (1571–1630) accepted the finite Copernican system with the Sun at its center and the sphere of fixed stars. *Kepler* inherited *Tycho Brahe*'s (1546–1601) careful and detailed observations of the planets and for years struggled to explain their motions, particularly that of Mars. At last he triumphed and succeeded in freeing astronomy from the paradigm of epicyclic motion (*Ptolemy* system). His important three laws of elliptical planetary motion served as the foundation stones in the Newtonian world system. *Galileo Galilei* (1564–1642) believed in the Copernican system. In the *Two Great Systems of the World* he contrasted geocentric and heliocentric systems and poured scorn on the physics of *Aristotle* and the astronomy of *Ptolemy*.

Robert Hooke (1635–1703), *Christopher Wren* and *Edmund Halley* outlined qualitatively what *I. Newton* later explained quantitatively. At about the time when *Newton* was silently pondering the Solar System, *Hooke* realized that the force controlling the Solar System, drawing the planets to the Sun and the Moon to the Earth as that which causes apples to fall from tree.

Isaac Newton (1642–1726) wrote in an unpublished manuscript referred to by its opening words (*De Gravitatione*) that *'an infinite and external divine power occupies all space and extends infinitely in all directions'*. In his "*Mathematical Principles of Natural Philosophy*" known as *Principia*, *Newton* said of space: *'Absolute space in its own nature, without relation to anything external remains always similar and immovable'*. On time he said: *'Absolute, true, and mathematical* *time, of itself and from its own nature, flows equably without relation to anything external'*. *Newton*'s celebrated three laws of motion are the corner stones of the Science.

The universal law of gravitation has been heralded as one of the great discoveries of the early scientific age; it is thought in freshman physics, and it is referred to in many popular expositions of science. Indeed, it must have taken enormous courage to link the laws of free fall with the laws that govern the dynamics of the Solar System.

But when the great law of universal gravitation is expounded, there is usually nothing said about the issue of action-at-a-distance versus contact interaction. That discussion is conveniently postponed to a much later and more sophisticated study of the laws of nature.

However, *Newton* himself was well aware of this problem (Letter from Newton to Bentley, pp. 302–303 in I. B. Cohen, *Isaac Newton's Paper and Letters on Natural Philosophy*, Harvard University Press, Cambridge, Massachusetts, 1978):

> *The Gravity should be innate inherent and essential to Matter, so that one Body may act upon another at a Distance thro' a Vacuum, without the Meditation of any thing else, by and through which their Action and Force may be conveyed from one to another, is to one so great an Absurdity that I believe no than who has in philosophical Matters a competent Faculty of thinking, can ever fall into it.*

The fact that *Newton* nevertheless went ahead with his theory as we know it indicates that he made a clear distinction between the *descrip-*

tion of the gravitational phenomena and their *causal explanation*; the latter, he felt, requires some intermediary that conveys that action (such as an *ether*). But he must have judged the former to be sufficient interest to warrant publication even without the causal explanation.

But while *Newton* was unable to find a solution to this problem, *he did not reject his law of universal gravitation* on account of that. Surely, the lack of explanation represented, at least in *Newton*'s mind, a serious objection to his gravitation theory. Yet, he decided to ignore it, and to present his theory anyway.

And so, to the great benefit of science, *Newton* published the *Principia* despite his knowledge that his gravitation theory is badly flawed in that *does not provide an explanation* of the force of gravitation but only a *description*. The solution to *Newton*'s problem did not come until about two and a half centuries later with the development of *Einstein*'s general relativity. In that theory, gravitation is propagated with the speed of light, and action-at-a distance is avoided completely.

1.3.2 Angular momentum and central forces

A central force is one whose direction is always along a radius, that is, either toward or away from a point we shall use as an origin (or force center) and whose magnitude depends solely upon the *distance* from origin, r. We can write this

$$\mathbf{F}(\mathbf{r}) = F(r)\hat{\mathbf{r}}, \tag{1.127}$$

where $\hat{\mathbf{r}}$ is a unit vector in the radial direction. Since $\hat{\mathbf{r}} = \mathbf{r}/r$ a central force can also be written as

$$\mathbf{F} = \frac{F(r)}{r}\mathbf{r}. \tag{1.128}$$

Central forces are important because we encounter them so often in physics. The gravitational force is a central force. The electrostatic force between two charges is a central force.

Much of physics – can be viewed as a careful application of *Newton*'s Second Law. So we may as well begin with that

$$\mathbf{F} = m\mathbf{a}. \tag{1.129}$$

We define the *torque* $\boldsymbol{\tau}$, we can write it as

$$\boldsymbol{\tau} = \mathbf{r} \times m\mathbf{a} \tag{1.130}$$

and angular momentum $\mathbf{L}$,

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \tag{1.131}$$

as the vector product of radius vector $\mathbf{r}$ and momentum $\mathbf{p}$. Now we calculate the time derivative of $\mathbf{L}$

$$\frac{d\mathbf{L}}{dt} = \frac{d}{dt}(\mathbf{r} \times \mathbf{p}) = \mathbf{r} \times m\mathbf{a} = \mathbf{r} \times \mathbf{F} = \boldsymbol{\tau}. \tag{1.132}$$

Since $\mathbf{v} \times \mathbf{v} = 0$. It is quite interesting that

$$\frac{d\mathbf{L}}{dt} = 0, \tag{1.133}$$

when $\mathbf{F}(r) \parallel \mathbf{r}$, i.e. when $\mathbf{F}(\mathbf{r})$ is a central force.

1.3.3 Inverse-square force (3-dimensional space)

If we investigate the force between two charges, such as the force binding an electron to its nucleus, or we investigate the force between two masses, such as the force binding the Moon in its orbit around Earth or Earth in her orbit around the Sun, we find that in both these cases the force varies inversely as the square of the distance. We can represent this behavior by writing:

$$F(r) = \frac{K}{r^2}, \tag{1.134}$$

where $K < 0$ for an attractive force and $K > 0$ for a repulsive force. In particular for gravitational forces K equals GMm where G is a universal constant. In SI units $G = 6.67\,10^{-11}\,({}^{\mathrm{Nm}^2}/_{\mathrm{kg}^2})$ and M and m are the two masses involved. For electrical forces $K = ({}^1/_{4\pi\epsilon_0})Qq$ where ϵ_0 is called the permittivity of free space. It has the value in SI units of $\epsilon_0 = 8.85 \cdot 10^{-12}\,({}^{\mathrm{C}^2}/_{\mathrm{Nm}^2})$ and Q and q are the electric charges involved. Since Q and q can each be positive or negative, K can be positive or negative. Hence, the electrical force can be either attractive or repulsive. The gravitational force for matter – matter interaction is attractive. The sign for matter – antimatter and antimatter –antimatter interaction is still an open problem for theoreticians and experimentalists.

In paragraph Xwe introduced the angular momentum **L**

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}. \tag{1.135}$$

It is interesting to calculate the time derivative for **L**. From formula (1.135) we obtain

$$\frac{d\mathbf{L}}{dt} = m\mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2}. \tag{1.136}$$

For central force $\mathbf{F}(r) = (F(r))/(r)\mathbf{r}$ we obtain from formula (1.136)

$$\frac{d\mathbf{L}}{dt} = 0; \qquad \mathbf{L} = \mathbf{constant}. \tag{1.137}$$

Now, what does constant angular momentum tell us about the motion? The angular momentum

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \tag{1.138}$$

is a vector perpendicular to the plane determined by the location vector $\mathbf{r}$ and the momentum $\mathbf{p}$. If $\mathbf{L}$ remains constant, this plane remains constant. The motion under a central force is confined to a plane. This is quite important. It is also very good news – the motion is describable in only two dimensions rather than the three that might well have been anticipated.

In a plane the acceleration can be written as

$$\begin{aligned} a_r &= \frac{d^2 r}{dt^2} - r\left(\frac{d\Theta}{dt}\right)^2, \\ a_\Theta &= r\frac{d^2\Theta}{dt^2} + 2\frac{dr}{dt}\frac{d\Theta}{dt} \end{aligned} \tag{1.139}$$

velocity

$$v_r = r\frac{dr}{dt}; \qquad v_\Theta = r\frac{d\Theta}{dt} \tag{1.140}$$

From formulae (1.138) and (1.140) we obtain for absolute value of L

$$L = mr^2\frac{d\Theta}{dt} = \text{constant}. \tag{1.141}$$

Considering formula (1.139) the Newton Second Law can be written as

$$\mathbf{F} = m\frac{d^2\mathbf{r}}{dt^2} = m\left[\frac{d^2r}{dt^2} - r\left(\frac{d\Theta}{dt}\right)^2\hat{\mathbf{r}} + \left(r\frac{d^2\Theta}{dt^2} + 2\frac{dr}{dt}\frac{d\Theta}{dt}\right)\hat{\Theta}\right] \quad (1.142)$$

As always, the single vector equation implies two ordinary scalar equations – one for each component

$$F(r) = m\frac{d^2r}{dt^2} - mr\left(\frac{d\Theta}{dt}\right)^2$$
$$0 = mr\frac{d\Theta}{dt} + 2m\frac{dr}{dt}\frac{d\Theta}{dt} \quad (1.143)$$

The second equation can be written as

$$\frac{d}{dt}\left(mr^2\frac{d\Theta}{dt}\right) = 0 \quad (1.144)$$

Comparing formulae (1.141) and (1.144) we conclude that formula (1.144) is the conservation of the angular momentum

$$\frac{dL}{dt} = 0. \quad (1.145)$$

In the first equation (formula (1.143)) we change of variable substituting $r = 1/u$. We also need formula (1.141)

$$\frac{d\Theta}{dt} = \frac{L}{mr^2} = \frac{L}{m}u^2. \quad (1.146)$$

After the calculation of the prescribed derivatives we obtain from equation (1.143)

$$\frac{d^2u}{d\Theta^2} + u = -\frac{m}{L^2}\frac{1}{u^2}F\left(\frac{1}{u}\right), \quad (1.147)$$
$$u = \frac{1}{r}.$$

Equation (1.147) is the master equation which describes the movement of the body with mass m in the field of central forces $F(^1/_u)$. We can imagine the following functions $F(^1/_u)$

$$F\left(\frac{1}{u}\right) = K_1 u^{\pi},\ K_2 u^3,\ K_3 u^2,\ K_4 u^{0.64},\ K_5 u^{-4.62}. \tag{1.148}$$

We can imagine the "other" universes for which the central forces have the different $F(^1/_u)$. But can life be originated and developed in all these universes? This question is answered by the anthropic principle and will be discussed later on. For the moment we can say the following:

Macroscopic structure of the Universe we live in can be understood with just two forces: *Newton* and *Coulomb*. For both forces

$$F\left(\frac{1}{u}\right) = K u^2. \tag{1.149}$$

Why?

With the forces described by formula (1.149) we obtain for equation (1.147)

$$\frac{d^2 u}{d\Theta^2} + u = -\frac{Km}{L^2} \tag{1.150}$$

This beautiful equation describes the classical motion of the planets, and electrons round the source of the force $F = Ku^2$. Moreover, the equation (1.149) in fact is the harmonic oscillator equation. The solution to the Eq. (1.150) can be written as

$$u = A\cos(\Theta - \Theta_0) - \frac{mK}{L^2}, \tag{1.151}$$

or

$$r = \frac{1}{A\cos(\Theta - \Theta_0) - \frac{mK}{L^2}}. \tag{1.152}$$

Equation (1.152) describes the *conic* curves: ellipse, parabola and hyperbola depending on constants A, Θ_0, m, K and L. We can choose our coordinate axes so that $\theta_0 = 0$ to simplify things just a little:

$$r = \frac{1}{A\cos\Theta - \left(\frac{mK}{L^2}\right)} \tag{1.153}$$

This is a *conic sections.* From plane geometry, any conic section can be written as

$$r = r_0 \frac{1+e}{1+e\cos\Theta} \tag{1.154}$$

where e is called the *eccentricity* of the orbit. For $e < 1$ the orbit is an ellipse. For the special case $e = 0$, this ellipse becomes a circle. For $e = 1$ the orbit is a parabola; for $e > 1$ a hyperbola. The geometry of these different conic sections is shown on Fig.

1.3.4 Kepler's Laws

Based on astronomical data taken by *Tycho Brahe*, early in the seventeenth century *Johannes Kepler* announced three general laws that described the motion of the planets around the Sun. *Newton*'s Law of Universal Gravitation, given shortly after *Kepler*'s laws, was readily accepted because it provided a description of the planets' motions entirely consistent with *Kepler*'s laws.

Kepler's law can be started as follows:

1. Planets move in orbits that are ellipses with the Sun at one focus (elliptical orbits)

2. Areas swept out by the radius vector from the Sun to a planet in equal times are equal.

3. The square of a planet's period is proportional to the *cube* of the semimajor axis of its orbit.

We have seen in the previous section that the first of these laws (that of elliptical orbits) follows directly from *Newton*'s Law of Universal Gravitation (i.e., from the inverse-square nature of the force of gravity).

Equal areas being swept in equal times is a consequence of the angular momentum's being constant. This can be seen if we start with Fig.which shows a body in an elliptical orbit about an origin 0.

For a small change in angle $d\Theta$ the area swept out as a body moves from r to $r + dr$ is

$$dA = \frac{1}{2} r dh \tag{1.155}$$

where dh is the perpendicular distance between the two radius vectors; this is just the area of a right triangle. But

$$dh = r d\Theta \tag{1.156}$$

Thus

$$dA = \frac{1}{2} r(r d\Theta) = \frac{1}{2} r^2 d\Theta \tag{1.157}$$

But $mr^2 d\Theta/(dt) = L$, the angular momentum which we know to be constant for any central force. Therefore,

$$\frac{dA}{dt} = \frac{1}{2}\frac{L}{m} = \text{constant} \tag{1.158}$$

again agreeing with *Kepler*'s second law.

We have discussed two of *Kepler*'s laws. *Kepler*'s Third Law is a consequence of inverse-square nature of the gravitational force. We can readily integrate Eq. (1.158) to find

$$A = \frac{LT}{2m} \tag{1.159}$$

or

$$T = \frac{2mA}{L} \tag{1.160}$$

where A is the area of the orbit and T is the period, the time necessary to complete one entire orbit. The area of an ellipse is

$$A = \pi a^2 \sqrt{1 - e^2}, \tag{1.161}$$

where a is the semimajor axis that is, half of the maximum diameter. From the preceeding section we find

$$a = -\frac{L^2}{mK} \frac{1}{1 - e^2} \tag{1.162}$$

Putting these together we have:

$$T = \frac{2m\pi}{\sqrt{m(-K)}} a^{\frac{3}{2}} \tag{1.163}$$

or

$$\frac{T^2}{a^3} = \left(\frac{2m\pi}{\sqrt{m(-K)}} \right)^2 = \text{constant}$$

which, indeed is just *Kepler*'s Third Law.

1.3.5 Systems of planets

So far we have considered the motion of a single body caused by the external forces acting upon it – for example the motion of an oscillating mass, or a planet in orbit around the sun. Now we shall inquire into the motion of several bodies – for example the several planets in our solar system. Earlier we introduced and used the idea of conservation of energy. Now with a system of planets, conservation principles take on a new and increased importance.

For the general case, consider N bodies (N planets) labelled 1, 2, 3, N. They are located at positions $\mathbf{r}_1, \mathbf{r}_2, \ldots \mathbf{r}_N$ and move with velocities $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_N$, all measured relative to some origin 0. The particles have masses $m_1, m_2, \ldots m_N$. This general situation is illustrated in Fig.Any particular particle, say, the ith particle experiences a *net* force that is the result of an external force on it, $\mathbf{F}_i$ and all the internal forces from all the other particles. If we write the internal force on particle i exerted by particle k as F_{ik} then

$$\mathbf{F}_i(\text{net}) = \mathbf{F}_i + \sum_{\substack{k=1 \\ k\neq i}}^{N} \mathbf{F}_{ik}. \tag{1.164}$$

Now we apply the *Newton*'s Second Law, $F(\text{net}) = m(d^2r)/(dt^2)$. In this situation we arrive at

$$m_i \frac{d^2r}{dt^2} = \mathbf{F}(\text{net})_i + \sum_{\substack{k=1 \\ k\neq i}}^{N} \mathbf{F}_{ik} \tag{1.165}$$

However, it occurs that the direct solution is out of the question for all but very special cases.

To start with we define the *center of mass* for the system of N planets. The center of mass $\mathbf{R}$ is just the average position of the mass and is given by:

$$\mathbf{R} = \frac{\sum m_i \mathbf{r}_i}{\sum m_i} \tag{1.166}$$

$M = \sum m_i$ is the total mass of the system. Differentiation of the center of mass position yields the velocity of the center of mass

$$\mathbf{V} = \frac{d\mathbf{R}}{dt} = \frac{1}{M} \sum m_i \frac{d\mathbf{r}_i}{dt} = \frac{1}{M} \sum m_i \mathbf{v}_i. \tag{1.167}$$

Further differentiate yields the acceleration of the center of mass:

$$\mathbf{A} = \frac{1}{M} \sum_i m_i \frac{d^2\mathbf{r}}{dt^2}. \tag{1.168}$$

We now return to Eq. (1.165) and sum up of the net force acting on all N planets

$$\sum_{i=1}^{N} m_i \frac{d^2\mathbf{r}}{dt^2} = \sum_{i=1}^{N} \sum_{\substack{k=1 \\ i \neq k}}^{N} \mathbf{F}_{ik}. \tag{1.169}$$

The last term is zero because every force $\mathbf{F}_{ji}$ will be cancelled by its counterpart $\mathbf{F}_{ij}$ and

$$\mathbf{F}_{ij} = -\bar{F}_{ij} \tag{1.170}$$

by *Newton*'s Third Law. This means that it is impossible to change the motion of center mass by internal force alone. The remaining force term is the resultant of all the *external forces* exerted on all the individual particles. The left hand side is the total mass multiplied by the acceleration of the center of mass.

That is,

$$M \frac{d^2\mathbf{R}}{dt^2} = \mathbf{F} \tag{1.171}$$

where

$$\mathbf{F} = \sum_{i=1}^{N} \mathbf{F}_i \tag{1.172}$$

Thus, the motion of the center of mass of any system of particles can be described by Eq. (1.171), which is identical to an equation describing the net force on a single object with mass M.

On the right hand side of the Eq. (1.167) we have the sum of the product $m_i\mathbf{v}_i$. The momentum $\mathbf{p}$ of a particle with mass m and velocity $\mathbf{v}$ is defined by

$$\mathbf{p} = m\mathbf{v} \tag{1.173}$$

For a constant mass the *Newton*'s Second Law can then be written as

$$\mathbf{F}_{\text{net}} = \frac{d\mathbf{p}}{dt} \tag{1.174}$$

If we apply Eq. (1.174) to ith particle we obtain

$$\frac{d\mathbf{p}_i}{dt} = \mathbf{F}_i(\text{net}) = \mathbf{F}_i + \sum_k \mathbf{F}_{ik} \tag{1.175}$$

Summing this over all N particles we have:

$$\frac{d\mathbf{P}}{dt} = \frac{d}{dt}\sum_i \mathbf{p}_i = \sum_i \mathbf{F}_i + \sum_i \sum_k F_{ik} \tag{1.176}$$

The *total momentum* of the system is the vector sum of the momenta of all the bodies in the system

$$\mathbf{P} = \sum_i \mathbf{p}_i = \sum_i m\mathbf{v}_i = M\mathbf{V} \tag{1.177}$$

That is, this vector sum of the momenta of the individual planets is the same as the momentum of a single particle whose mass is the total mass of the system and whose velocity is that of the center of mass of the system. Just as before we argue that the last term in Eq. (1.176) must be zero. Eq. (1.176) can now be written as

$$\frac{d\mathbf{P}}{dt} = \mathbf{F} \tag{1.178}$$

Again we have reduced the motion of the center of mass of a system of particles (planets) to that of a single body with mass M. In the case when $\mathbf{F} = 0$, we obtain from formula (1.178) the conservation of momentum for the system of N planets

$$\mathbf{P} = \mathbf{constant} \tag{1.179}$$

Which can be stated as follows.

In the absence of the external forces the total momentum of the system of N planets is **constant**. For example, when we neglect the interaction of α-Centauri star (which is in minimal distance from the Sun system) the total momentum of the Sun planetary system is constant and does not dependent on time.

1.3.6 Parade of planets – extrasolar planets

We humans are interested in the extrasolar planet systems. We are interested in more than just life in other planetary systems. We'd like to know how planetary systems form and evolve. Despite the astronomers successes, our planet search is still in its infancy. Still, with 33 planets already in hand (March 2000!), we see a few general trends. Planet formation appears to be a chaotic process that often tosses Earth-size planets out of their system entirely, leaving Jupiter-mass brutes behind in highly elongated (*eccentric*) orbits. Failed stars called *brown dwarfs* appear to be infrequent companions to solar-type stars. Astronomers have found the first full-fledged system of planets around a sunlike star, and other stars also show signs of multiple Jupiters.

The parade of extrasolar planets around normal stars began in 1995 when *M. Mayor* and *D. Queloz* of Geneva Observatory in Switzerland discovered a planet at a distance of only 0.05 AU from the star 51 Pegasi (one AU is the average Earth-Sun distance).

This was followed shortly thereafter by the discovery by *Marcy* and *Butler* of planets around 47 *Ursae Majoris* and 70 *Virgins*. The orbit of the 51 *Pegasi* planet and the eccentric orbit of 70 *Virgins* planet shook humans *heliocentric* expectation that planetary systems would be nearly identical to ours. Instead, planets fill orbital niches of unimagined diversity. Astronomers find planets by looking at how they yark their stars as they go around in their orbits. The jerky motion of a star is revealed in the star's spectrum. As the star approaches Earth in response to the planet's gravity its light is shifted toward the blue and of the spectrum. As the star recedes, its light is shifted toward the red. This subtle *Doppler*signature woven into the light of the planet star allows us to reconstruct the orbit and minimum mass of an otherwise hidden planet.

The nearly circular orbits of the planets in our solar system led astronomers to expect that planets around other stars would reside in nearly circular orbits too. After all, planets probably form in circular proplanetary disks, such as the disks seen in the Orion Nebula. The gas and dust in these disks follow circular orbits and friction within these disks would circularize the orbits of newly forming planets in much the same way that friction circularizes the flow of water going down in a bathtub drain.

But most of the extrasolar planets found so far reside in highly eccentric orbits, not circular. Indeed, the 18 extrasolar planets with the largest orbits all reside in eccentric orbits. Most of those orbits are more than twice as elongated as the orbits of Earth, Jupiter and Saturn.

Why are other planetary systems dominated by elliptical orbits rather than circular? The best clue comes from comets in our own solar system. Comets reside in orbits so elongated that they visit the inner solar system only rarely. But comets did not form in elliptical orbits. Rather they formed in circular orbits in the protosolar disk. Comets were gravitationally flung into their present-day elliptical orbits when they ventured too close to planets, in much the same way spacecraft receive gravitational assist from planets.

We now suspect that most planets themselves engage in this slingshot activity leaving than in disturbed, elliptical orbits. If two or more massive planets form in orbits a few AU apart, this fate is inevitable. One planet will be scattered inward, the other outward. If even one planet suffers this slingshot effect, it will likely travel close enough to neighboring planets to disturb them as well. This theory explains why the large majority of the extrasolar planets found to date reside in noncircular orbits.

Unfortunately, Jupiter - mass planets toss Earth – mass planets around. The resulting chaos may result in some planets being gravitationally eject from the planetary system altogether, leaving only the most massive survivors behind. Our Galaxy must be filled with trillions of Earth - size rogue planets - dark, rocky hulks wandering aimlessly through interstellar space.

The predominance of elliptical orbits implies that planetary systems with circular orbits may be the exception rather than the norm. Apparently, our nine planets were just far enough apart and low enough in mass to avoid this chaos. The nine planets do perturb one another, but not enough to cause close passages. Our planetary system may be one of the rare systems that remains just barely stable.

If our system is unusual in its circular orbits, we humans would seem to be extraordinarily lucky to be here. After all, the circular orbit of Earth keeps solar heating nearly constant, minimizing temperature fluctuations. Perhaps biological evolution would not have proceed to intelligence if Earth temperature were fluctuating wildly. It may be that Darwinian evolution toward complex organisms is enhanced by relatively quiescent climates enabled by circular orbit. If so, we owe our existence to Earth's stable orbit.

1.3.7 Why is space three-dimensional?

Without exception, the laws of physics are of such nature that they can be generalized to space of either more or fewer than three dimension. The statement that ordinary space has just three dimensions for the first time can be found out in *Ptolemy*: *Per diastaseos* (On the dimension). The problem of space dimension was already apparent to *Aristotle*, whose argument in support of three dimensions was recapitulated in *Galile*'os *Dialogue*. At the beginning of the XX century *P. Ehrenfest* (*Ann. Physics*, **61**, (1920), 440) pointed out that in a space of more than three dimensions the laws of physics do not allow stable planet orbits. Recently *G. J. Whitrow*, *The structure of the Universe*, Harper and Row, N. Y. 1959) rediscovered some features of *Ehrenfest*'s work and also advanced the interesting argument that the development of higher forms of life would be impossible in a space of fewer than three dimensions.

In any higher organism, a large number of cells must be intercounted by nerve fibers. If space had only two dimensions, an organism could be only a two-dimensional configuration and its nerve paths would cross. At the intersections, the nerves would have to penetrate each other, for absence of a third dimension would not permit a fiber to be led above or below another one. As a consequence nerve impulses would mutually interfere. *The existence of a highly developed organism having many non-intersecting nerve paths in thus possible only in a space having at least three dimensions.*

As we know both the Newtonian gravitational force and electrostatic force can be described in the three dimensional space [formula (1.134)]

$$F = \frac{K}{r^2}, \qquad n = 3, \tag{1.180}$$

where n is the number of dimension of space. For $n \neq 3$ the natural generalization of formula (1.180) is

$$F = (n-2)\frac{K}{r^{n-1}}, \qquad n \neq 2. \tag{1.181}$$

The impossibility of stable planet orbit for $n > 3$ can be seen in an elementary way. Let m be the mass of planet and L angular momentum (which is constant for the central force (1.181)) formula

$$L = mr^2\dot{\Theta}. \tag{1.182}$$

The gravitation potential for the *conservative* forcewill be

$$V = -\frac{K}{r^{n-2}}. \tag{1.183}$$

At the extreme distances from the central body for a planet with mass m, we have

$$\frac{dr}{dt} = 0. \tag{1.184}$$

The kinetic energy T at such points is then (formula (1.140))

$$T = \frac{p^2}{2m} = \frac{1}{2}mr^2\dot{\Theta}^2, \tag{1.185}$$

which by equation (1.182) becomes

$$T = \frac{L^2}{2mr^2}. \tag{1.186}$$

By conservation of mechanical energy (formula (1.126)) $T + V =$ constant, or

$$\frac{L^2}{2mr_1^2} - \frac{K}{r_1^{n-2}} = \frac{L^2}{2mr_2^2} - \frac{K}{r_2^{n-2}}, \tag{1.187}$$

where r_1 is the minimum distance from the central body and r_2 is the maximum distance, *perihelion* and *aphelion* respectively.

The equation (1.187) shows that for $n = 4$ there can be a finite, positive solution only if $r_1 = r_2$. For $n > 4$ it can be shown that an orbit in which r oscillates between two extremes is likewise ruled out.

In general the centripetal force in a circular orbit is

$$F_c = mr^2\dot{\Theta}^2. \tag{1.188}$$

Using Eq. (1.182) this becomes

$$F_c = \frac{L^2}{mr^3}. \tag{1.189}$$

In the actual eccentric orbit, the attractive force must be less than this centripetal force at *perihelion*, for then the planet is about to move outward. At *aphelion*, it is just the other way around.

These conditions can be expressed respectively by the following inequalities

(a)

$$F < F_c$$
$$\frac{(n-2)K}{r_1^{n-1}} < \frac{L}{mr_1^3} \quad \text{or} \quad \frac{K}{r_1^{n-2}} < \frac{L^2}{(n-2)mr_1^2}, \tag{1.190}$$

(b)

$$F > F_c$$
$$\frac{(n-2)K}{r_2^{n-1}} > \frac{L^2}{mr_2^3} \quad \text{or} \quad \frac{K}{r_2^{n-2}} > \frac{L^2}{(n-2)mr_2^2}. \tag{1.191}$$

Substituting formula (1.191) to formula (1.187) yields

$$\frac{L^2}{2mr_1^2} - \frac{L^2}{(n-2)mr_1^2} < \frac{L^2}{2mr_2^2} - \frac{L^2}{(n-2)mr_2^2}. \qquad (1.192)$$

Equation (1.192) can be factored

$$\frac{L^2}{mr_1^2}\left(\frac{1}{2} - (n-2)^{-1}\right) < \frac{L^2}{2mr_2^2}\left(\frac{1}{2} - (n-2)^{-1}\right). \qquad (1.193)$$

This relation obviously cannot be true for $n = 4$, for then each of the brackets becomes zero. Remembering that $r_2 > r_1$ it also cannot be true for any $n > 4$, which makes the values of the brackets less than $^1/_2$.

Thus, the existence of an elliptic orbit for $n \geq 4$ is ruled out. The results for planetary orbits are collected in Table

Table 1.3: Table

Phenomena	Cases thus excluded	
Bio-topology (existence of a highly developed organism)	$n < 3$	
Stability of planetary orbits	$n > 3$	
	$n = 4$	Possible only for circular orbit
	$n > 4$	
	$n < 3$	Excluded if the potential is to vanish at ∞

In conclusion, it may be said that stable elliptical planetary orbits can exist and support the existence of the highly developed organisms only in three dimensional space. Fine tuning of our universe for the existence of the highly developed organism also human being – the observes of the Universe, is well grounded on the Newton's description of the Nature.

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