# Introduction to spin waves

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## Contents



## Abstract

This work serves as a brief introduction to the theory of spin waves. We recall basic ideas and properties of spin algebra and spin states. We briefly discuss the connection between spin algebra and theory of representations of Lie algebras. We introduce the concept of spin waves and discuss its basic properties in the context of the Heisenberg ferromagnetic systems.

## Introduction

In this section, we will recall basic facts and formulas governing spin algebra. Since these topics are deeply discussed in every quantum mechanics handbook (see, for example, [2]), we will restrict ourselves mainly to listing the most important equations and definitions and brief comments, just for further reference. If the reader knows these topics and formulas, she/he is encouraged to skip this section.

## The spin

It is a well known fact that there is a quantum number $\mathrm{S} \in \frac{\mathbb{N}}{2} \cup \{0\}$ associated with each quantum particle. This number is called a *spin* of that particle. The spin itself can be intuitively viewed as an "intrinsic" angular momentum, but one has to be careful with thinking of a spin in that way, mainly because it has nothing to do with rotation. Nevertheless, algebra of spin possesses similar properties as the algebra of orbital angular momentum in quantum mechanics, with slight exception that spin quantum number can have half-integer value. Existence of spin introduces the additional degrees of freedom of a particle, and leads to various interesting physical phenomena. Particles with half-integer spin are called **fermions**, while these with integer spin - **bosons**. Table 1 presents the list of elementary particles together with their spin S. Non-elementary particles, like protons and neutrons, also posses a spin, but this case is more complicated and won't be discussed in detail. The reader should just keep in mind that for each quantum particle, the spin S is a fixed quantum number.

Table 1: Elementary particles and their spin.

| Fermions | | | | Bosons | |
|---|---|---|---|---|---|
| Quarks | | Leptons | | | |
| Particle: | Spin: | Particle: | Spin: | Particle: | Spin: |
| Up | 1/2 | Electron | 1/2 | Gluon | 1 |
| Down | 1/2 | Muon | 1/2 | Photon | 1 |
| Charm | 1/2 | Tau | 1/2 | Z Boson | 1 |
| Strange | 1/2 | Electron Neutrino | 1/2 | W Boson | 1 |
| Top | 1/2 | Muon Neutrino | 1/2 | Higgs Boson | 0 |
| Bottom | 1/2 | Tau Neutrino | 1/2 | | |

## The spin algebra

For a quantum particle with spin S, there is a Hilbert space $\mathbb{C}^{2\mathrm{S}+1}$ associated with that particle. Each vector from this space represents a spin state of that particle, and we can introduce spin operators defined on that space. The algebra of these operators is governed by the following commutation relations, which are defined in analogy to the commutation relations of orbital angular momentum:

$$[S^1, S^2] = iS^3, \qquad [S^2, S^3] = iS^1, \qquad [S^3, S^1] = iS^2. \tag{1}$$

In this paper, we will work with units such that $\hbar = 1$. The Hilbert space $\mathbb{C}^{2\mathrm{S}+1}$ is spanned by normalized, basis vectors denoted by

$$|\mathrm{S}, m\rangle, \qquad m \in \{-\mathrm{S}, -\mathrm{S}+1, ..., \mathrm{S}-1, \mathrm{S}\}. \tag{2}$$

The number $m$ is sometimes called a projection of a spin onto $z$-axis or $z$-component. For example, in the case of an electron, $\mathrm{S} = \frac{1}{2}$, the Hilbert space is $\mathbb{C}^2$ and we have two basis vectors: $|\frac{1}{2}, -\frac{1}{2}\rangle$ and $|\frac{1}{2}, \frac{1}{2}\rangle$, which are often denoted as $|\uparrow\rangle$, and $|\downarrow\rangle$ respectively, and called "up" and "down". These vectors can be identified with the standard basis vectors of $\mathbb{C}^2$, so that $|\uparrow\rangle = [1, 0]$ and $|\downarrow\rangle = [0, 1]$.

Let us go back to the general case. In analogy to the theory of orbital angular momentum, we have

$$S^3|\mathrm{S}, m\rangle = m|\mathrm{S}, m\rangle, \tag{3}$$

and hence

$$S^3|\mathrm{S}, \mathrm{S}\rangle = \mathrm{S}|\mathrm{S}, \mathrm{S}\rangle, \qquad S^3|\mathrm{S}, -\mathrm{S}\rangle = -\mathrm{S}|\mathrm{S}, -\mathrm{S}\rangle. \tag{4}$$

We also introduce the so called "raising" $(+)$ and "lowering" $(-)$ operators:

$$S^{\pm} = S^1 \pm iS^2. \tag{5}$$

These operators act on states $|\mathrm{S}, m\rangle$ in the following way:

$$S^{\pm}|\mathrm{S}, m\rangle = \sqrt{\mathrm{S}(\mathrm{S}+1) - m(m \pm 1)}|\mathrm{S}, m \pm 1\rangle, \tag{6}$$

so that the number $m$ in $|\mathrm{S}, m\rangle$ is increased by 1 or decreased by 1, hence the name of operators. In particular, the following property will be of great relevance for us:

$$S^-|\mathrm{S}, -\mathrm{S}\rangle = 0, \tag{7}$$

which simply means that the "lowest" possible state cannot be lowered without returning zero. Analogously, the "highest" possible state cannot be increased:

$$S^+|\mathrm{S}, \mathrm{S}\rangle = 0.$$

Using (1), one can prove the following, very important properties of rising and lowering operators, which will later be used frequently:

$$[S^-, S^+] = -2S^3, \qquad [S^3, S^{\pm}] = \pm S^{\pm}. \tag{8}$$

**Example: spin $\frac{1}{2}$**

For the easiest non-trivial case, $\mathrm{S} = \frac{1}{2}$ (for example, an electron), the Hilbert space is, as mentioned above, $\mathbb{C}^2$ and we have two basic states: $|\uparrow\rangle = [1, 0]$, $|\downarrow\rangle = [0, 1]$. Moreover, the spin operators $S^1, S^2, S^3$ are just Pauli matrices $\sigma^1, \sigma^2, \sigma^3$ multiplied by $\frac{1}{2}$ $(\hbar = 1)$, so that

$$S^1 = \frac{1}{2}\sigma^1 = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ S^2 = \frac{1}{2}\sigma^2 = \frac{1}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix},\ S^3 = \frac{1}{2}\sigma^3 = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

It follows that:

$$S^- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad S^+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

and one can check easily that properties (1) - (8) hold for the above operators and states.

# Representations and spins

All the facts we given in the previous section can be viewed as a direct result of representation theory of Lie groups and Lie algebras. The reader which is not familiar with these concepts is free to skip this entire section.

The real Lie algebra of a group SU(2) is denoted by $\mathfrak{su}(2)$. It is the real vector space of $2 \times 2$, anti-hermitian matrices of trace zero. Its complexification

$$\mathfrak{sl}_{\mathbb{C}}(2) = \mathfrak{su}(2) + i\mathfrak{su}(2)$$

is a complex vector space of $2 \times 2$, complex matrices with trace zero. This space is three-dimensional, and its basis vectors are the following matrices:

$$T = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad T_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad T_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

**Definition 1** *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra, $V$ a finite-dimensional vector space and $\mathfrak{gl}(V)$ be the space of all linear operators from $V$ to itself. Any linear map*

$$\rho : \mathfrak{g} \to \mathfrak{gl}(V)$$

*which preserves the commutator:*

$$\rho([g_1, g_2]) = [\rho(g_1), \rho(g_2)]$$

*is called a (finite-dimensional)* ***representation*** *of $\mathfrak{g}$ on $V$. We say that it is* ***irreducible*** *when the smallest non-zero subspace which is left invariant by all operators of the form $\rho(g), g \in \mathfrak{g}$, is $V$ itself.*

We have the following [1]:

**Theorem 1** *Every finite-dimensional, irreducible representation of* $\mathfrak{sl}_{\mathbb{C}}(2)$ *is unitarily equivalent to* $(\rho^{(n)}, V^{(n)})$*, where the dimension of a complex vector space* $V^{(n)}$ *is* $n+1$ *and, for a basis* $\{\mathbf{v}_i\}_{j=0}^{n}$ *of* $V^{(n)}$*, we have*

$$\begin{aligned}
&\rho^{(n)}(T)\mathbf{v}_j = \big(j - \frac{n}{2}\big)\mathbf{v}_j,\\
&\rho^{(n)}(T_+)\mathbf{v}_j = \mathbf{v}_{j+1},\\
&\rho^{(n)}(T_-)\mathbf{v}_j = j(n-j+1)\mathbf{v}_{j-1},
\end{aligned}$$

*with* $\mathbf{v}_{-1} = \mathbf{v}_{n+1} = \mathbf{0}$.

We now fix $\mathrm{S} \in \frac{\mathbb{N}}{2} \cup \{0\}$ and consider $(\rho^{(2\mathrm{S})}, V^{(2\mathrm{S})})$, where we define

$$\begin{aligned}
S^+ &= \rho^{(2\mathrm{S})}(T_+),\\
S^- &= \rho^{(2\mathrm{S})}(T_-),\\
S^3 &= \rho^{(2\mathrm{S})}(T).
\end{aligned}$$

If, in addition, we define a vector (or state) by the following rescaling:

$$|\mathrm{S}, m\rangle = (-1)^{\mathrm{S}+m}\sqrt{\frac{(\mathrm{S}-m)!}{(\mathrm{S}+m)!}}\mathbf{v}_{\mathrm{S}+m}, \qquad m = -\mathrm{S}, -\mathrm{S}+1, ..., \mathrm{S}-1, \mathrm{S},$$

then one can easily show that

$$\begin{aligned}
S^3|\mathrm{S}, m\rangle &= m|\mathrm{S}, m\rangle,\\
S^{\pm}|\mathrm{S}, m\rangle &= \sqrt{\mathrm{S}(\mathrm{S}+1) - m(m \pm 1)}|\mathrm{S}, m \pm 1\rangle,
\end{aligned}$$

which coincide with (3) and (6).

# Spin waves

In this section, we establish the notation and basic concepts of many-particle spin systems. We introduce the concept of spin waves and show that one-spin wave states are eigenstates of a Heisenberg Hamiltonian. We also prove that n-spin wave states are not eigenstates, but they provide a good approximation of true eigenstates in termodynamic limit.

### Introduction to spin waves

From now on, we will consider many-body spin systems on a finite, $d$-dimensional lattice $\Lambda = \{1, ..., N\}^d \subset \mathbb{Z}^d$. For a fixed spin $\mathrm{S} \in \frac{\mathbb{N}}{2} \cup \{0\}$, we associate with each site $\mathbf{x} \in \Lambda$ a Hilbert space $\mathbb{C}^{2\mathrm{S}+1}$. Figure 1 shows an example of one-dimensional spin system with $\mathrm{S} = \frac{1}{2}$ and $N = 6$, and Figure 2 presents a two-dimensional example.

The Hilbert space of the general, many-body system is thus:

$$\mathcal{H} = \bigotimes_{\mathbf{x} \in \Lambda} \mathbb{C}^{2\mathrm{S}+1}, \tag{9}$$

where we also assume **periodic boundary conditions**. Periodic boundary conditions can be intuitively considered as "the last particle is a neighbour of the first", see Figure 3, where this imaginary picture is displayed for one dimensional spin system.

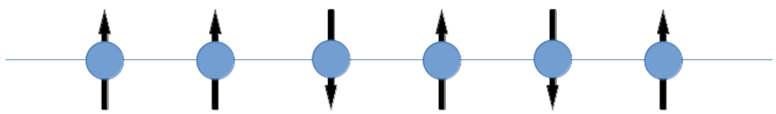

Figure 1: Example of a one-dimensional system for $\mathrm{S} = \frac{1}{2}$ and $N = 6$.

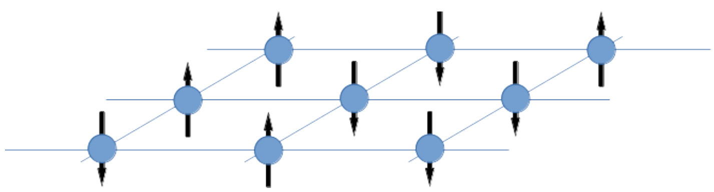

Figure 2: Example of a two-dimensional system for $\mathrm{S} = \frac{1}{2}$ and $N = 9$.

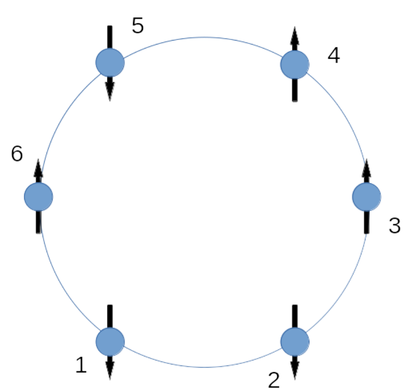


Figure 3: Example of a one-dimensional system with periodic boundary conditions for $\mathrm{S} = \frac{1}{2}$ and $N = 6$.

Since $\Lambda$ is a finite set, it's elements can be ordered, and, for given $\mathbf{x} \in \Lambda$, we can define operators

$$S^{\alpha}_{\mathbf{x}} = \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes S^{\alpha} \otimes \mathbb{1} \otimes \cdots \mathbb{1}, \qquad (N^d \text{ terms}). \tag{10}$$

with $\alpha$ being 1, 2, 3, + or −, and $S^{\alpha}$ acting exactly on site $\mathbf{x}$, without affecting other sites. We define the Heisenberg Hamiltonian acting on $\mathcal{H}$ via

$$\begin{aligned} H_{\Lambda} &= \sum_{<\mathbf{x},\mathbf{y}>\subset\Lambda} \Big( \mathrm{S}^2 - \mathbf{S}_{\mathbf{x}} \cdot \mathbf{S}_{\mathbf{y}} \Big) \\ &= \sum_{<\mathbf{x},\mathbf{y}>\subset\Lambda} \Big( \mathrm{S}^2 - (S^1_{\mathbf{x}} S^1_{\mathbf{y}} + S^2_{\mathbf{x}} S^2_{\mathbf{y}} + S^3_{\mathbf{x}} S^3_{\mathbf{y}}) \Big). \end{aligned} \tag{11}$$

In the above, $\mathbf{S}_{\mathbf{x}} = [S^1_{\mathbf{x}}, S^2_{\mathbf{x}}, S^3_{\mathbf{x}}]$ and $<\mathbf{x},\mathbf{y}>$ is an unordered pair of nearest neighbours, and the summation in (11) is taken over all such pairs, counting each pair only once.

Due to the "−" sign in front of the term $\mathbf{S}_{\mathbf{x}} \cdot \mathbf{S}_{\mathbf{y}}$, Hamiltonian $H_{\Lambda}$ describes ferromagnets: the lowest possible energy can be obtained by states where all spins are parallel, with $m$ being either S or −S, that is, by states of the form

$$\begin{aligned} &\bigotimes_{\mathbf{x}\in\Lambda} |\mathrm{S}, -\mathrm{S}\rangle, \\ &\bigotimes_{\mathbf{x}\in\Lambda} |\mathrm{S}, \mathrm{S}\rangle. \end{aligned} \tag{12}$$

In addition, Hamiltonian $H_{\Lambda}$ is normalized, by adding the constant $\mathrm{S}^2$, so that the lowest possible energy is 0.

According to (10), we see that operators of this form commute if sites on which they act do not coincide, that is

$$[S^{\alpha_1}_{\mathbf{x}}, S^{\alpha_2}_{\mathbf{y}}] = 0, \tag{13}$$

where $\alpha_1$, $\alpha_2 \in \{1, 2, 3, "+", "-"\}$, provided that $\mathbf{x} \neq \mathbf{y}$. Note that this holds when $\mathbf{x}$ and $\mathbf{y}$ are nearest neighbours. It turns out that, in general case, expressions (8) need to be modified to take into account lattice sites which operators act on:

$$\begin{aligned} [S^{-}_{\mathbf{x}}, S^{+}_{\mathbf{y}}] &= -2\delta_{\mathbf{x},\mathbf{y}} S^3_{\mathbf{x}}, \\ [S^3_{\mathbf{x}}, S^{\pm}_{\mathbf{y}}] &= \pm\delta_{\mathbf{x},\mathbf{y}} S^{\pm}_{\mathbf{x}}, \end{aligned} \tag{14}$$

For fixed $\mathbf{x} \in \Lambda$, we still have

$$\begin{aligned} S^+_{\mathbf{x}} &= S^1_{\mathbf{x}} + iS^2_{\mathbf{x}} \\ S^-_{\mathbf{x}} &= S^1_{\mathbf{x}} - iS^2_{\mathbf{x}} \end{aligned} \tag{15}$$

and thus

$$\begin{aligned} S^1_{\mathbf{x}} &= \frac{1}{2}(S^+_{\mathbf{x}} + S^-_{\mathbf{x}}) \\ S^2_{\mathbf{x}} &= \frac{1}{2i}(S^+_{\mathbf{x}} - S^-_{\mathbf{x}}) \end{aligned} \tag{16}$$

We can see that, for $\mathbf{x}$ and $\mathbf{y}$ being nearest neighbours:

$$\begin{aligned} S^1_{\mathbf{x}}S^1_{\mathbf{y}} + S^2_{\mathbf{x}}S^2_{\mathbf{y}} &= \frac{1}{4}(S^+_{\mathbf{x}}S^+_{\mathbf{y}} + S^+_{\mathbf{x}}S^-_{\mathbf{y}} + S^+_{\mathbf{y}}S^-_{\mathbf{x}} + S^-_{\mathbf{x}}S^-_{\mathbf{y}}) \\ &- \frac{1}{4}(S^+_{\mathbf{x}}S^+_{\mathbf{y}} - S^+_{\mathbf{x}}S^-_{\mathbf{y}} - S^+_{\mathbf{y}}S^-_{\mathbf{x}} + S^-_{\mathbf{x}}S^-_{\mathbf{y}}) \\ &= \frac{1}{2}(S^+_{\mathbf{x}}S^-_{\mathbf{y}} + S^+_{\mathbf{y}}S^-_{\mathbf{x}}). \end{aligned}$$

Plugging this result into (11), we obtain an expression of $H_\Lambda$ which is easier to work with:

$$H_\Lambda = \sum_{<\mathbf{x},\mathbf{y}>\subset\Lambda} \Big(\mathrm{S}^2 - \frac{1}{2}(S^+_{\mathbf{x}}S^-_{\mathbf{y}} + S^+_{\mathbf{y}}S^-_{\mathbf{x}}) - S^3_{\mathbf{x}}S^3_{\mathbf{y}}\Big). \tag{17}$$

Consider the first state in (12), namely

$$\Omega = \bigotimes_{\mathbf{x}\in\Lambda} |\mathrm{S}, -\mathrm{S}\rangle. \tag{18}$$

Clearly, for each $\mathbf{x} \in \Lambda$, we have

$$\begin{aligned} S^-_{\mathbf{x}}\Omega &= 0, \\ S^3_{\mathbf{x}}\Omega &= -\mathrm{S}\Omega, \end{aligned} \tag{19}$$

and hence we see that $H_\Lambda\Omega = 0$, so $\Omega$ is indeed a state of the lowest possible energy, or, in other words, a **ground state** of $H_\Lambda$. Similarly, the second state in (12) is also a ground state. However, it turns out that these are not the only possibilities:

**Theorem 2** *For $n \in \{0, 1, ..., 2\mathrm{S}|\Lambda|\}$, $(S_+)^n\Omega$ is a ground state of the Hamiltonian $H_\Lambda$ (17), that is, $H_\Lambda(S_+)^n\Omega = 0$, where*

$$S_+ = \sum_{\mathbf{x}\in\Lambda} S^+_{\mathbf{x}},$$

*and where we put $(S_+)^0 = \mathbb{1}_{\mathcal{H}}$.*

**Proof.** We have already seen that theorem is true for $n = 0$. To prove the result for $n > 0$, let us calculate the following commutator:

$$[H_\Lambda, S_+] = \sum_{<\mathbf{x},\mathbf{y}>\subset\Lambda} \sum_{\mathbf{z}\in\Lambda} \left[ \mathrm{S}^2 - \frac{1}{2}(S^+_\mathbf{x} S^-_\mathbf{y} + S^+_\mathbf{y} S^-_\mathbf{x}) - S^3_\mathbf{x} S^3_\mathbf{y}, S^+_\mathbf{z} \right]$$
$$= \sum_{<\mathbf{x},\mathbf{y}>\subset\Lambda} \sum_{\mathbf{z}\in\Lambda} \left( -\frac{1}{2}[S^+_\mathbf{x} S^-_\mathbf{y}, S^+_\mathbf{z}] - \frac{1}{2}[S^+_\mathbf{y} S^-_\mathbf{x}, S^+_\mathbf{z}] - [S^3_\mathbf{x} S^3_\mathbf{y}, S^+_\mathbf{z}] \right), \tag{20}$$

where

$$[S^+_\mathbf{x} S^-_\mathbf{y}, S^+_\mathbf{z}] = S^+_\mathbf{x} S^-_\mathbf{y} S^+_\mathbf{z} - S^+_\mathbf{z} S^+_\mathbf{x} S^-_\mathbf{y} = S^+_\mathbf{x} \left( S^+_\mathbf{z} S^-_\mathbf{y} - 2\delta_{\mathbf{y},\mathbf{z}} S^3_\mathbf{z} \right) - S^+_\mathbf{z} S^+_\mathbf{x} S^-_\mathbf{y}$$
$$= -2\delta_{\mathbf{y},\mathbf{z}} S^+_\mathbf{x} S^3_\mathbf{z}, \tag{21}$$

similarly to the above:

$$[S^+_\mathbf{y} S^-_\mathbf{x}, S^+_\mathbf{z}] = -2\delta_{\mathbf{x},\mathbf{z}} S^+_\mathbf{y} S^3_\mathbf{z}, \tag{22}$$

and finally:

$$\begin{aligned}
[S^3_\mathbf{x} S^3_\mathbf{y}, S^+_\mathbf{z}] &= S^3_\mathbf{x} S^3_\mathbf{y} S^+_\mathbf{z} - S^+_\mathbf{z} S^3_\mathbf{x} S^3_\mathbf{y} = S^3_\mathbf{x} \left( S^+_\mathbf{z} S^3_\mathbf{y} + \delta_{\mathbf{y},\mathbf{z}} S^+_\mathbf{z} \right) - S^+_\mathbf{z} S^3_\mathbf{x} S^3_\mathbf{y} \\
&= S^3_\mathbf{x} S^+_\mathbf{z} S^3_\mathbf{y} + \delta_{\mathbf{y},\mathbf{z}} S^3_\mathbf{x} S^+_\mathbf{z} - S^+_\mathbf{z} S^3_\mathbf{x} S^3_\mathbf{y} \\
&= \left( S^+_\mathbf{z} S^3_\mathbf{x} + \delta_{\mathbf{x},\mathbf{z}} S^+_\mathbf{z} \right) S^3_\mathbf{y} + \delta_{\mathbf{y},\mathbf{z}} S^3_\mathbf{x} S^+_\mathbf{z} - S^+_\mathbf{z} S^3_\mathbf{x} S^3_\mathbf{y} \\
&= S^+_\mathbf{z} S^3_\mathbf{x} S^3_\mathbf{y} + \delta_{\mathbf{x},\mathbf{z}} S^+_\mathbf{z} S^3_\mathbf{y} + \delta_{\mathbf{y},\mathbf{z}} S^3_\mathbf{x} S^+_\mathbf{z} - S^+_\mathbf{z} S^3_\mathbf{x} S^3_\mathbf{y} \\
&= \delta_{\mathbf{x},\mathbf{z}} S^+_\mathbf{z} S^3_\mathbf{y} + \delta_{\mathbf{y},\mathbf{z}} S^3_\mathbf{x} S^+_\mathbf{z}.
\end{aligned}$$

Note that, in our case, $\delta_{\mathbf{y},\mathbf{z}} S^3_\mathbf{x} S^+_\mathbf{z} = \delta_{\mathbf{y},\mathbf{z}} S^+_\mathbf{z} S^3_\mathbf{x}$, since the only possibility for this term to be non-zero is when $\mathbf{z} = \mathbf{y}$, but then $\mathbf{z} \neq \mathbf{x}$ because $\mathbf{x}$ and $\mathbf{y}$ are nearest neighbours. Thus, we can write:

$$[S^3_\mathbf{x} S^3_\mathbf{y}, S^+_\mathbf{z}] = S^3_\mathbf{x} S^3_\mathbf{y} S^+_\mathbf{z} - S^+_\mathbf{z} S^3_\mathbf{x} S^3_\mathbf{y} = \delta_{\mathbf{x},\mathbf{z}} S^+_\mathbf{z} S^3_\mathbf{y} + \delta_{\mathbf{y},\mathbf{z}} S^+_\mathbf{z} S^3_\mathbf{x} \tag{23}$$

Plugging these results into (20), we obtain:

$$[H_\Lambda, S_+] = \sum_{<\mathbf{x},\mathbf{y}>\subset\Lambda} \sum_{\mathbf{z}\in\Lambda} \left( \delta_{\mathbf{y},\mathbf{z}} S^+_\mathbf{x} S^3_\mathbf{z} + \delta_{\mathbf{x},\mathbf{z}} S^+_\mathbf{y} S^3_\mathbf{z} - \delta_{\mathbf{x},\mathbf{z}} S^+_\mathbf{z} S^3_\mathbf{y} - \delta_{\mathbf{y},\mathbf{z}} S^+_\mathbf{z} S^3_\mathbf{x} \right)$$
$$= \sum_{<\mathbf{x},\mathbf{y}>\subset\Lambda} \left( S^+_\mathbf{x} S^3_\mathbf{y} + S^+_\mathbf{y} S^3_\mathbf{x} - S^+_\mathbf{x} S^3_\mathbf{y} - S^+_\mathbf{y} S^3_\mathbf{x} \right) = 0 \tag{24}$$

It means that, for any power $n > 0$, we have

$$H_\Lambda (S_+)^n \Omega = (S_+)^n H_\Lambda \Omega = 0.$$

However, for $n > 2\mathrm{S}|\Lambda|$ we trivially have $(S_+)^n\Omega = 0$, because on each site $\mathbf{x} \in \Lambda$ we can act with $S^+_{\mathbf{x}}$ at most $2\mathrm{S}$ times to obtain non-zero result, and we have $|\Lambda|$ sites in total.

$\square$

Let us introduce the operator

$$S^+_{(\mathbf{k})} = \frac{1}{\sqrt{2\mathrm{S}|\Lambda|}} \sum_{\mathbf{x}\in\Lambda} e^{i\mathbf{k}\cdot\mathbf{x}} S^+_{\mathbf{x}},$$

and a state

$$|1_{\mathbf{k}}\rangle = S^+_{(\mathbf{k})}\Omega = \frac{1}{\sqrt{2\mathrm{S}|\Lambda|}} \sum_{\mathbf{x}\in\Lambda} e^{i\mathbf{k}\cdot\mathbf{x}} S^+_{\mathbf{x}}\Omega, \tag{25}$$

where $\mathbf{k} \in \frac{2\pi}{N}\mathbb{Z}^d$. Simple calculation shows that states of the form (25) are orthonormal in the sense that $\langle 1_{\mathbf{k}}|1_{\mathbf{k}}\rangle = \delta_{\mathbf{k},\mathbf{p}}$:

$$\begin{aligned}
\langle 1_{\mathbf{p}}|1_{\mathbf{k}}\rangle &= \left\langle \frac{1}{\sqrt{2\mathrm{S}|\Lambda|}} \sum_{\mathbf{y}\in\Lambda} e^{i\mathbf{p}\cdot\mathbf{y}} S^+_{\mathbf{y}}\Omega \middle| \frac{1}{\sqrt{2\mathrm{S}|\Lambda|}} \sum_{\mathbf{x}\in\Lambda} e^{i\mathbf{k}\cdot\mathbf{x}} S^+_{\mathbf{x}}\Omega \right\rangle \\
&= \frac{1}{2\mathrm{S}|\Lambda|} \sum_{\mathbf{y},\mathbf{x}\in\Lambda} e^{i(\mathbf{k}\cdot\mathbf{x}-\mathbf{p}\cdot\mathbf{y})} \langle S^+_{\mathbf{y}}\Omega | S^+_{\mathbf{x}}\Omega\rangle \\
&= \frac{1}{2\mathrm{S}|\Lambda|} \sum_{\mathbf{y},\mathbf{x}\in\Lambda} e^{i(\mathbf{k}\cdot\mathbf{x}-\mathbf{p}\cdot\mathbf{y})} \langle \Omega | S^-_{\mathbf{y}} S^+_{\mathbf{x}}\Omega\rangle \\
&= \frac{1}{2\mathrm{S}|\Lambda|} \sum_{\mathbf{y},\mathbf{x}\in\Lambda} e^{i(\mathbf{k}\cdot\mathbf{x}-\mathbf{p}\cdot\mathbf{y})} \langle \Omega | (S^+_{\mathbf{x}} S^-_{\mathbf{y}} - 2\delta_{\mathbf{x},\mathbf{y}} S^3_{\mathbf{x}})\Omega\rangle \\
&= -\frac{1}{\mathrm{S}|\Lambda|} \sum_{\mathbf{y},\mathbf{x}\in\Lambda} e^{i(\mathbf{k}\cdot\mathbf{x}-\mathbf{p}\cdot\mathbf{y})} \delta_{\mathbf{x},\mathbf{y}} \langle \Omega | S^3_{\mathbf{x}}\Omega\rangle \\
&= \frac{1}{|\Lambda|} \sum_{\mathbf{x}\in\Lambda} e^{i\mathbf{x}(\mathbf{k}-\mathbf{p})} = \delta_{\mathbf{k},\mathbf{p}}.
\end{aligned}$$

The states (25) are called **spin wave states**, or, more precisely, **one-spin wave states**, and vector $\mathbf{k} \in \frac{2\pi}{N}\mathbb{Z}^d$ is called a **momentum** of a spin wave state $|1_{\mathbf{k}}\rangle$. It turns out that spin wave states are crucial in spectral analysis of $H_\Lambda$. Indeed, one of the most important theorems in the theory of spin waves tells us that $|1_{\mathbf{k}}\rangle$ is in fact an eigenvector of Hamiltonian $H_\Lambda$:

**Theorem 3**

$$H_\Lambda |1_{\mathbf{k}}\rangle = \mathrm{S}\epsilon_{\mathbf{k}} |1_{\mathbf{k}}\rangle,$$

*where*

$$\epsilon_{\mathbf{k}} = \sum_{i=1}^{d} \left(2 - 2\cos\left(k^i\right)\right)$$

*with $k^i$ being the $i$-th component of a momentum vector* $\mathbf{k}$.

In the following proof, we will use facts that are implied by the periodic boundary conditions we assumed. Namely, for a function $f$ which takes two arguments $\mathbf{x}$, $\mathbf{y} \in \Lambda$, we can write

$$\sum_{<\mathbf{x},\mathbf{y}>\subset\Lambda} f(\mathbf{x},\mathbf{y}) = \sum_{\mathbf{x}\in\Lambda}\sum_{i=1}^{d} f(\mathbf{x},\mathbf{x}+\mathbf{e}_i) \tag{26}$$

where $\mathbf{e}_i$ is a basis, unit vector in $i$-th direction, and, for a one-argument function $g$:

$$\sum_{\mathbf{x}\in\Lambda} g(\mathbf{x}) = \sum_{\mathbf{x}\in\Lambda} g(\mathbf{x}+\mathbf{e}_i) \tag{27}$$

for any $i \in \{1,...,d\}$. The above formulas are just generalizations of the one-dimensional case. For example, (26) in $d = 1$ takes the form

$$\sum_{x=1}^{N} f(x, x+1),$$

and periodic boundary conditions provide that the last site is paired with the first site.

**Proof.**

$$\sqrt{2\mathrm{S}|\Lambda|}H_\Lambda S^+_{(\mathbf{k})}\Omega = \sum_{<\mathbf{x},\mathbf{y}>\subset\Lambda}\sum_{\mathbf{z}\in\Lambda} e^{i\mathbf{k}\cdot\mathbf{z}}\left(\mathrm{S}^2 S^+_{\mathbf{z}} - \frac{1}{2}S^+_{\mathbf{x}}S^-_{\mathbf{y}}S^+_{\mathbf{z}} - \frac{1}{2}S^+_{\mathbf{y}}S^-_{\mathbf{x}}S^+_{\mathbf{z}} - S^3_{\mathbf{x}}S^3_{\mathbf{y}}S^+_{\mathbf{z}}\right)\Omega.$$

We need to rewrite each term in the bracket using (21)(22) and (23):

$$\begin{aligned} S^+_{\mathbf{x}}S^-_{\mathbf{y}}S^+_{\mathbf{z}} &= S^+_{\mathbf{z}}S^+_{\mathbf{x}}S^-_{\mathbf{y}} - 2\delta_{\mathbf{y},\mathbf{z}}S^+_{\mathbf{x}}S^3_{\mathbf{z}} \\ S^+_{\mathbf{y}}S^-_{\mathbf{x}}S^+_{\mathbf{z}} &= S^+_{\mathbf{z}}S^+_{\mathbf{y}}S^-_{\mathbf{x}} - 2\delta_{\mathbf{x},\mathbf{z}}S^+_{\mathbf{y}}S^3_{\mathbf{z}} \end{aligned}$$

and

$$S^3_{\mathbf{x}}S^3_{\mathbf{y}}S^+_{\mathbf{z}} = S^+_{\mathbf{z}}S^3_{\mathbf{x}}S^3_{\mathbf{y}} + \delta_{\mathbf{x},\mathbf{z}}S^+_{\mathbf{z}}S^3_{\mathbf{y}} + \delta_{\mathbf{y},\mathbf{z}}S^+_{\mathbf{z}}S^3_{\mathbf{x}}.$$

Now, using these results:

$$
\begin{aligned}
\sqrt{2\mathrm{S}|\Lambda|}H_\Lambda S^+_{(\mathbf{k})}\Omega = &\sum_{<\mathbf{x},\mathbf{y}>\subset\Lambda}\sum_{\mathbf{z}\in\Lambda} e^{i\mathbf{k}\cdot\mathbf{z}}\Bigg(\mathrm{S}^2 S^+_{\mathbf{z}} - \frac{1}{2}\Big(S^+_{\mathbf{z}}S^+_{\mathbf{x}}S^-_{\mathbf{y}} - 2\delta_{\mathbf{y},\mathbf{z}}S^+_{\mathbf{x}}S^3_{\mathbf{z}}\Big) \\
&- \frac{1}{2}\Big(S^+_{\mathbf{z}}S^+_{\mathbf{y}}S^-_{\mathbf{x}} - 2\delta_{\mathbf{x},\mathbf{z}}S^+_{\mathbf{y}}S^3_{\mathbf{z}}\Big) - \Big(S^+_{\mathbf{z}}S^3_{\mathbf{x}}S^3_{\mathbf{y}} + \delta_{\mathbf{x},\mathbf{z}}S^+_{\mathbf{z}}S^3_{\mathbf{y}} + \delta_{\mathbf{y},\mathbf{z}}S^+_{\mathbf{z}}S^3_{\mathbf{x}}\Big)\Bigg)\Omega \\
&= \sum_{<\mathbf{x},\mathbf{y}>\subset\Lambda}\sum_{\mathbf{z}\in\Lambda} e^{i\mathbf{k}\cdot\mathbf{z}}\Bigg(\mathrm{S}^2 S^+_{\mathbf{z}} + \delta_{\mathbf{y},\mathbf{z}}S^+_{\mathbf{x}}S^3_{\mathbf{z}} + \delta_{\mathbf{x},\mathbf{z}}S^+_{\mathbf{y}}S^3_{\mathbf{z}} \\
&- S^+_{\mathbf{z}}S^3_{\mathbf{x}}S^3_{\mathbf{y}} - \delta_{\mathbf{x},\mathbf{z}}S^+_{\mathbf{z}}S^3_{\mathbf{y}} - \delta_{\mathbf{y},\mathbf{z}}S^+_{\mathbf{z}}S^3_{\mathbf{x}}\Bigg)\Omega \\
&= \sum_{<\mathbf{x},\mathbf{y}>\subset\Lambda}\sum_{\mathbf{z}\in\Lambda} e^{i\mathbf{k}\cdot\mathbf{z}}\Bigg(\delta_{\mathbf{y},\mathbf{z}}S^+_{\mathbf{x}}S^3_{\mathbf{z}} + \delta_{\mathbf{x},\mathbf{z}}S^+_{\mathbf{y}}S^3_{\mathbf{z}} - \delta_{\mathbf{x},\mathbf{z}}S^+_{\mathbf{z}}S^3_{\mathbf{y}} - \delta_{\mathbf{y},\mathbf{z}}S^+_{\mathbf{z}}S^3_{\mathbf{x}}\Bigg)\Omega \\
&= -\mathrm{S}\sum_{<\mathbf{x},\mathbf{y}>\subset\Lambda}\sum_{\mathbf{z}\in\Lambda} e^{i\mathbf{k}\cdot\mathbf{z}}\Bigg(\delta_{\mathbf{y},\mathbf{z}}S^+_{\mathbf{x}} + \delta_{\mathbf{x},\mathbf{z}}S^+_{\mathbf{y}} - \delta_{\mathbf{x},\mathbf{z}}S^+_{\mathbf{z}} - \delta_{\mathbf{y},\mathbf{z}}S^+_{\mathbf{z}}\Bigg)\Omega \\
&= -\mathrm{S}\sum_{<\mathbf{x},\mathbf{y}>\subset\Lambda}\Bigg(e^{i\mathbf{k}\cdot\mathbf{y}}S^+_{\mathbf{x}} + e^{i\mathbf{k}\cdot\mathbf{x}}S^+_{\mathbf{y}} - e^{i\mathbf{k}\cdot\mathbf{x}}S^+_{\mathbf{x}} - e^{i\mathbf{k}\cdot\mathbf{y}}S^+_{\mathbf{y}}\Bigg)\Omega \\
&\overset{(26)}{=} -\mathrm{S}\sum_{\mathbf{x}\in\Lambda}\sum_{i=1}^{d}\Bigg(e^{i\mathbf{k}\cdot(\mathbf{x}+\mathbf{e}_i)}S^+_{\mathbf{x}} + e^{i\mathbf{k}\cdot\mathbf{x}}S^+_{\mathbf{x}+\mathbf{e}_i} - e^{i\mathbf{k}\cdot\mathbf{x}}S^+_{\mathbf{x}} - e^{i\mathbf{k}\cdot(\mathbf{x}+\mathbf{e}_i)}S^+_{\mathbf{x}+\mathbf{e}_i}\Bigg)\Omega \\
&\overset{(27)}{=} -\mathrm{S}\sum_{\mathbf{x}\in\Lambda}\sum_{i=1}^{d}\Bigg(e^{i\mathbf{k}\cdot(\mathbf{x}+\mathbf{e}_i)}S^+_{\mathbf{x}} + e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{e}_i)}S^+_{\mathbf{x}} - e^{i\mathbf{k}\cdot\mathbf{x}}S^+_{\mathbf{x}} - e^{i\mathbf{k}\cdot\mathbf{x}}S^+_{\mathbf{x}}\Bigg)\Omega \\
&= -\mathrm{S}\sum_{i=1}^{d}\Bigg(e^{ik^i} + e^{-ik^i} - 2\Bigg)\sum_{\mathbf{x}\in\Lambda} e^{i\mathbf{k}\cdot\mathbf{x}}S^+_{\mathbf{x}}\Omega \\
&= \mathrm{S}\sum_{i=1}^{d}\Bigg(2 - 2\cos\left(k^i\right)\Bigg)\sqrt{2\mathrm{S}|\Lambda|}S^+_{(\mathbf{k})}\Omega = \mathrm{S}\epsilon_{\mathbf{k}}\sqrt{2\mathrm{S}|\Lambda|}S^+_{(\mathbf{k})}\Omega.
\end{aligned}
$$

The desired result is obtained simply by multiplying both sides of the above by the normalization factor $1/\sqrt{2\mathrm{S}|\Lambda|}$.

□

Note that, since

$$\cos(x) \approx 1 - \frac{x^2}{2}$$

for $x$ small, and each component of momentum $\mathbf{k} \in \frac{2\pi}{N}\mathbb{Z}^d$ is small for $N$ sufficiently large, we can see that for such $\mathbf{k}$:

$$\epsilon_{\mathbf{k}} \approx \sum_{i=1}^{d}(k^i)^2 = \mathbf{k}^2,$$

which is the kinetic energy for a free particle with momentum $\mathbf{k}$. Thus, for $\mathbf{k}$ small, an excitation of a form of a spin wave with momentum $\mathbf{k}$ can be viewed as appearance of a free quasi-particle called **magnon**.

Let us now turn our attention to the states of the form

$$S^+_{(\mathbf{k}_n)} S^+_{(\mathbf{k}_{n-1})} \cdots S^+_{(\mathbf{k}_1)} \Omega, \tag{28}$$

Looking at the structure of the above, one would hope that this state is an eigenstate of $H_\Lambda$ with its energy being simply $\mathrm{S}(\epsilon_{\mathbf{k}_n} + ... + \epsilon_{\mathbf{k}_1})$. States of this form are essentially a collective excitations of many non-interacting spin waves and are called **n-spin wave states**. Unfortunately, these states are in general no longer eigenstantes of $H_\Lambda$, as we shall see in a moment. The problem is that these states are "almost" eigenstates with expected energy $\mathrm{S}(\epsilon_{\mathbf{k}_n} + ... + \epsilon_{\mathbf{k}_1})$, but the remainder appears in eigenequation which in general is non-zero. However, these states can be considered as good approximations of the true eigenstates of $H_\Lambda$ for small momenta, and thus can be used to investigate the spectral properties of $H_\Lambda$ for low energies. Hence, the approximate description of the low-temperature behaviour of the system in thermodynamic limit may indeed be possible [3][4][5].

### Main theorem

Our next goal is to prove the following, crucial theorem:

**Theorem 4** *For a general state of the form* $S^+_{(\mathbf{k}_n)} S^+_{(\mathbf{k}_{n-1})} \cdots S^+_{(\mathbf{k}_1)} \Omega$, $\mathbf{k}_i \in \frac{2\pi}{N}\mathbb{Z}^d$ $\forall i \in \{1, ..., n\}$, $n > 1$, *we have*

$$\begin{aligned} &H_\Lambda S^+_{(\mathbf{k}_n)} S^+_{(\mathbf{k}_{n-1})} \cdots S^+_{(\mathbf{k}_1)} \Omega = \mathrm{S}\left(\sum_{i=1}^{n} \epsilon_{(\mathbf{k}_i)}\right) S^+_{(\mathbf{k}_n)} S^+_{(\mathbf{k}_{n-1})} \cdots S^+_{(\mathbf{k}_1)} \Omega \\ &+ \sum_{\substack{i,j=1 \\ i>j}}^{n} R^{'}_{\mathbf{k}_i,\mathbf{k}_j} S^+_{(\mathbf{k}_n)} S^+_{(\mathbf{k}_{n-1})} \cdots \widehat{S^+_{(\mathbf{k}_i)}} \cdots \widehat{S^+_{(\mathbf{k}_j)}} \cdots S^+_{(\mathbf{k}_1)} \Omega, \end{aligned} \tag{29}$$

*where the hat over an operator means that this particular operator is missing in the product of operators, and*

$$\begin{aligned} R^{'}_{\mathbf{k},\mathbf{q}} &= \frac{1}{2\mathrm{S}|\Lambda|} \sum_{<\mathbf{x},\mathbf{y}>\subset\Lambda} S^+_{\mathbf{x}} S^+_{\mathbf{y}} C_{\mathbf{x},\mathbf{y};\mathbf{k},\mathbf{q}}, \\ C_{\mathbf{x},\mathbf{y};\mathbf{k},\mathbf{q}} &= e^{i\mathbf{q}\cdot\mathbf{y}} e^{i\mathbf{k}\cdot\mathbf{y}} + e^{i\mathbf{q}\cdot\mathbf{x}} e^{i\mathbf{k}\cdot\mathbf{x}} - e^{i\mathbf{q}\cdot\mathbf{y}} e^{i\mathbf{k}\cdot\mathbf{x}} - e^{i\mathbf{q}\cdot\mathbf{x}} e^{i\mathbf{k}\cdot\mathbf{y}}. \end{aligned} \tag{30}$$

**Proof.** Let's start by calculating the remainder $R_{\mathbf{k}}$ defined via

$$R_{\mathbf{k}} = [H_\Lambda, S^+_{(\mathbf{k})}]. \tag{31}$$

This calculation is similar to (20). The only difference is the appearance of $1/\sqrt{2\mathrm{S}|\Lambda|}$ and $e^{i\mathbf{k}\cdot\mathbf{z}}$ factor after the sum over $\mathbf{z}$, so we can directly use (24) and write

$$\begin{aligned}
R_{\mathbf{k}} &= \frac{1}{\sqrt{2\mathrm{S}|\Lambda|}} \sum_{<\mathbf{x},\mathbf{y}>\subset\Lambda} \sum_{\mathbf{z}\in\Lambda} e^{i\mathbf{k}\cdot\mathbf{z}} \Big( \delta_{\mathbf{y},\mathbf{z}} S^+_{\mathbf{x}} S^3_{\mathbf{z}} + \delta_{\mathbf{x},\mathbf{z}} S^+_{\mathbf{y}} S^3_{\mathbf{z}} - \delta_{\mathbf{x},\mathbf{z}} S^+_{\mathbf{z}} S^3_{\mathbf{y}} - \delta_{\mathbf{y},\mathbf{z}} S^+_{\mathbf{z}} S^3_{\mathbf{x}} \Big) \\
&= \frac{1}{\sqrt{2\mathrm{S}|\Lambda|}} \sum_{<\mathbf{x},\mathbf{y}>\subset\Lambda} \Big( e^{i\mathbf{k}\cdot\mathbf{y}} S^+_{\mathbf{x}} S^3_{\mathbf{y}} + e^{i\mathbf{k}\cdot\mathbf{x}} S^+_{\mathbf{y}} S^3_{\mathbf{x}} - e^{i\mathbf{k}\cdot\mathbf{x}} S^+_{\mathbf{x}} S^3_{\mathbf{y}} - e^{i\mathbf{k}\cdot\mathbf{y}} S^+_{\mathbf{y}} S^3_{\mathbf{x}} \Big).
\end{aligned}$$

The rest $R_{\mathbf{k}}$ acts on $\Omega$ as follows:

$$\begin{aligned}
R_{\mathbf{k}}\Omega &= \frac{1}{\sqrt{2\mathrm{S}|\Lambda|}} \sum_{<\mathbf{x},\mathbf{y}>\subset\Lambda} \Big( e^{i\mathbf{k}\cdot\mathbf{y}} S^+_{\mathbf{x}} S^3_{\mathbf{y}} + e^{i\mathbf{k}\cdot\mathbf{x}} S^+_{\mathbf{y}} S^3_{\mathbf{x}} - e^{i\mathbf{k}\cdot\mathbf{x}} S^+_{\mathbf{x}} S^3_{\mathbf{y}} - e^{i\mathbf{k}\cdot\mathbf{y}} S^+_{\mathbf{y}} S^3_{\mathbf{x}} \Big)\Omega \\
&= -\frac{1}{\sqrt{2\mathrm{S}|\Lambda|}} \mathrm{S} \sum_{<\mathbf{x},\mathbf{y}>\subset\Lambda} \Big( e^{i\mathbf{k}\cdot\mathbf{y}} S^+_{\mathbf{x}} + e^{i\mathbf{k}\cdot\mathbf{x}} S^+_{\mathbf{y}} - e^{i\mathbf{k}\cdot\mathbf{x}} S^+_{\mathbf{x}} - e^{i\mathbf{k}\cdot\mathbf{y}} S^+_{\mathbf{y}} \Big)\Omega \\
&\overset{(26)}{=} -\frac{1}{\sqrt{2\mathrm{S}|\Lambda|}} \mathrm{S} \sum_{\mathbf{x}\in\Lambda} \sum_{i=1}^{d} \Big( e^{i\mathbf{k}\cdot(\mathbf{x}+\mathbf{e}_i)} S^+_{\mathbf{x}} + e^{i\mathbf{k}\cdot\mathbf{x}} S^+_{\mathbf{x}+\mathbf{e}_i} - e^{i\mathbf{k}\cdot\mathbf{x}} S^+_{\mathbf{x}} - e^{i\mathbf{k}\cdot(\mathbf{x}+\mathbf{e}_i)} S^+_{\mathbf{x}+\mathbf{e}_i} \Big)\Omega \\
&\overset{(27)}{=} -\frac{1}{\sqrt{2\mathrm{S}|\Lambda|}} \mathrm{S} \sum_{\mathbf{x}\in\Lambda} \sum_{i=1}^{d} \Big( e^{i\mathbf{k}\cdot(\mathbf{x}+\mathbf{e}_i)} S^+_{\mathbf{x}} + e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{e}_i)} S^+_{\mathbf{x}} - e^{i\mathbf{k}\cdot\mathbf{x}} S^+_{\mathbf{x}} - e^{i\mathbf{k}\cdot\mathbf{x}} S^+_{\mathbf{x}} \Big)\Omega \\
&= -\mathrm{S} \sum_{i=1}^{d} \Big( e^{ik^i} + e^{-ik^i} - 2 \Big) \frac{1}{\sqrt{2\mathrm{S}|\Lambda|}} \sum_{\mathbf{x}\in\Lambda} e^{i\mathbf{k}\cdot\mathbf{x}} S^+_{\mathbf{x}}\Omega \\
&= \mathrm{S} \sum_{i=1}^{d} \Big( 2 - 2\cos(k^i) \Big) S^+_{(\mathbf{k})}\Omega,
\end{aligned}$$

and we see that

$$R_{\mathbf{k}}\Omega = \mathrm{S}\epsilon_{\mathbf{k}} S^+_{\mathbf{k}}\Omega.$$

Basically, (31) tells us that we can commute $H_\Lambda$ with $S^+_{\mathbf{k}}$ but the remainder $R_{\mathbf{k}}$ would appear so that:

$$\begin{aligned}
&H_\Lambda S^+_{(\mathbf{k}_n)} S^+_{(\mathbf{k}_{n-1})} \cdots S^+_{(\mathbf{k}_1)}\Omega = (S^+_{(\mathbf{k}_n)} H_\Lambda + R_{\mathbf{k}_n}) S^+_{(\mathbf{k}_{n-1})} \cdots S^+_{(\mathbf{k}_1)}\Omega \\
&= S^+_{(\mathbf{k}_n)} H_\Lambda S^+_{(\mathbf{k}_{n-1})} \cdots S^+_{(\mathbf{k}_1)}\Omega + R_{\mathbf{k}_n} S^+_{(\mathbf{k}_{n-1})} \cdots S^+_{(\mathbf{k}_1)}\Omega.
\end{aligned}$$

Our goal is to move $H_\Lambda$ and $R_{\mathbf{k}_n}$ so that they act directly on $\Omega$ (recall that $H_\Lambda\Omega = 0$ and $R_{\mathbf{k}}\Omega = \mathrm{S}\epsilon_{\mathbf{k}} S^+_{\mathbf{k}}\Omega$). To do the next step of commuting operators in this way, it is clear that we need to calculate commutator of the form $[R_{\mathbf{k}}, S^+_{\mathbf{q}}]$.

We introduce

$$
\begin{aligned}
R'_{\mathbf{k},\mathbf{q}} &:= [R_{\mathbf{k}}, S^+_{\mathbf{q}}] \\
&= \frac{1}{2\mathrm{S}|\Lambda|} \sum_{<\mathbf{x},\mathbf{y}>\subset\Lambda} \sum_{\mathbf{z}\in\Lambda} e^{i\mathbf{q}\cdot\mathbf{z}} \Big[ e^{i\mathbf{k}\cdot\mathbf{y}} S^+_{\mathbf{x}} S^3_{\mathbf{y}} + e^{i\mathbf{k}\cdot\mathbf{x}} S^+_{\mathbf{y}} S^3_{\mathbf{x}} - e^{i\mathbf{k}\cdot\mathbf{x}} S^+_{\mathbf{x}} S^3_{\mathbf{y}} - e^{i\mathbf{k}\cdot\mathbf{y}} S^+_{\mathbf{y}} S^3_{\mathbf{x}}, S^+_{\mathbf{z}} \Big] \\
&= \frac{1}{2\mathrm{S}|\Lambda|} \sum_{<\mathbf{x},\mathbf{y}>\subset\Lambda} \sum_{\mathbf{z}\in\Lambda} e^{i\mathbf{q}\cdot\mathbf{z}} \Big( e^{i\mathbf{k}\cdot\mathbf{y}} [S^+_{\mathbf{x}} S^3_{\mathbf{y}}, S^+_{\mathbf{z}}] + e^{i\mathbf{k}\cdot\mathbf{x}} [S^+_{\mathbf{y}} S^3_{\mathbf{x}}, S^+_{\mathbf{z}}] \\
&\quad - e^{i\mathbf{k}\cdot\mathbf{x}} [S^+_{\mathbf{x}} S^3_{\mathbf{y}}, S^+_{\mathbf{z}}] - e^{i\mathbf{k}\cdot\mathbf{y}} [S^+_{\mathbf{y}} S^3_{\mathbf{x}}, S^+_{\mathbf{z}}] \Big).
\end{aligned}
$$

We see that the following commutator needs to be evaluated:

$$
\begin{aligned}
[S^+_{\mathbf{x}} S^3_{\mathbf{y}}, S^+_{\mathbf{z}}] &= S^+_{\mathbf{x}} S^3_{\mathbf{y}} S^+_{\mathbf{z}} - S^+_{\mathbf{z}} S^+_{\mathbf{x}} S^3_{\mathbf{y}} = S^+_{\mathbf{x}} (S^+_{\mathbf{z}} S^3_{\mathbf{y}} + \delta_{\mathbf{y},\mathbf{z}} S^+_{\mathbf{z}}) - S^+_{\mathbf{z}} S^+_{\mathbf{x}} S^3_{\mathbf{y}} \\
&= \delta_{\mathbf{y},\mathbf{z}} S^+_{\mathbf{x}} S^+_{\mathbf{z}},
\end{aligned}
$$

and, simply by exchanging $\mathbf{x}$ and $\mathbf{y}$ in the above,

$$
[S^+_{\mathbf{y}} S^3_{\mathbf{x}}, S^+_{\mathbf{z}}] = \delta_{\mathbf{x},\mathbf{z}} S^+_{\mathbf{y}} S^+_{\mathbf{z}}.
$$

Thus:

$$
\begin{aligned}
R'_{\mathbf{k},\mathbf{q}} &= \frac{1}{2\mathrm{S}|\Lambda|} \sum_{<\mathbf{x},\mathbf{y}>\subset\Lambda} \sum_{\mathbf{z}\in\Lambda} e^{i\mathbf{q}\cdot\mathbf{z}} \Big( e^{i\mathbf{k}\cdot\mathbf{y}} \delta_{\mathbf{y},\mathbf{z}} S^+_{\mathbf{x}} S^+_{\mathbf{z}} + e^{i\mathbf{k}\cdot\mathbf{x}} \delta_{\mathbf{x},\mathbf{z}} S^+_{\mathbf{y}} S^+_{\mathbf{z}} \\
&\quad - e^{i\mathbf{k}\cdot\mathbf{x}} \delta_{\mathbf{y},\mathbf{z}} S^+_{\mathbf{x}} S^+_{\mathbf{z}} - e^{i\mathbf{k}\cdot\mathbf{y}} \delta_{\mathbf{x},\mathbf{z}} S^+_{\mathbf{y}} S^+_{\mathbf{z}} \Big) \\
&= \frac{1}{2\mathrm{S}|\Lambda|} \sum_{<\mathbf{x},\mathbf{y}>\subset\Lambda} \Big( e^{i\mathbf{q}\cdot\mathbf{y}} e^{i\mathbf{k}\cdot\mathbf{y}} S^+_{\mathbf{x}} S^+_{\mathbf{y}} + e^{i\mathbf{q}\cdot\mathbf{x}} e^{i\mathbf{k}\cdot\mathbf{x}} S^+_{\mathbf{y}} S^+_{\mathbf{x}} - e^{i\mathbf{q}\cdot\mathbf{y}} e^{i\mathbf{k}\cdot\mathbf{x}} S^+_{\mathbf{x}} S^+_{\mathbf{y}} - e^{i\mathbf{q}\cdot\mathbf{x}} e^{i\mathbf{k}\cdot\mathbf{y}} S^+_{\mathbf{y}} S^+_{\mathbf{x}} \Big) \\
&= \frac{1}{2\mathrm{S}|\Lambda|} \sum_{<\mathbf{x},\mathbf{y}>\subset\Lambda} S^+_{\mathbf{x}} S^+_{\mathbf{y}} \Big( e^{i\mathbf{q}\cdot\mathbf{y}} e^{i\mathbf{k}\cdot\mathbf{y}} + e^{i\mathbf{q}\cdot\mathbf{x}} e^{i\mathbf{k}\cdot\mathbf{x}} - e^{i\mathbf{q}\cdot\mathbf{y}} e^{i\mathbf{k}\cdot\mathbf{x}} - e^{i\mathbf{q}\cdot\mathbf{x}} e^{i\mathbf{k}\cdot\mathbf{y}} \Big) \\
&= \frac{1}{2\mathrm{S}|\Lambda|} \sum_{<\mathbf{x},\mathbf{y}>\subset\Lambda} S^+_{\mathbf{x}} S^+_{\mathbf{y}} C_{\mathbf{x},\mathbf{y};\mathbf{k},\mathbf{q}}
\end{aligned}
$$

with

$$
C_{\mathbf{x},\mathbf{y};\mathbf{k},\mathbf{q}} = e^{i\mathbf{q}\cdot\mathbf{y}} e^{i\mathbf{k}\cdot\mathbf{y}} + e^{i\mathbf{q}\cdot\mathbf{x}} e^{i\mathbf{k}\cdot\mathbf{x}} - e^{i\mathbf{q}\cdot\mathbf{y}} e^{i\mathbf{k}\cdot\mathbf{x}} - e^{i\mathbf{q}\cdot\mathbf{x}} e^{i\mathbf{k}\cdot\mathbf{y}}
$$

Note that further commuting is trivial:

$$
[R'_{\mathbf{k},\mathbf{q}}, S^+_{(\mathbf{p})}] = \frac{1}{2\mathrm{S}|\Lambda|} \frac{1}{\sqrt{2\mathrm{S}|\Lambda|}} \sum_{<\mathbf{x},\mathbf{y}>\subset\Lambda} C_{\mathbf{x},\mathbf{y};\mathbf{k},\mathbf{q}} \sum_{\mathbf{z}\in\Lambda} e^{i\mathbf{p}\cdot\mathbf{z}} [S^+_{\mathbf{x}} S^+_{\mathbf{y}}, S^+_{\mathbf{z}}] = 0.
$$

Before proving (29), we need to show that

$$R_{\mathbf{k}_n} S^+_{(\mathbf{k}_{n-1})} \cdots S^+_{(\mathbf{k}_1)} \Omega = \mathrm{S}\epsilon_{\mathbf{k}_n} S^+_{(\mathbf{k}_n)} \cdots S^+_{(\mathbf{k}_1)} \Omega + \sum_{i=1}^{n-1} R'_{\mathbf{k}_n,\mathbf{k}_i} \widehat{S^+_{(\mathbf{k}_n)}} S^+_{(\mathbf{k}_{n-1})} \cdots \widehat{S^+_{(\mathbf{k}_i)}} \cdots S^+_{(\mathbf{k}_1)} \Omega, \tag{32}$$

Indeed, the above rule holds for $n = 2$:

$$R_{\mathbf{k}_2} S^+_{(\mathbf{k}_1)} \Omega = S^+_{(\mathbf{k}_1)} R_{\mathbf{k}_2} \Omega + R'_{\mathbf{k}_2,\mathbf{k}_1} \Omega = \mathrm{S}\epsilon_{\mathbf{k}_2} S^+_{(\mathbf{k}_2)} S^+_{(\mathbf{k}_1)} \Omega + R'_{\mathbf{k}_2,\mathbf{k}_1} \Omega.$$

Assume now that (32) holds for fixed $n > 1$. We have:

$$\begin{aligned}
&R_{\mathbf{k}_{n+1}} S^+_{(\mathbf{k}_n)} S^+_{(\mathbf{k}_{n-1})} \cdots S^+_{(\mathbf{k}_1)} \Omega = (S^+_{(\mathbf{k}_n)} R_{\mathbf{k}_{n+1}} + R'_{\mathbf{k}_{n+1},\mathbf{k}_n}) S^+_{(\mathbf{k}_{n-1})} \cdots S^+_{(\mathbf{k}_1)} \Omega \\
&= S^+_{(\mathbf{k}_n)} \Big( R_{\mathbf{k}_{n+1}} S^+_{(\mathbf{k}_{n-1})} \cdots S^+_{(\mathbf{k}_1)} \Omega \Big) + R'_{\mathbf{k}_{n+1},\mathbf{k}_n} S^+_{(\mathbf{k}_{n-1})} \cdots S^+_{(\mathbf{k}_1)} \Omega \\
&= S^+_{(\mathbf{k}_n)} \Big( \mathrm{S}\epsilon_{\mathbf{k}_{n+1}} S^+_{(\mathbf{k}_{n+1})} S^+_{(\mathbf{k}_{n-1})} \cdots S^+_{(\mathbf{k}_1)} \Omega + \sum_{i=1}^{n-1} R'_{\mathbf{k}_{n+1},\mathbf{k}_i} \widehat{S^+_{(\mathbf{k}_{n+1})}} S^+_{(\mathbf{k}_{n-1})} \cdots \widehat{S^+_{(\mathbf{k}_i)}} \cdots S^+_{(\mathbf{k}_1)} \Omega \Big) \\
&+ R'_{\mathbf{k}_{n+1},\mathbf{k}_n} S^+_{(\mathbf{k}_{n-1})} \cdots S^+_{(\mathbf{k}_1)} \Omega \\
&= \mathrm{S}\epsilon_{\mathbf{k}_{n+1}} S^+_{(\mathbf{k}_{n+1})} S^+_{(\mathbf{k}_n)} \cdots S^+_{(\mathbf{k}_1)} \Omega + \sum_{i=1}^{n-1} R'_{\mathbf{k}_{n+1},\mathbf{k}_i} \widehat{S^+_{(\mathbf{k}_{n+1})}} S^+_{(\mathbf{k}_n)} \cdots \widehat{S^+_{(\mathbf{k}_i)}} \cdots S^+_{(\mathbf{k}_1)} \Omega \\
&+ R'_{\mathbf{k}_{n+1},\mathbf{k}_n} \widehat{S^+_{(\mathbf{k}_{n+1})}} \widehat{S^+_{(\mathbf{k}_n)}} S^+_{(\mathbf{k}_{n-1})} \cdots S^+_{(\mathbf{k}_1)} \Omega \\
&= \mathrm{S}\epsilon_{\mathbf{k}_{n+1}} S^+_{(\mathbf{k}_{n+1})} S^+_{(\mathbf{k}_n)} \cdots S^+_{(\mathbf{k}_1)} \Omega + \sum_{i=1}^{n} R'_{\mathbf{k}_{n+1},\mathbf{k}_i} \widehat{S^+_{(\mathbf{k}_{n+1})}} S^+_{(\mathbf{k}_n)} \cdots \widehat{S^+_{(\mathbf{k}_i)}} \cdots S^+_{(\mathbf{k}_1)} \Omega,
\end{aligned}$$

hence, by induction, (32) holds for all natural $n > 1$. With this, we can now proceed to prove (29), again by induction. It holds for $n = 2$:

$$\begin{aligned}
H_\Lambda S^+_{(\mathbf{k}_2)} S^+_{(\mathbf{k}_1)} \Omega &= \Big( S^+_{(\mathbf{k}_2)} H_\Lambda + R_{\mathbf{k}_2} \Big) S^+_{(\mathbf{k}_1)} \Omega \\
&= S^+_{(\mathbf{k}_2)} H_\Lambda S^+_{(\mathbf{k}_1)} \Omega + R_{\mathbf{k}_2} S^+_{(\mathbf{k}_1)} \Omega \\
&= S^+_{(\mathbf{k}_2)} \Big( S^+_{(\mathbf{k}_1)} H_\Lambda + R_{\mathbf{k}_1} \Big) \Omega + \Big( S^+_{(\mathbf{k}_1)} R_{\mathbf{k}_2} + R'_{\mathbf{k}_2,\mathbf{k}_2} \Big) \Omega \\
&= \mathrm{S}\epsilon_{\mathbf{k}_1} S^+_{(\mathbf{k}_2)} S^+_{(\mathbf{k}_1)} \Omega + \mathrm{S}\epsilon_{\mathbf{k}_2} S^+_{(\mathbf{k}_2)} S^+_{(\mathbf{k}_1)} \Omega + R'_{\mathbf{k}_2,\mathbf{k}_2} \Omega \\
&= \mathrm{S} \Big( \epsilon_{\mathbf{k}_1} + \epsilon_{\mathbf{k}_2} \Big) S^+_{(\mathbf{k}_2)} S^+_{(\mathbf{k}_1)} \Omega + R'_{\mathbf{k}_2,\mathbf{k}_2} \Omega
\end{aligned}$$

Assume now that (29) holds for fixed $n > 1$. Using (32), we can show that it holds for $n + 1$:

$$
\begin{aligned}
H_\Lambda S^+_{(\mathbf{k}_{n+1})} S^+_{(\mathbf{k}_n)} \cdots S^+_{(\mathbf{k}_1)}\Omega &= \Big(S^+_{(\mathbf{k}_{n+1})} H_\Lambda + R_{\mathbf{k}_{n+1}}\Big) S^+_{(\mathbf{k}_n)} \cdots S^+_{(\mathbf{k}_1)}\Omega \\
&= S^+_{(\mathbf{k}_{n+1})}\Big(H_\Lambda S^+_{(\mathbf{k}_n)} \cdots S^+_{(\mathbf{k}_1)}\Omega\Big) + \Big(R_{\mathbf{k}_{n+1}} S^+_{(\mathbf{k}_n)} \cdots S^+_{(\mathbf{k}_1)}\Omega\Big) \\
&= S^+_{(\mathbf{k}_{n+1})}\Bigg(\mathrm{S}\bigg(\sum_{i=1}^{n} \epsilon_{\mathbf{k}_i}\bigg) S^+_{(\mathbf{k}_n)} \cdots S^+_{(\mathbf{k}_1)}\Omega \\
&+ \sum_{\substack{i,j=1\\ i>j}}^{n} R'_{\mathbf{k}_i,\mathbf{k}_j} S^+_{(\mathbf{k}_n)} S^+_{(\mathbf{k}_{n-1})} \cdots \widehat{S^+_{(\mathbf{k}_i)}} \cdots \widehat{S^+_{(\mathbf{k}_j)}} \cdots S^+_{(\mathbf{k}_1)}\Omega\Bigg) \\
&+ \Big(R_{\mathbf{k}_{n+1}} S^+_{(\mathbf{k}_n)} \cdots S^+_{(\mathbf{k}_1)}\Omega\Big) \\
&= \mathrm{S}\bigg(\sum_{i=1}^{n} \epsilon_{\mathbf{k}_i}\bigg) S^+_{(\mathbf{k}_{n+1})} S^+_{(\mathbf{k}_n)} \cdots S^+_{(\mathbf{k}_1)}\Omega \\
&+ \sum_{\substack{i,j=1\\ i>j}}^{n} R'_{\mathbf{k}_i,\mathbf{k}_j} S^+_{(\mathbf{k}_{n+1})} S^+_{(\mathbf{k}_n)} S^+_{(\mathbf{k}_{n-1})} \cdots \widehat{S^+_{(\mathbf{k}_i)}} \cdots \widehat{S^+_{(\mathbf{k}_j)}} \cdots S^+_{(\mathbf{k}_1)}\Omega \\
&+ \mathrm{S}\epsilon_{\mathbf{k}_{n+1}} S^+_{(\mathbf{k}_{n+1})} S^+_{(\mathbf{k}_n)} \cdots S^+_{(\mathbf{k}_1)}\Omega \\
&+ \sum_{j=1}^{n} R'_{\mathbf{k}_{n+1},\mathbf{k}_j} \widehat{S^+_{(\mathbf{k}_{n+1})}} \cdots \widehat{S^+_{(\mathbf{k}_j)}} \cdots S^+_{(\mathbf{k}_1)}\Omega \\
&= \mathrm{S}\bigg(\sum_{i=1}^{n+1} \epsilon_{\mathbf{k}_i}\bigg) S^+_{(\mathbf{k}_{n+1})} S^+_{(\mathbf{k}_n)} \cdots S^+_{(\mathbf{k}_1)}\Omega \\
&+ \sum_{\substack{i,j=1\\ i>j}}^{n+1} R'_{\mathbf{k}_i,\mathbf{k}_j} S^+_{(\mathbf{k}_{n+1})} S^+_{(\mathbf{k}_n)} \cdots \widehat{S^+_{(\mathbf{k}_i)}} \cdots \widehat{S^+_{(\mathbf{k}_j)}} \cdots S^+_{(\mathbf{k}_1)}\Omega.
\end{aligned}
$$

Thus, reasoning by induction shows that (29) holds for all $n > 1$. This ends the proof.

□

## Final considerations

Let us briefly come back to the constant $C_{\mathbf{x},\mathbf{y};\mathbf{k},\mathbf{q}}$ in (30). Note that

$$
C_{\mathbf{x},\mathbf{y};\mathbf{k},\mathbf{q}} = \big(e^{i\mathbf{q}\cdot\mathbf{y}} - e^{i\mathbf{q}\cdot\mathbf{x}}\big)\big(e^{i\mathbf{k}\cdot\mathbf{y}} - e^{i\mathbf{k}\cdot\mathbf{x}}\big). \tag{33}
$$

Since $\mathbf{k}, \mathbf{q} \in \frac{2\pi}{N}\mathbb{Z}^d$, we can choose $\mathbf{k}$ and $\mathbf{q}$ small for big $N$, in the sense that $|\mathbf{k}|, |\mathbf{q}| \sim \frac{1}{N}$. If it is so, then we can approximate exponents in (33) up to linear terms and find that

$$
|C_{\mathbf{x},\mathbf{y};\mathbf{k},\mathbf{q}}| \approx |(\mathbf{q}\cdot(\mathbf{y}-\mathbf{x}))(\mathbf{k}\cdot(\mathbf{y}-\mathbf{x}))| \leq |\mathbf{q}||\mathbf{y}-\mathbf{x}||\mathbf{k}||\mathbf{y}-\mathbf{x}| \sim \frac{1}{N^2},
$$

for $\mathbf{x}$ and $\mathbf{y}$ are nearest neighbours in (30), so $|\mathbf{y}-\mathbf{x}|=1$. Additionally, the number of terms in sum in (30) is $dN^d$, so that $R'_{\mathbf{k},\mathbf{q}}$ is of order $dN^{-2}$. Assuming that all momenta in (29) are small in the sense described above, and taking into account a factor of order $N^{-\frac{d}{2}}$ coming from every spin-wave operator in (29) (there are $n-2$ of them), we see that the second sum in (29) (the one including $R'$s) is of order $C(n)dN^{-2-\frac{d(n-2)}{2}}$, where $C(n)$ is a function depending only on the number of momenta $n \geq 2$. It turns out that this problematic term can be made arbitrarily small just by increasing the particle number.

The above discussion is of course by no means rigorous, but it presents the intuition lying behind the idea that, for small momenta, n-spin wave states are indeed good approximation of true eigenstates of the Heisenberg Hamiltonian in thermodynamic limit.

One can also analyze the problem in terms of energy expectations. Consider, for example, a two-spin wave state $S^+_{(\mathbf{k}_2)}S^+_{(\mathbf{k}_1)}\Omega$, where $\mathbf{k}_i = \frac{2\pi}{N}\mathbf{n}_i$ for $i \in \{1,2\}$ with $\mathbf{n}_i \in \mathbb{Z}^d$ fixed. In what follows, we skip the details of calculations, since they are tedious and long, but all methods used to deliver these results where presented throughout this work and are known to the reader. Let us calculate the energy expectation of state $S^+_{(\mathbf{k}_2)}S^+_{(\mathbf{k}_1)}\Omega$:

$$\langle S^+_{(\mathbf{k}_2)}S^+_{(\mathbf{k}_1)}\Omega|H_\Lambda S^+_{(\mathbf{k}_2)}S^+_{(\mathbf{k}_1)}\Omega\rangle = S(\epsilon_{\mathbf{k}_2}+\epsilon_{\mathbf{k}_1})\langle S^+_{(\mathbf{k}_2)}S^+_{(\mathbf{k}_1)}\Omega|S^+_{(\mathbf{k}_2)}S^+_{(\mathbf{k}_1)}\Omega\rangle + \langle S^+_{(\mathbf{k}_2)}S^+_{(\mathbf{k}_1)}\Omega|R'_{\mathbf{k}_2,\mathbf{k}_1}\Omega\rangle. \tag{34}$$

In the above,

$$\langle S^+_{(\mathbf{k}_2)}S^+_{(\mathbf{k}_1)}\Omega|S^+_{(\mathbf{k}_2)}S^+_{(\mathbf{k}_1)}\Omega\rangle = \frac{1}{4S^2|\Lambda|^2}\sum_{\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{t}\in\Lambda} e^{-i\mathbf{k}_2\cdot\mathbf{x}}e^{-i\mathbf{k}_1\cdot\mathbf{y}}e^{i\mathbf{k}_2\cdot\mathbf{z}}e^{i\mathbf{k}_1\cdot\mathbf{y}}\langle S^+_{\mathbf{x}}S^+_{\mathbf{y}}\Omega|S^+_{\mathbf{z}}S^+_{\mathbf{t}}\Omega\rangle \tag{35}$$

and

$$\langle S^+_{(\mathbf{k}_2)}S^+_{(\mathbf{k}_1)}\Omega|R'_{\mathbf{k}_2,\mathbf{k}_1}\Omega\rangle = \frac{1}{4S^2|\Lambda|^2}\sum_{\substack{\mathbf{x},\mathbf{y}\in\Lambda\\ <\mathbf{z},\mathbf{t}>\subset\Lambda}} e^{-i\mathbf{k}_2\cdot\mathbf{x}}e^{-i\mathbf{k}_1\cdot\mathbf{y}}C_{\mathbf{z},\mathbf{t};\mathbf{k}_2,\mathbf{k}_1}\langle S^+_{\mathbf{x}}S^+_{\mathbf{y}}\Omega|S^+_{\mathbf{z}}S^+_{\mathbf{t}}\Omega\rangle, \tag{36}$$

where

$$\langle S^+_{\mathbf{x}}S^+_{\mathbf{y}}\Omega|S^+_{\mathbf{z}}S^+_{\mathbf{t}}\Omega\rangle = 4S^2(\delta_{\mathbf{x},\mathbf{t}}\delta_{\mathbf{y},\mathbf{z}}+\delta_{\mathbf{x},\mathbf{z}}\delta_{\mathbf{y},\mathbf{t}}) - 4S\delta_{\mathbf{x},\mathbf{z}}\delta_{\mathbf{x},\mathbf{y}}\delta_{\mathbf{x},\mathbf{t}}. \tag{37}$$

Plugging this into (35) and (36), we obtain

$$\langle S^+_{(\mathbf{k}_2)} S^+_{(\mathbf{k}_1)} \Omega | S^+_{(\mathbf{k}_2)} S^+_{(\mathbf{k}_1)} \Omega \rangle = 1 + \delta_{\mathbf{k}_2,\mathbf{k}_1} - \frac{1}{S|\Lambda|}$$

and

$$\langle S^+_{(\mathbf{k}_2)} S^+_{(\mathbf{k}_1)} \Omega | R'_{\mathbf{k}_2,\mathbf{k}_1} \Omega \rangle = \frac{2}{|\Lambda|^2} \sum_{\mathbf{t}\in\Lambda} \sum_{i=1}^{d} \big(\cos(k_2^i) + \cos(k_1^i) - \cos(k_2^i - k_1^i) - 1\big).$$

With this, we conclude that, if $\mathbf{k}_1 \neq \mathbf{k}_2$:

$$\begin{aligned}\langle S^+_{(\mathbf{k}_2)} S^+_{(\mathbf{k}_1)} \Omega | H_\Lambda S^+_{(\mathbf{k}_2)} S^+_{(\mathbf{k}_1)} \Omega \rangle &= S(\epsilon_{\mathbf{k}_2} + \epsilon_{\mathbf{k}_1})(1 - \frac{1}{S|\Lambda|}) \\ &\quad + \frac{2}{|\Lambda|^2} \sum_{\mathbf{t}\in\Lambda} \sum_{i=1}^{d} \big(\cos(k_2^i) + \cos(k_1^i) - \cos(k_2^i - k_1^i) - 1\big),\end{aligned}$$

and, because

$$\left| \frac{2}{|\Lambda|^2} \sum_{\mathbf{t}\in\Lambda} \sum_{i=1}^{d} \big(\cos(k_2^i) + \cos(k_1^i) - \cos(k_2^i - k_1^i) - 1\big) \right| \leq \frac{8d}{|\Lambda|} \xrightarrow[|\Lambda|\to\infty]{} 0,$$

we see that

$$\langle S^+_{(\mathbf{k}_2)} S^+_{(\mathbf{k}_1)} \Omega | H_\Lambda S^+_{(\mathbf{k}_2)} S^+_{(\mathbf{k}_1)} \Omega \rangle \xrightarrow[|\Lambda|\to\infty]{} S(\epsilon_{\mathbf{k}_2} + \epsilon_{\mathbf{k}_1}). \tag{38}$$

Of course, the limit $|\Lambda| \to \infty$ means that we increase the particle number $N$ so that $N \to \infty$ (because $|\Lambda| = N^d$), but then also $\mathbf{k}_1, \mathbf{k}_2 \to 0$. Thus, (38) can be understood in a way that the energy of two-spin wave state is essencially a sum of energies of one-spin wave states in thermodynamic limit and when momenta are small.

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