MATHEMATICS RESEARCH CENTER, UNITED STATES ARMY
THE UNIVERSITY OF WISCONSIN

Contract No.: DA-11-022-ORD-2059

SOLVING FUNCTIONAL EQUATIONS BY CONTRACTIVE AVERAGING

E. H. Zarantonello

MRC Technical Summary Report #160
June, 1960

Madison, Wisconsin

ABSTRACT

The notion of monotonicity for operators from the Hilbert space into itself is introduced as follows: $T: \mathcal{D}(T) \to \mathcal{H}$ is said to be isotonic or antitonic according as

$$(Tx_1 - Tx_2, x_1 - x_2) \geq 0 \text{ or } (Tx_1 - Tx_2, x_1 - x_2) \leq 0, \quad \forall x_1, x_2 \in \mathcal{D}(T)$$

respectively. If an $\varepsilon > 0$ exists such that $T - (1+\varepsilon)I$ is isotonic then T is said to be strictly supraunitary, whereas if for such an ε, $T - (1-\varepsilon)I$ is antitonic then T is called strictly infraunitary. The following theorem is proved:

If a Lipschitzian mapping T defined on a closed ball about the origin vanishes at the origin and is either strictly supra- or infraunitary then the equation $x = y + Tx$ has a unique solution for any y in a suitable closed sphere about the origin. For a proper choice of an averaging factor α the solution can be obtained by iteration of the operator $x \to y + (1-\alpha)x + \alpha Tx$ on the initial approximation $x = 0$. Applications are given.

SOLVING FUNCTIONAL EQUATIONS BY CONTRACTIVE AVERAGING [1]

E. H. Zarantonello

"For any continuous function Tx of a real variable growing everywhere strictly faster or strictly slower than x, the equation

$$x = Tx$$

has a unique solution. This solution can be obtained by iteration of the function $(1-\alpha)x + \alpha Tx$, for a sufficiently small number α."

The purpose of this note is to show that the above theorem, and its corresponding local form, remain valid in Hilbert spaces, provided the notion of a function growing faster than another is adequately extended.

We shall work on a fixed Hilbert space $\mathcal{H}$, real but not necessarily separable, and all operators (denoted T, S, etc.) will have their domains and ranges in $\mathcal{H}$, and without exception will be single valued functions of their arguments. As usual, the scalar product in $\mathcal{H}$ will be denoted (x,y), and the norm $\|x\|$.

<u>Definition 1.</u> An operator T is said to satisfy Lipschitz conditions on a domain $\mathcal{D}$ if T is defined on $\mathcal{D}$ and there is a constant C such that

(1) $$\|Tx_1 - Tx_2\| \leq C\,\|x_1 - x_2\| \quad \text{for any } x_1, x_2 \in \mathcal{D}.$$

---

[1] Sponsored by the U. S. Army under Contract No. DA-11-022-ORD-2059, Mathematics Research Center, Madison, Wisconsin. Work partly done at the "Facultad de Ingenieria, Universidad Nacional de Cuyo", San Juan, Argentina.

The infimum of all possible constants C , that is,

$$\sup_{x_1, x_2} \frac{\| Tx_1 - Tx_2 \|}{\| x_1 - x_2 \|}$$

is called the Lipschitz norm of T on $\mathcal{D}$, and is denoted $|||T|||_{\mathcal{D}}$. Usually the indication of the domain will be dropped, letting it be inferred from the context.

If $|||T||| \leq 1$, T is said to be contractive, and strictly contractive if $|||T||| < 1$.

Analogously, we say that T is expansive or strictly expansive on $\mathcal{D}$ if for $x_1, x_2 \in \mathcal{D}$

$$\frac{\| Tx_1 - Tx_2 \|}{\| x_1 - x_2 \|} \geq 1 \quad \text{or} \quad \frac{\| Tx_1 - Tx_2 \|}{\| x_1 - x_2 \|} \geq k > 1 \quad \text{respectively.}$$

The ordering of functions by comparing their increments has the following natural extension to Hilbert space:

Definition 2. We shall say that an operator T is slower than an operator S on a domain $\mathcal{D}$, if both are defined on $\mathcal{D}$ and if

$$(Tx_1 - Tx_2, x_1 - x_2) \leq (Sx_1 - Sx_2, x_1 - x_2) \ , \quad \text{for all } x_1, x_2 \in \mathcal{D}$$

Such fact will be denoted $T < S$ or $S > T$, indifferently. Comparison with the zero operator leads to the notion of monotonicity. Thus we shall say that T is isotonic or antitonic if $T < 0$ or $T > 0$ respectively; an operator which is either isotonic or antitonic is said to be monotonic. On the other hand, comparison with the identity operation I yields the classes of supra- and infra-unitary operators, meaning that an operator T is supra- or infra-unitary if $T > I$ or $T < I$ respectively.

For a more precise description of the relations of an operator T with the identity it is convenient to introduce the following quantities:

$$\mu^+(T) = \sup_{x_1, x_2 \in} \frac{(Tx_1 - Tx_2, x_1 - x_2)}{(x_1 - x_2, x_1 - x_2)}$$

$$\mu^-(T) = \inf_{x_1, x_2 \in} \frac{(Tx_1 - Tx_2, x_1 - x_2)}{(x_1 - x_2, x_1 - x_2)}$$

Obviously $\mu^-(T) = -\mu^+(-T)$, $\mu^-(T) \le \mu^+(T)$ and $|\mu^-(T)|, |\mu^+(T)| \le |||T|||$. Now, T is said to be strictly supra- or strictly infra-unitary if $\mu^+(T) > 1$ or $\mu^+(T) < 1$ respectively.

A fundamental tool in our arguments is Banach's contraction principle, according to which a strictly contractive operator always has a fixpoint. We shall state and prove this theorem in a form suitable for our purposes.

Lemma 1 (Contraction Principle). If T is strongly contractive on a closed sphere $\overline{S}_a(y) : \|x - y\| \le a$ about the point y , and if

(2) $$\|Ty\| \le a(1 - |||T|||) ,$$

then the equation

(3) $$x = y + Tx$$

has a unique solution x in $S_a(y)$.

The solution can be obtained by iteration of the operator $T_{(y)} : x \to Tx = y + Tx$, on y .

Proof: Consider the sequence

$$y = \tilde{T}^0 y , \tilde{T}y , \tilde{T}^2 y, \ldots , \tilde{T}^n y , \ldots .$$

We shall first see that it can be continued indefinitely, that is, that all the $T^k y \in S_a(y)$. This is obviously true for $\widetilde{T}^0 y$. Proceeding by induction assume if for $T^0_{(y)} y, T^1_{(y)} y, \ldots, T^{n-1}_{(y)}$ and prove it for $\widetilde{T}^n y$. In the first place we have, if $1 \leq k \leq n$,

$$\|\widetilde{T}^k y - \widetilde{T}^{k-1} y\| = \|T\widetilde{T}^{k-1} y - T\widetilde{T}^{k-2} y\| \leq |||T||| \; \|\widetilde{T}^{k-1} y - \widetilde{T}^{k-2} y\| ,$$

whence

$$(4) \qquad \|\widetilde{T}^k y - \widetilde{T}^{k-1} y\| \leq |||T|||^{k-1} \|Ty\| , \quad k = 1, 2, \ldots, n .$$

Now, by the triangle inequality

$$(5) \quad \|\widetilde{T}^n y - y\| = \|\widetilde{T}^n y - \widetilde{T}^0 y\| = \|\widetilde{T}^n y - \widetilde{T}^{n-1} y\| + \|\widetilde{T}^{n-1} y - \widetilde{T}^{n-2} y\| + \ldots + \|\widetilde{T}^1 y - \widetilde{T}^0 y\|$$

$$\leq (|||T|||^{n-1} + |||T|||^{n-2} + \ldots + 1)\|Ty\|$$

$$\leq \frac{1 - |||T|||^n}{1 - |||T|||} \|Ty\| \leq a ,$$

which is the desired conclusion. Then, by induction, it follows that (4) is valid for all k's . Therefore, if $n > m$,

$$(6) \quad \|\widetilde{T}^n y - \widetilde{T}^m y\| \leq \sum_{k=1}^{n-m} \|\widetilde{T}^{m+k} y - \widetilde{T}^{m+k-1} y\| \leq \left(\sum_{k=1}^{n-m} |||T|||^{m+k-1} \right) \|Ty\|$$

$$\leq \frac{|||T^m||| - |||T|||^n}{1 - |||T|||} \|Ty\|$$

Hence, as $|||T||| < 1$, $\{\widetilde{T}^m y\}$ is a Cauchy sequence. Let x be its limit. Then, letting $n \to \infty$ in (5),

$$(7) \qquad \|x - y\| \leq \frac{\|Ty\|}{1 - |||T|||}$$

and

$$Tx + y = \widetilde{T}x = \lim_{n \to \infty} \widetilde{T}(\widetilde{T}^n y) = \lim_{n \to \infty} \widetilde{T}^{n+1} y = x ,$$

which proves that x is actually a solution of (3). It remains to check unicity. Should there be two solutions x_1 and x_2 in $S_a(y)$, then

$$\| x_1 - x_2 \| = \| Tx_1 - Tx_2 \| \leq |\!|\!| T |\!|\!| \, \| x_1 - x_2 \| \ ,$$

whence, since $|\!|\!| T |\!|\!| < 1$, it follows $\| x_1 - x_2 \| = 0$, that is, $x_1 = x_2$.

Remark. The contractivity of T has only been used on elements of the sequence $\widetilde{T}^n y$, which all belong to the range of $\widetilde{T}$, and hence it would be enough to assume it on that part of $S_a(y)$ belonging to the range of $\widetilde{T}$. In function spaces this applies, for instance, when the range of T contains only non-negative functions and y is itself a non-negative function. In such case the contractivity of T on non-negative functions is all one needs.

The hypotheses of Lemma 1 are hard to check because they are expressed with reference to the datum y which may itself be subjected to variations. The following form of the Contractions Principle obtained by restricting the hypotheses and relaxing the conclusions will be found to be more convenient.

Lemma 2. If T vanishes at the origin and is strictly contractive on a closed sphere $\overline{S}_r(0)$ about the origin, then the equation

$$x = y + Tx \tag{8}$$

has a unique solution x in $\overline{S}_r(0)$ for every $y \in \overline{S}_{r(1-|\!|\!| T |\!|\!|)}(0)$. The operator $(I - T)^{-1}$ is Lipschitzian and

$$\| I - T \|^{-1} \leq (1 - \| T \|)^{-1} \ . \tag{9}$$

The solution can be obtained by iteration of the operator $T^{(y)} = y + T$.

Proof: Under the hypotheses in order that T be contractive on a sphere $S_a(y)$ it is necessary that $y \in S_r(0)$, in which case one may take $a = r - \|y\|$. Therefore, since $\|Ty\| = \|Ty - T0\| \leq |||T|||\ \|y\|$ the requirement $\|Ty\| \leq a(1-|||T|||)$ of the previous Lemma would be fulfilled if $|||T|||\ \|y\| \leq a(1-|||T|||) = (r-\|y\|)(1-|||T|||)$, that is, if $\|y\| \leq r(1-|||T|||)$. Finally, if x_1 and x_2 are solutions corresponding to y_1 and y_2 $x_1 - x_2 = y_1 - y_2 + Tx_1 - Tx_2$ and $\|x_1 - x_2\| \leq \|y_1 - y_2\| + \|T\|\ \|x_1 - x_2\|$ whence (9) follows at once.

Lemma 3. If $T\alpha = (1-\alpha)I + \alpha T$, the equations

$$x = y + Tx$$
$$x = \alpha y + T_\alpha x$$

are equivalent for $\alpha \neq 0$.

Proof: Obvious.

Lemma 4 (Averaging Principle). If $1 \leq |||T||| < \infty$ in a domain $\mathcal{D}$ and if either

a) T is strictly infra-unitary, or

b) T is strictly supra-unitary,

then $T_\alpha = (1-\alpha)I + \alpha T$ is strictly contractive for

(10) $$0 < \alpha < 2\alpha^+ = \frac{2(1-\mu^+(T))}{(1-\mu^+(T))^2 + |||T|||^2 - (\mu^+(T))^2}, \quad \text{in case (a)},$$

and for

(11) $$\frac{2(1-\mu^-(T))}{(1-\mu^-(T))^2 + |||T|||^2 - (\mu^-(T))^2} = 2\alpha^- < \alpha < 0, \quad \text{in case (b)}.$$

Moreover,

(12) $$|||T_{\alpha^+}||| \leq \sqrt{\frac{|||T|||^2 - \mu^+(T)^2}{(1-\mu^+(T))^2 + |||T|||^2 - \mu^+(T)^2}} \quad \text{in case (a)}$$

(13) $$|||T_a - ||| \le \sqrt{\frac{|||T|||^2 - \mu^-(T)^2}{(1-\mu^-(T))^2 + |||T|||^2 - \mu^-(T)^2}} \quad \text{in case (b).}$$

Proof: We have

$$\|T_a x_1 - T_a x_2\|^2 = (1-a)^2 \|x_1 - x_2\|^2 + a^2 \|Tx_1 - Tx_2\|^2 + 2a(1-a)(x_1 - x_2, Tx_1 - Tx_2) \ ,$$

whence

$$\|T_a x_1 - T_a x_2\|^2 \le \begin{cases} [(1-a)^2 + a^2 |||T|||^2 + 2a(1-a)\mu^+(T)] \|x_1 - x_2\|^2 \ , & \text{if } \ 0 < a < 1 \ , \\ [(1-a)^2 + a^2 |||T|||^2 + 2a(1-a)\mu^-(T)] \|x_1 - x_2\|^2 \ , & \text{if } \ -1 < a < 0 \ . \end{cases}$$

The proof concludes by remarking that if $\mu^+(T) < 1$ the first term in square brackets is < 1 for $0 < a < 2a^+$ and attains its minimum (equal to the square of right member of (13)) at $a = a^+$, while if $\mu^-(T) > 1$ the second is < 1 for $2a^- < a < 0$ with the minimum at $a = a^-$.

Combining now Lemmas 2, 3 and 4 we obtain our main result:

Theorem 1. If T is Lipschitzian and either strictly infra- or supra-unitary on a sphere $S_r(0)$ and if $T(0) = 0$, the equation

(14) $$x = y + Tx$$

has a unique solution x in $S_r(0)$ obtainable by iteration, for each $y \in S_{r_1}(0)$, where

(15) $$r_1 = \begin{cases} r(1 - |||T|||) & \text{if } |||T||| < 1 \\ \dfrac{r(1-\mu^+(T))}{1 + \sqrt{\dfrac{|||T|||^2 - \mu^+(T)^2}{(1-\mu^+(T))^2 + |||T|||^2 - (\mu^+(T))^2}}} & \text{if } |||T||| \ge 1 \ , \ \text{and } \mu^+(T) < 1 \\ \dfrac{r(\mu^-(T) - 1)}{1 + \sqrt{\dfrac{|||T|||^2 - (\mu^-(T))^2}{(1-\mu^-(T))^2 + |||T|||^2 - (\mu^-(T))^2}}} & \text{if } |||T||| \ge 1 \ \text{and } \mu^-(T) > 1 \ . \end{cases}$$

Correspondingly one has

$$(16)\quad \|x-y\| \le \begin{cases} \dfrac{|||T|||}{1-|||T|||}\ \|y\| & \text{if } |||T||| < 1 \\[2ex] \dfrac{|||T|||}{1-\mu^+(T)}\left(1+\dfrac{1-\mu^+(T)}{\sqrt{2(1-\mu^+(T))+|||T|||^2-1}}\right), & \text{if } |||T||| \ge 1\ ,\ \mu^+(T)<1 \\[2ex] \dfrac{|||T|||}{\mu^-(T)-1}\left(1+\dfrac{\mu^-(T)-1}{\sqrt{2(\mu^-(T)-1)+|||T|||^2-1}}\right), & \text{if } |||T||| \ge 1\ ,\ \mu^-(T)>1\ . \end{cases}$$

The actual determination of the quantities $|||T|||$, $\mu^+(T)$ and $\mu^-(T)$ becomes simpler when T is a differentiable operator. In such cases these quantities are directly related to the norm of the gradient of T and to the greatest and smallest eigenvalues of its symmetrical part.

Theorem 2. If T is differentiable on a convex domain $\mathcal{D}$ and has a bounded gradient $\nabla T(x)$ at each point $x \in \mathcal{D}$, then

$$(17)\qquad |||T||| = \sup_{x \in \mathcal{D}} \|\nabla T(x)\|$$

$$(18)\qquad \mu^+(T) = \sup_{x \in \mathcal{D}}\ \sup_{\|\omega\|=1} (\nabla T(x)\omega,\omega)$$

$$(19)\qquad \mu^-(T) = \inf_{x \in \mathcal{D}}\ \inf_{\|\omega\|=1} (\nabla T(x)\omega,\omega)\ .$$

Proof: For fixed x_1 , x_2 and z the function

$$f(t) = (T((1-t)x_1 + tx_2),\ z)$$

is differentiable in t and

$$f'(t) = (\nabla T((1-t)x_1 + tx_2)(x_2 - x_1),\ z)\ .$$

Hence, by the mean value theorem

$$(20)\ (Tx_2 - Tx_1,\ z) = f(1) - f(0) = (\nabla T((1-\theta)x_1 + \theta x_2)(x_1 - x_2),\ z)\ ,\quad 0<\theta<1\ .$$

Letting now, $z = Tx_2 - Tx_1$, one obtains

$$\| Tx_1 - Tx_2 \|^2 \leq \sup_{x \in \mathcal{D}} \| \nabla T(x) \| \; \| x_2 - x_1 \| \; \| Tx_2 - Tx_1 \| ,$$

which implies

$$||| T ||| \leq \sup_{x \in \mathcal{D}} \| \nabla T(x) \| .$$

As the opposite inequality is obvious, (18) is proved. Similar proofs based on (21) hold for (19) and (20).

For transforms defined in the whole space one may surmise, arguing from the one-dimensional case, that for existence it is sufficient that the hypotheses of the previous theorems about the growth of the operator be fulfilled at infinity only. This conjecture may easily be proved for completely continuous operators by an application of Schauder's extension of Brouwer's Fix Point Theorem. Naturally the localization of the order relation $T < S$ at infinity should be understood as meaning the existence of a sphere outside which the above relation holds.

Theorem 3. If T is completely continuous in the whole space and if it is Lipschitzian and either strictly infra- or strictly supra-unitary at infinity, then the equation

$$x = y + Tx \tag{21}$$

has a solution at least for every y .

Proof: Based on Lemma 3, let us choose an α_0 and an r such that T_{α_0} is strictly contractive outside a sphere of radius r , and replace (22) by the equivalent equation

$$x = \alpha_0 y + T_{\alpha_0} x \tag{22}$$

Let $\widetilde{T}_{a_0}$ denote the operator $a_0 y + T_{a_0}$. Then if x_0 is a point with $\|x_0\| = r$ and if $\|x\| > r$,

$$\|\widetilde{T}_{a_0} x\| \leq \|\widetilde{T}_{a_0} x - \widetilde{T}_{a_0} x_0\| + \|\widetilde{T}_{a_0} x_0\| \leq |||T_{a_0}||| \; \|x - x_0\| + \|\widetilde{T}_{a_0} x_0\|$$

$$\leq |||T_{a_0}||| \; \|x\| + |||T_{a_0}||| \; \|x_0\| + \|\widetilde{T}_{a_0} x_0\| \; .$$

So if $\|x\| \geq (|||T_{a_0}||| \; \|x_0\| + \|\widetilde{T}_{a_0} x_0\|)/(1 - |||T_{a_0}|||)$, $\|\widetilde{T}_{a_0} x\| \leq \|x\|$.

That means that points at a distance from origin greater than some R are brought closer to the origin by $\widetilde{T}_{a_0}$. The sphere $S_R(0)$ is not necessarily mapped into itself by $\widetilde{T}_{a_0}$, but since $\widetilde{T}_{a_0}$ is bounded, its image will be contained in some $S_{R_1}(0)$ with $R_1 > R$. By this construction it is clear then that $S_{R_1}(0)$ is mapped into itself by $\widetilde{T}_{a_0}$. Finally, $\widetilde{T}_{a_0}$ being a linear combination of the identity and a completely continuous operator and mapping $S_{R_1}(0)$ into itself must have a fixed point in $S_{R_1}(0)$ by virtue of Schauder's Fix Point Theorem.

Before closing up this brief theory of functional equations we would like to make a few comments about its discretization. Let's suppose that $\mathfrak{B}$ is separable and let P_n be an increasing sequence of projections approaching the identity . Consider then the operators

$T_n = P_n T P_n$, constructed from a fixed one T . It is easily checked that $\mu^+(T_n) \leq \mu^+(T)$, $\mu^-(T_n) \geq \mu^-(T)$ and $|||T_n||| \leq |||T|||$. So if conditions of Theorem 1 are verified for T they are also verified for T_n and the equations

(23) $$x_n = y_n + T_n x_n \; , \quad y_n = P_n y \; ,$$

have unique solutions x_n within $S_r(0)$ for $y \in S_{r_1}(0)$, and $\|x_n - y_n\|$ admit estimates (17) uniformly. Moreover, if $\|T_n z - Tz\| \to 0$ uniformly over $S_r(0)$ (for instance if T is completely continuous), then a little reflection shows that the solutions x_n tend to the solution x of $x = y + Tx$. As equations (22) are really equations on finite dimensional spaces, they provide, within the frame of this theory, a discretization scheme for equations of the form $x = y + Tx$.

We wish also to say a few words about how could one proceed to extend this theory to spaces other than Hilbert spaces. Reflexive Banach spaces seem to offer the proper ground for generalizations. For one thing, it is possible there to furnish what looks like a natural extension of the notion of order of operators in Hilbert space. Simply notice that in reflexive Banach spaces there is a unique one-one mapping $x \longleftarrow \alpha^*$ onto their conjugates with the property that $(x,x^*) = \|x\|^2 = \|x^*\|^2$; with this operation construct a pseudo scalar product $[x,y] = \frac{1}{2}(x,y^*) + \frac{1}{2}(x^*,y)$ and use this in place of the ordinary scalar product to define $S < T$. On the other hand, the proofs of Lemmas 1, 2 and 3 go through unchanged, as they don't make use of the notion of order. The crucial point is Lemma 4, which, if at all provable, seems to call for a careful study of the way the operation $x \to x^*$ acts on sums of vectors.

Applications

We shall now exhibit a few applications of the theory just presented. Our first application is of an entirely general nature and gives sufficient conditions for the local invertibility of an operator. Next we especialize our theory to deal with equations of Hammerstein type (Acta Math. 54 (1930), 117-76), and at the end we give an application to a specific case originated in the theory of free boundaries.

Application 1. A Lipschitzian operator is locally invertible at any point where it is strictly monotonic (isotonic or antitonic)[2].

In fact, the local invertibility of T at x_0 amounts to say that the equation

$$y = T(x_0 + x) - Tx_0 \tag{24}$$

has an unique solution x in a neighborhood of the origin for each y in another such neighborhood. Now (25) is equivalent to either one of the equations

$$x = [x + (T(x_0 + x) - Tx_0)] - y \tag{25}$$

$$x = [x - (T(x_0 + x) - Tx_0)] + y \tag{26}$$

If T is isotonic, take equation (26), remark that the operator in square brackets is strictly supra-unitary, and conclude from Theorem 1 the local invertibility of T. If T is antitonic, the operator in square brackets in (26) is strictly infra-unitary, and the invertibility of T follows as before from Theorem 1.

On the real line, a milder notion of strict monotonicity, namely that $(Tx_1 - Tx_2, x_1 - x_2)$ be $\neq 0$ and of the same sign for $x_1 \neq x_2$, is both necessary and sufficient. The necessity of this condition is obviously not true for dimensions greater than 1, but one wonders if such condition

---

[2] T is strictly isotonic (antitonic) at x_0 if $(Tx_1 - Tx_2, x_1 - x_2)/\|x_1 - x_2\|^2 \geq k > 0$ $(\leq k < 0)$, in a neighborhood of x_0.

is sufficient in Hilbert space. For completely continuous operators, the sufficiency is easily proved with the help of Schauder's Fix Point Theorem.

Application 2. Let us take now $\mathscr{H} = L^2(0,1)$. To any function of two variables $f(t,u)$, $0 \le t \le 1$, $-\infty < u < +\infty$ one may assign the operator $\mathfrak{f}$ which transforms the function $u(t)$ into $f(t,u(t))$. Conditions can be given under which $\mathfrak{f}$ maps L^2 into itself (consult for instance the recent work of Krasnozelski "Topological methods in the theory of non-linear integral equations", of which an English translation is due to appear soon). We shall not enter into this question here, but shall assume that $f(t,0) \equiv 0$ and that the incremental ratio $(f(t,u_1)-f(t,u_2))/(u_1-u_2)$ remains uniformly bounded for all u_1 and u_2 , conditions which certainly guarantee that $\mathfrak{f}$ is an operator of L^2 into L^2 .

By combining the above type of operators with linear ones, a large class of non-linear operators is generated, among which those of Hammerstein type are counted. We shall consider here operators of the form $H^*\mathfrak{f}H$, where H is linear and bounded, and H^* is its adjoint. Noticing that

$$(H^*\mathfrak{f}Hx_2 - H^*\mathfrak{f}Hx_1, x_2 - x_1) = \int_0^1 \frac{f(Hx_2(t))-f(Hx_1(t))}{Hx_2(t) - Hx_1(t)} (Hx_2(t)-Hx_1(t))^2\, dt \quad ,$$

one derives at once the inequalities

$$\left[\inf_{t,u_1,u_2} \frac{f(t,u_1)-f(t,u_2)}{u_1-u_2}\right] \inf_{\|\omega\|=1} \|H\omega\|^2 \le \lambda^-(H^*\mathfrak{f}H) \le \lambda^+(H^*\mathfrak{f}H) \le \tag{27}$$

$$\left[\sup_{t,u_1,u_2} \frac{f(t,u_1)-f(t,u_2)}{u_1-u_2}\right] \sup_{\|\omega\|=1} \|H\omega\|^2 \quad .$$

These inequalities make it plain what conditions should be imposed on the function $f(t,u)$ and the operator H in order to make $H^{*}\mathfrak{f}H$ either supra- or infra-unitary. We prefer however to dress these conditions in a different guise more often met in the applications.

Theorem 4. If D is a self-adjoint and positive definite operator in L^2, $f(t,u)$ a differentiable function vanishing for $u = 0$, and if either

a) (28) $$\left(\sup_{t,u} \frac{\partial f(t,u)}{\partial u}\right) \sup_{\|\omega\|=1} (D\omega,\omega) < 1 ,$$

or

b) (29) $$1 < \left(\inf_{t,u} \frac{\partial f(t,u)}{\partial u}\right) \inf_{\|\omega\|=1} (D\omega,\omega) ,$$

then the equation

(30) $$x(t) = y(t) + f(t, Dx(t))$$

has an unique solution $x(t)$ obtainable by iteration.

Proof: Take $z = \sqrt{D}x$ as the unknown function and replace (1) by the equivalent equation

(31) $$z = \sqrt{D}y + \sqrt{D}\,\mathfrak{f}\,\sqrt{D}z .$$

By inequality (28), $\sqrt{D}\,\mathfrak{f}\sqrt{D}z$ is either infra- or supra-unitary according to whether a) or b) is valid respectively, and Theorem 1 can be applied to reach the desired conclusion.

This Corollary can also be phrased so as to include systems of functional equations. We shall simply state the results, trusting that the reader will have no difficulty in filling in the details. Before, let us recall that for linear self-adjoint operators $\lambda^+(D)$ and $\lambda^-(D)$ simply coincide with the greatest and the smallest eigenvalues of D.

Theorem 5. Let $D_1, D_2, \ldots, D_n$ be n self-adjoint, positive definite operators in L^2, and $f_i(t, u_1, \ldots, u_n)$ be n differentiable functions of n + 1 variables vanishing when $u_i = 0$, $i = 1, \ldots, n$. Then, if either

a) The greatest eigenvalue of the matrix $\frac{1}{2}\left(\frac{\partial f_i}{\partial u_j} + \frac{\partial f_j}{\partial u_i}\right)$ multiplied by $\max(\lambda^+(D_1), \lambda^+(D_2), \ldots, \lambda^+(D_n))$ remains below a constant less than one, or

b) The smallest eigenvalue of the matrix $\frac{1}{2}\left(\frac{\partial f_i}{\partial u_j} + \frac{\partial f_j}{\partial u_i}\right)$ multiplied by $\min(\lambda^-(D_1), \lambda^-(D_2), \ldots, \lambda^-(D_n))$ remains above a constant greater than one,

the system of equations

$$x_i(t) = y_i(t) + f_i\Big(t, D_1 x_1(t), D_2 x_2(t), \ldots, D_n x_n(t)\Big) \quad i = 1, 2, \ldots n . \tag{32}$$

has an unique solution which can be obtained by iteration.

Application 3. This application is devoted to a constructive existence proof of a cavity flow past a symmetric, convex arc of circle. The determinations of such a flow can be reduced to the solution of the single integral equation (cf. Birkhoff-Zarantonello, "Wakes, jets, and cavities", chapters VI and VII),

$$\lambda(\sigma) = \Theta \frac{\nu(\sigma) e^{-D\lambda(\sigma)}}{\int_0^{\pi} \nu(\sigma) e^{-D\lambda(\sigma)} d\sigma} , \tag{33}$$

where

a) Θ is the angular extent of the arc of circle.

b) $\nu(\sigma)$ is a known function describing the geometrical structure of the flow in question. It is always non-negative and bounded.

c) D is Dini's integral operator for the half-circle. It is associated with the kernel

$$D(s,\sigma) = \frac{1}{2\pi}\log\left[\frac{\tan\frac{1}{2}\sigma+\tan\frac{1}{2}s}{\tan\frac{1}{2}\sigma+\tan\frac{1}{2}s}\right]^2 = \frac{2}{\pi}\sum_1^{\infty}\frac{\sin js\,\sin j\sigma}{j}\ , \quad \begin{matrix}0\leq\sigma\leq\pi\\ 0\leq s\leq\pi\ .\end{matrix}$$

D is positive definite and completely continuous. It maps functions in $L^2(0,\pi)$ into continuous ones. Its eigenvalues are j^{-1}, and the corresponding eigenfunctions $\sqrt{\frac{2}{\pi}}\sin j\sigma$, $j = 1, 2, \ldots$.

d) $\lambda(\sigma)$ is the unknown function. We shall assume $\lambda \in L^2(0,\pi)$.

To apply our theory to (32) we shall first replace it by the equivalent equation

$$\sqrt{D}\lambda(\sigma) = \Theta\sqrt{D}\left[\frac{\nu(\sigma)e^{-D\lambda(\sigma)}}{\int_0^{\pi}\nu(\sigma)e^{-D\lambda(\sigma)}d\sigma}\right] \quad , \tag{34}$$

and take here $x(\sigma) = \sqrt{D}\lambda(\sigma)$ as the unknown function. Thus (34) becomes

$$x = \Theta\sqrt{D}\left[\frac{\nu e^{-\sqrt{D}x}}{\int_0^{\pi}\nu e^{-\sqrt{D}x}d\sigma}\right] . \tag{35}$$

The proof consists in showing that the operator

$$Tx = \Theta\sqrt{D}\left[\frac{\nu e^{-\sqrt{D}x}}{\int_0^{\pi}\nu e^{-\sqrt{D}x}d\sigma}\right]$$

is antitonic, and hence, a fortiori, infra-unitary. To this effect, we compute $\nabla T(x)$, or rather $(\nabla T(x)\omega,\omega)$. Letting ψ be the function $\nu e^{-\sqrt{D}x}$, one finds at once

$$(\nabla T(x)\omega,\omega) = \frac{\partial}{\partial\varepsilon}(T(x+\varepsilon\omega),\omega)_{\varepsilon=0} = \frac{\Theta}{(\psi,1)^2}\{(\psi,\sqrt{D}\omega)^2 - (\psi\sqrt{D}\omega,\sqrt{D}\omega)(\psi,1)\} \ .$$

Now, by Schwarz inequality

$$(\psi,\sqrt{D}\omega)^2 = \left(\int_0^{\pi}\psi\sqrt{D}\omega\,d\sigma\right)^2 = \left(\int_0^{\pi}(\sqrt{\psi}\sqrt{D}\omega)(\sqrt{\psi})d\sigma\right)^2 \leq \int_0^{\pi}(\sqrt{\psi}\sqrt{D}\omega)^2 d\sigma\int_0^{\pi}(\sqrt{\psi})^2 d\sigma$$

$$\leq (\psi\sqrt{D}\omega,\sqrt{D}\omega)(\psi,1) \ ,$$

and

$$(\nabla T(x)\omega, \omega) \leq 0 \quad .$$

By Theorem 2, this is equivalent to say that T is antitonic. Q.D.E.

AR 70-31 Unclassified

Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Mathematics Research Center, U. S. Army University of Wisconsin, Madison, Wis. 53706	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP None

3. REPORT TITLE

SOLVING FUNCTIONAL EQUATIONS BY CONTRACTIVE AVERAGING

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Summary Report: no specific reporting period.

5. AUTHOR(S) (First name, middle initial, last name)

E. H. Zarantonello

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
June 1960	17	

8a. CONTRACT OR GRANT NO. Contract No. DA-31-124-ARO-D-462	9a. ORIGINATOR'S REPORT NUMBER(S) 160
b. PROJECT NO. None c. d.	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report) none

10. DISTRIBUTION STATEMENT

Distribution of this document is unlimited.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
None	Army Research Office-Durham, N.C.

13. ABSTRACT

The notion of monotonicity is introduced and it is proved that Lipschitzian strictly monotone operators in Hilbert space are invertible. Applications are given.

DD FORM 1 NOV 65 1473

Unclassified

Security Classification

AGO 5698A