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On Lorentz-Sharpley spaces

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ABSTRACT : We study the Lorentz-Sharpley and Lorentz-Orlicz spaces from point of view of their convexity, concavity and interpolation properties.

1. Introduction

The spaces $\Lambda(w,p)$ were introduced 1951 by G.G.Lorentz [L1] as the class of measurable functions $f : \mathbb{R}_+ \to \mathbb{R}$ such that :

$$\|f\|_{w,p}^p = \int_0^\infty f^*(s)^p w(s)\,ds < \infty \tag{1}$$

where f^* is the non-increasing rearrangement of f. Here w is a given positive, non-increasing weight, such that $\int_0^1 w(s)\,ds < \infty$ and $\int_0^\infty w(s)\,ds = \infty$. (It was shown in [L1] that the condition "w non-increasing" is necessary and sufficient for $\|\ \|_{w,p}$ to be a norm, i.e. to satisfy the triangle inequality).

This generalization of $L_{p,q}$ Lorentz spaces was studied by authors interested either in their applications to interpolation theory or in their Banach spaces properties.

To the first family belongs the paper of R.Sharpley [S] (1972), which studies the spaces $\Lambda(w,p)$ with non monotone weights. Sharpley's spaces $\Lambda_\alpha(X)$ are connected to rearrangement invariant spaces X, but in fact are nothing but spaces $\Lambda(w,p)$, defined by (1), with $p = 1/\alpha$ and $w(s) = \frac{\lambda_X(s)^p}{s}$ (where λ_X is the fundamental function of the r.i. space X). The monotonicity condition for w is replaced in [S] by two conditions on λ_X :

$$\sup_{\substack{\rho>1\\u>0}} \frac{\lambda_X(\rho u)}{\rho^\gamma \lambda_X(u)} < \infty \quad (2) \quad \text{and :} \quad \sup_{\substack{\rho<1\\u>0}} \frac{\lambda_X(\rho u)}{\rho^\beta \lambda_X(u)} < \infty \quad (3)$$

where $0 < \beta \le \gamma < 1$. A remarkable fact showed in [S] is that the class of $\Lambda_\alpha(X)$ spaces ($0 < \alpha < 1$; λ_X verifying (2) and (3)) is self-dual, and more precisely $\Lambda_\alpha(X)^* = \Lambda_{\alpha_*}(X^*)$, with $\alpha_* = 1-\alpha$, and equivalent norms. In particular $\Lambda_\alpha(X)$ is reflexive (extending in this case a result of [L1] for monotone weights, where an isometric description of the dual is given).

The authors of the second aforementioned category were generally interested in Lorentz spaces with non-increasing weights. Apart from Halperin [H] (who investigated uniform convexity of functional Lorentz spaces) they studied mostly the sequential version, often denoted by $d(w,p)$. We may quote in particular Ruckle, Sargent, Garling in the 60's (see the references in [G]) ; and Altshuler, Bor-Luh-Lin, Casazza (together or separately) in the 70's, who were interested in particular in symmetric bases in spaces $d(w,p)$: see f.i. [ACB].

Let us also quote the paper [CD] (1988) which studies the geometry of $L_{p,q}$ spaces.

Here we are more concerned with convexity or concavity properties of $\Lambda(w,p)$ spaces, which were studied by Reisner [Re] (1981), Novikov [N] (1982) and Schütt [Sch] (1989), generalizing results of J.Creekmore [C] (1981) relative to $L_{p,q}$ spaces. Recall that a Banach (or quasi-Banach) lattice is said *p-convex* if :

$$\exists C, \forall x_1, ..., x_n \in X \quad \left\| (\sum_{i=1}^{n} |x_i|^p)^{1/p} \right\|_X \leq C(\sum_{i=1}^{n} \|x_i\|^p)^{1/p})$$

It is said *q-concave* if :

$$\exists c > 0, \forall x_1, ..., x_n \in X \quad \left\| (\sum_{i=1}^{n} |x_i|^q)^{1/q} \right\|_X \geq c(\sum_{i=1}^{n} \|x_i\|^q)^{1/q})$$

In particular, $\Lambda(w,p)$ is p-convex (resp p-concave) iff w is, up to an admissible change (see the precise definition below), non-increasing (resp non-decreasing).

Let us recall now f.i. the characterization of 2-concave Lorentz spaces given by [Sch] :

Theorem. *Let w be a non increasing weight and $1 \leq p < 2$. Then $\Lambda(w,p)$ is 2-concave iff there exists a constant $C > 0$ such that :*

$$\forall x > 0, \qquad \int_0^x w(t)t^{-p/2}\, dt \leq Cx^{-p/2} \int_0^x w(t)\, dt.$$

Of similar nature is the result of Ariño, Eldeeb, Peck [AEP] (1988) which studied non locally convex spaces with *increasing* weight : when $p > 1$, $d(w,p)$ is convexifiable (*i.e.* has an equivalent quasi-norm which is a norm) iff :

$$\sup_n \frac{1}{n}(\sum_{k=1}^{n} w_k).(\sum_{k=1}^{n} w_k^{-q/p}) < \infty, \quad \frac{1}{p} + \frac{1}{q} = 1$$

(which is interpreted as an "(A_p) condition").

In section 2 we extend this kind of result to spaces $\Lambda(w,p)$ with arbitrary (i.e. non-monotone) weight and then in section 4 to Lorentz-Orlicz spaces $L(W,\varphi)$, where φ is a moderate Orlicz function. Unlike the preceding ones (except that of [N]), the proof here is almost "calculous-free". (The very concise proof of [N] does not seem to extend to the present case).

The preceding result of Schütt was used by him to characterize 2-concave Lorentz spaces $\Lambda(w,p), 1 \leq p < 2$ (with decreasing weights) as those $\Lambda(w,p)$ spaces which isomorphically embed as Banach spaces in L_p. His argument involves an extension of a result of C.Merucci [M] (1983) who represents the space $\Lambda(w,p)$ as an interpolation space with function parameter $(L_{p_0}, L_{p_1})_{f,p;K}$ when the Boyd indices of $\Lambda(w,p)$ are strictly between p_0 and p_1. In [Sch], the space $\Lambda(w,p)$ is represented as $(L_{p_0}, L_{p_1})_{\lambda,p;K}$, an interpolation space with "measure parameter" $\lambda = dw(t^r)$, $\frac{1}{r} = \frac{1}{p} - \frac{1}{2}$.

In section 3 we give other representations of $\Lambda(w,p)$ as interpolation spaces (of Orlicz spaces) with applications to duality. In view of this kind of results, this generalization of the "usual" Lorentz spaces appears as far from being artificial.

In Section 5 we identify the spaces of multiplicators between Lorentz-Sharpley spaces and apply to (Calderon) interpolation of Lorentz-Sharpley spaces.

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2. Convexity and concavity of Lorentz-Sharpley spaces.

We set $W(t) = \int_0^t w(s)\,ds$ and remark that if W_1, W_2 are equivalent functions ($W_1 \sim W_2$) then $\Lambda(w_1,p) = \Lambda(w_2,p)$ with equivalent norms : for $\|f\|_{w,p}^p = -\int_0^\infty W(s)\,d(f^*(s)^p)$ .We say that the change from w_1 to w_2 is *admissible* if W_1 and W_2 are equivalent.

Note that the fundamental function of the r.i. space $\Lambda(w,p)$ is $\|\mathbb{1}_{[0,t]}\| = W(t)^{1/p}$.

We say that a non-decreasing function λ, which vanishes at 0, is γ-quasi-concave (resp β-quasi-convex) if it verifies condition (2) (resp (3)) of section 1 (in place of λ_X), which means that it is equivalent to an γ-concave, resp. β-convex function.

Theorem 1. *Let $p > 1$.The following assertions are equivalent :*
i) $\Lambda(w,p)$ is convexifiable;
ii) For any $f \in \Lambda(w,p)$, $\|f\| \sim \|f^{**}\|$;
iii) There exists $\epsilon > 0$ such that $W(t)^{1/p}$ is $(1-\epsilon)$-quasi-concave.

Proof : (ii) $\Rightarrow$ (i) is trivial. (iii) $\Rightarrow$ (ii) : it is well known that the quasilinear operator $M : f \to f^{**}$ is bounded on $L_r, 1 < r \leq \infty$. Note that (iii) implies that the lower Boyd index of $\Lambda(w,p)$ is strictly greater than 1. A suitable version of Boyd interpolation theorem ([LT II],2.b.11) proves the implication.

The main point of the Theorem is thus (i) $\Rightarrow$ (iii). To simplify we will consider the sequence case $d(w,p)$ but the proof is the same for the function space case.

If $d(w,p)$ is convex, the sequence $W_n^{1/p} = \|\mathbb{1}_{[0,n]}\|$ is (up to equivalence) concave (i.e. the restriction of a concave function on $\mathbb{R}_+$). We now use the following observation of Lindberg [L] (given for Orlicz functions).

Lemma 2. *If φ is a concave increasing function on $[0,\infty]$ which is never q-quasiconcave on $[1,\infty]$, for any $q < 1$, then there exists a sequence (x_i) of real positive numbers such that :*

$$\forall v > 0, \quad \frac{\varphi(v.x_i)}{\varphi(x_i)} \to v \quad as \quad i \to \infty$$

Coming back to the Lorentz space $d(w,p)$, suppose that $\frac{W_n}{n^{p-\varepsilon}}$ is not equivalent to a decreasing sequence, for any $\varepsilon > 0$.

Applying the Lindberg lemma to the sequence $(W_n^{1/p})_n$, we find a sequence $(m_j)_j$ of integers, $m_j \to \infty$, such that :

$$\forall k \in \mathbb{N}\ , \quad \frac{W_{m_j.k}}{W_{m_j}} \underset{j\to\infty}{\to} k^p\ .$$

Let us show that $d(w,p)$ contains the n-dimensional Lorentz spaces $\ell^n_{1,p}$ uniformly, as sublattices ; which means that for any $n \in \mathbb{N}$, and any $\varepsilon > 0$ we can find disjoint vectors $x^{(1)},\ldots,x^{(n)}$ in $d(w,p)$, such that :

$$\left\|\sum_{i=1}^{n} \alpha_i x^{(i)}\right\|_{d(w,p)} \underset{1+\varepsilon}{\sim} \|(\alpha_i)_{i=1}^n\|_{1,p}$$

with equivalence constants less than $1+\varepsilon$.

For any $x = (x_n)_{n\geq 1} \in \mathbb{R}^{(\mathbb{N})}$ and $m \in \mathbb{N}, m > 0$, we put : $D_m x(k) = x_{[\frac{k}{m}]}$.

Then :

$$\sum_{n=1}^{\infty}(D_{m_j}\, x)^{*p}_n . w_n = \sum_n (D_{m_j}\, x^*)^p_n . w_n =$$

$$= \sum_n x^{*p}_n [W_{nm_j} - W_{(n-1).m_j}] = \sum_n (x^{*p}_n - x^{*p}_{n+1}) W_{nm_j}$$

$$\underset{j\to\infty}{\sim} \quad W_{m_j} \sum_n (x^{*p}_n - x^{*p}_{n+1}).n^p = W_{m_j} \sum_n x^{*p}_n (n^p - (n-1)^p) = W_{m_j}\, \|x\|^p_{1,p}$$

hence :

$$\forall x \in \mathbb{R}^{\mathbb{N}}, \quad \left\|\frac{1}{W^{1/p}_{m_j}} D_{m_j}\, x\right\|_{d(w,p)} \underset{j\to\infty}{\to} \|x\|_{1,p}.$$

which by compacity can be obtained uniformly for x in the unit ball of $\ell^n_{1,p}$.

Now, as is well known, $\ell_{1,p}$ is not convex, or equivalently the $\ell^n_{1,p}$ are not uniformly isomorphic to normed spaces (see [LT], II, ex.1.f.9, or [C]), hence the contradiction. □

For the sake of completeness we give the proof of Lindberg's lemma 2. By hypothesis :

$$\forall q < 1, \; \inf_{\substack{\lambda>0\\ x<1}} \frac{\varphi(\lambda x)}{\lambda^q \varphi(x)} = 0.$$

Let $\psi(x) = \frac{\varphi(x)}{x}$, which defines a non-increasing function.

We have :

$$\forall \varepsilon > 0 , \quad \inf_{\substack{\lambda>0\\ x>1}} \frac{\lambda^\varepsilon \psi(\lambda x)}{\psi(x)} = 0$$

this implies :

$$\forall \delta > 0 , \quad \inf_{x>1} \frac{\psi(\delta x)}{\psi(x)} \leq 1$$

[for, if $\theta := \inf_{x>1} \frac{\psi(\delta x)}{\psi(x)} > 1$, set $\varepsilon = -\frac{\log\theta}{\log\delta}$, which is positive as we have necessarily $\delta < 1$. Then for $\lambda = \delta^n, n \in \mathbb{Z}$, we have : $\lambda^\varepsilon = \theta^{-n}$,and :

$$\frac{\lambda^\varepsilon \psi(\lambda x)}{\psi(x)} = \theta^{-n} \prod_{j=1}^{n} \frac{\psi(\delta^j x)}{\psi(\delta^{j-1} x)} \geq \theta^{-n}.\theta^n = 1$$

in the general case $\lambda = \delta^n.\rho$, with $\delta \leq \rho \leq 1$, hence :

$$\inf_{\substack{\lambda>0\\x>1}} \frac{\lambda^\varepsilon \psi(\lambda x)}{\psi(x)} \geq \inf_{\substack{x>1\\\delta\leq\rho\leq 1}} \frac{\rho^\varepsilon \psi(\rho x)}{\psi(x)} \geq \delta^\varepsilon = \frac{1}{\theta} > 0 \qquad].$$

So for each i we can find x_i such that :

$$1 \leq \frac{\psi\left(\frac{1}{i}x_i\right)}{\psi(x_i)} \leq 1 + \frac{1}{i}.$$

As ψ is non-increasing, we obtain :

$$\forall u \in \left[\frac{1}{i}, i\right], \ 1 \leq \frac{\psi(ux_i)}{\psi(x_i)} \leq 1 + \frac{1}{i}.$$

Finally :

$$\forall u \in (0,\infty), \ \frac{\psi(ux_i)}{\psi(x_i)} \underset{i\to\infty}{\to} 1, \ i.e. \ \frac{\varphi(ux_i)}{\varphi(x_i)} \underset{i\to\infty}{\to} u.$$

hence lemma 2. □

Theorem 1 has the following analog, with plainly analogous proof :

Proposition 3. *Let $p > r$. The following assertions are equivalent :*
i) $\Lambda(w,p)$ *is r-convex;*
ii) For any $f \in \Lambda(w,p)$, $\|f\| \sim \|M_r f\|$ *, where* $M_r f(t) := \left(\frac{1}{t}\int_0^t f^*(s)^r\, ds\right)^{1/r}$ *;*
iii) There exists $\varepsilon > 0$ *such that* $W^{1/p}$ *is* $(\frac{1}{r} - \varepsilon)$*-quasi-concave.*

Proposition 3 has in turn the following companion :

Proposition 4. *Let $p < q$ and suppose that W has some concavity (is α-quasi-concave, for some $\alpha > 0$). The following assertions are equivalent :*
i) $\Lambda(w,p)$ *is q-concave.*
ii) For any $f \in \Lambda(w,p)$, $\|f\| \sim \|N_q f\|$ *, where* $N_q f(t) = \left(\frac{1}{t}.\int_t^\infty f^*(s)^q\, ds\right)^{1/q}$ *;*
iii) There exists $\varepsilon > 0$ *such that the function* $W^{1/p}$ *is* $(\frac{1}{q} + \varepsilon)$*-quasi-convex.*

Proof : (i) ⇒ (iii) has a very analogous proof to that of the same implication in Th.1 ; one uses the convex function version of Lindberg lemma instead of the concave one. For (iii) ⇒ (ii), we may suppose that $\Lambda(w,p)$ is convex (after suitable convexification procedure, see [LT] II.1.d) ; note that $N_q f \geq D_{1/2} f^*$, hence $\|N_q f\| \geq \frac{1}{2}\|f\|$. Then :

$$N_q f(t) = \sum_{k\geq 0}\left(\frac{1}{t}\int_{2^k.t}^{2^{k+1}.t} f^*(s)^q\, ds\right)^{1/q} \leq \sum_{k\geq 0} 2^{k/q} f^*(2^k.t)$$

hence :

$$\|N_q f\| \leq \sum_{k\geq 0} 2^{k/q}\|D_{2^{-k}} f\| \leq \left(\sum_{k\geq 0} 2^{k(\frac{1}{q}-\frac{1}{q_1})}\right).\|f\| \leq C\|f\|$$

where $q_1 < q$ is any real strictly greater than the second Boyd index q_Λ of $\Lambda(w,p)$ (which is strictly less than q). Finally for (ii) $\Rightarrow$ (i), we note that, f.i. in the case $q = 1$, we have for any integrable positive functions f, g :

$$t.N_1(f+g) = \int_t^\infty (f+g)^*(s)\,ds = \int_0^\infty (f+g)^*(s)\,ds - \int_0^t (f+g)^*(s)\,ds$$

$$\geq (\int_0^\infty f^*(s)\,ds + \int_0^\infty g^*(s)\,ds) - (\int_0^t f^*(s)\,ds + \int_0^t g^*(s)\,ds)$$

$$= t.(N_1 f + N_1 g)\ ,$$

hence for any $f, g \in \Lambda(w,p),\ f, g \geq 0$, we have $N_1(f+g) \geq N_1 f + N_1 g$ which implies as $p \leq 1$ that $\|N_1(f+g)\| \geq \|N_1 f\| + \|N_1 g\|$, by reverse Minkowski inequality. □

Remark. It follows clearly from the preceding propositions that if $\Lambda(w,p)$ is convex (resp. r-convex, resp. q-concave) and $p > 1$ (resp. $p > r$, resp. $p > q$) then it is $(1+\varepsilon)$-convex (resp. $(r+\varepsilon)$-convex, resp $(q-\varepsilon)$-concave) for some positive ε.

3. Representation of spaces $\Lambda(w,p)$ by interpolation of Orlicz spaces.

By Orlicz function we mean here a non-decreasing continuous function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, with $\varphi(0) = 0$ and $\varphi(t) \to \infty$ as $t \to \infty$. We also suppose that φ has "some non-trivial convexity", i.e. is α-quasi-convex for some $\alpha > 0$. In this case the Orlicz space L_φ is α-convex. We denote by λ_φ the fundamental function of L_φ. Recall that $\lambda_\varphi(t) = \frac{1}{\varphi^{-1}(\frac{1}{t})}$. To any function $f : \mathbb{R}_+ \mapsto \mathbb{R}_+$ we associate the function $\tilde{f} : \mathbb{R}_+ \mapsto \mathbb{R}_+$ defined by $\tilde{f}(t) = \frac{1}{f(\frac{1}{t})}$

Theorem 5. *Suppose that φ, ψ are Orlicz functions such for some $\varepsilon > 0$, the function $\frac{\lambda_\varphi^{1-\varepsilon}}{\lambda_\psi}$ is equivalent to a non-decreasing function. Then $(L_\varphi, L_\psi)_{\theta,p} = \Lambda(w,p)$ (with equivalent norms), where w is defined by its primitive W : $W(t)^{1/p} = \lambda_\varphi(t)^{1-\theta}.\lambda_\psi(t)^\theta$.*

Proof : We use a result of J.Arazy [A] concerning the K-functional for rearrangement invariant spaces (see also [Ma 2] for a refinement of this result). If E, F are two r.i. spaces, let $M = \mathrm{Mult}(F,E)$ be the space of multipliers from F to E, i.e. of functions g such that the associated multiplication operator $M_g :\ f \mapsto f.g$ is bounded from F into E. M is equipped with the norm $\|g\|_M = \sup\{\|g.f\| \,/\, \|f\|_F \leq 1\}$. Suppose that $E = M.F$ and that M is a r.i. space (in the sense of [L.T.]II) with fundamental function λ_M verifying $\lim_{t\to 0} \lambda_M(t) = 0$, $\lim_{t\to\infty} \lambda_M(t) = \infty$. Then the K-functional of interpolation $K(t, f; E, F)$ is given by :

$$K(t,f) \ \sim\ \left\|\mathbb{1}_{[0,u(t)]}.f^*\right\|_E + t.\left\|\mathbb{1}_{[u(t),\infty]}.f^*\right\|_F \tag{2.1}$$

$u(t)$ being defined by the equation $\lambda_E(u(t)) = t.\lambda_F(u(t))$, i.e. $\lambda_M(u(t)) = t$.

We apply Arazy's result to the case $E = L_\varphi,\ F = L_\psi$. In this case $M(L_\varphi, L_\psi) = L_M$, where the Orlicz function M is defined (up to equivalence) by the relation :

$$\lambda_\varphi(t)\ \sim\ \lambda_\psi(t).\lambda_M(t)$$

and moreover $L_\varphi = L_\psi . L_M$. This seems to have been observed first by B.Maurey [Mau] (1974). In this case the function u appearing in (2.1) is defined by : $u(t) = \tilde{M}(t) := \frac{1}{M(\frac{1}{t})}$.

Write $A(t,f) = \left\| \mathbb{1}_{[0,\tilde{M}(t))} \cdot f^* \right\|_\varphi$ and $B(t,f) = \left\| \mathbb{1}_{[\tilde{M}(t),\infty)} \cdot f^*(t) \right\|_\psi$, and let us first evaluate $A = (\int_0^\infty (t^{-\theta} A(t,f))^p \frac{dt}{t})^{1/p}$.

If φ is convex, we have : $A(t,f) \leq \sum_{k=0}^{\infty} \left\| \mathbb{1}_{[\tilde{M}(2^{-(k+1)}t), \tilde{M}(2^{-k}t)]} \cdot f^* \right\|_\varphi$. (If φ is only α-convex, $\alpha < 1$, we would use the α-norm triangle inequality). Then, if $p \geq 1$, we have :

$$\begin{aligned} A &\leq \sum_k \left[\int_0^\infty (t^{-\theta} \left\| \mathbb{1}_{[\tilde{M}(2^{-(k+1)}t), \tilde{M}(2^{-k}t)]} \cdot f^* \right\|_\varphi)^p \frac{dt}{t} \right]^{1/p} \\ &= (\sum_k 2^{-k\theta}). \left[\int_0^\infty (t^{-\theta} \left\| \mathbb{1}_{[\tilde{M}(\frac{t}{2}), \tilde{M}(t)]} \cdot f^* \right\|_\varphi)^p \frac{dt}{t} \right]^{1/p}. \end{aligned}$$

Conversely we have clearly $A \geq \left[\int_0^\infty (t^{-\theta} \left\| \mathbb{1}_{[\tilde{M}(\frac{t}{2}), \tilde{M}(t)]} \cdot f^* \right\|_\varphi)^p \frac{dt}{t} \right]^{1/p}$.

Now we estimate this last quantity. We have :

$$\begin{aligned} \int_0^\infty (t^{-\theta} \left\| \mathbb{1}_{[\tilde{M}(\frac{t}{2}), \tilde{M}(t)]} \cdot f^* \right\|_\varphi)^p \frac{dt}{t} &\leq \int_0^\infty [t^{-\theta} f^*(\tilde{M}(\frac{t}{2})) \lambda_\varphi(\tilde{M}(t))]^p \frac{dt}{t} \\ &\leq C^p \int_0^\infty [t^{-\theta} f^*(\tilde{M}(\frac{t}{2})) \lambda_\varphi(\tilde{M}(\frac{t}{2}))]^p \frac{dt}{t} \\ &= 2^{\theta p} C^p \int_0^\infty [t^{-\theta} f^*(\tilde{M}(t)) \lambda_\varphi(\tilde{M}(t))]^p \frac{dt}{t} \end{aligned}$$

where $C = \sup_{t>0} \frac{\lambda_\varphi \circ \tilde{M}(2t)}{\lambda_\varphi \circ \tilde{M}(t)} < \infty$, as the hypothesis implies that $\frac{\lambda_\varphi(\tilde{M}(s))^{1-\varepsilon}}{\lambda_\psi(\tilde{M}(s))} = \frac{s}{\lambda_\varphi(\tilde{M}(s))^\varepsilon}$ is a non-increasing function of s (hence $C \leq 2^{1/\varepsilon}$).

On the other hand,we have :

$$A^p = \int_0^\infty [t^{-\theta} \left\| \mathbb{1}_{[0,\tilde{M}(t)]} f^* \right\|_\varphi]^p \frac{dt}{t} \geq \int_0^\infty [t^{-\theta} f^*(\tilde{M}(t)) \lambda_\varphi(\tilde{M}(t))]^p \frac{dt}{t}$$

and finally we obtain :

$$\begin{aligned} A^p &\sim \int_0^\infty [t^{-\theta} f^*(\tilde{M}(t)) \lambda_\varphi(\tilde{M}(t))]^p \frac{dt}{t} = \int_0^\infty [\lambda_M(u)^{-\theta} \lambda_\varphi(u) f^*(u)]^p \frac{d\lambda_M(u)}{\lambda_M(u)} \\ &= \int_0^\infty [\lambda_\varphi(u)^{1-\theta} \lambda_\psi(u)^\theta f^*(u)]^p \frac{d\lambda_M(u)}{\lambda_M(u)}. \end{aligned}$$

Set $\lambda(u) = \lambda_\varphi(u)^{1-\theta}\lambda_\psi(u)^\theta$. The hypothesis implies (up to a change of φ, ψ to equivalent functions) that :

$$(1-\varepsilon)\frac{d\lambda_\varphi}{\lambda_\varphi} - \frac{d\lambda_\psi}{\lambda_\psi} \geq 0$$

hence : $\frac{d\lambda_M}{\lambda_M} = \frac{d\lambda_\varphi}{\lambda_\varphi} - \frac{d\lambda_\psi}{\lambda_\psi} \geq \varepsilon\frac{d\lambda_\varphi}{\lambda_\varphi}$, thus $\frac{d\lambda_M}{\lambda_M} \sim \frac{d\lambda_\varphi}{\lambda_\varphi}$, and finally :

$$\frac{d\lambda}{\lambda} = (1-\theta)\frac{d\lambda_\varphi}{\lambda_\varphi} + \theta\frac{d\lambda_\psi}{\lambda_\psi} \sim \frac{d\lambda_\varphi}{\lambda_\varphi} \sim \frac{d\lambda_M}{\lambda_M}.$$

Coming back to the preceding expression of A^p, we find :

$$A^p \sim \int_0^\infty (\lambda(u)f^*(u))^p\,\frac{d\lambda(u)}{\lambda(u)} = \frac{1}{p}\int_0^\infty f^*(u)^p\,d(\lambda(u)^p) = \|f\|^p_{w,p}.$$

It remains to evaluate the contribution B of the term $B(t,f)$ of $K(t,f)$ in $\|f\|_{\theta,p}$. We have :

$$B(t,f) \leq t.\sum_{k=0}^\infty \left\|\mathbb{1}_{[\tilde{M}(2^k t),\tilde{M}(2^{k+1}t]}f^*\right\|_\psi$$

Hence :

$$\begin{aligned}
B &\leq \left[\int_0^\infty (t^{1-\theta}\sum_{k=0}^\infty \left\|\mathbb{1}_{[\tilde{M}(2^k t),\tilde{M}(2^{k+1}t)]}f^*\right\|_\psi)^p\,\frac{dt}{t}\right]^{1/p}\\
&\leq (\sum_{k=0}^\infty 2^{k(1-\theta)}).\left[\int_0^\infty (t^{1-\theta}\left\|\mathbb{1}_{[\tilde{M}(t),\tilde{M}(2t)]}f^*\right\|_\psi)^p\,\frac{dt}{t}\right]^{1/p}\\
&\leq C\left[\int_0^\infty (t^{1-\theta}f^*(\tilde{M}(t))\lambda_\psi(\tilde{M}(2t)))^p\,\frac{dt}{t}\right]^{1/p}\\
&\sim C\left[\int_0^\infty (t^{-\theta}f^*(\tilde{M}(t))\lambda_\varphi(\tilde{M}(2t)))^p\,\frac{dt}{t}\right]^{1/p}\\
&\leq C'\left[\int_0^\infty (t^{-\theta}f^*(\tilde{M}(t))\lambda_\varphi(\tilde{M}(t)))^p\,\frac{dt}{t}\right]^{1/p}
\end{aligned}$$

Finally we obtain $B \lesssim A$, hence $\|f\|_{\theta,p} \sim A \sim \|f\|_{w,p}$. □

Remark 6. The preceding proof remains clearly valid if one replaces Orlicz spaces by r.i. spaces E, F verifying $E = M.F$, $\lambda_E(t) \underset{t\to 0}{\to} 0$, $\lambda_E(t) \underset{t\to\infty}{\to} \infty$ and $\frac{\lambda_E^{1-\varepsilon}}{\lambda_F}$ equivalent to a non-decreasing function. (I was told by M.Mastyło that a similar result was known to him [private communication]).

Corollary 7. *If the Orlicz function φ is r-quasi-convex, for some $r > 1$ then :* $(L_1, L_\varphi)_{\theta,p} = \Lambda(w,p)$, *with* $W^{1/p}(t) = t^{1-\theta}\lambda_\varphi(t)^\theta$.

Corollary 8. *For any Orlicz function (not necessarily convex), such that* $\lambda_\varphi(t) \underset{t\to 0}{\longrightarrow} 0$, $\lambda_\varphi(t) \underset{t\to\infty}{\longrightarrow} \infty$, *we have :*
$(L_\varphi, L_\infty)_{\theta,p} = \Lambda(w,p)$, *with* $W^{1/p}(t) = \lambda_\varphi(t)^{1-\theta}$.

Corollary 9. *a) Every convex and q-concave space* $\Lambda(w,p)$, $1 < p < q < \infty$, *is a Lions-Peetre interpolation space between* L_1 *and a convex, q-concave Orlicz space.*
b) Every convex space $\Lambda(w,p)$, $p > 1$, *is a Lions-Peetre interpolation space between a convex Orlicz space and* L_∞.

Corollary 10. *The space* $\Lambda(w,p)$, $p > 1$, *when convex, is reflexive.*

Proof : By the preceding corollary, $\Lambda(w,p) = (L_\varphi, L_\infty)_{\theta,p}$ for some convex non-degenerate Orlicz function φ ($0 < \varphi(t) < \infty$, $\forall t \in (0,\infty)$). By [B], th.III.1.1, the reflexivity of $L_\varphi, L_\infty)_{\theta,p}$ is independant of p, $1 < p < \infty$. But choosing $r = \frac{1}{1-\theta}$ we have $(L_\varphi, L_\infty)_{\theta,r} = \Lambda(u,r)$ whith $U \sim \lambda_\varphi$,which is quasiconcave : hence u can be supposed non-increasing, and $\Lambda(u,r)$ is thus reflexive by [L1]. □

Using the duality theorem for Lions-Peetre interpolation method, and the preceding corollaries, we obtain immediately Sharpley's duality formula : $\Lambda(w,p)^* = \Lambda(w_*,q)$ ($\frac{1}{p} + \frac{1}{q} = 1$, $W(t)^{1/p}.W_*(t)^{1/q} = t$), when $p > 1$ and $\Lambda(w,p)$ has some non-trivial concavity (this last condition is necessary, at least for non-increasing weights, as was shown in [Re2]).

Let us derive $\Lambda(w,p)^*$ when no concavity condition on $\Lambda(w,p)$ is given. We first introduce the following :
Notation. $L^{w,p}$ is the space of measurable functions f such that $f^{**} \in \Lambda(w,p)$; this space is equipped with the norm : $\|f\|_{L^{w,p}} = \|f^{**}\|_{\Lambda(w,p)}$.

Then Corollary 7 has the following extension :

Proposition 11. *Let $p > 1$ and φ be an Orlicz function such that* $\frac{\varphi(t)}{t} \underset{t\to 0}{\longrightarrow} 0$, *and* $\frac{\varphi(t)}{t} \underset{t\to\infty}{\longrightarrow} \infty$. *Then :* $(L_1, L_\varphi)_{\theta,p} = L^{\bar{w},p}$,*where $\bar{w}$ is connected to the function W of cor.7* $(W(t)^{1/p} = t^{1-\theta}\lambda_\varphi(t)^\theta)$ *by the relation :*

$$\bar{w}(t) = p\frac{W(t)}{t} - \frac{dW}{dt}(t) .$$

Proof : We have :

$$K(t,f;L_1,L_\varphi) \sim A(t,f) + B(t,f)$$

with : $A(t,f) = \left\|\mathbb{1}_{[0,\tilde{\varphi}_*(t))}f^*\right\|_1$ and $B(t,f) = t.\left\|\mathbb{1}_{[\tilde{\varphi}_*(t),\infty)}f^*\right\|_\varphi$,
where φ_* is the Young conjugate of φ.

We shall prove that the contribution B of $B(t,f)$ in the interpolation norm is dominated by that of $A(t,f)$. As in the proof of Th.5, we have :

$$B^p \leq C_\theta \int_0^\infty (t^{1-\theta} \left\|\mathbb{1}_{[\tilde\varphi_*(t),\tilde\varphi_*(2t))} f^*\right\|_\varphi)^p \frac{dt}{t}.$$

Set : $C(t,f) = \left\|\mathbb{1}_{[\tilde\varphi_*(t),\tilde\varphi_*(2t)]}\right\|$; then :

$$K(t, \mathbb{1}_{[0,\tilde\varphi_*(2t))} f^*) \sim A(t,f) + C(t,f) \text{ ;}$$

on the other hand :

$$K(t, \mathbb{1}_{[0,\tilde\varphi_*(2t))} f^*) \leq \left\|\mathbb{1}_{[0,\tilde\varphi_*(2t))} f^*\right\|_1 = A(2t,f).$$

Hence $C(t,f) \lesssim A(2t,f)$ and finally :

$$\int_0^\infty (t^{-\theta} C(t,f))^p \frac{dt}{t} \lesssim \int_0^\infty (t^{-\theta} A(2t,f))^p \frac{dt}{t}$$
$$\leq \int_0^\infty (t^{-\theta} A(t,f))^p \frac{dt}{t} = A$$

Now we have :

$$\|f\|^p_{\theta,p} \sim A = \int_0^\infty \left(\int_0^{\tilde\varphi_*(t)} f^*(s)\,ds\right)^p \frac{dt}{t^{1+\theta p}} = \int_0^\infty \left(\int_0^t f^*(s) ds\right)^p \frac{d\lambda_{\varphi_*}(t)}{\lambda_{\varphi_*}(t)^{1+\theta p}}$$
$$= \int_0^\infty f^{**}(t)^p t^p \frac{d\lambda_{\varphi_*}(t)}{\lambda_{\varphi_*}(t)^{1+\theta p}} = \int_0^\infty f^{**}(t)^p \, d\bar W(t)$$

with :

$$\bar W(t) = \int_0^t s^p \frac{d\lambda_{\varphi_*}(s)}{\lambda_{\varphi_*}(s)^{1+\theta p}} = \frac{1}{\theta p}\left[-\frac{s^p}{\lambda_\varphi(s)^{\theta p}}\right]_0^t + \frac{p}{\theta p}\int_0^t \frac{s^{p-1}}{\lambda_{\varphi_*}(s)^{\theta p}} ds$$

and as $\dfrac{s^p}{\lambda_{\varphi_*}(s)^{\theta p}} \sim s^{(1-\theta)p} \lambda_\varphi(s)^{\theta p} = W(t)$, we obtain :

$$\bar W(t) \sim -W(t) + p\int_0^t W(s)\frac{ds}{s} = \int_0^t (p\frac{W(s)}{s} - w(s))ds.$$

Note that the modified weight $\bar w(s) = p\dfrac{W(s)}{s} - w(s)$ is non-negative because $\dfrac{W(s)}{s^p}$ is non-increasing. □

Corollary 12. *If $1 < p < \infty$ and if $\Lambda(w,p)$ is convex, then $\Lambda(w,p)^* = L^{\bar w_*, q}$ where $\bar w_*$ is the modified weight associated to $W_*(t) = \dfrac{t^q}{W(t)^{q/p}}$.*

Proof : Use cor.7, the duality theorem for Lions-Peetre interpolation and prop.10. □

As an application we give the following :

Corollary 13. *The Banach envelope* $\ell_{1,p}^{\sharp}$ *of the Lorentz space* $\ell_{1,p}$ *is the space* $\ell^{\bar{w},p}$ *with* $\bar{w}_k = \dfrac{k^{p-1}}{(1+\log\ k)^p}$.

Proof : We have $\ell_{1,p}^* = d(w_*, q)$ with $w_{*k} = w_k^{-q/p} = \frac{1}{k}$ (see [AEP]) ; hence $W_{*k} \sim \log k$ and $W_{**k} \sim \dfrac{k^p}{(\log k)^{p/q}}$ and finally : $\bar{w}_{**k} = p\frac{W_{**k}}{k} - w_{**k} \sim \dfrac{p}{q}\dfrac{k^{p-1}}{(\log k)^p}$. □

Let us mention that prop.11 has a simpler counterpart in the case $p = 1$.

Proposition 14. *a) For any convex Orlicz function* φ *one has* $(L_1, L_\varphi)_{\theta,1} = \Lambda(w,1)$ *with* $W(s) = s^{1-\theta}\lambda_\varphi(s)^\theta$.

b) Consequently any $\Lambda(w,1)$ *which is convex and has some non-trivial concavity can be represented as some interpolation space* $(L_1, L_\varphi)_{\theta,1}$.

Proof : We first suppose that $\frac{\varphi(t)}{t} \underset{t\to 0}{\longrightarrow} 0$ and $\frac{\varphi(t)}{t} \underset{t\to\infty}{\longrightarrow} \infty$.

As in the case of Prop 11, only the contribution of $A(t,f)$ in the interpolation norm has to be computed. We have :

$$\int_0^\infty t^{-\theta} A(t,f)\,\frac{dt}{t} = \int_0^\infty t^{-\theta}\left(\int_0^{\tilde{\varphi}_*(t)} f^*(s)\,ds\right)\frac{dt}{t} = \int_0^\infty \left(\int_{\lambda_{\varphi_*}(s)}^\infty \frac{dt}{t^{\theta+1}}\right) f^*(s)\,ds$$
$$= \frac{1}{\theta}\int_0^\infty \lambda_{\varphi_*}(s)^{-\theta} f^*(s)\,ds = \frac{1}{\theta}\int_0^\infty f^*(s)w(s)\,ds$$

with $w(s) = s^{-\theta}\lambda_\varphi(s)^\theta$. Then $sw(s) = s^{1-\theta}\lambda_\varphi(s)^\theta$ is equivalent to a concave, $(1-\theta)$-convex function ; hence :

$$W(t) = \int_0^t w(s)\,ds \sim tw(t) = t^{1-\theta}\lambda_\varphi(t)^\theta.$$

Now if f.i. $\frac{\varphi(t)}{t} \underset{t\to 0}{\longrightarrow} a > 0$, we may suppose (passing to an equivalent Orlicz function) that $\varphi(t) = t$ for $t \le 1$. In this case it is possible to prove that :

$$\begin{cases} K(t,f) \sim \left\|\mathbb{1}_{[0,\tilde{\varphi}_*(t))} f^*\right\|_1 + t.\left\|\mathbb{1}_{[\tilde{\varphi}(t),\infty)} f^*\right\|_\varphi & \text{for } t \le 1 \\ K(t,f) \sim \|f\|_1 \qquad \text{for } t > 1 \end{cases}$$

and the same calculous than before leads to :

$$\|f\|_{\theta,1} \sim \int_0^1 f^*(s)w(s)\,ds + \|f\|_1 \sim \int_0^\infty f^*(s)w(s)\,ds\ .$$

The proof of assertion b) of the proposition is easy and left to the reader. □

As usual, prop 14 has easy analogs in the p-convex or q-concave cases :

Proposition 15. *a) For any p-quasi-convex (resp p-quasi-concave) Orlicz function φ one has $(L_p, L_\varphi)_{\theta,p} = \Lambda(w,p)$ with $W(t) = t^{\frac{1-\theta}{p}}\lambda_\varphi(t)^\theta$.*

b) Consequently any space $\Lambda(w,p)$ which is p-convex and r-concave (resp p-concave and r-convex) can be represented as $(L_p, L_\varphi)_{\theta,p}$ for some $0 < \theta < 1$ and some p-convex, r-concave (resp p-concave, r-convex) Orlicz function.

Corollary 9, its p-convex analog and Proposition 15 have the following immediate consequence on the embeddability of Lorentz-Sharpley spaces in Lebesgue spaces.

Proposition 16. *If $p_0 \le p < p_1$ (resp. $p_0 < p = p_1$), and $\Lambda(w,p)$ is p_0-convex, p_1-concave, then it is lattice embeddable in $\ell_p(L_{p_0}(\ell_{p_1}))$ (resp. in $\ell_p(L_{p_0}(L_p))$.*

In particular if $\Lambda(w,p)$ is 2-concave and p_0-convex, it embeds (as Banach space) in $\ell_p(L_{p_0})$.

Proof : Suppose f.i. $p_0 < p < p_1$. We can choose $p_0 < p_0' < p < p_1' < p_1$ such that, again, $\Lambda(w,p)$ is p_0'-convex, p_1'-concave. Then, using the p-convex version of cor.9, we can find a p_0-convex, p_1-concave φ such that $\Lambda(w,p) = (L_{p_0}, L_\varphi)_{\theta,p}$ (choose $\theta = \frac{\frac{1}{p_0} - \frac{1}{p_1'}}{\frac{1}{p_0} - \frac{1}{p_1}}$). But now $\Lambda(w,p)$ identifies to an ℓ_p-direct sum of spaces X_j normed by the K-functionals $K(2^j, ., L_{p_0}, L_\varphi)$, $(j \in \mathbb{Z})$. The X_j's can be shown to be uniformly isomorphic to Orlicz spaces L_{ψ_j}, where the ψ_j's are (uniformly) p_0-convex, p_1-concave Orlicz function given up to equivalence by :

$$\psi_j(s) \sim s^{p_0} \wedge \varphi(2^j s).$$

Such Orlicz spaces are sublattices of $L_{p_0}(L_{p_1})$ (and even of $L_{p_0}(\ell_{p_1})$, if one manages to choose φ $p"_1$-concave, with $p"_1 < p_1$) : see [R] for this result, which is an easy consequence of a famous result of Bretagnolle and Dacunha-Castelle [BDC]. The cases $p = p_0$, $p = p_1$ are treated using prop 15. ☐

4. Extension of the results to Orlicz-Lorentz spaces

Lorentz-Orlicz spaces are, roughly speaking, generalizations of the spaces $\Lambda(w,p)$ when "p is replaced by an Orlicz function φ". Two versions of this generalization (at least) appear in the litterature.

The first one was introduced by A.Torchinsky [T] :

$$L(W,\varphi) = \{f \in L_0 \ / \ \int_0^\infty \varphi(W(t).f^*(t))\,\frac{dt}{t} < \infty\}$$

where W is a given continuous increasing function, with $W(0) = 0, W(t) \underset{t\to\infty}{\to} \infty$. Note that W is moreover concave in Torchinsky's setting. $L(W,\varphi)$ is equipped with the quasinorm :

$$\|f\|_{W,\varphi} = \|W.f^*\|_{L_\varphi(\frac{dt}{t})}$$

which is associated to the "modular" Φ :

$$\Phi(f) = \int_0^\infty \varphi(W(t).f^*(t))\,\frac{dt}{t}\ .$$

The second version of Lorentz-Orlicz spaces appears in [Ma 1] (see also [K1], [K2], [Mas]) :

$$\Lambda(w,\varphi) = \{f \in L_0 \ / \ \int_0^\infty \varphi(f^*(t))w(t)\,dt < \infty\}$$

where w is a weight function such that $W(t) = \int_0^t w(s)\,ds$ exists, $W(t) \underset{t\to\infty}{\to} \infty$. ($w$ is moreover non-increasing in [Ma 1]). We do not conserve here the notations of [T] or [Ma 1]. We will consider below only the first version of Lorentz-Orlicz spaces, which is (by far) easier to handle with. We denote by $\ell_{W,\varphi}$ the sequential version of $L(W,\varphi)$.

From now on we suppose the following hypothesis (H) on W, φ to be satisfied :

(H) : *W is α-quasiconvex, β-quasiconcave ($0 < \alpha \le \beta < \infty$) and φ has some non-trivial convexity.*

Lemma 17. *The fundamental function of $L(W,\varphi)$ is equivalent to W.*

Proof : Under hypothesis (H), we have $\int_0^1 \varphi(u)\,\frac{du}{u} < \infty$, and we may suppose $\alpha\frac{ds}{s} \le \frac{dW}{W}(s) \le \beta\frac{ds}{s}$. Let $a, b > 0$ be defined by $\int_0^a \varphi(u)\,\frac{du}{u} = \alpha$ and $\int_0^b \varphi(u)\,\frac{du}{u} = \beta$. Then :

$$\Phi\left(\frac{a}{W(t)}.\mathbb{1}_{[0,t]}\right) = \int_0^t \varphi(\frac{a.W(s)}{W(t)})\,\frac{ds}{s} \lesssim \frac{1}{\alpha}\int_0^t \varphi(\frac{a.W(s)}{W(t)})\frac{dW(s)}{W(s)}$$
$$= \frac{1}{\alpha}\int_0^a \varphi(u)\frac{du}{u} = 1$$

hence $\lambda_{W,\varphi}(t) \le \frac{W(t)}{a}$. Similarly $\lambda_{W,\varphi}(t) \ge \frac{W(t)}{b}$. □

Lemma 18 . *Under hypothesis (H), if the <u>sequential</u> Orlicz spaces ℓ_φ and ℓ_ψ are equal (i.e. φ and ψ are equivalent over $[0,1]$), then $L(W,\varphi) = L(W,\psi)$ (with equivalent norms).*

Proof : We can suppose that $\varphi(t) = \psi(t)$ for $0 \le t \le A$, where A is determined below.

If $\|f\|_{W,\varphi} \le 1$, we have : $\int_0^\infty \varphi(f^*(s)W(s))\,\frac{ds}{s} \le 1$, hence in particular :

$$1 \ge \int_{\frac{t}{2}}^t \varphi(f^*(s)W(s))\,\frac{ds}{s} \ge \varphi(f^*(t)W(\frac{t}{2})).\ln 2$$

hence $f^*(t)W(\frac{t}{2}) \le \varphi^{-1}(\frac{1}{\ln 2})$ and $f^*(t)W(t) \le A = \sup_{t>0}\frac{W(t)}{W(\frac{t}{2})}\varphi^{-1}(\frac{1}{\ln 2})$. Then we have :

$$\|f\|_{W,\varphi} = 1 \Longrightarrow \Psi(f) = \Phi(f) = 1 \Longrightarrow \|f\|_{W,\psi} = 1. \ \square$$

The same reasoning proves that if $\ell_\varphi \subset \ell_\psi$ then $L(W,\varphi) \subset L(W,\psi)$.

Lemma 19. *Under condition (H), ℓ_φ embeds in both $\ell_{W,\varphi}$ and $L(W,\varphi)$ as sublattice.*

Proof : This is the analog of a classical result for the case $\varphi(t) = t^p$.

a) We consider first the sequential version $\ell_{W,\varphi}$.

For any N there exists K_N such that $\sup_{t>0} \frac{W(t)}{W(K_N t)} \leq 2^{-N}$ (as W has some convexity).Set $A_N = \prod_{j\leq N} K_j$ and $x_N = \frac{1}{W(A_N)} \mathbb{1}_{[A_{N-1},A_N]}$. We have :

$$\left(\sum_{N=1}^{m} a_N x_N\right)^* = \sum_{N=1}^{m} a_N x_N \tag{$*$}$$

as soon as $\forall N,\ \frac{a_{N+1}}{a_N} \leq 2^N$.

In case $(*)$ is verified, then for each $\rho > 0$:

$$\Phi(\rho \sum_N a_N x_N) = \sum_N \int_{A_{N-1}}^{A_N} \varphi(\rho \frac{W(t)}{W(A_N)} a_N) \frac{dt}{t}$$

which, as φ and W have some convexity, can be majorized by :

$$C \sum_N \int_{A_{N-1}}^{A_N} \frac{d}{dt} \varphi(\rho \frac{W(t)}{W(A_N)} a_N)\, dt = C \sum_N \left[\varphi(\rho a_N) - \varphi(\rho \frac{W(A_{N-1})}{W(A_N)} a_N)\right]$$
$$\leq C \sum_N \varphi(\rho a_N) \leq \sum_N \varphi(C_1.\rho.a_N)$$

and conversely :

$$\Phi(\rho \sum_N a_N x_N) \geq \sum_N \int_{\frac{A_N}{e}}^{A_N} \varphi(\rho \frac{W(t)}{W(A_N)} a_N) \frac{dt}{t} \geq \sum_N \varphi(\rho \frac{W(A_N/e)}{W(A_N)}.a_N)$$
$$\geq \sum_N \varphi(C_2.\rho.a_N).$$

If $(*)$ is not verified for certain $N \in \{1,...,m\}$, we modify the a_N's by setting $a'_N = \sup_{k\geq 0}(2^{-kN} a_{N+k})$ which is easily seen not to modify substantially the norm of a normalized vector of the space spanned by the x_N's either in ℓ_φ or in $\ell_{W,\varphi}$.

b) case of $L(W,\varphi)$

Here we set $f_N = \frac{1}{W(1/A_N)} \mathbb{1}_{[1/A_{N+1},1/A_N]}$ and we repeat the preceding argument. □

Corollary 20. *Under condition (H), if $L(W,\varphi)$ is convex (resp p-convex, resp q-concave) so is ℓ_φ.*

We denote by $L_{1,\varphi}$ the space $L(W,\varphi)$ with $W(t) = t$.

Lemma 21. *If φ is (equivalent to) a convex function on $[0,1]$ and $\lim_{t\to 0}\frac{\varphi(t)}{t}=0$, then $L_{1,\varphi}$ is not convexifiable.*

Proof : The counterexample is the same as in the case of $L_{1,p}$ or $\ell_{1,p}$.

We consider f_N defined by : $f_N(t)=(\frac{1}{t}\wedge N).\mathbb{1}_{[0,1]}$ in the case of $L_{1,\varphi}([0,1])$, and $x^{(N)}$ defined by : $x_k^{(N)}=\frac{1}{k}$ if $k\leq N$, $=0$ if $k>N$ in the case of $\ell_{1,\varphi}$.

If $L_{1,\varphi}$ is convexifiable, we have :

$$\|f\|_{1,\varphi}\geq c\|f_N\|_1$$

for some constant $c>0$.But :

$$\|f_N\|_1=\int_0^1(\frac{1}{t}\wedge N)\,dt=1+\ln N$$

while :

$$\Phi\left(\frac{f_N}{c\ln N}\right)=\int_0^1\varphi\left(\frac{1\wedge tN}{c\ln N}\right)\frac{dt}{t}=\int_0^{1/N}\varphi\left(\frac{tN}{c\ln N}\right)\frac{dt}{t}+\int_{1/N}^1\varphi\left(\frac{1}{c\ln N}\right)\frac{dt}{t}$$

where the first term is majorized (by convexity of φ) by :

$$\frac{1}{c\ln N}\int_0^{1/N}\varphi(tN)\frac{dt}{t}=\frac{1}{c\ln N}\int_0^1\varphi(u)\frac{du}{u}\underset{N\to\infty}{\to}0$$

and the second by :

$$(\ln N)\varphi(\frac{1}{c\ln N})=c.\frac{\varphi(1/c\ln N)}{1/c\ln N}$$

which tends to 0 as N goes to infinity by hypothesis. □

Proposition 22. *Let φ be a convex Orlicz function such that $\frac{\varphi(t)}{t}\underset{t\to 0}{\to}0$, and assume condition (H). The following assertions are equivalent :*
i) $L(W,\varphi)$ is convexifiable ;
ii) For any $f\in L(W,\varphi)$, $\|f^{**}\|_{W,\varphi}\sim\|f\|_{W,\varphi}$;
iii) W is $(1-\varepsilon)$-quasiconcave, for some $\varepsilon>0$.

Proof : (i) $\Longrightarrow$ (iii) : As for proposition 1 we prove that if it is not the case then $L_{1,\varphi}$ is "crudely" finitely representable in $L(W,\varphi)$ (i.e. finite dimensional subspaces of the first space are uniformly isomorphic to subspaces of the second one), which is impossible by lemma 21. Let us show it in the case of $\ell_{W,\varphi}$. Note that by lemma 17, W is quasiconcave. If W is quasiconcave, but not $(1-\varepsilon)$-quasiconcave for any $\varepsilon>0$, we can find a sequence $a_j\to\infty$ in $\mathbb{N}$ such that $\forall u\in\mathbb{R}_+$ $\frac{W_{[ua_j]}}{W_{a_j}}\underset{j\to\infty}{\to}u$. For any $x\in\mathbb{R}^{(\mathbb{N})}$, consider the vector $x^{(j)}=\frac{1}{W_{a_j}}D_{a_j}x$. We have :

$$\Phi(x^{(j)})=\sum_{k\geq 1}\sum_{\ell=(k-1)a_j+1}^{ka_j}\frac{1}{\ell}\varphi\left(\frac{W_\ell}{W_{a_j}}x_k^*\right)$$

which, as φ and W have some convexity, can be majorized as follows :

$$\begin{aligned}
&\leq C\sum_k \sum_{\ell=(k-1)a_j+1}^{ka_j} \left[\varphi\left(\frac{W_\ell}{W_{a_j}}x_k^*\right) - \varphi\left(\frac{W_{\ell-1}}{W_{a_j}}x_k^*\right)\right] \\
&= C\sum_k \left[\varphi\left(\frac{W_{ka_j}}{W_{a_j}}x_k^*\right) - \varphi\left(\frac{W_{(k-1)a_j}}{W_{a_j}}x_k^*\right)\right] \\
&\leq C\sum_k \frac{W_{ka_j} - W_{(k-1)a_j}}{W_{ka_j}}\varphi\left(\frac{W_{ka_j}}{W_{a_j}}x_k^*\right) \underset{j\to\infty}{\to} C\sum_k \frac{1}{k}\varphi(kx_k^*).
\end{aligned}$$

Hence

$$\limsup_{j\to\infty}\ \Phi(x^{(j)}) \leq \sum_k \frac{1}{k}\varphi(Ckx_k^*).$$

Conversely :

$$\begin{aligned}
\Phi(x^{(j)}) &\geq \sum_{k\geq 1}\sum_{\ell=(k-1)a_j+1}^{ka_j} \frac{1}{\ell}\varphi\left(\frac{W_\ell}{W_{a_j}}x_k^*\right) \\
&\geq \varphi\left(\frac{W_{a_j/e}}{W_{a_j}}x_1^*\right)\sum_{\ell=a_j/e}^{a_j}\frac{1}{\ell} + \sum_{k\geq 2}\varphi\left(\frac{W_{(k-1)a_j}}{W_{a_j}}x_k^*\right)\cdot \sum_{\ell=(k-1)a_j+1)}^{ka_j}\frac{1}{\ell} \\
&\underset{j\to\infty}{\to} \varphi(\frac{1}{e}x_1^*) + \sum_{k\geq 2}\varphi((k-1)x_k^*)\ln\frac{k}{k-1}
\end{aligned}$$

Hence :

$$\liminf_{j\to\infty} \Phi(x^{(j)}) \geq \sum_{k\geq 1}\frac{1}{k}\varphi(\frac{1}{e}kx_k^*).$$

(iii) $\Longrightarrow$ (ii) : Because the lower Boyd index of $L(W,\varphi)$ is strictly larger than 1. $\square$

Proposition 23. *(*) Suppose that ψ is a p-quasiconvex function, with $p > 1$. Then $(L_1, L_\psi)_{\theta,\varphi} = L(W,\varphi)$, with $W(t) = t^{1-\theta}\lambda_\psi(t)^\theta$.*

Consequently every space $L(W,\varphi)$, where W is α-quasiconvex, β-quasiconcave, $0 < \alpha \leq \beta < 1$, is representable as a space $(L_1, L_\psi)_{\theta,\varphi}$.

Proof : As in section 3, we decompose the K-functional : $K(t,f) = A(t,f) + B(t,f)$ hence $\|t^{-\theta}K(t,f)\|_\varphi \sim \|t^{-\theta}A(t,f)\|_1 + \|t^{-\theta}B(t,f)\|_\varphi$ and we show as for the usual (θ,p)-interpolation that the second term is controlled by the first one and that $\|t^{-\theta}A(t,f)\|_\varphi \sim \|t^{-\theta}C(t,f)\|_\varphi$ with : $C(t,f) = \tilde{\psi}_*(t)f^*(\tilde{\psi}_*(t))$. Hence the interpolation norm is determined by the modular :

$$\bar{\Phi}(f) = \int_0^\infty \varphi(t^{-\theta}\tilde{\psi}_*(t)f^*(\tilde{\psi}(t))\,\frac{dt}{t}$$

(*) I was advised by L. Maligranda that a similar result is stated in the paper [EFM] (in proof at the time of writing this footnote) but as $\tilde{\psi}_*$ is convex and has some concavity, $\frac{dt}{t} \sim \frac{d\tilde{\psi}_*}{\tilde{\psi}_*}$, hence :

$$\bar{\Phi}(f) \sim \int_0^\infty \varphi(t^{-\theta}\tilde{\psi}_*(t)f^*(\tilde{\psi}_*(t))\,\frac{d\tilde{\psi}_*}{\tilde{\psi}_*}$$
$$= \int_0^\infty \varphi(\lambda_{\psi_*}(u)^{-\theta}uf^*(u))\,\frac{du}{u} \sim \int_0^\infty \varphi(W(u)f^*(u))\,\frac{du}{u}$$

□

Proposition 24. *Suppose that ψ is a p-convex, q-concave Orlicz function, with $0 < p \leq q < \infty$. Then $(L_\psi, L_\infty)_{\theta,\varphi} = L(W,\varphi)$, with $W(t) \sim \lambda_\psi(t)^{1-\theta}$.*
Consequently under the same assumptions on W as in the preceding proposition 21, $L(W,\varphi)$ is representable as an interpolation space $(L_\psi, L_\infty)_{\theta,\varphi}$.

Proof : We follow the same lines as in the preceding proposition, but the role of $\tilde{\psi}_*$ is now played by $\tilde{\psi}$. □

Remark 24a The proof of prop.24 suggests that the "good" definition of the modular of $L(W,\varphi)$ for a W without non-trivial convexity would be :

$$\Phi(f) = \int_0^\infty \varphi(W(s)f^*(s))\,\frac{dW(s)}{W(s)}$$

5. Multiplicators and Interpolation of Lorentz-Sharpley Spaces

Recall that if F, G are two quasi-normed Köthe function spaces, the space of multiplicators $\mathrm{Mult}(G,F)$ is the space of measurable functions h such that the associated multiplication operator $M_h : g \mapsto h.g$ is bounded from G to F (a systematic study of these spaces is done in [MaP]).

Proposition 25. *Consider three Lorentz-Sharpley spaces $\Lambda(w,r), \Lambda(w_0,p_0), \Lambda(w_1,p_1)$ with $0 < r, p_0, p_1 < \infty$, $\frac{1}{r} = \frac{1}{p_0} + \frac{1}{p_1}$, and $w^{1/r} = w_0^{1/p_0}.w_1^{1/p_1}$. If moreover $\Lambda(w_0,p_0)$ and $\Lambda(w,r)$ have non-trivial concavity then the space $\mathrm{Mult}(\Lambda(w_0,p_0),\Lambda(w,r))$ of multiplicators from $\Lambda(w_0,p_0)$ into $\Lambda(w,r)$ equals $\Lambda(w_1,p_1)$.*

Proof : a) We have $\|g.h\|_{w,r} \leq C\,\|g\|_{w_0,p_0} \cdot \|h\|_{w_1,p_1}$ as a simple Hölder argument shows :

$$\int_0^\infty (g.h)^{*r}(s)w(s)\,ds \leq \int_0^\infty g^*(\frac{s}{2})^r h^*(\frac{s}{2})^r w(s)\,ds$$
$$\leq C\int_0^\infty g^*(s)^r h^*(s)^r w(s)\,ds \qquad (\text{if } W(2s) \leq CW(s))$$
$$= C\int_0^\infty g^*(s)^r h^*(s)^r w_0^{r/p_0} w_1(s)^{r/p_1}\,ds$$
$$\leq C\left[\int_0^\infty g^*(s)^{p_0} w_0(s)\,ds\right]^{r/p_0} \cdot \left[\int_0^\infty h^*(s)^{p_1} w_1(s)\,ds\right]^{r/p_1}.$$

b) Conversely let $h \in \Lambda(w_1, p_1)$ and set :

$$g(s) = h^*(s)^{\frac{r}{p_0 - r}} \left(\frac{w(s)}{w_0(s)} \right)^{\frac{1}{p_0 - r}}$$

(with the usual convention 0/0=0). We have :

$$\int_0^\infty (g(s)h^*(s))^r \, ds = \int_0^\infty h^*(s)^{\frac{p_0 r}{p_0 - r}} \left(\frac{w(s)}{w_0(s)} \right)^{\frac{r}{p_0 - r}} w(s) \, ds$$
$$= \int_0^\infty h^*(s)^{p_1} w_1(s) \, ds$$

as $p_1 = \frac{p_0 r}{p_0 - r}$; similarly :

$$\int_0^\infty g(s)^{p_0} w_0(s) \, ds = \int_0^\infty h^*(s)^{\frac{p_0 r}{p_0 - r}} \left(\frac{w(s)}{w_1(s)} \right)^{\frac{p_0}{p_0 - r}} w_0(s) \, ds$$
$$= \int_0^\infty h^*(s)^{p_1} w_1(s) \, ds.$$

Hence :

$$\left(\int_0^\infty (gh^*)^r(s) w(s) \, ds \right)^{1/r} = \left(\int_0^\infty g^{p_0}(s) w_0(s) \, ds \right)^{1/p_0} \cdot \left(\int_0^\infty h^*(s)^{p_1} w_1(s) \, ds \right)^{1/p_1}$$

Now we regularize g to a non-increasing $\bar{g}$ (following a procedure of [S]), setting :

$$\bar{g}(s) = \left[\int_s^\infty \frac{g(t)^r}{t} \, dt \right]^{1/r} =: u(s)^{1/r}.$$

We obtain :

$$\begin{aligned}
\int_0^\infty \bar{g}(s)^{p_0} w_0(s) \, ds &= \int_0^\infty \bar{g}(s)^{p_0} \, dW_0(s) = -\int_0^\infty W_0(s) \, d\bar{g}(s)^{p_0} \\
&= -\int_0^\infty W_0(s) \, du(s)^{p_0/r} = -\frac{p_0}{r} \int_0^\infty W_0(s) u(s)^{\frac{p_0}{r} - 1} \, du(s) \\
&= \frac{p_0}{r} \int_0^\infty W_0(s) u(s)^{\frac{p_0}{r} - 1} \frac{g(s)^r}{s} \, ds \sim \int_0^\infty \bar{g}(s)^{p_0 - r} g(s)^r w_0(s) \, ds \\
&\le \left[\int_0^\infty \bar{g}(s)^{p_0} w_0(s) \, ds \right]^{\frac{p_0 - r}{p_0}} \left[\int_0^\infty g(s)^{p_0} w_1(s) \, ds \right]^{\frac{r}{p_0}}
\end{aligned}$$

hence : $\|\bar{g}\|_{w_0, p_0} \le C . \|g\|_{w_0, p_0}$. On the other hand :

$$\int_0^\infty (\bar{g}(s) h^*(s))^r w(s) \, ds = \int_0^\infty \left(\int_s^\infty g(t)^r \frac{dt}{t} \right) h^*(s) w(s) \, ds$$

$$\geq \int_0^\infty \left(\int_s^\infty g(t)^r h^*(t)^r \frac{dt}{t} \right) dW(s) = \int_0^\infty W(s) g(s)^r h^*(s)^r \frac{ds}{s}$$

$$\sim \int_0^\infty g(s)^r h^*(s)^r w(s)\, ds$$

which shows that :

$$\|\bar{g}h^*\|_{w,r} \geq C. \left(\int_0^\infty g(s)^r h^*(s)^r w(s)\, ds \right)^{1/r},$$

hence : $\|\bar{g}h^*\|_{w,r} \geq C\, \|\bar{g}\|_{w_0,p_0} \cdot \|h^*\|_{w_1,p_1}$. □

Now recall that the equality $F = G.H$ between three quasi-Banach Köthe function spaces F, G, H means that :

$$F = \{g.h \;/\; g \in G,\ h \in H\}$$

and F is equipped with the quasi-norm :

$$\|f\|_F = \inf\{\|g\|_G \cdot \|h\|_H \;/\; f = g.h,\ g \in G,\ h \in H\}.$$

Note that if F, G, H are rearrangement invariant, and verify the preceding relation $F = G.H$, then their fundamental functions verify $\lambda_F = \lambda_G.\lambda_H$.

Proposition 26. *Suppose that the spaces $\Lambda(w_0, p_0)$ and $\Lambda(w_1, p_1)$ have non-trivial concavity. Then $\Lambda(w_0, p_0).\Lambda(w_1, p_1) = \Lambda(w, r)$, where $\frac{1}{r} = \frac{1}{p_0} + \frac{1}{p_1}$ and $W^{1/r} = W_0^{1/p_0} W_1^{1/p_1}$.*

Remark 27 The relation between the functions W, W_0, W_1 is nothing but the afore mentioned multiplication relation between fundamental functions of the three r.i. spaces, and is thus a necessary condition. It identifies to the condition on weights w, w_0, w_1 given in prop.25 provided that the three spaces have non-trivial concavity, but if $W_1(s) \not\sim s w_1(s)$, one obtains under the hypotheses of prop.25 that : $\Lambda(w, r) \neq \Lambda(w_0, p_0).\Lambda(w_1, p_1)$.

Proof of prop.26 : We obtain prop.26 from prop.25 using the following duality argument whose proof is straightforward :

Lemma 28. *Let F, G two Köthe Banach function spaces and E=F.G . Denote by E', F', G' their Nakano conjugates (spaces of order-continuous linear forms). Then $E' = \mathrm{Mult}(F, G') = \mathrm{Mult}(G, F')$.*

We may, by standard convexification procedure, suppose that all the Lorentz spaces here have Lorentz spaces as duals. Then by prop.25 :

$$\Lambda(w_*, r_*) = \mathrm{Mult}(\Lambda(w_0, p_0).\Lambda(w_{1*}, p_{1*})),$$

where $\frac{1}{r} + \frac{1}{r_*} = 1 = \frac{1}{p_1} + \frac{1}{p_{1*}}$, $w_* = w^{-r_*/r}$, $w_{1*} = w_1^{-p_{1*}/p_1}$. Then dualize by Lemma 28. □

Corollary 29. *If $\Lambda(w_0,p_0)$ and $\Lambda(w_1,p_1)$ have some non-trivial concavity, then :*

$$\Lambda(w,p) = \Lambda(w_0,p_0)^{1-\theta}\Lambda(w_1,p_1)^{\theta}$$

(with equivalent norm), where $w = w_0^{1-\theta}w_1^{\theta}$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$

Proof : This is obtained from prop 27 by a standard convexification argument. □

Remark Corollary 29 could also be deduced from [Ca],sec.13.5 (where the hypothese are slightly too restrictive). Note also that as the Lorentz spaces are order-continuous the preceding result is in the convex case equivalent to a result on complex interpolation (by equivalence of complex interpolation and Calderon's lattice interpolation methods, see [Ca]). This last one was also obtained (and generalized) by E.Hernandez and J.Soria (this same conference) by a different method (using commutation of complex interpolation and real interpolation with function parameter).

A remark on equality between Lorentz-Sharpley spaces and Orlicz spaces.

G.G.Lorentz [L2] gave a necessary and sufficient condition for the space $\Lambda(w,p)$ to be equal to an Orlicz space, when w is a decreasing weight. Supposing that w has an inverse function w^{-1}, this condition may be written :

$$\exists\gamma > 0 \quad s.t. \quad \int_0^\infty \frac{1}{w^{-1}(\gamma w(t))}\,dt < \infty \tag{4.1}$$

Now using the duality between $\Lambda(w,2)$ and $\Lambda(\frac{1}{w},2)$, we see that for increasing weights the same condition holds. Note also that if w verifies (4.1) then so does any power $w^\alpha \quad (\alpha \in \mathbb{R},\ \alpha > 0)$.

It is natural to ask for a characterization of Lorentz-Sharpley spaces that are equal to Orlicz spaces in the case of non-monotone weights.

The preceding results give us a simple sufficient criterion. (See also [MS]).

Corollary 30. *The following conditions are equivalent and are sufficient for $\Lambda(w,p)$, $0 < p < \infty$) to be equal to an Orlicz space :*
i) $\frac{W(t)}{t} \sim w_1(t).w_2(t)$ *where w_1 is non-increasing, w_2 is non-decreasing, and each one veri fies Lorentz condition (4.1).*
ii) $\Lambda(w,p) = \Lambda(w_1,p_1)^{\theta}.\Lambda(w_2,p_2)^{1-\theta}$, *where $0 < \theta < 1$ and $\Lambda(w_1,p_1)$, $\Lambda(w_2,p_2)$ are both equal to some Orlicz space.*

The proof of this corollary is straightforward using the preceding results.

We formulate now the following :

Question. Are the sufficient conditions of corollary 30 also necessary ? (*)

(*) A positive answer to this question was given after the first draft of this paper by S. J. Montgomery-Smith [MS]

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