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MATHEMATICS
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Representation Theorem for Semi-Boolean Algebras. II

by

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Summary. The present paper is a continuation of [2]. In this paper a new kind of semi-Boolean algebras — which are here called semi-field of sets — are introduced and investigated. The next theorem is proved: *Every semi-Boolean algebra is isomorphic to a semi-field of sets.* Following semi-Boolean algebras with infinite joins and meets are considered. For these algebras the representation theorem analogue to the Rasiowa—Sikorski lemma is formulated and proved.

The present paper is a continuation of [2] the knowledge of which is here assumed. The terminology and notation in this paper are the same as in [2].

Let $(\mathcal{B}(X), \cup, \cap, -, I, C)$ be a bi-topological field of sets i.e. $\mathcal{B}(X)$ is a field of subsets of a bi-topological space $\langle X, I, C\rangle$ such that for every $Y \in \mathcal{B}(X)$ the following condition is fulfilled:

$$(**)\qquad \begin{aligned} IY &= CIY \\ CI &= ICY. \end{aligned}$$

The algebra $\mathfrak{B} = (G_I(\mathcal{B}(X)), \cup, \cap, \Rightarrow, \dot{-})$ and every its subalgebra will be called a semi-field of sets (more exactly: a semi-field of subsets of $\langle X, I, C\rangle$). On account of [2] $G_I(\mathcal{B}(X))$ denotes the class of I-open sets in $\mathcal{B}(X)$ the operations $\Rightarrow$, $\dot{-}$ are defined as follows: for every $Y, Z \in G_I(\mathcal{B}(X))$

$$(1)\qquad Y \Rightarrow Z = I((X-Y) \cup Z)$$

$$(2)\qquad Y \dot{-} Z = C(Y \cap (X-Z)).$$

The following Theorem explains the connection between semi-Boolean algebras and semi-field of sets.

THEOREM 1. *For every semi-Boolean algebra $\mathfrak{A}$ there exists a semi-field of sets $\mathfrak{B}$ and an isomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$.*

Let $\mathfrak{A} = (A, \cup, \cap, \Rightarrow, \dot{-})$ be an arbitrary semi-Boolean algebra. Let us denote by X the set of all prime filters ∇ of the lattice $(A, \cup, \cap)$ and for every $a \in A$ let $h(a)$ denote the set of all $\nabla \in X$ such that $a \in \nabla$. Let $\mathcal{R}$ be the class of all $h(a)$, $a \in A$ i.e. $\mathcal{R} = \{h(a)\}_{a \in A}$.

Let us define an interior operation I and a closure operation C in X in the following way: for every $Y \subset X$

$$IY = \bigcup_{\substack{h(a) \in \mathcal{R} \\ h(a) \subset Y}} h(a)$$

$$CY = \bigcap_{\substack{h(b) \in \mathcal{R} \\ Y \subset h(b)}} h(b).$$

Let $\mathcal{B}(X)$ be the field of subsets of X generated by $\mathcal{R}$ such that the following condition is satisfied

$$\text{if } Y \in \mathcal{B}(X) \text{ then } IY \in R \text{ and } CY \in R.$$

It is easy to see that the operations I and C are conjugate over $\mathcal{B}(X)$ i.e. for every $Y \in \mathcal{B}(X)$ the condition (**) is satisfied. Thus the algebra $(\mathcal{B}(X), \cup, \cap, -, I, C)$ is a bi-topological field of subsets of the bi-topological space $\langle X, I, C\rangle$. The class of all I-open elements in $\mathcal{B}(X)$ coincides with $\mathcal{R}$, i.e. $G_I(\mathcal{B}(X)) = \mathcal{R}$. In consequence, the algebra $\mathfrak{B} = (\mathcal{R}, \cup, \cap, \Rightarrow, \dot{-})$ where the operations $\cup, \cap$ are the set-theoretical union and intersection respectively, and $\Rightarrow$, $\dot{-}$ are defined by (1) and (2) is a semi-field of sets. It will be proved that the mapping h is the required isomorphism of the semi-Boolean algebra $\mathfrak{A}$ onto the semi-field of sets $\mathfrak{B}$. It is known that the mapping h is one-to-one and

(2) $$h(a \cup b) = h(a) \cup h(b)$$

(3) $$h(a \cap b) = h(a) \cap h(b).$$

It is sufficient to prove that the following conditions are satisfied

(4) $$h(a \Rightarrow b) = h(a) \Rightarrow h(b)$$

(5) $$h(a \dot{-} b) = h(a) \dot{-} h(b)$$

for $a, b \in A$.

On account of (1) and (2) we have to show that

(6) $$h(a \Rightarrow b) = I\big((X - h(a)) \cup h(b)\big)$$

(7) $$h(a \dot{-} b) = C\big(h(a) \cap (X - h(b))\big).$$

The condition (6) follows from [3].

Clearly the condition (7) is equivalent to the following two conditions:

(8) $$h(a) \cap (X - h(b)) \subset h(a \dot{-} b)$$

(9) $$\text{if } h(a) \cap (X - h(b)) \subset h(c) \text{ then } h(a \dot{-} b) \subset h(c).$$

It is easy to verify that for $a, b \in A$, $a \cup b = b \cup (a \dot{-} b)$. Hence $h(a) \cup h(b) = h(b) \cup h(a \dot{-} b)$. This implies that $h(a) \cap X - h(a) \subset h(a \dot{-} b)$ which proves (8). Let us suppose that for some $c \in A$, $h(a) \cap X - h(b) \subset h(c)$. Thus $h(a) \subset h(c) \cup h(b) = h(b \cup c)$. It is easy to show that

$$h(x) \subset h(y) \text{ if and only if } h(x \dot{-} y) = \emptyset$$

and

$$x \dot{-} (y \cup z) = (x \dot{-} y) \dot{-} z$$

for any $x, y, z \in A$.

Hence we obtain $h(a \dot{-} b) \subset h(c)$, and the condition (9) is fulfilled. We infer from (2), (3), (4) and (5) that h is the required isomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$, which completes the proof of Theorem 1.

In the sequel, we will consider the semi-Boolean algebras with infinite joins and meets. Our aim is to give a representation theorem for these algebras.

THEOREM 2. *Let $\mathfrak{B} = (B, \cup, \cap, \rightarrow, -, I, C)$ be a bi-topological Boolean algebra. Denote by $\mathfrak{A} = (A, \cup, \cap, \Rightarrow, \dot{-})$ the semi-Boolean algebra such that $A = G_I(B)$. For every $t \in T$ let $a_t \in A$. Then the join $\bigcup_{t \in T}^{\mathfrak{A}} a_t$ exists if and only if the join $\bigcup_{t \in T}^{\mathfrak{B}} a_t$ exists and*

$$\bigcup_{t \in T}^{\mathfrak{A}} a_t = \bigcup_{t \in T}^{\mathfrak{B}} a_t .$$

Similary the meet $\bigcap_{t \in T}^{\mathfrak{A}} a_t$ exists if and only if the meet $\bigcap_{t \in T}^{\mathfrak{B}} a_t$ exists and

$$\bigcap_{t \in T}^{\mathfrak{A}} a_t = \bigcap_{t \in T}^{\mathfrak{B}} a_t .$$

The proof of this theorem is omitted.

Let $\mathfrak{A} = (A, \cup, \cap, \Rightarrow, \dot{-})$ be a semi-Boolean algebra and let Q be a set of infinite joins and meets in $\mathfrak{A}$ i.e. elements of the form

$$a_s = \bigcup_{t \in T_s}^{\mathfrak{A}} a_{s,t} \quad (s \in S)$$

$$b_s = \bigcap_{t \in T'_s}^{\mathfrak{A}} b_{s,t} \quad (s \in S').$$

We will say that an isomorphism h from a semi-Boolean algebra $\mathfrak{A}$ into a semi-Boolean algebra $\mathfrak{B}' = (B, \cup, \cap, \Rightarrow, \dot{-})$ is a Q-isomorphism of $\mathfrak{A}$ into $\mathfrak{B}'$ provided it preserves all the infinite joins and meets in Q i.e. if

$$h(a_s) = \bigcup_{t \in T_s}^{\mathfrak{A}'} h(a_{s,t}) \quad (s \in S) \tag{9}$$

$$h(b_s) = \bigcap_{t \in T'_s}^{\mathfrak{B}'} h(b_{s,t}) \quad (s \in S'). \tag{10}$$

THEOREM 3. *For every semi-Boolean algebra $\mathfrak{A} = (A, \cup, \cap, \Rightarrow, \dot{-})$, if the set Q is at most enumerable,* (i.e. *if the sets S and S' are at most enumerable*) *then there exists a semi-field of sets $\mathfrak{B}$ and a Q-isomorphism h from $\mathfrak{A}$ onto $\mathfrak{B}$ such that the infinite joins and meets on the right-hand sides of the equations* (9) *and* (10) *coincide with the set-theoretical unions and intersections respectively.*

Let $\mathfrak{A} = (A, \cup, \cap, \Rightarrow, \dot{-})$ be a semi-Boolean algebra. On account of theorem 2 [2] we can assume that $A = G_{I^*}(B)$ where B is the set of all elements of a bi-topological Boolean algebra $\mathfrak{B} = (B, \cup, \cap, \Rightarrow, -, I^*, C^*)$. Denote by $\mathfrak{B}_0 = (B, \cup, \cap, \rightarrow, -)$ the Boolean algebra of $\mathfrak{B}$. It is known [1] that there exists a Boolean Q-isomorphism h of the Boolean algebra $\mathfrak{B}_0$ into the field $B(X)$ of all subsets of space X. We define an interior operation I in X and a closure operation C in X as follows:

$$IY = \bigcup_{\substack{h(a) \subset Y \\ a = I^* a}} h(a)$$

$$CY = \bigcup_{\substack{h(b) \supset Y \\ b = C^* b}} h(b).$$

It is easy to verify that these operations are conjugate over $h(B)$. Thus the algebra $(h(B), \cup, \cap, -, I, C)$ is a bi-topological field of subsets of the bi-topological space $\langle X, I, C\rangle$. Let $G_I(h(B))$ denotes the class of all I-open elements of $h(B)$. The algebra $\mathfrak{B}=(G_I(h(B)), \cup, \cap, \Rightarrow, \dot{-})$ where the operations $\Rightarrow, \dot{-}$ are defined by (1) and (2) is a semi-field of sets of the bi-topological space $\langle X, I, C\rangle$. We will prove that the mapping h is the required Q-isomorphism of the semi-Boolean algebra $\mathfrak{A}$ onto the semi-field of sets $\mathfrak{B}$. It is sufficient to prove that the equations corresponding to (9) and (10) — where the sets S and S' are at most enumerable and the infinite joins $\bigcup_{t\in T_s}^{\mathfrak{B}'} h(a_{s,t})$ and meets $\bigcap_{t\in T_s'}^{\mathfrak{B}'} h(b_{s,t})$ on the right-hand sides of equations coincide with the set-theoretical unions and intersections, respectively — are satisfied. Suppose that $a_s=\bigcup_{t\in T_s}^{\mathfrak{A}} a_{s,t}$ for every $t\in T_s$, $a_{s,t}\in A$. By Theorem 2 we can write that $a_s=\bigcup_{t\in T_s}^{\mathfrak{B}} a_{s,t}$. Since h is a Boolean Q-isomorphism of the Boolean algebra $\mathfrak{B}_0=(B, \cup, \cap, \rightarrow, -)$ into the field $B(X)$ of all subsets of X we infer that $h(a_s)=\bigcup_{t\in T_s} h(a_{s,t})$, where $\bigcup_{t\in T_s} h(a_{s,t})$ denotes the set-theoretical union. It remains to show that $\bigcup_{t\in T_s} h(a_{s,t})$ is an I-open element in $h(B)$ i.e. $\bigcup_{t\in T_s} h(a_{s,t})\in G_I(h(B))$. This follows immediately from the fact that for every $s\in S$ and $t\in T_s$ $h(a_{s,t})$ is an I-open set. Thus the condition (9) is satisfied. The condition (10) can be proved in a similar way.

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Ц. Раушэр, **Теорема о представлении для полу-Булевых алгебр. II часть**

Содержание. Представленная работа является продолжением работы [2]. В этой работе рассматривается специальный тип алгебр, так называемых полу-поля множеств. Показано также что каждая полу-Булева алгебра изоморфна с некоторым полу-полем множеств. Кроме того, исследованы также полу-Булевы алгебры с операциями бесконечной суммы и бесконечного частного. Для алгебр такого типа сформулирована и доказана теорема о представлении, которая аналогична лемме Расева—Сикорски.