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A SEMANTICS FOR THE CALCULUS E OF ENTAILMENT

In this paper a semantics for the calculus E of entailment is constructed. This semantics is based on a notion of E-structure which is a generalization of Kripke S4 models. We use this semantics to prove that a hypothesis of D. Prawitz [3] and R.K.Meyer [5] on the equivalence of calculus E and NR-theory of entailment is false.

We begin from the definition of E_{ICD}-structure which gives a semantics for the calculus E_{ICD}- the positive fragment of E. Axioms for E and E_{ICD} can be found for example in [1].

Let S be a set, P be a non-empty subset of S, R be a ternary relation on S. The system $< S,P,R >$ is called an E_{ICD} - structure, if it satisfies the following conditions for all U,V,W,X,Y in S.

E 1. $(\exists t \in P)\, R(t,v,v)$

E 2. $x \in P \wedge R(x,u,v) \Longrightarrow (\exists t \in P)\, R(u,t,v)$

E 3. $R(u,w,x) \wedge R(x,v,y) \Longrightarrow (\exists z)(R(w,z,y) \wedge R(u,v,z))$

E 4. $R(u,v,x) \Longrightarrow (\exists z)(R(u,v,z) \wedge R(z,v,x))$

E 5. $x \in P \wedge R(x,v,u) \wedge R(u,w,y) \Longrightarrow R(v,w,y)$

E 6. $x \in P \wedge R(x,v,u) \wedge R(w,u,y) \Longrightarrow R(w,v,y)$

Let $\underline{S} = < S,P,R >$ be an E_{ICD} - structure. By forcing on $\underline{S}$ we mean a relation $\models$ between elements of S and formulas of E_{ICD} with the following properties:

(i) $u \in P \wedge R(u,v,w) \Longrightarrow (v \models p \Longrightarrow w \models p)$ for every propositional variable p,

(ii) $v \models (\alpha \,\&\, \beta) \Longleftrightarrow v \models \alpha$ and $v \models \beta$,

(iii) $v \models (\alpha \vee \beta) \Longleftrightarrow v \models \alpha$ or $v \models \beta$,

(iv) $v \models (\alpha \rightarrow \beta) \Longleftrightarrow (\forall u,w)\ (R(v,u,w) \wedge u \models \alpha \Longrightarrow w \models \beta)$

(where α and β are arbitrary formulae and v,u,w are elements of S).

We say that a formula α is true in the E_{ICD} - structure $\underline{S} = < S,P,R >$ by forcing $\models$ on $\underline{S}$, if $x \models \alpha$ for every x P. A formula α is valid in $\underline{S}$ if α is true in $\underline{S}$ by every relation of forcing on $\underline{S}$.

THE COMPLETENESS THEOREM FOR E_{ICD}. For any formula α of the calculus E_{ICD}, α is a theorem of E_{ICD} iff α is valid in all E_{ICD} - structures.

Recall that the calculus NR [5] is defined as the result of adding S4-style theory of necessity to the calculus R of relevant implication. Let us introduce an entailment connective on the definition $\alpha \Longrightarrow \beta = N(\alpha \rightarrow \beta)$, where $\rightarrow$ is relevant implication and N is necessity. D.Prawitz and R.K.Meyer conjectured that NR-theory of entailment coincides exactly with that of E.

Let us consider the formula:

$(p \Longrightarrow (q \Longrightarrow r)) \,\&\, (q \Rightarrow p \vee r) \Longrightarrow (q \Longrightarrow r)$

G.E.Minc showed that this formula is a theorem of NR and conjectured that it is not provable in E. We prove this conjecture by using of our semantics. The following system $\underline{S} = < S,P,R >$ is an E_{ICD} - structure, where $S = \{ e,a,b,c \}$, $P = \{e\}$, R contains all the triples $< e,x,x >$, $< x,e,x >$, $< x,x,x >$, as well as $< a,b,c >$, $< b,c,c, >$, $< a,c,c >$, $< a,b,b >$, $< b,b,c >$, $< a,e,b >$, $< a,e,c >$, $< b,e,c >$. Then the formula $(p \rightarrow (q \rightarrow r)) \& (q \rightarrow p \vee r) \rightarrow (q \rightarrow r)$ is not true in $\underline{S}$ by $\models$, where $\models$ is the least relation of forcing on $\underline{S}$ such that $c \models p$, $b \models g$, $b \models r$. Therefore this formula is not a theorem of E_{ICD}. It follows from [2], that this formula is not provable also in E.

Now we extend the semantics for the whole E. The system $\underline{S} = < S,P,R,g >$ is called an E - structure, if $< S,P,R >$ is an E_{ICD} - structure, g is an involution on S (i.e. a function such that $g(g(x)) = x$) and the following conditions are fulfilled:

E 7. $R(u,v,w) \Rightarrow R(u,g(w),g(v))$

E 8. $R(u,g(u),u)$.

The forcing on the E - structure $\underline{S} = < S,P,R,g >$ is to satisfy the requirements (i) - (iv) of the definition of forcing on the E_{ICD} - structure and moreover.

(v) $v \models \alpha \Leftrightarrow g(v) \not\models \alpha$.

The notion of validity is analogous to the corresponding notion for E_{ICD} - structures. We have then

THE COMPLETENESS THEOREM FOR E. For any formula α of the calculus E, α is a theorem of E iff α is valid in all E - structures.

REMARK. R.Routley and R.K.Meyer proved [4], that it is enough to consider E_{ICD} - structures in which the set P contains only one element. It does not hold for E - structures. If $\underline{S}$ = < S,P,R,g > is an E - structure and P contains the only element, then the formula pv((p ⟶ p) ⟶ p) underivable in E is valid in $\underline{S}$.

We note also, that there is a direct connection between Kripke S4 models and E structures. Recall that S4 model is a system < S, ≤ > where ≤ is a transitive, reflexive relation on S. Set P = S, g(x) = x, and R(x,y,z) ≡ x ≤ y = z. Then < S,P,R,g > is an E - structure.

R E F E R E N C E S

[1] L.L.Maksimowa: Intérprêtaciá i téorémy otdéléniá dlá insislênij E i R. Algêbra i logika, 10 No 4 (1971), pp.376-392.

[2] R.K.Meyer: Some problems no longer open for E and related logics. Journal of Symbolic Logic 35, No 2 (1970), pp.353.

[3] D.Prawitz: Natural deduction, a proof-theoretical study. Stockholm 1965.

[4] R.Routley, R.K.Meyer: The semantics of entailment II. Journal of Philosophical Logic, 1, No 1 (1972), pp.53-73.

[5] R.Routley, R.K.Meyer: The semantics of entailment III. Journal of Philosophical Logic, 1, No 2 (1972), pp.192-208.

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