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# ROUGH DRAFT

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## Intrinsic Boundaries in Certain Mathematical Toposes exemplify "Logical" operators not passively preserved by Substitution

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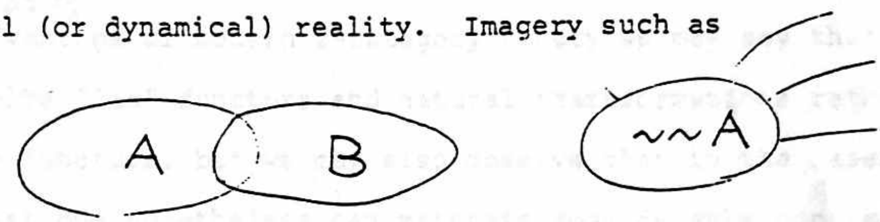
In a co-Heyting algebra where $\sim A$ denotes the smallest element for which $A \vee {\sim} A = 1$, the fact that the boundary

$$\partial A \underset{\text{def}}{=} A \wedge {\sim} A$$

may be non-zero is not a mere logical curiosity, but often a geometrical (or dynamical) reality. Imagery such as




directly suggests such formulas as

$$\partial(A \wedge B) = (\partial A) \wedge B \vee A \wedge (\partial B)$$

$$A = \partial A \vee {\sim\sim} A$$

which in fact can be formally proved to hold in any co-Heyting algebra. Better than mere formal dualization, the ubiquity of naturally-arising mathematical examples which have this structure demands its study. Beyond some degree of completeness, it is crucially certain distributive laws which make it possible. Logic should aim at comprehending not only Heyting algebra, but also this companion refinement of the "classical" first approximation

$$\text{regular open} \xrightarrow[\sim]{} \text{regular closed}$$

and also the larger framework within which the two interact.

Any lex category with exponentiation and a subobject-classifier $\Omega$ *is* a topos; in particular one can algebraically define outright the propositional operations $\Omega \times \Omega \longrightarrow \Omega$ and prove that any $\Omega^X$ is naturally a Heyting algebra. This may be interpreted to mean that the natural logic which is passively conserved by substitution along all $X \longrightarrow Y$ is indeed the Heyting one. Mere tricks with formal duality do not lead to anything of mathemataical substance. However, it has been known for centuries that interesting logical operators such as ~~possibility~~ "necessity" are often not passively conserved by substitution; from the vantage of modern 2-category theory we may say that they involve "lax" functors and natural transformations rather than mere functors, but we may also observe that in the cases of interest one nonetheless can maintain considerable control in that the lax operators are often *adjoint* to other operators which do behave conservatively under substitution. For example, as we'll see below

$$(A \vee B)(f) \underset{X}{\equiv} A(f) \vee B(f)$$

for $X \xrightarrow{f} Y$ with $A,B$ attributes of type $Y$, but merely

$$\sim(Af) \underset{X}{\subsetneq} (\sim A)(f),$$

while at both $X$ and $Y$, $\sim$ is adjoint to $\vee$.

There are at least two ways in toposes where co-Heyting algebras arise. By a theorem of Isbell-Mikkelsen, the lattice of all subtoposes of any topos naturally has such structure; as I pointed out in [1974], there is thus a boundary operator, assigning to every subtopos another subtopos, which needs to be computed in some concrete examples in order to illuminate the model theory of positive theories (for via the "classifying topos" construction, the classes of models of positive theories also have the co-Heyting property) and in order to reciprocally provide actual mathematical tempering of speculations about "paraconsistency". Not very many examples have actually been calculated so far, but see the appendix. The second way, upon which we mostly concentrate below, is that in certain toposes (notably presheaf toposes and somewhat more generally semi-continuous sheaves) the lattice of all subobjects of any object has the requisite stronger distributive property so that it is both Heyting and co-Heyting. A third possibility, not yet investigated, is that a sublattice of "closed" subobjects could be defined in a spirit partly dual to that of Penon's powerful intrinsic notion of "open" subobjects.

If $\mathbb{C}$ is any small category and X any object in the topos $\mathcal{S}^{\mathbb{C}^{op}}$ of "presheaves" (=right actions on abstract sets), then the lattice of all subobjects of X in this topos is isomorphic to the poset of functors to $\mathbf{2}$

$$\mathbf{2}^{(\mathbb{C}/X)^{op}}$$

and hence satisfies both infinite distributive laws since $\mathbf{2}$ does; since it is also complete, it therefore has not only "$\Rightarrow$" right adjoint to $\wedge$ (with the special case $\neg A = (A \Rightarrow 0)$) as in any topos, but also "logical subtraction" left adjoint to $\vee$ (with the special case $\sim A = (1 \setminus A))$. Since $\sim A \vee A = 1$ must hold and unions in presheaves are computed as in $\mathcal{S}$ for each $C \in \mathbb{C}$, while $\sim A$ must also be a subfunctor of $X$, it is easily seen that

<u>Proposition</u> The elements of $\sim A$ of type $C$ are given in terms of the base topos $\mathcal{S}$ by

$$(\sim A)(C) = \left\{ x \in X(C) \;\middle|\; \exists\ C \xrightarrow{u} D \text{ in } \mathbb{C}\ \exists\ \bar{x} \in X(D) \quad \bar{x} \notin A(D) \text{ and } x = \bar{x}u \right\}$$

For example, if $\mathbb{C}$ is the three-element monoid consisting of the identity and two "constants" $u = \partial_0$, $u = \partial_1$, then $X$ is an arbitrary reflexive directed graph and $A$ an arbitrary subgraph; the above implies that $\partial A$ is the discrete subgraph of $A$ consisting of those points $x$ for which there exist "arrows" $\bar{x} \notin A$ connecting $x$ in one or the other of the two possible (in/out) directions with another point of $X$ not in $A$. Similarly, if $\mathbb{C}$ is the monoid of non-negative time durations so that $X$ is a (not-necessarily reversible) dynamical system and $A$ is a property of states which is stable under evolution, then $\partial A \subset A$ consists of those states which can arise from histories which were not always in $A$ in the past.

While for the Heyting negation we always have (for arbitrary subobjects) the passive compatability $\neg B(f) = (\neg B)f$ for any substitution $X \xrightarrow{f} Y \supset B$, for the co-Heyting negation $\sim B$ we only have the lax inequality.

More precisely

Proposition If $\sim B(f) = (\sim B)f$ for all $X \xrightarrow{f} Y \supset B$ in $\mathcal{S}^{\mathbb{C}^{op}}$, then $\mathbb{C}$ is a groupoid. (Hence the logic is in fact Boolean and the two negations agree).

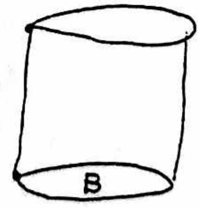
Thus, except in the case where all operators in $\mathbb{C}$ are invertible, the co-Heyting negation is not induced by a single map $\Omega \dashrightarrow \Omega$ on the truth-value object ; rather for each object $X$ there is a map $\sim_X : \mathcal{E}(X,\Omega) \longrightarrow \mathcal{E}(X,\Omega)$ in the base topos $\mathcal{S}$ (where in our examples $\mathcal{E} = \mathcal{S}^{\mathbb{C}^{op}}$) but for a map $X \xrightarrow{f} Y$ the square

$$\begin{array}{ccc} \mathcal{E}(Y,\Omega) & \xrightarrow{\sim_Y} & \mathcal{E}(Y,\Omega) \\ \downarrow{\scriptstyle -f} & & \downarrow{\scriptstyle -f} \\ \mathcal{E}(X,\Omega) & \xrightarrow[\sim_X]{} & \mathcal{E}(X,\Omega) \end{array}$$

does not commute but only lax-commutes in the sense that there is an inequality.

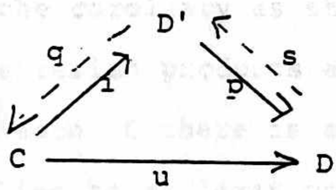
However, in some cases there are special f's, namely the projection maps of cartesian products, for which such diagrams do commute. Though even this is not true for our directed graph example, it does hold in some presheaf categories of the "gros topos" (=category of "all spaces" of a certain kind). The intuition pointing toward these cases is the one fundamental to the computation of the tin required to make a tin can, namely

$$\partial(A \times B) = (\partial A) \times B \cup A \times (\partial B)$$



where in the case pictured the boundary of the altitude is $\partial A = 2$ while the boundary of the base is the closed curve $\partial B$. This Leibniz rule for (boundaries of) cartesian products will follow from the generally-valid (in co-Heyting algebras) Leibniz rule for intersections via such facts as $A \times B = A \times Y \cap X \times B$ (where $A \subset X$, $Y \subset B$) provided only that $\sim(A \times Y) = (\sim A) \times Y$ and $\sim(X \times B) = X \times (\sim B)$ (where the left-hand $\sim = \sim_{X \times Y}$ in both cases), in other words, provided the two special projection f's $X \times Y \longrightarrow X$, $X \times Y \longrightarrow Y$ do passively substitute into $\sim_X, \sim_Y$.

__Theorem__ Suppose $\mathbb{C}$ has the property that every map $C \xrightarrow{u} D$ can be factored into a split mono followed by a split epi




$$u = pi$$
$$1_C = qi$$
$$1_D = ps$$

Then in $\mathcal{S}^{\mathbb{C}^{op}}$, substitution along any projection map commutes with the co-Heyting negation, so in particular the Leibniz rule for the boundary of a cartesian product holds.

Proof: Suppose $\langle x,y \rangle \in \sim(A) \times Y$. We must show $\langle x,y \rangle \in \sim(A \times Y)$, i.e. that there exist $u', \bar{\bar{x}}, \bar{\bar{y}}$ with $\bar{\bar{x}} \notin A$ but $x = \bar{\bar{x}} u'$, $y = \bar{\bar{y}} u'$. We are given $u, \bar{x}$ with $\bar{x} \notin A$ and $\bar{x} u = x$ so consider a factorization of $u$ as stated above and let $u' = i$, $\bar{\bar{x}} = \bar{x} p$, $\bar{\bar{y}} = yq$. Then

$$\langle \bar{\bar{x}}, y\rangle u' = \langle \bar{x}\, p\, u', y\, q\, u'\rangle = \langle \bar{x}\, u, y\rangle = \langle x,y\rangle$$

but we still have $\bar{\bar{x}} \notin A$; for if $\bar{\bar{x}} \in A$ were true then $\bar{x} = \bar{\bar{x}}s$ would also be in $A$ contrary to hypothesis.

Corollary If $\mathbb{C}$ has binary cartesian products and if every set $\mathbb{C}(D,D')$ is non empty, then $\mathcal{S}^{\mathbb{C}^{op}}$ satisfies the Leibniz rule for the boundary of cartesian products of presheaves.

For then we can verify the hypothesis of the theorem by setting $D' = C \times D$ and letting $i$ be the graph of $u$ and letting $s$ be the graph of any convenient $D \dashrightarrow C$ ($p,q$ being the product projections). Since the hypothesis of the theorem is self-dual, another corollary would have binary coproducts and non-empty hom-sets as sufficient condition on $\mathbb{C}$. A special case of the corollary as stated applies to categories $\mathbb{C}$ with finite cartesian products and non-empty objects in the sense that for each $C$ there is a point $1 \dashrightarrow C$. But the theorem also applies to at least one very important example which has neither products nor coproducts, as follows

Corollary In the topos of simplicial sets $\mathcal{S}^{\Delta^{op}}$, the Leibniz rule for the co-Heyting boundary of a cartesian product is valid.

For the category $\Delta$ of finite totally-ordered sets and order-preserving maps can be shown to have the property that every map $C \xrightarrow{u} D$ factors as a split mono followed by a split epi, namely the third object can be taken as the ordered sum

$$D' = \sum_{d \in D}^{\theta} C_d$$

where

$$C_d = \begin{cases} u^{-1}(d) & \text{if the latter is non empty} \\ \mathbb{1} & \text{otherwise} \end{cases}$$

## A P P E N D I X

The lattice of actual subtoposes of a given topos is well-known to be a co-Heyting algebra (dual to the Heyting algebra of Grothendieck topologies); it corresponds to classes of models of positive extensions of the theory for which the given topos is classifying. In the very special case of a presheaf topos $\mathcal{S}^{\mathbb{C}^{op}}$ in which $\mathbb{C}$ is a locally finite category with idempotents split, this lattice can be identified with

$$2^{\mathcal{G}^{op}}$$

where $\mathcal{G}$ is the poset of objects of $\mathbb{C}$ ordered by the existence of a retract, in other words, with the lattice of those classes $\mathbb{A}$ of objects of $\mathbb{C}$ such that $C \underset{i}{\overset{p}{\leftrightarrows}} A$, $pi = 1_C$, $A \in \mathbb{A}$ imply $C \in \mathbb{A}$. Thus in this special case the lattice of subtoposes is not only co-Heyting, but also Heyting. However, we want to note the

<u>Proposition</u> If $\mathbb{C}$ is locally finite and $\mathbb{A} \subset \mathbb{C}$ full, then the boundary of the subtopos $\mathcal{S}^{\mathbb{A}^{op}} \hookrightarrow \mathcal{S}^{\mathbb{C}^{op}}$ is $\mathcal{S}^{(\partial\mathbb{A})^{op}}$ where $\partial\mathbb{A}$ is the set of all objects of $\mathbb{C}$ which are both retracts of objects of $\mathbb{A}$ but also retracts of objects of $\mathbb{C}-\mathbb{A}$.

For example, any positive extension of the theory of distributive lattices (such as the theory of Boolean algebras, the theory of totally ordered sets with endpoints, etc.) can be correlated with a subcategory $\mathbb{A}$ of the category $\mathbb{C}$ of finite posets, and as such has a "boundary" which is another theory.

ADDENDUM

20 March 1990

Three further remarks about the intrinsic boundaries in presheaf toposes

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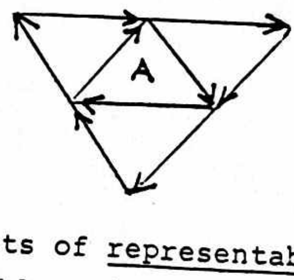
It was pointed out that in the category of reflexive graphs, any object arising as a boundary is discrete (i.e. 0-dimensional, agreeing with the intuition that graphs are one-dimensional and that $\partial$ should decrease dimension by one; perhaps a general result of that kind could be proved, using the definition of "dimension" as the Hegelian negation of an essential subtopos). Similarly, in the petit topos $\mathcal{S}$ for any $A \subset X$ one has $(1, \partial A) = \emptyset$, i.e. boundaries as objects are special objects. By contrast, Schanuel observed that in the petit topos $\mathcal{S}^{\omega^{op}}$, any object $A$ can be embedded in a suitable $X$ so that $A = \partial A$ for the boundary within $X$; namely define

$$X_n \overset{\text{def}}{=} A_n + A_{n-1} \qquad (A_{-1} = \emptyset)$$

with $X_{n+1} \longrightarrow X_n$ taking everything into $A_n$. Then using the explicit formula for $\partial A$ in any $\mathcal{S}^{\mathbb{C}^{op}}$ one sees that $\sim\sim A = \emptyset$ and hence $A = \partial A \cup \sim\sim A = \partial A$. that $\sim A = X$ so

The second remark concerns the representable functors in $\mathcal{S}^{\mathbb{C}^{op}}$. The intuition that the sphere is the boundary of the ball inside a suitable $X$ can indeed be realized, for example in simplicial sets $\mathcal{S}^{\Delta^{op}}$ (which as pointed out in the paper is one of those where the good Leibniz rule holds) by taking $X$ to be




But boundaries of subobjects of representable $X$ (and more generally of quotients of representables, i.e. those objects $X$ having a biggest proper subobject $X_0$ (= all elements of $X$ whose corresponding Yoneda map is not surjective)) are especially transparent (as pointed out by Gustavo Arenas): if $A \neq X$ then $\partial A = A$.

Addendum 2

Perhaps not sufficiently explicit in the paper is the following: The fact that the de dicto/de re distinction is a serious one even for substitution along projections $X^{n+1} \longrightarrow X^n$ underlines the obligatory nature of the formal precision which I have advocated for over 25 years but which most logicians still blithely ignore even when it leads to absurdities such as the alleged "non-transitivity of entailment": I refer to the precision which Lambek calls the "declaration of variables" in order to have meaningful formulas. I do not know whether the sufficient condition given in my theorem (namely that every map in $\mathbb{C}$ factors as a map having a retraction followed by a map having a section) is necessary for conservation of $\sim$ along projections.