THE PROBLEM OF DUALISM IN THE INTUITIONISTIC LOGIC AND BROUWERIAN LATTICES

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At different times a number of authors (G.Moisil, V.Kuznetsov, A.Muchnik, C.Rauszer) had suggested "symmetrical" formulations of the intuitionistic propositional calculus, with every connection $\&, \vee, \Rightarrow, \neg$ having duals $\vee, \&, \Leftarrow, \ulcorner$ and in which a principle analogious to the duality principle of the classical logic was restored. On the other hand, J.C.C. McKinsey and A.Tarski in their basic paper [1] on Brouwerian algebras (i.e. the algebras commonly associated with the intuitionistic calculus) paid a special attention to differences between Boolean and Brouwerian algebras, connected with the violation of the duality law in Brouwerian algebras. The authors write ([1], p.141) "The problem of dualism in Brouwerian algebras is not yet clear". In the same paper the notion of double-Brouwerian algebras is introduced, which was first mentioned by Th.Skolem [2]. Considering this we shall call double-Brouwerian algebras Skolem algebras.

The problem 33 and 76 of Birkhoff [3] should be attributed to the mentioned questions.

Some our results on this direction are given here.

<u>Skolem lattice</u> is a distributive lattice $(T,\vee,\wedge)$ with the smallest element 0 and the largest element 1, such that for any two elements a and b of T.

(1) the pseudo-complement of a relative b $(a \rightarrow b)$, defined to be the largest $c \in T$ such that $a \wedge c \leq b$, exists and dually

(2) the pseudo-difference of a and b $(a \dot{-} b)$, defined to be the smallest $c \in T$ such that $a \leq b \vee c$, exists.

From the point of view of universal algebras, Skolem lattices are regarded as algebras $(T,\vee,\wedge,\rightarrow,\dot{-},0,1)$. Accordingly homomorphisms of Skolem algebras are functions which preserve the operations and $0,1$. Let SK be the category of Skolem algebras and homomorphisms. Let $(\mathfrak{X},\leq)$ and $(\mathfrak{X}',\leq')$ be two ordered sets; a map $f\colon \mathfrak{X} \rightarrow \mathfrak{X}'$ will be called rigid map if

$$x \leq' f(z) \leq' y \Leftrightarrow (\exists z'; z'')(z' \leq z \leq z'' \,\&\, f(z') = x \,\&\, f(z'') = y)$$

We say $(\mathfrak{X},R)$ is an ordered Stone space, if it is o-dimensional Hausdorff Compact space, R is an ordering relation such that for any $A \subseteq \mathfrak{X}$ $\ddot{R}\,\mathrm{cl}A = \mathrm{cl}\,\ddot{R}A$ and $R\,\mathrm{cl}A = \mathrm{cl}\,RA$, where cl is the topological operation of closure.

Using the results from [5] we can prove the following: (1) The category $\mathcal{Sk}$ of Skolem algebras and homomorphisms is (dually) equivalent to the category $\mathcal{N}$ of ordered Stone spaces and rigid continuous maps;

(2) Representation Theorem. Every Skolem lattice $T \in \mathcal{Sk}$ is isomorphic to the lattice of all open-closed cones of the ordered Stone space $T^{*} = (\mathfrak{X}, \leq) \in \mathcal{N}$ (We recall [5] that a set $A \subseteq \mathfrak{X}$ is called cone if $x \in A$ and $x \leq y$ imply $y \in A$).

(3) The lattice of all congruence relations of the Skolem algebra T is isomorphic to the lattice of all closed quasi-components of the ordered Stone space $T^{*} = (\mathfrak{X}, R)$. A set $A \subseteq \mathfrak{X}$ is called a component of $\mathfrak{X}$ if A is maximal (w.r.t. $\subseteq$) R-connected subset of $\mathfrak{X}$; quasi-components are unions of components.

As consistent with Birhoff's Problem 33([3], p.131) we take note of the related result:

The complete distributive lattice T satisfies both lows

(a) $\bigvee_{\alpha} (a \wedge a_{\alpha}) = a \wedge \bigvee_{\alpha} a_{\alpha}$

(b) $\bigwedge_{\alpha} (a \vee a_{\alpha}) = a \vee \bigwedge_{\alpha} a_{\alpha}$

(i.e. is a Skolem lattice) iff T is isomorphic to the lattice of all open-closed cones of an ordered Stone space $(\mathfrak{X}, R)$, where $\mathfrak{X}$ is extremally disconnected.

We denote by $C(T)$ the center of a lattice T, i.e. the set of all complemented elements of T. We have:

(4) The lattice of congruence relations of the Skolem algebra T is isomorphic to the lattices of all filters of the center $C(T)$;

(5) Every Skolem algebras T is semi-simple algebra, i.e. isomorphic a subdirect product of simple Skolem algebras;
(6) Skolem algebra T is simple iff the centre is two-element Boolean lattice.

We'd like to add a few words about the lattice L of all nontrivially equational closses of Skolem algebras. L is a distributive lattice; the smallest element of L is the class of all Boolean algebras. The class of all Lukasiewicz algebras is an atom of L.

Birkhoff's Problem 76 ([3],P.229): "Find necessary and sufficient conditions on a Brouwerian lattice for be isomorphic to the lattice of all closed elements in a suitable closure algebra".

We suggest the following answer:
Perfect Kripke model $(\mathfrak{X},R)$ (see [5]) is called symmetrical if $(\mathfrak{X},\breve{R})$ is perfect Kripke model (where $\breve{R}(x,y) \Leftrightarrow R(y,x)$). A Brouwerian lattice T is isomorphic to the lattice of all closed elements in some closure algebra iff perfect Kripke model $T^{*} = (\mathfrak{X},R)$ is symmetrical.

In conclusion it would be desirable to say a few words about the symmetrical intuitionistic calculus (in short, SI_n). The semantics of the calculus SI_n like that of the intuitionistic calculus can be defined in terms of Kripke-style frames, i.e.

in terms of triples $(M, \leq, f)$, where M is a non-empty set (of times), $\leq$ is a temporal ordering, f is a valuation, i.e. a function assigning the truth value $f(A,x) = A(x) \in \{0,1\}$ to a formula A and $x \in M$.

For the purpose of comparison let's consider only two items of the definition of the function f related to the negations $\neg, \ulcorner$:

(a) The value of a formula $\neg A$ at a time x is true (i.e. $\neg A(x) = 1$) if A is false at all times <u>later</u> than the time x $((\forall y)(x \leq y \Rightarrow A(y) = 0))$

(b) The value of a formula $\ulcorner A$ at a time x is true (i.e. $\ulcorner A(x) = 1$) if A is false at some time, which the time x later than $((\exists y)(y \leq x \;\&\; A(y) = 0)$.

So $\ulcorner A$ is a "precedent" negation as opposed to a "prognostic" negation, $\neg A$.

It is not difficult to see that the calculus SI_n is complete with respect to the semantics of mentioned style.

Finally, it can be shown, mutatis mutandis, that the theorem of translation of the calculus SI into the system of tense-logic K^2C4T (see [4]) with the operators P ("it has been that...") and F ("it will be that...") holds (analogical to the well-known McKinsey-Tarski's theorem of translation of the intuitionistic calculus into the Lewis modal system S4).

REFERENCES

1. McKinsey T.C.C., Tarski A. "On closed elements in closure algebras"
   Ann.Math 1946,47, 122-162.

2. Skolem Th."Untersuchungen über die Axiome des KlassenKalküls und über Productations-und Summationsprobleme, welche gewisse Klassen von Aussagen betreffen"
   Skrifter utigit av Videnkapsselskapet i Kristiania,I.Matematisk-naturvidenskabelig klasse 1919, no.3(1919), 37 pp.

3. Birkhoff G,"Lattice theory" (3rd.ed.), Providence, P.I., 1967

4. Segerberg K,"Modal logics with linear alternative relations"

Theoria , 1970, 36, N3 301-322.

5. Эсвкия Л.Л. О топологических моделях Крипке.
Доклады АН СССР, 2I4, № 2, I974, стр.298-301.