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TOPICS IN NONLINEAR FUNCTIONAL ANALYSIS

by

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College Park, Maryland

THE INSTITUTE FOR FLUID DYNAMICS

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FOREWORD

These notes are essentially the lectures given by the author at the Institute for Fluid Dynamics and Applied Mathematics during the Fall of 1966.

We are most thankful to the faculties of the Institute for Fluid Dynamics and Applied Mathematics and the Mathematics Department of the University of Maryland for continuing interest in my lectures and much encouragement in writing them up. We wish to thank Professor Les Karlovitz for many helpful discussions during the actual writing of these notes. We wish also to thank Professors Felix Browder and Walter Petryshyn for supplying us with preprints and information on unpublished results of their work, which constitutes the bulk of these notes.

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Djairo Guedes deFigueiredo

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CHAPTER I

EXISTENCE OF FIXED POINTS FOR NONEXPANSIVE MAPPINGS IN BANACH SPACES

I. INTRODUCTION

Let E be a Banach space with norm $\|\cdot\|$, C a subset of E. A mapping $T : C \to E$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in E$. Isometries (i.e., $\|Tx - Ty\| = \|x - y\|$ for all $x,y \in C$) and contractions (i.e., $\|Tx - Ty\| \leq k\|x - y\|$ for all $x,y \in C$ and some fixed k, $0 < k < 1$) are examples of nonexpansive mappings.

It is of great importance in the applications (see Browder [5] or section 11 below) to find out if nonexpansive mappings have fixed points, that is, points $x \in C$ such that $Tx = x$. Of course, this is too general a question to have some reasonable answer; see examples below, section 2. In order to obtain existence of fixed points for such mappings some restrictions have to be made on the Banach space E and on the subset C.

At this point we should remark that if T is a contraction in any closed subset C of a (general) Banach space with $T(C) \subseteq C$, then T has a unique fixed point in C. This is a particular case of the well known principle of contraction mappings: "Let (M,d) be a complete metric space and $T : M \to M$ a contraction mapping, i.e., $d(Tx, Ty) \leq kd(x,y)$ for all $x,y \in M$ and some fixed k, $0 < k < 1$. Then T has a unique fixed point. For a proof see, for example, Kolmogorov and Fomin, "Elements of Functional Analysis", vol. I.

Coming back to the general nonexpansive mappings we realize that, in view of the examples in section 2, we should consider subsets C of E that are <u>bounded</u>, <u>closed</u> and <u>convex</u>. Moreover, we consider mappings T from C <u>into</u> C. The more general situation of nonexpansive mappings T from C to E has been considered by Petryshyn [28] for the Hilbert space case; see also Browder and Petryshyn [10]. The Banach space E should also be restricted somehow, as example 4 indicates. Brodsky and Milman [4] have considered isometries T of a bounded closed convex set C of a Banach space E into itself. They were able to prove existence of a fixed point for T provided E is reflexive and C has normal structure (see definition immediately below). Moreover [4] contains a theorem about existence of a common fixed point for all continuous mapping of C into itself which do not decrease distances.

<u>Definitions</u>. Let B be a bounded set in a Banach space E and let $\delta(B)$ be its diameter. A point $x \in B$ is said to be a <u>diametral point</u> of B if $\sup_{y \in B} \|x - y\| = \delta(B)$. A convex set S of E is said to have <u>normal structure</u> if every bounded convex subset S_1 of S, which contains more than one point, has a point that is not a diametral point of S_1.

An argument similar to the one in Brodsky and Milman [4] was used by Kirk [24] to obtain a proof for the following theorem.

<u>Theorem I.1</u> (Kirk). <u>Let E be a reflexive Banach space, and C a bounded closed convex subset of E</u>. <u>Furthermore suppose that C has normal structure</u>. <u>Then a nonexpansive mapping T of C into itself has a fixed point</u>.

Theorem I.1 in the case of a uniformly convex Banach space was also discovered independently by Browder [6] and Göhde [18]. Corollary I.1 below establishes their result as a consequence of Theorem I.1. And in the remark after that corollary we show why Theorem I.1 is more general. We should also remark that the proof in Browder [6] involves an argument similar to the one in Brodsky and Milman. However Göhde's proof works through a specially chosen sequence of fixed points for the contractions $T_r = rT$, $0 < r < 1$.

2. EXAMPLES

We present now three examples to show that the restrictions put on C in theorem I.1 are necessary. The fourth example indicates that one cannot expect existence of fixed points for nonexpansive mappings in the most general class of Banach spaces.

Example 1. (Necessity for boundedness of C). A translation in a Banach space is an isometry and obviously has no fixed points.

Example 2. (Necessity for closedness of C). Let E be a Hilbert space (E = R will suffice). Let C be the interior of the unit ball, i.e., $C = \{x : \|x\| < 1\}$. Consider T the mapping of C into itself defined by

$$Tx = x/2 + a/2$$

where a is any vector in E with unit norm. It is easy to see that T has no fixed point in C.

Example 3. (Necessity for convexity of C). Let E be a Hilbert space ($E = R$ will suffice again). Let C be the set containing just two distinct points a and b. Define $T : C \to C$ as $Ta = b$ and $Tb = a$. Clearly T is an isometry and has no fixed point.

Example 4. (Kirk) Let E be the space $C[0,1]$ of real-valued continuous functions defined in $[0,1]$ with the maximum norm. It is well known that $C[0,1]$ is not a reflexive Banach space. Now consider C as the unit ball about the origin in $C[0,1]$, and let T be the mapping defined as follows:

$$T : C \to E$$

$$f(x) \to xf(x)$$

It is easy to verify that $T(C) \subset C$ and T has no fixed point.

3. PROOF OF THEOREM I.1.

Theorem I.1 will be proved using Theorem I.2 below. The proof presented here is essentially the one given by Kirk [24]. Let C be a bounded closed convex set in a Banach space E, and T a nonexpansive mapping of C into itself. Let us denote by $\mathcal{C}$ the the collection of all closed convex subsets C_1 of C which are invariant under T, i.e., $T(C_1) \subset C_1$.

Theorem I.2. Let E be a reflexive Banach space, C a bounded closed convex subset of E and T a nonexpansive mapping of C into itself.

$\mathcal{C}$ is the collection defined above. Suppose that given $C_1 \in \mathcal{C}$, either C_1 consists precisely of one point or there exists $C_2 \in \mathcal{C}$ such that $C_2 \subset C_1$ and $C_2 \neq C_1$. Then T has a fixed point in C.

Proof of theorem I.2. Let us consider the collection $\mathcal{C}$ preordered by inclusion, and let $C_1 \supset C_2 \supset \ldots$ be a chain in $\mathcal{C}$. We prove that this chain has a lower bound. In fact, let $C_o = \bigcap_{j=1}^{\infty} C_j$. C_o is closed and convex. Since the sets C_j are bounded closed convex subsets of a reflexive Banach space, they are weakly compact. Moreover they have the finite intersection property. Hence C_o is non-empty. It is also immediate that C_o is invariant under T. So $C_o \in \mathcal{C}$ and it is a lower bound for the given chain. Now by Zorn's lemma (see, for example, Dugundji, "Topology"), it follows that $\mathcal{C}$ has a minimal element $\hat{C}$. If $\hat{C}$ has just one point the proof is finished. Otherwise, it follows by assumption that there exists a proper subset of $\hat{C}$ in $\mathcal{C}$. But this contradicts the minimality of $\hat{C}$. So $\hat{C}$ must have only one point, which will be fixed by T. This completes the proof of theorem I.2.

Before proving Theorem I.1 we introduce the notion of kernel of a set.

Definitions. Let B be a bounded set in a Banach space E. Let $r_x(B) = \sup_{y \in B} \|x - y\|$ and $r(B) = \inf_{x \in B} r_x(B)$. The kernel of B is the set $K(B) = \{x \in B : r_x(B) = r(B)\}$. Let us remark that in general $K(B)$ could be either empty, or a proper part of B, or the whole of B. The next two lemmas show that both of the extreme cases above will not occur in important cases.

Lemma I.1. Let E be a reflexive Banach space and C a bounded closed convex subset. Then the kernel $K(C)$ of C is closed convex and non-empty.

Proof. From the definition of kernel we can write immediately

$$K(C) = \bigcap_{n=1}^{\infty} K_n(C) \quad ,$$

$$K_n(C) = \{x \in C : r_x(C) \leq r(C) + \tfrac{1}{n}\} \quad .$$

where each set $K_n(C)$ is non-empty by definition. It is closed and convex because

$$K_n(C) = \bigcap_{y \in C} \{x \in C : \|x - y\| \leq r(C) + \tfrac{1}{n}\} \quad .$$

Using reflexivity, it follows that $K(C)$ is non-empty.

Lemma I.2. Let E be a Banach space. Let B be a bounded set which contains a point which is not diametral of B. Then if the kernel $K(B)$ is non-empty, its diameter is strictly less than the diameter of B. In particular this is true if B has normal structure and E is reflexive.

Proof. Since B has a point x, which is not diametral of B it follows that $r_x(B) < \text{diam}\,(B)$. So

$$\text{diam}[K(B)] = \sup_{y,z \in K(B)} \|y - z\| \leq r(B) \leq r_x(B) < \text{diam}\,(B) \quad ,$$

which proves the lemma.

Proof of theorem I.1. The proof makes use of Theorem I.2. Let $C_1 \in \mathcal{C}$. If C_1 consists of precisely one point, this point will be fixed under T and the theorem will be proved. Let us assume then that C_1 has more than one point and let us prove that C_1 contains a proper subset $C_2 \in \mathcal{C}$. If we do that, Theorem I.1 is proved using Theorem I.2. By Lemmas I.1 and I.2 the kernel $K(C_1)$ of C_1 is a proper closed convex non-empty subset of C_1. If $K(C_1)$ is invariant under T, take $C_2 = K(C_1)$. Otherwise there exists $x \in K(C_1)$ such that $Tx \notin K(C_1)$. On the other hand

$$\|Tx - Ty\| \leq \|x - y\| \leq r(C_1) , \quad \text{for all} \quad y \in C_1 .$$

This shows that $T(C_1) \subset B_{r(C_1)}(Tx)$, i.e. $T(C_1)$ is contained in the ball of radius $r(C_1)$ about Tx. This together with the fact that C_1 is invariant under T implies

$$T(C_1 \cap B_{r(C_1)}(Tx)) \subset C_1 \cap B_{r(C_1)}(Tx) .$$

Since $Tx \notin K(C_1)$ it follows that $C_1 \cap B_{r(C_1)}(Tx) \neq C_1$. So in the case that $K(C_1)$ is not invariant under T we take $C_2 = C_1 \cap B_{r(C_1)}(Tx)$. We have just seen that such a set is in $\mathcal{C}$ and is a proper subset of C_1. The theorem is proved.

4. THEOREM I.1 FOR UNIFORMLY CONVEX SPACES.

A Banach space is said to be uniformly convex (or uniformly rotund) if given $\varepsilon > 0$ there exists $\delta(\varepsilon)$ such that $\|x - y\| \geq \varepsilon$

for $\|x\| \leq 1$ and $\|y\| \leq 1$ implies $\|(x + y)/2\| \leq 1 - \delta(\varepsilon)$. This notion was introduced by Clarkson [13]. In that paper he proved that L^p spaces, $1 < p < \infty$, are uniformly convex. Later Milman [26], Pettis [27] and Kakutani [20] proved that every uniformly convex Banach space is reflexive. Of course the converse is not true, i.e., there are reflexive Banach spaces that are not uniformly convex. Indeed, consider a finite dimensional Banach space E in which the surface of the unit ball has a "flat" part. Such a Banach space E is reflexive because of finite dimensionality. But the flat portion in the surface of the ball destroys uniform convexity.

The following result is useful in the applications.

<u>Proposition I.1</u>. <u>Let E be a uniformly convex Banach space. Suppose that there are given two sequences $\{x_n\}$ and $\{y_n\}$ such that $\|x_n\| \to 1$, $\|y_n\| \leq \|x_n\|$ and $\|(x_n + y_n)/2\| \to 1$ as $n \to \infty$. Then $\|x_n - y_n\| \to 0$ as $n \to \infty$.</u>

<u>Proof</u>. Construct two new sequences

$$z_n = \frac{x_n}{\|x_n\|} \quad \text{and} \quad w_n = \frac{y_n}{\|x_n\|}$$

It is easy to see that $\|z_n\| = 1$, $\|w_n\| \leq 1$ and $\|(z_n + w_n)/2\| \to 1$. So by uniform convexity it follows that $\|z_n - w_n\| \to 0$, which implies readily that $\|x_n - y_n\| \to 0$ as $n \to \infty$.

<u>Remark</u>. In the preceding proposition the condition on $(x_n + y_n)/2$

could be replaced by the analogous one on $\alpha x_n + (1 - \alpha)y_n$ where $0 < \alpha < 1$.

Corollary I.1 (Browder [6], Göhde [18]). *Let E be a uniformly convex Banach space and C a bounded closed convex set in E . Let T be a nonexpansive mapping of C into itself. Then T has a fixed point.*

This corollary is a consequence of Theorem I.1 and the following lemma.

Lemma I.3. *In a uniformly convex Banach space E , every bounded closed convex set has normal structure.*

Proof. All we have to do is to prove that a bounded closed convex set C in E which consists of more than one point has a point which is not diametral. Let $x,y \in C$ such that $\|x - y\| \geq 1/2 \text{ diam } (C)$, and let $u = (x + y)/2$. We claim that u is not a diametral point. Indeed, suppose that there exists a sequence of points $v_n \in C$ such that $\|u - v_n\| \to \text{diam } (C)$. Clearly $\|x - v_n\|$ and $\|y - v_n\|$ are $\leq \text{diam } (C)$. So by uniform convexity it follows that $\|x - y\| = \|(x - v_n) - (y - v_n)\| \to 0$, which contradicts the assumption that $\|x - y\| \geq 1/2 \text{ diam } C > 0$

Remark. A Banach space is said to be *strictly convex* (or *rotund*) if $\|\lambda x + (1 - \lambda)y\| < 1$ for all λ, $0 < \lambda < 1$, and all $x,y \in E$ such that $\|x\| = \|y\| = 1$. It is easy to see that every uniformly convex Banach space is strictly convex. The converse is not true.

Brodsky and Milman [4] have obtained a sufficient condition for a convex set C to have normal structure. To describe it we need the following quantity. Let $\rho(E)$ be the supremum of the numbers

ρ defined as follows. Given an integer n and positive numbers ε and η, there exists a n-dimensional simplex $S(n,\varepsilon,\eta)$ such that $|\rho-(\text{length of any edge of } S(n,\varepsilon,\eta))| < \eta$ and $\left| \|x\| - 1 \right| < \varepsilon$ for all $x \in S(n,\varepsilon,\eta)$. Brodsky and Milman sufficient condition for normal structure can be stated as follows: "If $\rho(E) < 1$, then every convex set C in E has normal structure". As this result shows E might not be strictly convex but Theorem I.1 holds as long as the "flat parts" of the surface of the unit ball are "small". This shows the distinction between Theorem I.1 and Corollary I.1. Namely Theorem I.1 holds for some class of Banach spaces which are not strictly convex.

5. NONEXPANSIVE MAPPINGS WHOSE SEQUENCE OF ITERATES $T^n x$ HAS A SMALL DIAMETER

In this section we consider nonexpansive mappings acting in a general reflexive Banach space E. Let C be a bounded closed convex subset of E and T a nonexpansive mapping of C into itself. We remark that the set C is not required to have normal structure.

As in Section 3 let us denote by $\mathcal{C}$ the collection of all closed convex subsets of C, which are invariant under T. The following basic assumption is made:

(A) For every $C_1 \in \mathcal{C}$, which consists of more than one point, there exists a point $x \in C_1$ such that

$$\text{diam } (\{T^n x\}_{n=1}^{\infty}) < 1/2 \text{ diam } (C_1) .$$

Theorem I.1. Let C be a bounded closed convex set in a reflexive Banach space and let T be a nonexpansive mapping of C into itself. If (A) holds, then T has a fixed point.

Proof. The proof uses Theorem I.2. Let $C_1 \in \mathcal{C}$. If the set C_1 contains just one point, this point will be fixed and the theorem is proved. Suppose that C_1 consists of more than one point; we claim that C_1 has a proper subset $C_2 \in \mathcal{C}$. By hypothesis (A) there exists $x \in C_1$ such that $d = \text{diam}(\{T^n x\}) < 1/2 \text{ diam}(C_1)$. The set $F_n = \bigcap_{j=n}^{\infty} B_d(T^j x) \cap C_1$ is closed, convex and non-empty (because $T^n x, T^{n+1} x, \ldots$ all are in F_n). Moreover $F_1 \subset F_2 \subset \ldots$ and $T(F_n) \subset F_{n+1}$. Consequently $C_2 = \bigcup_{n=1}^{\infty} F_n$ is a closed convex subset of C_1 which is invariant under T. Moreover $C_2 \neq C_1$ because $\text{diam}(C_2) \leq 2d < \text{diam } C_1$. The proof is finished.

6. THE KAKUTANI-YOSIDA ERGODIC THEOREM AND CONSEQUENCES

The basic result in this section is the classical mean ergodic theorem (Theorem I.4) of Kakutani and Yosida for continuous linear mappings in Banach spaces. This theorem will be used to establish an ergodic theorem (Theorem I.5) for continuous affine mappings in Banach spaces. For later use, in section 8, we shall prove corollaries to theorems I.4 and I.5, which are due to Edelstein [16]. Theorem I.5 will be used in the proof of Theorem I.6, which strengthens results of Browder [7] and Browder and Petryshyn [11]. See deFigueiredo and Karlovitz [30].

Theorem I.4 (Kakutani-Yosida). Let A be a linear operator mapping a Banach space E into itself. Suppose that

(a) there exists a constant C such that $\|A^n\| \leq C$ for all $n = 1, 2, \ldots$

(b) for some point $x \in E$, the sequence $x_n = \frac{1}{n} \sum_{j=1}^{n} A^j x$ contains a subsequence $\{x_{n(j)}\}$ that converges weakly to some point $\bar{x} \in E$.

Then the whole sequence $\{x_n\}$ converges in the norm to $\bar{x}$.

Proof. 1°) First we prove that $\bar{x}$ is a fixed point for A, i.e. $A\bar{x} = \bar{x}$. Since every continuous linear mapping in a Banach space is also weakly continuous we have

$$x_{n(j)} \rightharpoonup \bar{x} \Rightarrow Ax_{n(j)} \rightharpoonup A\bar{x} \quad . \tag{1}$$

(We use " $\rightharpoonup$ " to denote weak convergence and " $\rightarrow$ " to denote strong convergence, i.e. convergence in the norm). On the other hand, using hypothesis (a) we have the following estimate:

$$\|Ax_n - x_n\| = \left\| \frac{1}{n} (A^{n+1}x - Ax) \right\| \leq \frac{2c}{n} \|x\|$$

Using this in (1) we conclude that $\bar{x} = A\bar{x}$.

2°) Now let us prove that $x_n \rightarrow \bar{x}$. Since $\bar{x}$ is a fixed point for A we have

$$x_n = \bar{x} + \frac{1}{n} \sum_{j=1}^{n} A^j (x - \bar{x}) \quad .$$

So we have only to prove that

(*) $$\frac{1}{n} \sum_{j=1}^{n} A^j(x - \bar{x}) \to 0 \quad , \quad \text{as} \quad n \to \infty \, .$$

On the other hand, since a subsequence of

$$\{I - \frac{1}{n} \sum_{j=1}^{n} A^j\}x = \{(I - A) \sum_{j=0}^{n-1} \frac{n-j}{n} A^j\}x$$

converges weakly to $x - \bar{x}$, it follows that $x - \bar{x} \in \bar{R}$ where $\bar{R}$ is the closure of the range R of $I - A$. (Observe that, since R is a subspace, the weak closure and the strong closure are the same). So given $\varepsilon > 0$ there exist $z \in E$ and $h \in E$ with $||h|| < \varepsilon$ such that $x - \bar{x} = (I - A)z + h$. Then

$$\frac{1}{n} \sum_{j=1}^{n} A^j(x - \bar{x}) = \frac{1}{n} \sum_{j=1}^{n} A^j(I - A)z + \frac{1}{n} \sum_{j=1}^{n} A^j h$$

$$= \frac{1}{n} (A - A^{n+1})z + \frac{1}{n} \sum_{j=1}^{n} A^j h \quad .$$

Using hypothesis (a) we see that

$$||\frac{1}{n} (A - A^{n+1})z + \frac{1}{n} \sum_{j=1}^{n} A^j h|| \leq \frac{2c}{n} ||z|| + c\varepsilon \quad .$$

From this relation, (*) follows immediately and the proof is finished.

Corollary I.2 (Edelstein). Let A be a linear operator mapping a Banach space E into itself. Suppose that

(a) There exists a constant $c > 0$ such that $\|A^n\| \leq c$, $n = 1, 2, \ldots$

(b) There exists a point $x \in E$ such that E is the closure of the affine manifold generated by $\{A^n x\}_{n=0}^{\infty}$ that is, the closure of the set

$$\{y \in E : y = \sum_{j=0}^{m} \lambda_j A^j x , \sum_{j=0}^{m} \lambda_j = 1 , \quad m = 0, 1, 2, \ldots\}$$

(c) <u>for the element x of hypothesis (b) the sequence $x_n = \frac{1}{n} \sum_{j=1}^{n} A^j x$ contains a subsequence that converges weakly to some point $\bar{x}$</u>.

<u>Then, for all $y \in E$, the sequence $y_n = \frac{1}{n} \sum_{j=1}^{n} A^j y$ converges in the norm to 0. Moreover $\bar{x} = 0$ and $Ay \neq y$ for all $y \neq 0$</u>.

<u>Proof of Corollary I.2.</u> By theorem I.4 we conclude that x_n converges in the norm to $\bar{x}$ and $\bar{x}$ is a fixed point for A. If we prove that for any $y \in E$ the sequence $\{y_n\}$ converges strongly to $\bar{x}$, then the result follows; for $(y - x)_n = y_n - x_n$ by the linearity of A. So let us prove that $y_n \to \bar{x}$. Given $\varepsilon > 0$ there exists $h \in E$, with $\|h\| < \varepsilon$, and real numbers $\lambda_o, \lambda_1, \ldots \lambda_m$, with $\sum_{j=0}^{m} \lambda_j = 1$, such that $y = \sum_{j=0}^{m} \lambda_j A^j x + h$. Then

$$y_n = \frac{1}{n} \sum_{i=1}^{n} A^i (\sum_{j=0}^{m} \lambda_j A^j x) + \frac{1}{n} \sum_{i=1}^{n} A^i h$$

$$= \sum_{j=0}^{m} \lambda_j A^j (\frac{1}{n} \sum_{i=1}^{n} A^i x) + \frac{1}{n} \sum_{i=1}^{n} A^i h$$

by the linearity of A. From this we obtain the following estimate using the fact that $\bar{x}$ is a fixed point of A

$$\|y_n - \bar{x}\| \leq \|\sum_{j=0}^{m} \lambda_j A^j (x_n - \bar{x})\| + c\varepsilon .$$

Taking limits it follows

$$\overline{\lim_{n\to\infty}} \|y_n - \bar{x}\| \leq c\varepsilon$$

Since ε is arbitrary we obtain $y_n \to \bar{x}$, as we have claimed.

Theorem I.5. Let T be an affine mapping of a Banach space E into itself. Assume that

(a) There exists a constant $c > 0$ such that $\|T^n x - T^n y\| \leq c\,\|x - y\|$ for all x,y in E, and all integers $n > 0$.

(b) For some point $y \in E$, the sequence $y_n = \frac{1}{n}\sum_{j=1}^{n} T^j y$ contains a subsequence $\{y_{n(j)}\}$ that converges weakly to a point $\bar{y}$.

(c) For some point $z \in E$, the sequence $\{T^{n(j)}z\}$ is bounded, where $\{n(j)\}$ is the subsequence of hypothesis (b).

Then the whole sequence $\{y_n\}$ converges strongly to $\bar{y}$ and $\bar{y}$ is a fixed point of T.

Remark. In view of (a), hypothesis (c) implies that for every $w \in E$ the sequence $\{T^{n(j)}w\}$ is bounded.

Proof. 1°) First we prove that $\bar{y}$ is a fixed point for T. By definition, the mapping T being affine, it implies that there exist a linear mapping A and a vector $a \in E$ such that $Tx = Ax + a$. It then follows that $a = T0$. By induction we obtain that $T^n x = A^n x + T^n 0$. Combining this with hypothesis (a) we get $\|A^n\| \leq C$. Since an affine continuous mapping is also weakly continuous we obtain

(3) $$y_{n(j)} \rightharpoonup \bar{y} \Rightarrow Ty_{n(j)} \rightharpoonup T\bar{y}$$

On the other hand since T is affine and satisfies hypothesis (a) :

$$\|Ty_n - y_n\| = \|\frac{1}{n}(T^{n+1}y - Ty)\| \leq \frac{c}{n}\|T^n y - y\| .$$

Using hypothesis (c) we obtain

$$(3) \qquad \|Ty_{n(j)} - y_{n(j)}\| \leq \frac{c}{n(j)}(K + \|y\|)$$

where K is such that $\|T^{n(j)}y\| \leq K$ for all $n(j)$. Finally (2) and (3) together imply $T\bar{y} = \bar{y}$.

2°) We have

$$(4) \qquad \frac{1}{n}\sum_{j=1}^{n} A^j y = \frac{1}{n}\sum_{j=1}^{n} T^j y - \frac{1}{n}\sum_{j=1}^{n} T^j 0 ,$$

and

$$(5) \qquad \frac{1}{n}\sum_{j=1}^{n} A^j \bar{y} = \bar{y} - \frac{1}{n}\sum_{j=1}^{n} T^j 0 ,$$

where we have used the fact that $\bar{y}$ is a fixed point. From (4) and (5) we obtain, by subtraction,

$$\frac{1}{n}\sum_{j=1}^{n} A^j (y - \bar{y}) = \frac{1}{n}\sum_{j=1}^{n} T^j y - \bar{y} .$$

Using hypothesis (b) we have that a subsequence of $\frac{1}{n}\sum_{j=1}^{n} A^j (y - \bar{y})$ converges weakly to 0 . Since the operator A satisfies all the hypotheses of theorem I.4, we can apply that theorem and conclude that the whole sequence $\frac{1}{n}\sum_{j=1}^{n} A^j(y - \bar{y})$ converges strongly to 0. It then follows that the whole sequence $\frac{1}{n}\sum_{j=1}^{n} T^j y$ converges strongly to $\bar{y}$. The proof of the theorem is complete.

Corollary I.3 (Edelstein). Let T be an affine mapping in a Banach space E. Assume that

(a) There exists $c > 0$ s.t. $\|T^n x - T^n y\| \leq c \|x - y\|$ for all x, y in E and all integers $n > 0$.

(b) There exists a point $x \in E$ such that E is the closure of the affine manifold generated by $\{T^n x\}_{n=0}^{\infty}$. (See Corollary I.2).

(c) For the point x of hypothesis (b), the sequence $x_n = \frac{1}{n}\sum_{j=1}^{n} T^j x$ contains a subsequence $\{x_{n(j)}\}$ that converges weakly to $\bar{x}$.

(d) For some point $w \in E$ the sequence $\{T^{n(j)} w\}$ is bounded.

Then the whole sequence $\{x_n\}$ converges strongly to $\bar{x}$. Moreover for every $y \in E$ the sequence $y_n = \frac{1}{n}\sum_{j=1}^{n} T^j y$ converges strongly to $\bar{x}$.

Remark. This corollary has been stated by Edelstein [16] without hypothesis (d). We believe that this hypothesis is essential for the validity of this result. Hypotheses (a), (b) and (c) do not imply (d) as it can be shown by means of a counterexample.

Proof. By the previous theorem it follows that the whole sequence $\{x_n\}$ converges strongly to $\bar{x}$. To prove that $y_n \to \bar{x}$, for every $y \in E$, we will apply Corollary I.2. Let z be a point in the affine manifold L generated by $\{T^n x\}$, i.e.

$$z = \sum_{j=0}^{m} \lambda_j T^j x , \quad \sum_{j=0}^{m} \lambda_j = 1 .$$

Since $\bar{x}$ is a fixed point of T we obtain

(6) $$z - \bar{x} = \sum_{j=0}^{m} \lambda_j (T^j x - T^j \bar{x}) = \sum_{j=0}^{m} \lambda_j A^j (x - \bar{x})$$

where A is the linear operator such that $Tx = Ax + T0$. (Cf. proof of Theorem I.5). From (6) we see that the affine manifold generated by $\{A^n(x - \bar{x})\}_{n=0}^{\infty}$ is precisely $L - \bar{x}$. Since L is dense in E it follows that $L - \bar{x}$ is also dense in E. So hypothesis (b) of Corollary I.2 is satisfied for the point $x - \bar{x}$. Hypothesis (a) is also satisfied (cf. proof of Theorem I.5). And finally as in the proof of Theorem I.5 we have

(7) $$\frac{1}{n} \sum_{j=1}^{n} A^j (x - \bar{x}) = \frac{1}{n} \sum_{j=1}^{n} T^j x - \bar{x}$$

from what follows that hypothesis (c) of Corollary I.2 is also verified. So applying Corollary I.2 it follows that $\frac{1}{n} \sum_{j=1}^{n} A^j u$ converges strongly to 0 for any $u \in E$. Then $\frac{1}{n} \sum A^j (y - \bar{x}) \to 0$. Using this result in (7) we conclude that $\frac{1}{n} \sum_{j=1}^{n} T^j y \to \bar{x}$, which completes the proof of the corollary.

7. AN ITERATIVE METHOD OF SOLUTION OF LINEAR FUNCTIONAL EQUATIONS IN BANACH SPACES

This section contains an iterative method of solution of the functional equation

(1) $$x = Ax + a$$

where A is a linear operator in a Banach space E and a is a given vector in E. The iteration method of Picard-Poincaré-Neumann

$$x_{n+1} = Ax_n + a$$

where the initial value x_o is given, has been recently studied by Browder and Petryshyn [11]. The method presented here works for a wider class of operators than the one considered in [11].

The following method is considered

(2) $$y_{n+1} = \frac{n}{n+1} Ay_n + \frac{1}{n+1} Ay_o + a$$

where y_o is a given initial value.

Let T be the affine mapping in E defined by $Tx = Ax + a$. Finding a solution of (1) is equivalent to finding a fixed point of T. It is easy to see that y_n as defined in (2) can be expressed as

(3) $$y_n = \frac{1}{n} \sum_{j=1}^{n} T^j y_o \quad .$$

Indeed, let us denote this arithmetic mean by z_n, i.e.

$$z_n = \frac{1}{n} \sum_{j=1}^{n} T^j y_o \quad ,$$

Since T is affine it follows

$$Tz_n = \frac{1}{n} \sum_{j=1}^{n} T^{j+1} y_o$$

This together with the definition of z_{n+1} gives

$$z_{n+1} = \frac{n}{n+1} Tz_n + \frac{1}{n+1} Ty_o = \frac{n}{n+1} Az_n + \frac{1}{n+1} Ay_o + a \ .$$

From this expression for z_{n+1} we obtain first $z_1 = y_1$ and then $z_n = y_n$ by induction.

It is also useful to observe that

$$(4) \qquad T^n x = A^n x + A^{n-1} a + \dots + Aa + a \ .$$

Now we can state our theorems.

Theorem I.6. *Let E be a (general) Banach space and A a linear mapping in E satisfying the following conditions*

(a) $\|A^n\| \leq C$ *for all* $n = 1, 2, \dots$.

(b) $A^n x$ *converges weakly, as* $n \to \infty$, *for all* $x \in E$.

(c) *The equation* $x = Ax + a$ *has a solution*.

Then, for each given y_o, *the sequence* $\{y_n\}$ *defined in (2) converges strongly to a solution of equation (1)* .

Remark. This theorem extends Theorem 1, part a, of Browder and Petryshyn [11] . Observe that here we are working with the arithmetic means of the sequence of the iterates considered in [11] . So it is reasonable to expect convergence in some cases where the original sequence of iterates fail to converge.

Proof of theorem I.6. The idea is to apply the ergodic theorem of affine mappings (Theorem I.5) proved in section 6. First we observe that hypothesis (a) here implies hypothesis (a) of Theorem I.5.

Since (1) has a solution $\bar{x}$, this solution is a fixed point for T. So by (4)

$$\bar{x} = A^n \bar{x} + \sum_{j=0}^{n-1} A^j a$$

from which follows, by virtue of hypothesis (b), that the sequence $\{\sum_{j=0}^{n-1} A^j a\}$ converges weakly. This together with hypothesis (b) in (4) gives that $T^n y_o$ converges weakly. Thus $\{T^n y_o\}$ is bounded and hypothesis (c) of Theorem I.5 is satisfied. Also the sequence of the arithmetic means of $T^n y_o$, (i.e., y_n) converges weakly, which gives hypothesis (b) of Theorem I.5. Using that theorem we conclude that the sequence $\{y_n\}$ converges strongly to a fixed point of T, i.e. to a solution of equation (1). The theorem is proved.

<u>Theorem I.7</u>. <u>Let E be a reflexive Banach space and A a linear mapping in E satisfying the following conditions</u>.

(a) <u>$\|A^n\| \leq C$ for all $n = 1, 2, \ldots$</u> .

(b) <u>$\|a + Aa + \ldots + A^n a\| \leq k$ for all $n = 1, 2, \ldots,$ where k is some constant</u>.

<u>Then, for each given y_o, the sequence $\{y_n\}$ defined in (2) converges strongly to a solution of equation (1)</u> .

<u>Remark</u>. This result strengthens theorem 1(c) of Browder and Petryshyn [11] , where they assumed instead of hypothesis (a) the convergence of $A^n x$ for every x. (This last hypothesis implies (a) by the Uniform Boundedness Principle) . Cf also a previous result of Browder [7] .

Proof. Hypotheses (a) and (b) used in (4) imply that $\{T^n y_o\}$ is bounded, which implies hypothesis (c) of Theorem I.5. Hypothesis (a) of Theorem I.5 is obviously a consequence of hypothesis (a) of this theorem. Finally $\{y_n\}$ being a bounded sequence in a reflexive Banach space contains a weakly convergent subsequence. This gives hypothesis (b) of Theorem I.5 . Applying this theorem we obtain the strong convergence of the sequence $\{y_n\}$ to a fixed point of T , that is, to a solution of equation (1) .

Theorem I.8. Let E be a Banach space and A a linear mapping in E satisfying the following conditions.

(a) $\|A^n\| \leq C$ for all $n = 1, 2, \ldots,$

(b) $\|a + Aa + \ldots + A^n a\| \leq k$ for all $n = 1, 2, \ldots;$ k is some constant.

In addition, suppose that, for a given y_o , the sequence $\{y_n\}$ defined in (2) contains a subsequence that converges weakly. Then the whole sequence $\{y_n\}$ converges strongly to a solution of equation (1) .

Remark. Hypothesis (b) can be relaxed by assuming only that $a + Aa + \ldots + A^{n(j)}a$ is uniformly bounded for the sequence $n(j)$ such that $y_{n(j)}$ converges weakly. This theorem is an improvement over theorem 1(b) of Browder and Petryshyn [11] in the sense that hypothesis (a) and (b) here are weaker than the ones in [11], and the weak convergence of a subsequence of $\{y_n\}$ is far less restrictive than the strong convergence of a subsequence of $T^n x$.

Proof of theorem I.8. Hypotheses (a) and (b) when used in (4) give that the sequence $\{T^n x\}$ is bounded for every $x \in E$. So condition (c) of theorem I.5 is satisfied. Condition (a) of theorem I.5 is also satisfied.

Our assumption about the weak convergence of a subsequence of $\{y_n\}$ is precisely condition (b) of theorem I.5. So using that theorem we conclude that the whole sequence $\{y_n\}$ converges to a solution of (1), which proves the theorem.

8. EDELSTEIN'S RESULTS ON FIXED POINTS OF NONEXPANSIVE MAPPINGS

In previous sections we have considered nonexpansive mappings on reflexive Banach spaces. Now we relax this restriction on the space, but as a counterpart somewhat stronger conditions are required on the mapping. The results of the present section are due to Edelstein [16].

Let E be a strictly convex Banach space, and T a nonexpansive mapping in E. The following basic assumption is made:

(A) There exists a point $x \in E$ such that the sequence $\{T^n x\}_{n=1}^{\infty}$ has a convergent subsequence.

Lemma I.4. Suppose (A) is satisfied. Then there exists $y \in E$ and a sequence of integers $n_1 < n_2 < \ldots$ such that $T^{n_j} y \to y$.

Proof. By (A) there exists $y \in E$ and a sequence of integers $m_1 < m_2 < \ldots$ such that $T^{m_j} x \to y$. We may assume that $m_{j+1} - m_j$ increases as $j \to \infty$. Let $n_j = m_{j+1} - m_j$. Claim $T^{n_j} y \to y$. In fact

$$\|y - T^{n_j} y\| \leqslant \|y - T^{m_{j+1}} x\| + \|T^{m_{j+1}} x - T^{n_j} y\|$$

$$\leqslant \|y - T^{m_{j+1}} x\| + \|T^{m_j} x - y\| ,$$

where nonexpansiveness was used to get the last inequality. From the above inequality the result follows immediately.

In basis of this lemma, a nonexpansive mapping T satisfies (A) if and only if it satisfies the following condition:

(A′) There exists a $y \in E$ and a sequence of integers $n_1 < n_2 < \ldots$ such that $T^{n_j} y \to y$.

Now let us state the main theorem of this section.

Theorem I.9. Let E be a strictly convex Banach space, and T a nonexpansiveness mapping in E. Let us assume (A′) above. Suppose also that the sequence

$$y_n = \frac{1}{n} \sum_{j=1}^{n} T^j y$$

has a subsequence $y_{n(j)}$ that converges weakly to a point $\bar{y}$. Assume further that $T^{n(j)} y$ is bounded. Then $\bar{y}$ is a fixed point of T.

Remark. This theorem has been stated by Edelstein without the assumption that $\{T^{n(j)} y\}$ is bounded. We believe that this theorem is not true without this hypothesis. See remark after Corollary I.3 in section 6.

The proof of Theorem I.9 will be preceded by a series of lemmas.

Lemma I.5. Let E be a strictly convex Banach space and T a nonexpansive mapping in E. Let us assume (A′) above. Let $S = \{x \in E : T^{n_j} x \to x\}$. Then T is an affine isometry of S into itself. Moreover S is a closed and convex subset of E.

Proof 1°) $T(S) \subset S$. For $T^{n_j}(Tx) \to Tx$.

2°) T is an isometry in S . By nonexpansiveness $\|Tx_1 - Tx_2\| \leq \|x_1 - x_2\|$. On the other hand

$$\|x_1 - x_2\| = \lim \|T^{n_j}x_1 - T^{n_j}x_2\| \leq \|Tx_1 - Tx_2\| \quad .$$

3°) T is affine. Let $x_1, x_2 \in S$ and let $x = \lambda x_1 + (1 - \lambda)x_2$. Then using 2°) and the nonexpansiveness of T we obtain

$$\|x_1 - x_2\| = \|Tx_1 - Tx_2\| \leq \|Tx_1 - Tx\| + \|Tx - Tx_2\|$$

$$\leq \|x_1 - x\| + \|x - x_2\| = \|x_1 - x_2\| \quad .$$

Since E is strictly convex, the inequality above implies $Tx - Tx_1 = \alpha(Tx_1 - tx_2)$ where α is some real number. So Tx is on the line through Tx_1 and Tx_2 . Finally we observe that $\|Tx - Tx_j\| \leq \|x - x_j\|$, $j = 1, 2,$ and conclude that $Tx = \lambda Tx_1 + (1 - \lambda) Tx_2$. This proves that T is affine in S .

4°) S is convex. Let $x_1, x_2 \in S$. We want to prove that $x = \lambda x_1 + (1 - \lambda)x_2 \in S$. From 3°) we have $T^{n_j}x = \lambda T^{n_j}x_1 + (1 - \lambda)T^{n_j}x_2$. Taking limits we obtain

$$\lim T^{n_j}x = \lambda \lim T^{n_j}x_1 + (1 - \lambda) \lim T^{n_j}x_2 = \lambda x_1 + (1 - \lambda)x_2 = x \quad .$$

5°) S is closed. Let $x_n \in S$ such that $x_n \to x$. From the estimate

$$\|T^{n_j}x - x\| \leq \|T^{n_j}x - T^{n_j}x_n\| + \|T^{n_j}x_n - x_n\| + \|x_n - x\|$$

$$\leq 2\,\|x - x_n\| + \|T^{n_j}x_n - x_n\|$$

We conclude that $T^{n_j}x \to x$. This proves that $x \in S$, and so S is closed.

The proof of Lemma I.5 is complete.

<u>Lemma I.6</u>. <u>Let C be a convex set in a normed space E, and let T be an affine isometry of C into E</u>. <u>Let $L(C)$ be the affine manifold spanned by C</u>, i.e.,

$$L(C) = \{y \in E : y = \textstyle\sum_{j=1}^{m} \lambda_j x_j \ , \ \sum_{j=1}^{m} \lambda_j = 1 \ , \ x_j \in C\}$$

<u>Then there exists an affine isometry $\tilde{T} : L(C) \to L(C)$ such that $\tilde{T}|C = T$</u> .

We omit the proof of Lemma I.6. See [16] .

<u>Proof of theorem I.9</u>. Let $L(y)$ be the affine manifold spanned by $\{T^n y\}_{n=1}^{\infty}$. Let Z be the (possibly empty) set of all fixed points of T in the closure $\overline{L(y)}$ of $L(y)$. It is easy to see that the set K = convex closure of $Z \cup \{T^n y\}_{n=1}^{\infty}$ is invariant under T , on account of the fact that T is affine. Since K is contained in S (see Lemma I.5) , it follows that T is an affine isometry of K into itself. By Lemma I.6 there exists an extension $\tilde{T}$ of T to $\overline{L(y)}$. Without loss of generality we may assume that $0 \in \overline{L(y)}$, and so we

consider $\overline{L(y)}$ as a Banach space. Now $\tilde{T}$ satisfies the hypothesis of the Theorem I.5. So the sequence $\{y_n\}$ converges to a fixed point $\bar{y}$ of $\tilde{T}$. Since K is closed it follows that $\bar{y} \in K$ and so $\bar{y}$ is a fixed point of T. The proof is complete.

Remark. Using Corollary I.3 we conclude that $\bar{y}$ is the unique fixed point of T in the manifold $L(K)$.

9. A THEOREM OF GÖHDE

In this section we present a theorem about the existence of fixed points for a nonexpansive mapping T in a general Banach space E. As one should expect, some drastic restriction has to be put on the mapping T. The following basic assumption on T is made.

(G) There exists a compact set M in E such that, for every $x \in E$, the closure of the sequence of iterates $\{T^n x\}_{n=1}^{\infty}$ contains a point of M.

Before stating the main result of this section we need the following definition. A set C in a linear space is said to be star-shaped about a point $y \in C$ if for every $z \in C$ there exists a nonnegative number t_z, $0 \leq t_z \leq +\infty$ such that the set $\{y + tz : 0 \leq t < t_z\}$ is in C and the set $\{y + tz : t_z < t\}$ is outside of C. It is clear that every convex set is star-shaped about any of its points.

Theorem I.10. Let E be a Banach space and T a nonexpansive mapping of a bounded closed star-shaped subset C into itself. Suppose that assumption (G) is satisfied. Then T has a fixed point.

Remark. This theorem is also valid in an incomplete normed space. Theorem I.10 for general normed spaces under the assumption that C is a convex set has been proved by Göhde [17]. His proof works equally well for the case when C is only assumed to be star-shaped.

Proof. We may suppose without loss of generality that the origin 0 is in C, and that C is star-shaped about 0. Then the mapping $T_r = rT$, $0 < r < 1$, is a contraction in C, and consequently it has a fixed point $x_r \in C$. We have the following estimate

$$\|x_r - Tx_r\| = \|rTx_r - Tx_r\| \leq (1 - r)\|Tx_r\| \leq d(1 - r)$$

where d is the diameter of C.

On the other hand, by (G), there exists an integer $n(r)$ and a point $y_r \in M$ such that $\|y_r - T^{n(r)}x_r\| \leq 1 - r$.
Then

$$\|y_r - Ty_r\| \leq \|y_r - T^{n(r)}x_r\| + \|T^{n(r)}x_r - T^{n(r)+1}x_r\| + \|T^{n(r)+1}x_r - Ty_r\|$$

$$\leq 2\|y_r - T^{n(r)}x_r\| + \|x_r - Tx_r\|$$

by the nonexpansiveness of T. Using the estimates above this inequality yields

(1) $$\|y_r - Ty_r\| \leq (d + 2)(1 - r) .$$

Let now $\{r\}$ be a sequence converging to 1. Using the compactness of M it follows that there exists a subsequence of $\{y_r\}$ (denote it again by $\{y_r\}$) that converges to $y \in M$, as $r \to 1$. From (1) it follows immeditately that y is a fixed point of T.

Remark. Assumption (G) is stronger than Edelstein's condition (A) of section 8. That is the following proposition holds.

Proposition I.2. *Let E be a Banach space and T a mapping of E into itself satisfying the following condition:*

(G_x) *There exist a compact set M and a point x such that the closure of the sequence of iterates $\{T^n x\}_{n=1}^{\infty}$ contains a point of M.*

Then $\{T^n x\}_{n=1}^{\infty}$ contains a subsequence that converges strongly.

Proof. By (G_x) there exist an integer $m(n)$ and $y_n \in M$ such that $\|T^{m(n)}x - y_n\| < 1/n$. Using the fact that M is compact, it follows that there exists a subsequence $\{y_{n_j}\}$ that converges strongly to $y \in M$. It is then immediate that $T^{m(n_j)}x$ converges to y. Q.E.D.

This shows that Theorem I.10 is a special case of theorem I.9 in the case of strictly convex Banach spaces and a weakly compact C.

10. COMMON FIXED POINTS FOR COMMUTING NONEXPANSIVE MAPPINGS.

In this section we consider a family Γ of commuting nonexpansive mappings in a Banach space E, (i.e., $T_\alpha T_\beta = T_\beta T_\alpha$ for all T_α, T_β in Γ), and we will establish conditions under which these mappings have a common fixed point, (i.e., a point $x \in E$ such that $T_\alpha x = x$ for all T_α in Γ). Before looking at the case of nonexpansive mappings we state, without proof, the following theorem due to Kakutani [21] and

Markov [25] . A proof of it can be found in Bourbaki ["Espaces Vectoriels Topologiques", Chap. II, Appendice] .

Theorem I.11. *Let E be a Hausdorff topological vector space over the reals, and K a convex compact subset of E . Let Γ be a family of commuting continuous affine mappings in E . Suppose that $T(K) \subset K$ for all T in Γ . Then there exists a point $x_o \in K$ which is fixed for all T in Γ .*

Now let us come back to the case of nonexpansive mappings. First let us consider the case of *strictly convex* Banach spaces, and afterwards the general case. The case of a strictly convex Banach space is much easier to handle than the general case because of the following proposition.

Proposition I.3. *Let E be a strictly convex Banach space and T a nonexpansive mapping in E . Then the set F of fixed points of T is convex.*

Proof. We may assume that F consist of more than one point; otherwise the result is proved. Suppose that x_1 and x_2 are in F and let us prove that $x = \lambda x_1 + (1 - \lambda)x_2$, $0 < \lambda < 1$, is also in F . In fact, by nonexpansiveness we obtain

$$\|x_1 - x_2\| = \|Tx_1 - Tx_2\| \leq \|Tx_1 - Tx\| + \|Tx - Tx_2\|$$

$$\leq \|x_1 - x\| + \|x - x_2\| = \|x_1 - x_2\| .$$

Since E is strictly convex it follows that the vectors $Tx_1 - Tx$ and

$Tx - Tx_2$ are linearly dependent. But this implies that the vector Tx is in the straight line through $Tx_1 (=x_1)$ and $Tx_2 (=x_2)$. On the other hand $||Tx - Tx_j|| \leq ||x - x_j||$, $j = 1, 2$. Thus Tx must coincide with x. The proposition is proved.

Remark. This proposition is not true in the most general class of Banach spaces, as the following example shows. Let $E = \ell^\infty(2)$, i.e., the spaces of pairs $x = (a,b)$ with the max-norm $||x|| = \max(|a|, |b|)$. Let T be the mapping defined as follows

$$Tx = T\{(a,b)\} = (|b|,b) .$$

It is easy to see that T is nonexpansive and that $(1,1)$ and $1,-1)$ are fixed points of T. However no other point in the segment joining these two points is a fixed point of T. This example is due to deMarr [15].

This proposition will be used to prove the following theorem.

Theorem I.12. *Let E be a strictly convex Banach space. Let Γ be a family of commuting nonexpansive mappings T, from a weakly compact subset C of E into E. We assume that each T has a nonempty set $F(T)$ of fixed points. Then there exists a point $x_0 \in E$ such that $Tx_0 = x_0$ for all $T \in \Gamma$.*

Remark. This theorem is essentially theorem 2 of Belluce and Kirk [2]. It includes theorem 2 of Browder [6] and theorem 1 of Kasahara [23]. The proof presented here is due to Belluce and Kirk [2].

We shall need the following two lemmas in the proof of Theorem I.12.

Lemma I.7. Let E be a Banach space and K a weakly compact subset of it. Let $\phi : E \to R^1$ be a weakly lower semicontinuous function in E. Then the infimum of ϕ is achieved in K, i.e., there exists $x_o \in K$ such that $\phi(x_o) = \inf_{x \in K} \phi(x)$.

This lemma is a particular case of the following.

Proposition I.4. Let E be a topological space and K a compact subset of it. Let $\phi : E \to R^1$ be a lower semicontinuous function in E. Then there exists $x_o \in K$ such that $\phi(x_o) = \inf_{x \in K} \phi(x)$.

Proof. First we show that ϕ is bounded from below in K. Suppose by contradiction that this is not so. That is, let $\{x_n\}$ be a sequence in K such that $\phi(x_n) \leq -n$. By compactness of K there exists a subsequence $\{x_{n(j)}\}$ converging to some point $\bar{x} \in K$. By the definition of lower semicontinuous function it then follows

$$\phi(\bar{x}) \leq \lim\inf \phi(x_{n(j)}) .$$

However this contradicts the assumption $\phi(x_n) \leq -n$. So ϕ is bounded from below in K. Let $a = \inf_{x \in K} \phi(x)$. Then there exists a sequence $\{y_n\} \subset K$ such that $\phi(y_n) \to a$. Again by compactness we have $y_{n(j)} \to \bar{y}$, which implies $\phi(\bar{y}) \leq \lim\inf \phi(y_{n(j)}) = a$. Thus it follows $\phi(\bar{y}) = a$. The proposition is proved.

Lemma I.8. Let E be a strictly convex Banach space and K a weakly compact convex subset of it. Then, for every $y \notin K$, there exists a unique $x_o \in K$ such that $||x_o - y|| = \inf_{x \in K} ||x - y||$.

Proof. Applying Lemma I.7 with $\phi(x) = \|x - y\|$, it follows that there exists at least one point in K which minimizes $\|x - y\|$ over K. Suppose that there are two such points, x_1 and x_2. Denoting by $d = \inf_{x \in K} \|x - y\|$ we have $\|x_1 - y\| = \|x_2 - y\| = d$. By the convexity of K it follows that $\lambda x_1 + (1 - \lambda)x_2 \in K$. So

$$d \leq \|\lambda x_1 + (1 - \lambda)x_2 - y\| \leq \lambda \|x_1 - y\| + (1 - \lambda)\|x_2 - y\| = d$$

which implies that $\|\lambda x_1 + (1 - \lambda)x_2 - y\| = d$ for all $0 \leq \lambda \leq 1$. This however contradicts the fact that the Banach space is strictly convex. So $x_1 = x_2$, and the lemma is proved.

Proof of Theorem I.12. By Proposition I.3 it follows that the set F(T) of fixed points of the mapping T is convex. The continuity of T implies immediately that F(T) is closed. So F(T) is a closed and convex subset of E. Since every closed convex set in a Banach space is also weakly closed, it follows that F(T) is weakly closed. On the other hand F(T) is a subset of a weakly compact set C. So F(T) is also weakly compact, for every $T \in \Gamma$. Thus we need only to prove now that the collection of sets F(T), $T \in \Gamma$, has the property of finite intersection. Once this is done it will follow that $\bigcap F(T)$ is a non-empty closed convex subset of C, and the theorem will be proved.

The remaining part of the proof is by induction. Let $T_1, \ldots, T_n$ be mappings of Γ, such that $F_n = \bigcap_{j=1}^{n} F(T_j) \neq \phi$. Let T be any other element of Γ. It is easy to see that $T(F_n) \subset F_n$. (Indeed let $x \in F_n$, then $T_j(Tx) = T(T_j x) = Tx$ for

$j = 1, \ldots, n$. This shows that $Tx \in F_n$) . Now let $y \in F(T)$ and let $z \in F_n$ be the unique point in F_n which is closest to y(cf. Lemma I.8). Since $Tz \in F_n$ and $\|Tz - y\| \leq \|z - y\|$ it follows that $Tz = z$. The proof is complete.

The next theorem is an extension of a result of DeMarr [15] . It is due to Belluce and Kirk [2].

<u>Theorem I.13</u>. <u>Let C be a bounded closed convex set in a Banach space E, and Γ a family of commuting nonexpansive mappings of C into itself. Let M be a compact subset of C with the property that $M \cap \{T_1^n x : n = 1, 2, \ldots\} = \phi$ for some $T_1 \in \Gamma$ and all $x \in C$. Then there exists $x_o \in C$ such that $Tx_o = x_o$ for all $T \in \Gamma$.</u>

<u>Remark</u>. This final observation has some interest in itself, although it is not concerned with nonexpansive mappings. For some years the following question remained unanswered. If f and g are two commuting continuous functions of interval $[0,1]$ into itself, do they have a common fixed point x_o, i.e., $f(x_o) = g(x_o) = x_o$? This problem was treated by some authors. Cohen [12] answered the question in the affirmative for certain special classes of functions. See also Baxter [1]. Shields [29] considered commuting continuous functions of the closed unit disc into itself which are analytic in the open disc. He then proved the existence of a common fixed point.

Recently Boyce [3] and Huneke [19] by means of counter-examples answered the question in the negative for the most general class of continuous functions in the interval [0,1] .

11. PERIODIC SOLUTIONS OF NONLINEAR DIFFERENTIAL EQUATIONS

Let H be a Hilbert space and f a function mapping $R^+ \times H$ into H. Here $R^+ = \{t \in R^1, t \geq 0\}$. We consider differential equations of the form

$$(1) \qquad \frac{du}{dt} = f(t,u) \quad .$$

A function $u : R^+ \to H$ is a solution of (1) if it is a C^1-function (i.e. it has derivative du/dt at every point $t \in R^+$) and satisfies (1):

$$\frac{du}{dt}(t) = f(t,u(t)) \ , \quad t \in R^+ \ .$$

The initial value problem consists in finding a solution of (1) such that for $t = 0$ equals certain given value $u_o \in H$.

The existence and uniqueness of solutions for the initial value problem for (1) is a well established theory by now; see for example Dieudonné ("Foundations of Modern Analysis", Chap X, §4). Namely the following result is well known. If f satisfies a local Lipschitz condition in u, then the initial value problem has a unique local solution. For finite dimensional spaces, one still has existence (but not uniqueness) if it is only required that f be continuous. This last result is false in infinite dimension, see example in Dieudonné's book, p. 287.

In [9], Browder proved that the following two assumptions are sufficient for existence of a unique solution (defined for all $t \in R^+$) for the initial value problem.

(I) f is a continuous mapping of $R^+ \times H$ into H, carrying bounded sets into bounded sets.

(II) There exists a continuous real-valued function $c(t)$ on R^+ such that for all $u, v \in H$ we have

$$\mathrm{Re}(f(t,u) - f(t,v), u - v) \leq c(t) \|u - v\|^2$$

for all $t \in R^+$.

In later work Browder has been able to prove the same result under a weaker hypothesis replacing (I), namely that f is continuous but not necessarily bounded.

In this section we assume that f is periodic in t with period ξ. We are interested in periodic solutions for the initial value problem

$$\frac{du}{dt} = f(t,u \qquad (2)$$

$$u(0) = u_o \quad ,$$

where f is such that the initial value problem (2) has a unique solution. In particular this will be the case if f satisfies hypotheses (I) and (II) above.

The following result was proved by Browder [5]. In later work [8] he extended these results for Banach spaces.

Theorem I.14. Let H be a Hilbert space and f a function mapping $R^+ \times H$ into H. Suppose that the initial value problem (2) has a unique solution $u : R^+ \to H$. Assume that f is periodic in t of period ξ and satisfies the following two conditions:

(A) For each $t \in R^+$ and $u,v \in H$ we have

$$\operatorname{Re}(f(t,u) - f(t,v)\ ,\ u - v) \leq 0\ .$$

(B) There exists $r > 0$ such that

$$\operatorname{Re}(f(t,u),u) < 0$$

for all $t \in R^+$ and all u such that $\|u\| = r$. Then (2) has a periodic solution with period ξ.

Proof. By hypothesis, to each $v \in H$ there corresponds a unique solution of (2) with $u(0) = v$. Define a mapping $T : H \to H$ which assigns to each $v \in H$ the value $u(\xi)$. In this way the problem of finding a periodic solution of (2) with period ξ is reduced to the one of determining a fixed point of T.

First we observe that T maps the ball $B_r(0)$ of radius r about the origin in H into itself. In fact

$$\frac{1}{2}\ \frac{d}{dt}\{\ \|u(t)\|^2\} = \operatorname{Re}\ (\frac{du}{dt}\ ,u) = \operatorname{Re}(f(t,u(t)),u(t))$$

From this and hypothesis (B) it follows that, if for any value of t in $[0,\xi]$ the solution $u(t)$ is such that $\|u(t)\| = r$, then $\frac{d}{dt}\{\|u(t)\|^2\} < 0$. This means that the solution $u(t)$ cannot leave the ball $B_r(0)$.

Next we prove that T is a nonexpansive mapping. Indeed

$$\frac{1}{2}\frac{d}{dt}\{\|u(t) - u_1(t)\|^2\} = \mathrm{Re}(f(t,u(t)) - f(t,u_1(t)),\ u(t) - u_1(t)) \leq 0$$

using hypothesis (A) . This implies $\|u(\xi) - u_1(\xi)\| \leq \|u(0) - u_1(0)\|$ which gives the nonexpansiveness of T .

The proof is then completed if we recall that a nonexpansiveness mapping of a ball in a Hilbert space into itself has a fixed point.

REFERENCES

[1] Baxter, G. - On fixed points of the composite of commuting functions. Proc. Amer. Math. Soc. 15(1964) pp.851-855.

[2] Belluce, L.P. and Kirk, W.A. - Fixed-point theorems for families of contraction mappings. Pac. J. Math., Vol 18 (1966), pp.213-217.

[3] Boyce, W.M. - Commuting functions with no fixed points. Notices Amer. Math. Soc. 14(1967), p. 280.

[4] Brodsky, M.S. and Milman, D.P. - On the center of a convex set. Dokl. Akad. Nauk SSSR (N.S.) 59(1948), pp. 937-840. Math. Rev. 9(1948), p. 448.

[5] Browder, F.E. - Existence of periodic solutions for nonlinear equations of evolution. Proc. N.A.S. 53(1965), pp. 1100 - 1103.

[6] Browder, F.E. - Nonexpansive nonlinear operators in a Banach space. Proc. Nat. Acad. Sci. 54(1965), pp. 1041-1044.

[7] Browder, F.E. - On the iteration of transformations in noncompact minimal dynamical system. Proc. Am. Math. Soc. 9(1958), pp. 773-780.

[8] Browder, F.E. - Periodic solutions of nonlinear equations of evolution in infinite dimensional spaces. Lecture series in differential equations, session 3, University of Maryland (1966).

[9] Browder, F.E. - Nonlinear equations of evolution. Ann. of Math. 80(1964), pp. 485-523.

[10] Browder, F.E. and Petryshyn, W.V. - Construction of fixed points of nonlinear mappings in Hilbert space. (to appear).

[11] Browder, F.E. and Petryshyn, W.V. - The solution by iteration of linear functional equations in Banach spaces. Bull. Am. Math. Soc. (1966).

[12] Cohen, H. - On fixed points of commuting functions. Proc. Amer. Math. Soc. 15(1964), pp. 293-296.

[13] Clarkson, J.A. - Uniformly convex spaces. Trans. Am. Math. Soc. 40(1936), pp. 296-414.

[14] Day, M.M. - Normed Linear Spaces. Springer (1962).

[15] DeMarr, R. - Common fixed points for commuting contraction mappings. Pac. J. Math. 13(1963), pp. 1139-1141.

[16] Edelstein, M. - On nonexpansive mappings of Banach spaces. Proc. Cambridge Phylos. Soc. 60(1964), pp. 439-447.

[17] Göhde, D. - Über fixpunkte bei stetigen Selbstabbildungen mit kompakten Iterierten. Math. Nach. 28(1964), pp. 45-55.

[18] Göhde, D. - Zum Prinzip der kontraktiven Abbildung. Math. Nach. 30(1965), pp. 251-258.

[19] Huneke, J.P. - Two counterexamples to a conjecture on commuting continuous functions on the closed unit interval. Notices Amer. Math. Soc. 14(1967), p. 284.

[20] Kakutani, S. - Weak topology and regularity of Banach spaces. Proc. Imp. Acad. Tokyo 15(1939), pp. 169-173.

[21] Kakutani, S. - Two fixed-point theorems concerning bicompact convex sets. Proc. Imp. Acad. Tokyo 14(1938), pp. 242-245.

[22] Kakutani, S. and Yosida, K. - Operator theoretical treatment of Markov's process and mean ergodic theorem. Ann. of Math. 42(1941), pp. 188-228.

[23] Kasahara, S. - A fixed point theorem for contraction mappings in a uniformly convex normed space. Proc. Imp. Acad. Tokyo 42(1966), pp. 593-595.

[24] Kirk, W.A. - A fixed point theorem for mappings which do not increase distance. Am. Math. Monthly 72(1965), pp. 1004-1006.

[25] Markov, A. - Quelques théorèmes sur les ensembles abeliens. Dokl. Akad. Nauk SSSR (N.S.) 10(1936), pp. 311-314.

[26] Milman, D.P. - On some criteria for regularity of spaces of the type (B). Dokl. Akad. Nauk. SSSR (N.S.) 20(1938), pp. 243-246.

[27] Pettis, B.J. - A proof that every uniformly convex space is reflexive. Duke Math. J. 5(1939), p. 249-253.

[28] Petryshyn, W.V. - Construction of fixed point of demicompact mappings in Hilbert space. Journal Math. Anal. Appl. 14(1966) pp.

[29] Shields, A. - On fixed points of commuting analytic functions. Proc. Amer. Math. Soc. 15(1964),pp. 703-706.

[30] deFigueiredo, D.G. and Karlovitz, L.A. - An application of the Kakutani-Yosida ergodic theorem to the approximate solution of linear functional equations. University of Maryland Tech. Note, June 1967.

CHAPTER II

ITERATION METHODS FOR OBTAINING FIXED POINTS OF NONEXPANSIVE MAPPINGS

1. INTRODUCTION

Let E be a Banach space and T a nonexpansive mapping of E into itself. We are concerned with the question of obtaining fixed points of T by the iteration method

$$x_{n+1} = Tx_n \ , \quad n = 0, 1, 2, \ldots, \tag{1}$$

where x_o is some given initial approximation. The basic problem here is the convergence of the sequence $\{x_n\}$ defined in (1) . It is well known that if the mapping T is a contraction (i.e., $\|Tx - Ty\| \leq k \|x - y\|$ for all $x,y \in E$ and some fixed k, $0 < k < 1$), then the sequence $\{x_n\}$ defined in (1) converges to the unique fixed point of T . However, if T is only a nonexpansive mapping (i.e., $\|Tx - Ty\| \leq \|x - y\|$) there is no guarantee that the sequence $\{x_n\}$ converges. In fact, it does not in general. This is shown by the following two simple examples. (i) T is a translation in the real line. (ii) T is a rotation of the plane around the origin. In the last example the sequence $\{x_n\}$ does not converge as long as the initial approximation x_o is not the origin.

__Definition__. A mapping T from a Banach space E into itself is said to be __asymptotically regular__ if $T^{n+1}x - T^n x$ converges to 0 as $n \to \infty$, for all points x in E . (Browder and Petryshyn [3])

Using this definition and the fact that $x_n = T^n x_o$ we obtain the following necessary condition for the convergence of the sequence $\{x_n\}$. "If the sequence $\{x_n\}$ defined in (1) is convergent for each initial approximation x_o, then the mapping T is asymptotically regular".

A nonexpansive mapping T is not necessarily asymptotically regular. However, the determination of the fixed points of T can, in some cases, be replaced by the same problem for an asymptotically regular mapping. Namely, the following result holds, Browder and Petryshyn [3] .

Proposition II.1. Let E be a uniformly convex Banach space and T a nonexpansive mapping of E into itself. Suppose that the set $F(T)$ of fixed points of T is non-empty. Then the mapping $T_\lambda = \lambda I + (1 - \lambda)T$, for $0 < \lambda < 1$, is nonexpansive and asymptotically regular. Moreover $F(T) = F(T_\lambda)$.

Proof. It is immediate that $F(T) = F(T_\lambda)$. It is also easy to see that T_λ is nonexpansive as a consequence of the nonexpansiveness of T. Now let $u \in F(T)$. Then

$$\|T_\lambda^{n+1}x - u\| = \|T_\lambda^{n+1}x - T_\lambda u\| \leq \|T_\lambda^n x - u\| .$$

So the sequence $\{\|T_\lambda^n x - u\|\}$ is non-increasing. Thus it converges to some $d \geq 0$. If $d = 0$ it follows immediately that $T_\lambda^{n+1}x - T_\lambda^n x \to 0$. Suppose now that $d > 0$. The following identity holds

$$(2) \quad T_\lambda^{n+1}x - u = \lambda T_\lambda^n x + (1 - \lambda)TT_\lambda^n x - u = \lambda(T_\lambda^n x - u) + (1 - \lambda)(TT_\lambda^n x - u) .$$

Since $\|T_\lambda^{n+1}x - u\| \to d$, $\|T_\lambda^n x - u\| \to d$ and $\|TT_\lambda^n x - u\| \leq \|T_\lambda^n x - u\|$, it follows using Proposition I.1 in (2) that $\|(T_\lambda^n x - u) - (TT_\lambda^n x - u)\| \to 0$. Therefore $\|T_\lambda^{n+1}x - T_\lambda^n x\|$ also converges to zero.

__Remark__. The use of T_λ in place of T for the determination of fixed points was considered by Krasnoselsky [6] for $\lambda = 1/2$ and compact mappings. A general λ was considered by Schaefer [10] in the case of compact mappings and Petryshyn [9] in the case of demicompact mappings, (see definition in Section 3 of this Chapter) .

2. LIMITS OF SUBSEQUENCES $\{T^{n(j)}x\}$.

In this section we shall prove that under certain conditions a limit of a subsequence of iterates $\{T^{n(j)}x\}$ is a fixed point for T . Precisely, the two following results hold. See Browder and Petryshyn [3].

__Proposition II.2__. __Let T be a nonexpansive asymptotically regular mapping in a Banach space E . Suppose that a subsequence $\{T^{n(j)}x_o\}$ converges strongly to some point y. Then y is a fixed point of T and the whole sequence $T^n x_o$ converges strongly to y .__

__Proof__. First we prove that y is a fixed point. Indeed

$$T^{n(j)}x_o \to y \quad \text{implies} \quad (I - T)T^{n(j)}x_o \to (I - T)y \ .$$

On the other hand $(I - T)T^{n(j)}x_o = T^{n(j)}x_o - T^{n(j)+1}x_o \to 0$ because T is asymptotically regular. Thus $(I - T)y = 0$, i.e., y is a fixed point of T . Next we see that the whole sequence converges to y because

$$\|T^{n+1}x_o - y\| \leq \|T^n x_o - y\| \quad \text{for all} \quad n = 1, 2, \ldots$$

To state the next theorem we need the following notion.

Definition. A mapping S of a Banach space into itself is said to be demiclosed if for any sequence $\{x_n\}$ such that $x_n \rightharpoonup x$ (i.e. x_n converges weakly to x) and $Sx_n \to y$ then $y = Sx$.

Proposition II.3. Let T be an asymptotically regular mapping in a Banach space E. Suppose that $I - T$ is demiclosed and there exists a subsequence $\{T^{n(j)}x_o\}$ which converges weakly to some point y. Then y is a fixed point of T.

Proof. Using the fact that T is asymptotically regular we have that $(I - T)T^{n(j)}x_o \to 0$. Since $T^{n(j)}x_o \rightharpoonup y$ and $I - T$ is demiclosed it follows that $(I - T)y = 0$, i.e. y is a fixed point of T.

3. STRONG CONVERGENCE OF THE ITERATES $T^n x$.

In this section we discuss conditions that insure the strong convergence of the iterates $T^n x$. We present first a result of Browder and Petryshyn [3], Theorem II.1, and then a theorem of Edelstein [4], Theorem II.2, for compact operators.

Theorem II.1. Let T be a nonexpansive asymptotically regular mapping in a Banach space E. Suppose that the set $F(T)$ of fixed points of T is non-empty. Let us assume further that T satisfies the following condition:

(α) I-T maps bounded closed sets into closed sets.

Then, for each point x_o *in* E , *the sequence* $\{T^n x_o\}$ *converges strongly to some point in* $F(T)$.

Proof. If y is a fixed point of T it follows that

$$||T^{n+1}x_o - y|| \leq ||T^n x_o - y|| \quad , \quad n = 1, 2, \ldots \quad .$$

So the sequence $\{T^n x_o\}$ is bounded. Let G be the strong closure of $\{T^n x_o\}$. By condition (α) it follows that $(I - T)G$ is closed. This together with the fact that T is asymptotically regular gives that $0 \in (I - T)G$. So there exists $z \in G$ such that $(I - T)z = 0$. But this implies that either $z = T^n x_o$ for some n , or there exists a sequence $\{T^{n(j)}x_o\}$ converging to z . Since z is a fixed point of T we can then conclude that, in either case, the whole sequence $\{T^n x_o\}$ converges to z . The proof is complete.

Remark. Let λ be such that $0 < \lambda < 1$. Let $T_\lambda = \lambda I + (1 - \lambda)T$. T satisfies condition (α) if and only if T_λ also does. To see that just observe that $I - T_\lambda = (1 - \lambda)(I - T)$.

Using this remark and Proposition II.1 we have immediately the following corollary of Theorem II.1.

Corollary II.1. *Let* T *be a nonexpansive mapping of a uniformly convex Banach space* E *into itself*. *Suppose that the set* $F(T)$ *of fixed points of* T *is non-empty*. *Let us also assume that* T *satisfies the following condition*

(α) *$I - T$ maps bounded closed sets into closed sets*.

$$\|T^{n+1}x_o - y\| \leq \|T^n x_o - y\| \quad \text{for all} \quad n = 1, 2, \ldots$$

To state the next theorem we need the following notion.

Definition. A mapping S of a Banach space into itself is said to be demiclosed if for any sequence $\{x_n\}$ such that $x_n \rightharpoonup x$ (i.e. x_n converges weakly to x) and $Sx_n \to y$ then $y = Sx$.

Proposition II.3. *Let T be an asymptotically regular mapping in a Banach space E. Suppose that $I - T$ is demiclosed and there exists a subsequence $\{T^{n(j)}x_o\}$ which converges weakly to some point y. Then y is a fixed point of T.*

Proof. Using the fact that T is asymptotically regular we have that $(I - T)T^{n(j)}x_o \to 0$. Since $T^{n(j)}x_o \rightharpoonup y$ and $I - T$ is demiclosed it follows that $(I - T)y = 0$, i.e. y is a fixed point of T.

3. STRONG CONVERGENCE OF THE ITERATES $T^n x$.

In this section we discuss conditions that insure the strong convergence of the iterates $T^n x$. We present first a result of Browder and Petryshyn [3], Theorem II.1, and then a theorem of Edelstein [4], Theorem II.2, for compact operators.

Theorem II.1. *Let T be a nonexpansive asymptotically regular mapping in a Banach space E. Suppose that the set $F(T)$ of fixed points of T is non-empty. Let us assume further that T satisfies the following condition:*

(α) *I-T maps bounded closed sets into closed sets.*

Then, for each point x_o *in* E , *the sequence* $\{T^n x_o\}$ *converges strongly to some point in* $F(T)$.

Proof. If y is a fixed point of T it follows that

$$\|T^{n+1}x_o - y\| \leq \|T^n x_o - y\| \quad , \quad n = 1, 2, \ldots \quad .$$

So the sequence $\{T^n x_o\}$ is bounded. Let G be the strong closure of $\{T^n x_o\}$. By condition (α) it follows that $(I - T)G$ is closed. This together with the fact that T is asymptotically regular gives that $0 \in (I - T)G$. So there exists $z \in G$ such that $(I - T)z = 0$. But this implies that either $z = T^n x_o$ for some n , or there exists a sequence $\{T^{n(j)} x_o\}$ converging to z . Since z is a fixed point of T we can then conclude that, in either case, the whole sequence $\{T^n x_o\}$ converges to z . The proof is complete.

Remark. Let λ be such that $0 < \lambda < 1$. Let $T_\lambda = \lambda I + (1 - \lambda)T$. T satisfies condition (α) if and only if T_λ also does. To see that just observe that $I - T_\lambda = (1 - \lambda)(I - T)$.

Using this remark and Proposition II.1 we have immediately the following corollary of Theorem II.1.

Corollary II.1. *Let* T *be a nonexpansive mapping of a uniformly convex Banach space* E *into itself*. *Suppose that the set* $F(T)$ *of fixed points of* T *is non-empty*. *Let us also assume that* T *satisfies the following condition*

(α) *$I - T$ maps bounded closed sets into closed sets*.

Then, for each point x_o in E, the sequence $\{x_n\}$ defined by

$$x_{n+1} = \lambda x_n + (1 - \lambda)Tx_n , \quad 0 < \lambda < 1 ,$$

converges strongly to a fixed point of T.

It is very reasonable, at this point, to ask which classes of operators satisfy condition (α). To answer it we introduce the following concept.

Definition. A continuous mapping T from a Banach space E into itself is said to be demicompact if every bounded sequence $\{x_n\}$, such that $\{(I - T)x_n\}$ converges strongly, contains a strongly convergent subsequence $\{x_{n(j)}\}$. See Petryshyn [9], where it is proved that the class of demicompact operators contains, among others, all compact operators. By compact operator we mean a continuous operator mapping bounded sets into relatively compact sets.

Proposition II.4. A demicompact mapping T of a Banach space E into itself satisfies condition (α).

Remark. It was stated in [3] that the converse of Proposition II.4 holds. We believe this is not true. For example, the mapping $T = I$ satisfies trivially condition (α), but it is not demicompact.

Proof of Proposition II.4. Let B be a bounded closed set in E. Let $\{(I - T)x_n\}$ be a sequence in $(I - T)B$ such that $(I - T)x_n \to y$. We claim that there exists a point $x \in B$ such that $(I - T)x = y$. To prove this we use the demicompactness of T to conclude that $\{x_n\}$

contains a convergent subsequence $\{x_{n(j)}\}$. Since B is closed we have that $x_{n(j)} \to x$, where x is some point of B . By the continuity of T it follows that $(I - T)x_{n(j)} \to (I - T)x$. Then $(I - T)x = y$ and the proposition is proved.

In [4] Edelstein established the following result, which had been previously proved by Krasnoselsky [6] and Schaefer [10] for uniformly convex Banach spaces.

<u>Theorem II.2</u>. <u>Let E be a strictly convex Banach space and C a closed convex set in E . Let T be a non-expansive mapping defined in C such that T(C) is a relatively compact set contained in C . Let $T_\lambda = \lambda I + (1 - \lambda)T$, where $0 < \lambda < 1$. Then, for each point x_o in C , the sequence $\{T_\lambda^n x_o\}$ converges strongly to a fixed point of T .</u>

<u>Remark</u>. Observe that we cannot conclude, in this case, that T_λ is asymptotically regular. The reason being that Proposition II.1 does not hold for the wider class of strictly convex spaces. So Theorem II.2 is not a special case of Theorem II.1.

<u>Proof of Theorem II.2</u>. 1°) By the Schauder's fixed point theorem (see Remark 1 below) it follows that the set F(T) of fixed points of T is non-empty. Let $y \in F(T)$. It is easy to see that $F(T) = F(T_\lambda)$. Then, for every point $x \in C$ such that $x \notin F(T)$, we have that

(1) $$\|T_\lambda x - y\| < \|x - y\| .$$

This follows from $T_\lambda x - y = \lambda(x - y) + (1 - \lambda)(Tx - y)$ using strict convexity.

2°) Now we show that the sequence $\{T_\lambda^n x_o\}$ contains a strongly convergent subsequence. Indeed, let K be the convex closure of the set $\overline{T(C)} \cup \{x_o\}$. By Mazur's theorem (see Remark 2 below) it follows that K is compact. Since $\{T_\lambda^n x_o\} \subset K$ it follows that there exists a subsequence $\{T_\lambda^{n(j)} x_o\}$ which converges strongly to some point z.

2°) Finally we claim that $z \in F(T)$. If this is proved, then the theorem will be proved; for, it will follow from (1) that the whole sequence $\{T_\lambda^n x_o\}$ converges to z. Suppose that $z \notin F(T)$. Then we have from (1) that

(2) $$d = ||z - y|| - ||T_\lambda z - y|| > 0$$

for each $y \in F(T)$. From the conclusion of 2°) it follows that

$$T_\lambda^{n(j)+1} x_o - y \to T_\lambda z - y \quad .$$

So, for all $n(j)$ sufficiently large we have

$$||T_\lambda^{n(j)+1} x_o - y|| \leq ||T_\lambda z - y|| + \frac{d}{2} \quad .$$

This together with (1) gives

(3) $$||T_\lambda^k x_o - y|| \leq ||T_\lambda z - y|| + \frac{d}{2}$$

for all k sufficiently large. From (2) and (3) follows

$$||T_\lambda^k x_o - y|| \leq ||z - y|| - \frac{d}{2}$$

for all k sufficiently large. This however contradicts the fact that $T^{n(j)}x_o \to z$.

Remark 1. Schauder fixed point theorem states that a compact mapping from a bounded closed convex set of a Banach space into itself has a fixed point. For a proof, see Schauder [11], or Cronin [13] or Chapter IV of these Lecture Notes. Observe that we have considered above the special case of a compact nonexpansive mapping. For this case, a simple proof of the existence of fixed points was given in Section 10 of Chapter I.

Remark 2. Mazur's theorem used above states: "Let K be a compact subset of a Hausdorff complete locally convex space (in particular, a Banach space). Then the convex closure of K is compact. See Bourbaki, "Espaces Vectoriels Topologiques", Chapter II, §4.

4. WEAK CONVERGENCE OF THE ITERATES $T^n x_o$

In this section we study conditions that give the weak convergence of the sequence $\{T^n x_o\}$ of the iterates of a nonexpansive mapping T .

Theorem II.3. Let E be a reflexive Banach space and T a nonexpansive asymptotically regular mapping from E into itself. Suppose that $I - T$ is demiclosed, and that the set $F(T)$ of fixed points of T is non-empty. Then, for each point x_o in E , every subsequence of $\{T^n x_o\}$ contains a further subsequence which converges weakly to a fixed point of T . In particular, if $F(T)$ consists of precisely one point then the whole sequence $\{T^n x_o\}$ converges weakly to this point. (Browder-Petryshyn [3])

Proof. Let $y \in F(T)$. By the nonexpansiveness of T we have $\|T^n x_o - y\| \leq \|x_o - y\|$. So the sequence $\{T^n x_o\}$ is bounded. Then the reflexivity of E implies that every subsequence of $\{T^n x_o\}$ contains a further subsequence which converges weakly. By Proposition II.3 we have that the limit of this last subsequence is a fixed point of T. If $F(T)$ contains only one point y then the whole sequence must converge weakly to y.

Remark. If E is a Hilbert space the hypothesis that $I - T$ is demiclosed in the previous theorem is superfluous. In fact the following result holds.

Proposition II.5. Let E be a Hilbert space and T a nonexpansive mapping of E into itself. Then $I - T$ is demiclosed.

This proposition is a special case of the following theorem. To state it we need the following notion.

Definition. A mapping S from a Hilbert space E into itself is said to be monotone if $(Sx - Sy, x - y) \geq 0$ for all $x,y \in E$.

Remark. Monotone mappings have been studied extensively by Browder; see, for example, [1] and [2]. This notion was apparently introduced independently by many authors, Kachurovsky and Vainberg [5], Zarantonello [12], Minty [7] and Duffin [14].

Example. If T is nonexpansive then $I - T$ is monotone.

__Theorem II.4.__ __Let S be a monotone continuous mapping of a Hilbert space E into itself.__ __Then S is demiclosed.__

__Proof.__ Let $x_n \in E$ such that $x_n \rightharpoonup x$ (i.e. x_n converges weakly to x) and $Sx_n \to y$. We wish to prove that $Sx = y$. By monotonicity

$$(Sx_n - Sz \, , \; x_n - z) \geq 0$$

for all $z \in E$. Taking limits we obtain

$$(y - Sz, \; x - z) \geq 0 \quad . \tag{2}$$

Now let w be an arbitrary vector in E and $t > 0$. Taking $z = x + tw$ in (2) we obtain

$$(y - S(x + tw), \; w) \geq 0 \quad .$$

Making $t \to 0$ and using the continuity of S (note: this is the only place where continuity is used) we have $(y - Sx, w) \geq 0$ for all $w \in E$. This implies $(y - Sx, w) = 0$ and finally $y = Sx$.

__Remark.__ The continuity of S in the preceding theorem can be replaced by the weaker condition that S is continuous from line segments in H to the weak topology of E . This type of continuity has been named __hemicontinuity__ by Browder. It so happens that in the applications to partial differential equations it is not hard to verify that certain mappings are hemicontinuous.

Theorem II.3 above has been considerably strengthened in the case of Hilbert spaces by Opial [8] .

Theorem II.5. *Let E be a Hilbert space and T a nonexpansive asympototically regular mapping of E into itself. Suppose that the set F of fixed points of T is non-empty. Then, for each point x_o in H, the sequence $\{T^n x_o\}$ converges weakly to a point of F.*

Before proving this theorem we establish the following result.

Proposition II.6. *Let E be a Banach space and ϕ a convex continuous real-valued function in E. Then ϕ is weakly lower semicontinuous.*

Proof. We have to prove that, for each given real number a, the set

$$V_a = \{x \in E : \phi(x) \leq a\}$$

is weakly closed. This however is a consequence of the fact that V_a is closed and convex.

Proof of Theorem II.5. 1°) Since F is non-empty we see that a ball B about some fixed point and containing x_o is mapped into itself by T ; consequently B contains the sequence of iterates $T^n x_o$. So, we restrict ourselves to mappings of a ball into itself. The set F of fixed points is then bounded, closed and convex. (See Section 10 of Chapter I). So the set F is weakly compact.

2°) Let us define in F the following mapping

$$\phi : F \to R^+ , \quad (R^+ = \text{non-negative real numbers})$$

(3) $$\phi(y) = \inf_n \|T^n x_o - y\| = \lim_{n\to\infty} \|T^n x_o - y\| .$$

(In (3) lim = inf because the sequence $\{\|T^n x_o - y\|\}$ is non-increasing). The mapping ϕ so defined is continuous. Indeed,

$$\phi(y') = \lim \|T^n x_o - y'\| \leq \lim \|T^n x_o - y\| + \|y - y'\| = \phi(y) + \|y - y'\|$$

from this inequality follows $|\phi(y) - \phi(y')| \leq \|y - y'\|$. On the other hand, ϕ is a convex function. In fact

$$\begin{aligned}\phi(\lambda y + (1-\lambda)y') &= \lim \|T^n x_o - (\lambda y + (1-\lambda)y')\| \\ &\leq \lambda \lim \|T^n x_o - y\| + (1-\lambda) \lim \|T^n x_o - y'\| \\ &= \lambda\phi(y) + (1-\lambda)\,\phi(y') \quad .\end{aligned}$$

So, using Proposition II.6, we conclude that ϕ is weakly lower semi-continuous.

3°) In view of the conclusions of 1°) and 2°) we can apply Proposition I.4. Then we conclude that there exists a point $\bar{y}$ in F such that

$$\phi(\bar{y}) = d = \inf_{y \in F} \phi(y) \quad .$$

Now we claim that $\bar{y}$ is unique. In fact, suppose this is not so. i.e., there exists another point $y' \in F$ such that $\phi(y') = d$. By the convexity of ϕ it follows that $\phi(\lambda\bar{y} + (1-\lambda)y') = d$ for all $0 \leq \lambda \leq 1$. So $\|x_n - \bar{y}\| \to d$, $\|x_n - y'\| \to d$ and $\|x_n - (\lambda\bar{y} + (1-\lambda)y')\| \to d$. By uniform convexity it follows that $\|(x_n - \bar{y}) - (x_n - y')\| \to 0$, i.e. $\bar{y} = y'$.

4°) Finally we prove that the sequence $\{T^n x_o\}$ converges weakly to $\bar{y}$. To this effect we prove that given any subsequence of $\{T^n x_o\}$, it contains a further subsequence which converges to $\bar{y}$. In fact, given any subsequence of $\{T^n x_o\}$, it follows that it contains a further subsequence $\{T^{n(j)} x_o\}$ which converges weakly to some point z. We claim that $z = \bar{y}$. Indeed, we have

$$\|T^{n(j)}x_o - \bar{y}\|^2 = \|T^{n(j)}x_o - z\|^2 + \|z - \bar{y}\|^2 + 2\,\mathrm{Re}(T^{n(j)}x_o - z, z - \bar{y}) .$$

Taking limits we obtain

$$\phi(\bar{y}) = \phi(z) + \|z - \bar{y}\|$$

which is possible only if $z = \bar{y}$. The proof of the theorem is complete.

Remark. Opial [8] has proved this theorem for the class of Banach spaces that have a weakly continuous duality mapping. This class contains all Hilbert spaces but it does not include all uniformly convex Banach spaces. For example, L^p, $p \neq 2$, does not have a weakly continuous duality mapping. (See Browder and deFigueiredo, reference [12] of Chapter III)

5. ON THE CONVERGENCE OF THE FIXED POINTS OF THE CONTRACTIONS rT

Let T be a nonexpansive mapping of a Banach space E into itself. Let $\{r_n\}$ be a sequence of real numbers such that $r_n \to 1$ and $0 < r_n < 1$. For each n the mapping $T_n = r_n T$ is a contraction, and consequently it has a unique fixed point x_n, i.e. $T_n x_n = x_n$. It is natural to ask if the sequence $\{x_n\}$ converges to a fixed point of T. One cannot expect, in general, an affirmative answer to this question. For there are nonexpansive mappings which do not have a fixed point. Theorem I.10 provides some answer for a certain class of nonexpansive mappings in Banach spaces. The situation in the Hilbert space case is completely settled by the following result of F.E. Browder.

Theorem II.6. Let C be a bounded closed convex set in a Hilbert space H, and T a nonexpansive mapping of C into itself. Suppose that $0 \in C$. Let $\{r_n\}$ be a sequence of real numbers such that $r_n \to 1$, and $0 < r_n < 1$. Let x_n be the unique fixed point of $T_n = r_n T$. Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Remark. It is easy to see that the sequence $\{x_n\}$ contains a subsequence which converges weakly to a fixed point of T. In fact, by the weak compactness of C it follows that there exists a subsequence $\{x_{n(j)}\}$ which converges weakly to some point z. On the other hand, since

$$(I - T)x_n = r_n Tx_n - Tx_n = (r_n - 1)Tx_n$$

it follows that $(I - T)x_n$ converges strongly to 0. Since $I - T$ is demiclosed (see Proposition II.5) we conclude that $(I - T)z = 0$, i.e. z is a fixed point of T.

Proof of theorem II.6. 1°) Let F be the set of fixed points of T. It has been already proved (Proposition I.3) that F is closed and convex. Let w_o be the (unique) point in F closest to the origin.

We claim that the sequence x_n converges strongly to w_o.

2°) First we prove that if $x_{n(j)} \rightharpoonup z$ then $z = w_o$ and $x_{n(j)} \to w_o$. For any s, $0 < s < 1$, we have

$$(1 - s)x_s + sWx_s = 0 , \quad \text{where} \quad W = I - T ,$$

and

$$(1 - s)w + sWw = (1 - s)w , \quad \text{for} \quad w \in F$$

By subtraction

$$(1 - s)(x_s - w) + s(Wx_s - Ww) = -(1 - s)w .$$

This last expression implies

(1) $$(1 - s) \|x_s - w\|^2 + s(Wx_s - Ww, x_s - w) = -(1 - s)(w, x_s - w)$$

Since W is monotone, it follows from (1) that

$$\|x_s - w\|^2 \leq (w, w - x_s) ,$$

for all $w \in F$. Then for $w = w_o$ we get

(2) $$\|x_s - w_o\|^2 \leq (w_o, z - x_s) + (w_o, w_o - z)$$

In the preceding Remark we have seen that $z \in F$. So $\|w_o\| \leq \|z\|$. On the other hand we have for $0 \leq \lambda \leq 1$

$$\|w_o\|^2 \leq \|\lambda z + (1 - \lambda)w_o\|^2 = \|w_o\|^2 + \lambda^2 \|z - w_o\|^2 + 2 \text{ Re } \lambda(w_o, z - w_o) .$$

From this inequality it follows that $(w_o, w_o - z) \leq 0$. This used in (2) gives

$$\text{(3)} \qquad \|x_s - w_o\| \leq (w_o, z - x_s) .$$

Using (3) with x_s replaced by $x_{n(j)}$ and taking limits we obtain $\|x_{n(j)} - w_o\| \to 0$.

3°) Now we prove that the whole sequence $\{x_n\}$ converges strongly to w_o. Given any subsequence of $\{x_n\}$ we have seen by the above remark that it contains a weakly convergent subsequence. By 2°) above this last subsequence converges strongly to w_o. Thus the whole sequence has to converge strongly to w_o.

6. AN ITERATION METHOD FOR GENERAL NONEXPANSIVE MAPPINGS IN HILBERT SPACES.

In the previous section we have considered a non-expansive mapping T of a bounded closed convex subset C of a Hilbert space H into itself. We have seen that the sequence of the fixed points x_n of the mappings $T_n = r_n T$, $0 < r_n < 1$, converges strongly to a fixed point of T, as $r_n \to 1$. In the present section we propose an iteration method for obtaining this fixed point without the knowledge of the fixed points x_n .

Theorem II.7. Let C be a bounded closed convex subset of a Hilbert space H . Suppose $0 \in C$. Let T be a nonexpansive mapping of C into itself, and let $T_n = \frac{n}{n+1} T$, $n = 1, 2, \ldots$. Then for each $y_o \in C$ the sequence

$$(1) \qquad y_n = T_n^{n^2} y_{n-1} \qquad n = 1, 2, \ldots$$

converges strongly to a fixed point of T.

Proof. For each n the mapping T_n has a unique fixed point x_n . For a given initial approximation x we have the following estimate for the error

$$||T_n^k x - x_n|| \leq \frac{\left(\frac{n}{n+1}\right)^k}{1-\frac{n}{n+1}} ||T_n x - x||$$

This implies

$$(2) \qquad ||T_n^k x - x_n|| \leq K \frac{n^k}{(1+n)^{k-1}} \quad ,$$

where K is the diameter of C .

By Theorem II.6, the sequence $\{x_n\}$ converges strongly to some fixed point w_o of T . We claim that the sequence $\{y_n\}$ defined in (1) converges strongly to w_o . In fact, we have

$$||y_n - w_o|| \leq ||T_n^{n^2} y_{n-1} - x_n|| + ||x_n - w_o|| \quad .$$

Using the estimate (2) we obtain

$$\|y_n - w_o\| \leq K\, n^{n^2}/(1+n)^{n^2-1} + \|x_n - w\| \quad . \tag{3}$$

It is easy to check that the right hand side of (3) converges to 0 as $n \to +\infty$.

Remark. It can be seen that the sequence, for given $z_o \in C$,

$$z_n = T_n^n \, z_{n-1} \qquad n = 1, 2, \ldots$$

does not necessarily converge.

REFERENCES

[1] Browder, F.E., Problèmes Non-linéaires. Seminaire de Mathématiques Supérieures. Université de Montréal 1965.

[2] Browder, F.E., Existence and uniqueness theorems for solutions of nonlinear boundary value problems. Proc. Symp. Appl. Math. (AMS) vol. 17(1965) pp. 24-49

[3] Browder, F.E. and Petryshyn, W.V., The solution by iteration of non-linear functional equations in Banach spaces. Bull. Am. Math. Soc. 72(1966) pp. 571-575.

[4] Edelstein, M., A remark on a theorem of Krasnoselsky. Am. Math. Monthly 73(1966) pp. 509-510.

[5] Kachurovsky, R.I. and Vainberg, M.M., On the variational theory of nonlinear operators and equations. Dokl. Akad. Nauk. 129(1959) pp. 1199-1202.

[6] Krasnoselsky, M.A., Two observations about the method of successive approximations. Uspehi Mat. Nauk. 19(1955) pp. 123-127.

[7] Minty, G.J., Monotone (nonlinear) operators in Hilbert spaces. Duke Math. Journal, vol. 29(1962) pp. 341-346.

[8] Opial, Z., Weak convergence of the sequence of successive approximations for nonexpansive mappings. (to appear in the Bull. Am. Math. Soc.).

[9] Petryshyn, W.V., Construction of fixed points of demicompact mappings in Hilbert spaces. Journal Math. Annal. Appl. 14(1966) pp. 276-284.

[10] Schaefer, H., Über die Methode sukzessiver Approximationen. Jahresbericht Deutsch. Math. Verein. 59(1957) pp. 131-140.

[11] Schauder, J., Der fixpunktsatz in Funktionalraümen. Studia Math. 2(1930) pp. 171-180.

[12] Zarantonello, Solving functional equations by contractive averaging. MRC Technical Report (Univ. of Wisconsin at Madison) No.160(1960).

[13] Cronin, J., Fixed points and topological degree in nonlinear analysis. Math. Surveys No.11. Amer. Math. Soc. (1964).

[14] Duffin, R., Nonlinear networks, IV, Proc. Amer. Math. Soc., 1(1950) pp. 233-240.

CHAPTER III

MONOTONE OPERATORS

1. INTRODUCTION

This chapter is devoted to the theory of the monotone operators (see definition below). We will be concerned with the existence of solutions for functional equations of the form $Tu = f$, where T is a monotone operator. The basic result of this chapter is Theorem III.1 due to Browder and Minty. In order to state it we need some definitions.

Let E be a real Banach space, E^* its dual space and T a non-linear (or rather: a not necessarily linear) operator mapping E into E^* . The operator $T : E \to E^*$ is monotone if $(Tx - Ty, x - y) \geq 0$ for all $x,y \in E$. The parenthesis $(\; , \;)$ denotes the duality pairing between E^* and E . The operator $T : E \to E^*$ is hemicontinuous if it is continuous from line segments of E to the weak topology of E^* . The operator $T : E \to E^*$ is said to be coercive if $(Tx,x)/\|x\| \to +\infty$ as $\|x\| \to +\infty$.

Theorem III.1. *Let E be a reflexive Banach space, E^* its dual space and T a (not necessarily linear) operator mapping E into E^* . Suppose that T is monotone, hemicontinuous and coercive. Then the mapping T is surjective, i.e., $T(E) = E^*$.*

Remark 1. Let us consider the special case of Theorem III.1 when E is the set of real numbers. Then the mapping T is a real-valued function defined over the reals. In this case hemicontinuity coincides with continuity in the usual sense. Monotonicity in the sense defined above means that the function T is monotonically non-decreasing, i.e. $Tx \geq Ty$ for $x \geq y$. The hypothesis that T is coercive implies that $Tx \to +\infty$ as $x \to +\infty$ and $Tx \to -\infty$ as $x \to -\infty$. The above theorem merely says that such a function T maps the reals onto the reals.

Remark 2. If E is a Hilbert space, we have $E = E^*$ by the usual identification of E^* with E using the Riesz-Fréchet representation theorem. Then, in this case, the duality pairing is the inner product in E. For Theorem III.1 in Hilbert spaces see Browder [5], [6], [7] and [8], Minty [27] and Dolph and Minty [14].

Remark 3. A mapping $T : E \to E^*$ is said to be strongly monotone if there is a constant $c > 0$ such that

$$(1) \qquad (Tx - Ty, x - y) \geq c\|x - y\|^2$$

for all $x,y \in E$. It is easy to see that a strongly monotone mapping is coercive. In fact, using (1) with $y = 0$ we have $(Tx - T0, x) \geq c\|x\|^2$, which implies $(Tx,x) \geq (c\|x\| - \|T0\|)\|x\|$. The last inequality implies the coerciveness of T. For strongly monotone mappings we have the following corollary of Theorem III.1.

Corollary III.1. Let E be a reflexive Banach space and E^* its dual. Let $T : E \to E^*$ be a hemicontinuous strongly monotone operator mapping E into E^*. Then the operator T is bijective and T^{-1} is continuous from the strong topology of E^* to the strong topology of E.

Proof. By Theorem III.1 and Remark 3 it follows immediately that T is surjective. On the other hand (1) implies

$$(2) \qquad \|Tx - Ty\| \geqslant c\,\|x - y\|$$

for all $x,y \in E$. So the mapping T is injective. Consequently it is bijective. The continuity of T^{-1} is immediate from (2).

Remark 4. Suppose that, in Corollary III.1, we replace the hemicontinuity of T by the continuity of T from the strong topology of E to the strong topology of E^*. Then the conclusion of that corollary is that T is a homeomorphism between E and E^* with the norm topology.

2. THEOREM III.1 FOR LINEAR MAPPINGS

In this section we consider an analogue of Theorem III.1 in the special case of a linear mapping T. First we observe that if T is linear then the hypothesis of hemicontinuity of T is entirely superfluous. Indeed, the mere linearity of T implies that $T(x + tz)$ converges to Tx as $t \to 0$. In Theorem III.2 below the continuity of the mapping T is replaced by the requirement that T be closed. The proof of Theorem III.2 makes use of the adjoint operator T^* in a very essential way. So such a proof does not extend to the case of a nonlinear T.

We remark that, for simplicity, we work in a real Banach space. The method works also for complex Banach spaces.

A mapping $T : E \to F$ from a Banach space E into another Banach space F is said to be closed if $x_n \to x$ in E and $Tx_n \to y$ in F implies $Tx = y$. Every continuous mapping is also closed; but not conversely.

Theorem III.2. Let E be a reflexive Banach space, and E^* its strong dual, i.e. the dual space with the norm topology. Let T be a closed linear mapping from E into E^*. Suppose that there is a positive constant c such that $|(Tx,x)| \geq c\|x\|^2$ for all $x \in E$. Then T is a homeomorphism between E and E^*.

To prove this theorem we need some facts about the adjoint of a continuous linear mapping. For completeness we present succinctly these facts in the sequel.

Let X and Y be two Banach spaces and T a continuous linear mapping from X into Y. Let X^* and Y^* denote the dual spaces of X and Y respectively. The adjoint operator $T^* : Y^* \to X^*$ is defined, for each $y^* \in Y^*$, by the relation $(T^*y^*,x) = (y^*,Tx)$ for all $x \in X$. It is an easy matter to check that T^*y^* is actually an element of X^*; so T^* is well defined. It also follows that T^* is a linear mapping. Moreover T^* is a continuous mapping from the strong topology of Y^* to the strong topology of X^*.

Let A be a linear subspace of a Banach space X. The annihilator A° of A is defined as the set

$$\{y^* \in X^* : (y^*,x) = 0 \quad \text{for all} \quad x \in A\} .$$

It is easy to see that A° is a linear subspace. Moreover if X is reflexive then $A^{\circ\circ} = \bar{A}$. The following result is also readily proved.

Lemma III.1. Let X and Y be two Banach spaces, X^* and Y^* their dual spaces. Let T be a continuous linear mapping from X into Y, and T^* its adjoint operator. Then the null space $N(T^*)$ of T^* is equal to $R(T)^\circ$, where $R(T)$ is the range of T.

Proof of Theorem III.2. 1°) From the inequality $|(Tx,x)| \geq c\,\|x\|^2$ it follows that $\|Tx\| \geq c\,\|x\|$ for all $x \in E$. This implies that T is injective, T^{-1} is continuous and $R(T)$ is closed. Thus by the Closed Graph Theorem it follows that the mapping T is continuous. So the proposition will be proved if we show that T is surjective, i.e. $R(T) = E^*$.

2°) We first observe that the adjoint T^* also maps E into E^*, because E is reflexive. Moreover T^* satisfies the inequality $|(T^*x,x)| \geq c\,\|x\|^2$, which implies that T^* is injective, i.e., $N(T^*) = \{0\}$).

3°) By Lemma III.1 we have $N(T^*) = R(T)^\circ$, which gives $N(T^*)^\circ = R(T)^{\circ\circ}$. Using the conclusion of 2°) and the fact that $R(T)^{\circ\circ} = \overline{R(T)}$ in the last relation we obtain $E^* = \overline{R(T)}$. Finally from the fact that $R(T)$ is closed it follows $R(T) = E^*$.

Remark 5. The basic assumption of Theorem III.2, namely

$$|(Tx,x)| \geq c\,\|x\|^2, \quad \text{for all} \quad x \in E,$$

will be fulfilled if we assume that $(Tx,x) \geq c\,\|x\|^2$.

Remark 6. Many results of the type of Theorem III.2 may be found in Browder [10].

Remark 7. An extremely interesting result, akin to Theorem III.2, is the following theorem due to Friedrichs [16]. See also Mikhlin [26].
"Let T be a linear operator defined in a dense subspace $D(T)$ of a Hilbert space E. Suppose that $(Tx,x) \geq \|x\|^2$ for all $x \in D(T)$.

Then T is surjective". This theorem is not contained in our Theorem III.2.

We now present an application of Theorem III.2 to the problem of representation of continuous linear functionals in a Banach space by means of certain bilinear forms.

Theorem III.3. *Let E be a reflexive Banach space and B a bilinear form in E. Suppose that*

(i) B is continuous, i.e. $|B(u,v)| \leq K \|u\| \|v\|$ for all $u,v \in E$

(ii) B is coercive, i.e. $|B(u,v)| \geq c \|u\|^2$ for all $u \in E$.

Then, for each $\ell \in E^*$, there exist a unique $x \in E$ such that $\ell(u) = B(u,x)$.

Remark 8. A bilinear form $B : E \in E \to C$ is said to be *strongly coercive* if there exists a constant $c > 0$ such that $B(u,u) \geq c \|u\|^2$ for all $u \in E$. It is immediate that every strongly coercive bilinear form is also coercive. Theorem III.3 with the hypothesis of coerciveness replaced by strong coerciveness and E = Hilbert space reduces to the well-known Lax-Milgram lemma. See Lax-Milgram [22], Nirenberg [28] or Yosida [36].

Remark 9. A constructive proof of Theorem III.3 in the Hilbert space case has been given by Petryshyn [29] and Hildebrandt and Wienholtz [17].

Remark 10. A proof of Theorem III.3, distinct from the one given here, has been recently given by Sauer [31]. In that paper Sauer shows that the assumption that E be reflexive is actually an essential one.

Proof of Theorem III.3. Let E^* be the dual space of E. Let us consider E^* equipped with the norm topology. We define a mapping T from E into E^* in the following way. To each $x \in E$ we associate the functional $\ell_x : E \to R^1$, $\ell_x(y) = B(y,x)$. It is easily seen that ℓ_x is linear. Furthermore Hypothesis (i) implies that ℓ_x is continuous. So the mapping $T : E \to E^*$ is well defined. The linearity of T follows from the bilinearity of the form B. And Hypothesis (i) implies that T is continuous. Finally Hypothesis (ii) implies that T satisfies the inequality

$$|(Tx,x)| \geq c\,||x||^2$$

for all $x \in E$. Thus applying Theorem III.2 we conclude that the mapping $T : E \to E^*$ is bijective. This means that given $\ell \in E^*$ there exists a unique $x \in E$ such that $Tx = \ell$. This implies $\ell(u) = (Tx,u) = B(u,x)$. The proof is complete.

3. THEOREM III.1 : THE FINITE DIMENSIONAL CASE

In this section we assume that E is a finite dimensional Banach space. Before stating Theorem III.1 for this case we prove the following result.

Lemma III.2. Let E be a finite dimensional Banach space, and E^* its dual space. Let T be a hemicontinuous monotone mapping from E into E^*. Then T is continuous.

Remark. We consider E^* endowed with the norm-topology. (In fact we could consider any other locally convex topology. For all these topologies are the same, as a consequence of the finite dimensionality of E.)

Proof. 1°) We first prove that if $\{x_n\}$ is a convergent sequence in E then there exists $M > 0$ such that $\|Tx_n\| \leq M$ for all n. In fact, assume this is not the case; then there exists a subsequence of $\{x_n\}$, that we denote by $\{x_n\}$ again, such that $\|Tx_n\| \to +\infty$. Let x be an arbitrary vector in E. By monotonicity we have

$$(1) \qquad (Tx_n - Tx, x_n - x) \geq 0 .$$

Setting $y_n = Tx_n / \|Tx_n\|$ we have from (1)

$$(2) \qquad \left(y_n - \frac{Tx}{\|Tx_n\|}, x_n - x\right) \geq 0 .$$

Since $\{y_n\}$ is a sequence of vectors with unit norm in a finite dimensional Banach space, it follows that there exists a convergent subsequence. Denote this subsequence by $\{y_n\}$ again and its limit by y. Taking limits in (2) we obtain

$$(3) \qquad (y - 0, x_o - x) \geq 0$$

where $x_o = \lim x_n$. Since x is arbitrary, (3) implies that $y = 0$, which contradicts the fact that $\|y\| = 1$.

2°) We now prove that T is continuous, i.e., if $x_n \to x_o$ it follows that $Tx_n \to Tx_o$. To do this we show that every subsequence of $\{Tx_n\}$ contains a further subsequence which converges to Tx_o. Indeed, given any subsequence of $\{Tx_n\}$ it follows from 1°) that this subsequence is bounded. By finite dimensionality it follows that it contains a further subsequence which converges. Denote by $\{Tx_n\}$ this last subsequence and by w its limit. To complete the proof we shall prove that $w = Tx_o$. Let x be an arbitrary vector in E. By monotonicity

$$(Tx - Tx_n , \ x - x_n) \geq 0$$

Taking limits it follows

$$(Tx - w , \ x - x_o) \geq 0 \tag{4}$$

Let x be of the form $x = x_o + tv$, $t \geq 0$. So in (4) we obtain

$$(T(x_o + tv) - w , \ tv) \geq 0$$

which gives

$$(T(x_o + tv) - w, \ v) \geq 0 .$$

Making $t \to 0$ and using the hemicontinuity of T we have

$$(Tx_o - w, \ v) \geq 0 . \tag{5}$$

Since (5) holds for all $v \in E$, it follows $Tx_o = w$. The proof is complete.

In view of Lemma III.2, Theorem III.4 below implies Theorem III.1 in the finite dimensional case.

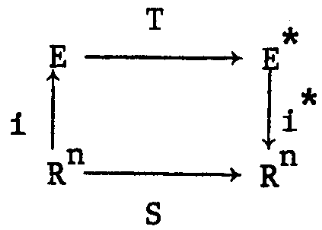
<u>Theorem III.4</u>. <u>Let E be a finite dimensional Banach space, and E^*</u>

its dual space. Let T be a continuous monotone coercive operator mapping E into E^*. Then T is surjective, i.e., $T(E) = E^*$.

Proof. 1°) We need to prove the theorem only in the case when E is the n-dimensional Euclidean space R^n. To prove this claim we assume that the theorem has been proved in this special case (i.e. $E = R^n$) and show that it is true in general. First we observe that any n-dimensional Banach space is linearly isomorphic to R^n. That is, there exists a linear mapping $i : R^n \to E$ such that

$$c_1|x| \leq \|i(x)\| \leq c_2|x| \tag{6}$$

for all $x \in R^n$. Here c_1 and c_2 are constants and $|\cdot|$ ($\|\cdot\|$) denotes the Euclidean norm in R^n (the norm in E). Now given a continuous monotone coercive mapping $T : E \to E^*$ we define the mapping $S = i^* \circ T \circ i$, where i^* is the adjoint of i.



The mapping S is continuous as a composition of continuous mappings. The mapping S is monotone; in fact for x and y in R^n we have

$$(Sx - Sy, x - y) = ((i^* \circ T \circ i)x - (i^* \circ T \circ i)y, x - y)$$

$$= (T(i(x)) - T(i(y)), i(x) - i(y))$$

which is ≥ 0 ; because T is monotone. The mapping S is coercive; in fact, for $x \in R^n$ we have

$$(Sx,x) = (T(i(x)), i(x)) .$$

This together with (6) and the coerciveness of T gives the result. Since the theorem is supposed true in the case $E = R^n$, it follows that S is surjective. And this obviously impies that T is also surjective.

2°) It suffices to prove that $Tx = 0$ has a solution or, equivalently, that $W = I - T$ has a fixed point. By coerciveness there exists $r > 0$ such that $(Tx,x) > 0$ for all $x \in S_r$, where $S_r = \{x \in E : \|x\| = r$. (We may assume that $r > 1$) . Then

$$(7) \qquad (Wx,x) = (x,x) - (Tx,x) < \|x\|^2$$

for all $x \in S_r$.

Now define a new mapping $W' : R^n \to R^n$ as follows

$$W'x = \begin{cases} Wx & \text{if} \quad \|Wx\| < r \\ \dfrac{Wx}{\|Wx\|} & \text{if} \quad \|Wx\| \geq r \end{cases}$$

It is clear that W' is continuous and $W'(B_r) \subset B_r$, where B_r is the ball of radius r about the origin. So, by the Brouwer fixed point theorem, it follows that W' has a fixed point x_o . Now there are two possibilities: either x_o belongs to the interior of B_r or x_o is on the boundary S_r . In the first case, it follows that $W'x_o = Wx_o = x_o$, i.e. x_o is a fixed point of W . In the second case, we have that

$$x_o = W'x_o = \frac{Wx_o}{\|Wx_o\|} \quad .$$

from this follows

$$(x_o,x_o) = \frac{1}{\|Wx_o\|} (Wx_o,x_o) \quad .$$

Using (7) we obtain

$$\|x_o\|^2 < \frac{1}{\|Wx_o\|} \|x_o\|^2$$

which gives $\|Wx_o\| < 1$. But this contradicts the fact that $\|Wx_o\| \geq r$ with $r > 1$. The theorem is proved.

4. PROOF OF THEOREM III.1.

In this section we prove Theorem III.1 stated in the Introduction of this Chapter.

Let Λ be the collection of all finite dimensional subspaces of the Banach space E . We consider this collection preordered (partially ordered) by conclusion. For $F \in \Lambda$ we denote by $j_F : F \to E$ the inclusion mapping. Endowing the subspace F with the induced topology, it follows that j_F is a continuous mapping. Let us denote by j_F^* the adjoint of j_F .

$$
\begin{array}{ccc}
E & \xrightarrow{\;T\;} & E^* \\
\uparrow{\scriptstyle j_F} & & \downarrow{\scriptstyle j_F^*} \\
F & \xrightarrow[\;T_F\;]{} & F^*
\end{array}
$$

Let us denote by T_F the composite mapping $j_F^* \circ T \circ j_F$. Using the fact that T is hemicontinuous, monotone and coercive, it follows readily that T_F is also hemicontinuous, monotone and coercive. Using Lemma III.2 and Theorem III.4 it follows that, for every $F \in \Lambda$, there exists $x_F \in F$ such that $T_F x_F = 0$.

Now we claim that there exists $M > 0$ such that $\|x_F\| \leq M$ for all $F \in \Lambda$. Suppose that this is not the case; so there exists a sequence $\{F_n\}$ such that $\|x_{F_n}\| \to +\infty$. Then by coerciveness it follows that $(Tx_{F_n}, x_{F_n}) \to +\infty$. But this is a contradiction because

$$
(Tx_{F_n}, x_{F_n}) = (T(j_{F_n}(x_{F_n})), j_{F_n}(x_{F_n}))
$$

$$
= ((j_F^* \circ T \circ j_F)x_{F_n}, x_{F_n}) = (T_{F_n} x_{F_n}, x_{F_n}) = 0 .
$$

For each $F_o \in \Lambda$ we define the set

$$
V_{F_o} = \bigcup_{F \supset F_o} \{x_F\}
$$

By reflexivity it follows that the weak closure $\bar{V}_{F_o}$ of V_{F_o} is weakly compact. It is easy to see that the collection of sets $V_{F_o} . F_o \in \Lambda$, has the finite intersection property. So the intersection of all of them is non-empty. Let x_o be a point in this intersection. We claim that $Tx_o = 0$.

Let x be an arbitrary point in E . Let $F_o \in \Lambda$ be such that $x \in F_o$. For $F \supset F_o$, we have by monotonicity

(1) $$(Tx - Tx_F, x - x_F) \geq 0 \quad .$$

Since $(Tx_F, x - x_F) = (Tx_F, j_F(x - x_F)) = ((j_F^* o T o j_F)x_F, x - x_F) = 0$, we obtain from (1)

$$(Tx, x - x_F) \geq 0 \text{ ,}$$

for all $F \supset F_o$. Consequently

(2) $$(Tx, x - x_o) \geq 0$$

for all $x \in E$. Now let y be arbitrary in E and $t > 0$. For $x = x_o + ty$ we obtain from (2)

$$(T(x_o + ty), ty) \geq 0$$

which gives $(T(x_o + ty), y) \geq 0$. By hemicontinuity it follows that $(Tx_o, y) \geq 0$ for all $y \in E$. This implies that $Tx_o = 0$. The theorem is proved.

5. AN APPLICATION OF THEOREM III.1 TO ELLIPTIC EQUATIONS

Let Ω be an open set in R^n. We consider real-valued functions $u(x)$ defined in Ω. In this section we investigate the Dirichlet problem for a quasi-linear equation of the type

$$Au = \sum_{|\alpha| \leq m} D^{\alpha} A_{\alpha}(x, u, \dots, D^m_u) = f \quad . \tag{1}$$

Here we use standard notation. $\alpha = (\alpha_1, \dots, \alpha_n)$ is a vector with non-negative integral components. $|\alpha| = \alpha_1 + \dots + \alpha_n$. $D^{\alpha} = D_1^{\alpha_1} \dots D_n^{\alpha_n}$, where $D_j = \frac{\partial}{\partial x_j}$.

The method employed here is an analogue of the variational method used for linear elliptic partial differential equations. So one starts with a generalized Dirichlet problem. This generalized problem is a natural extension of the classical problem in the following sense. If the classical problem has a solution, then this is also a solution of the generalized problem. If some proper restrictions are made, one hopes that the solution to the generalized problem be also a solution to the classical one. In the linear case this question is very much settled. See, for example, the book of Agmon, "Lectures on Elliptic Boundary Value Problems". In the non-linear case this is a very much open problem.

In order to formulate the generalized problem we introduce the so-called <u>Sobolev spaces</u> $W^{m,p}(\Omega)$. We are interested only in the case where m is a non-negative integer and $p > 1$. Let us denote by $\mathcal{D}'(\Omega)$ the space of distributions in Ω, see, for example, Schwartz

[32] or Tréves [33] . Then we define

$$W^{m,p}(\Omega) = \{u \in \mathcal{D}'(\Omega) : D^\alpha u \in L^p(\Omega),\ |\alpha| \leq m\} .$$

The following expression

$$||u||_{m,p} = \sum_{|\alpha|\leq m}\left(\int_\Omega |D_u^\alpha|^p dx\right)^{1/p}$$

defines a norm in $W^{m,p}(\Omega)$. With this norm the space $W^{m,p}(\Omega)$ is a reflexive Banach space. Let $\mathcal{D}(\Omega)$ denote the space of the infinitely differentiable functions with compact support in Ω . We denote by $W_o^{m,p}(\Omega)$ the closure of $\mathcal{D}(\Omega)$ in $W^{m,p}(\Omega)$. The dual of $W_o^{m,p}(\Omega)$ is also a space of distributions, which is denoted by $W^{-m,p'}(\Omega)$, $p' = p/(p-1)$. See Lions[24] for a clear exposition on Sobolev spaces.

In the framework of the Sobolev spaces the generalized Dirichlet problem is formulated as follows:

"Given f in the dual $W^{-m,p'}(\Omega)$ of $W_o^{m,p}(\Omega)$, find $u \in W_o^{m,p}(\Omega)$ such that

$$a(u,\phi) = \int_\Omega \sum_{|\alpha|\leq m}(-1)^{|\alpha|} A_\alpha(x,u,\ldots D_u^m) D^\alpha \phi\, dx = \langle f,\phi\rangle$$

for all $\phi \in W_o^{m,p}(\Omega)$" . ($\langle , \rangle$ denotes the duality pairing between $W_o^{m,p}(\Omega)$ and its dual.). We shall prove below that this problem has a solution provided some hypotheses are made on the operator A.

<u>Assumptions on the operator A</u>.

(I) The functions $A_\alpha(x,\xi)$, where $\xi = \{\xi_\beta : |\beta| \leq m\}$ is a vector is some Euclidean space R^M , are continuous in ξ for fixed x and measurable in x for fixed ξ. That is, the functions $A_\alpha : \Omega \times R^M \to R^1$, $|\alpha| \leq m$, satisfy the so-called Caratheodory conditions.

(II) For all x in Ω and $|\alpha| \leq m$ we have the following estimate

$$|A_\alpha(x,\xi)| \leq \text{const } (1 + |\xi|^{p-1})$$

(III) For all u and v in $W_o^{m,p}(\Omega)$

$$a(u, u - v) - a(v, u - v) \geq 0$$

(IV) For all u in $W_o^{m,p}(\Omega)$

$$a(u,u) \geq c(||u||_{m,p}) \, ||u||_{m,p} \quad ,$$

where $c(r)$ is a real-valued function such that

$$c(r) \to +\infty \quad \text{as} \quad r \to +\infty \quad .$$

<u>Remark</u>. We shall see below that the above conditions insure the existence of solution for the generalized Dirichlet problem. These conditions are however too restrictive. Much weaker assumptions have been shown to be sufficient for existence of solution. See, for example, Browder [3], Leray and Lions [23] and Vishik [35]. In these papers assumptions of the type (III) and (IV) are made only on "terms of higher order", in analogy with the linear case.

It follows from Assumption (II), using Hölder's inequality, that

(2) $$|a(u,v)| \leq g(\|u\|_{m,p}) \|v\|_{m,p} \quad ,$$

where $g(r)$ is some real valued function.

Now we observe that the form $a(u,v)$ is linear in v. Inequality (2) above implies that, for fixed $u \in W_o^{m,p}(\Omega)$, $v \to a(u,v)$ is a continuous linear functional on $W_o^{m,p}(\Omega)$. So there exists a unique Tu in $W^{-m,p'}(\Omega)$ such that

(3) $$a(u,v) = \langle Tu,v \rangle$$

for all $v \in W_o^{m,p}(\Omega)$. So, the generalized Dirichlet problem is equivalent to solving the functional equation

(4) $$Tu = f \quad ,$$

where T is a non-linear mapping from $W_o^{m,p}(\Omega)$ into $W^{-m,p'}(\Omega)$ defined by (3). However, the existence of solution for equation (4) is guaranteed by Theorem III.1, because the operator T is

a) hemicontinuous--consequence of Assumptions (I) and (II),

b) monotone--consequence of Assumption (III)

c) coercive--consequence of Assumption (IV)

6. LIPSCHITZIAN MONOTONE MAPPINGS. THE GRADIENT METHOD.

In previous sections we have proved the existence of solutions for the functional equation $Tx = f$, where T is a monotone hemicontinuous coercive mapping and f is a given vector. The method used in the existence proof does not provide a constructive way of obtaining a solution of the functional equation. In this section we assume a stronger continuity hypothesis on T and obtain an iterative method for solution of the equation $Tx = f$ in the Hilbert space case.

Theorem III.5. Let H be a Hilbert space and T a strongly monotone mapping of H into itself, i.e.,

$$(Tx - Ty, x - y) \geq c \|x - y\|^2$$

for all x,y in H and some positive constant c. Suppose further that T is Lipschitzian, i.e., there is a positive constant k such that

$$\|Tx - Ty\| \leq k \|x - y\|$$

for all x,y in H. Let us denote by T_f the mapping defined by $T_f x = Tx - f$. Let λ be a positive number such that

(1) $$0 < k^2\lambda^2 - 2c\lambda + 1 < 1 .$$

Then, for any given x_o, the sequence $\{x_n\}$ defined by

(2) $$x_{n+1} = (I - \lambda T_f)x_n \qquad n = o, 1, 2, \ldots$$

converges strongly to the unique solution $\bar{x}$ of $Tx = f$. Moreover, we have the following error estimate

(3) $$\|x_n - \bar{x}\| \leqslant \frac{\lambda q^n}{1-q} \|Tx_o - f\| ,$$

<u>where</u> $q = \sqrt{k^2\lambda^2 - 2c\lambda + 1}$.

<u>Proof</u>. 1°) <u>The mapping $I - \lambda T_f$ is a contraction.</u> Indeed, using the assumptions that T is Lipschitzian and strongly monotone we obtain:

$$\|(I - \lambda T_f)x - (I - \lambda T_f)y\|^2 = \|x - y\|^2 + \lambda^2 \|Tx - Ty\|^2 - 2\lambda(Tx - Ty, x - y)$$

$$\leqslant (1 + k^2\lambda^2 - 2c\lambda) \|x - y\|^2 .$$

Since λ satisfies the inequality (1) , it follows that $I - \lambda T_f$ is a contraction.

2°) The result of 1°) implies that the sequence $\{x_n\}$ defined in (1) converges strong to the unique fixed point of $I - \lambda T_f$. However it is immediate that a fixed point of $I - \lambda T_f$ is a solution of $Tx = f$, and conversely. So the sequence $\{x_n\}$ converges strongly to a point $\bar{x}$, which is then the unique solution of $Tx = f$.

3°) <u>The error estimate</u>. Since $I - \lambda T_f$ is a contraction we have

$$\|x_n - \bar{x}\| \leqslant \frac{q^n}{1-q} \|(I - \lambda T_f)x_o - x_o\| = \frac{\lambda q^n}{1-q} \|Tx_o - f\|$$

<u>Remark 1</u>. The existence of a unique solution for $Tx = f$, under the conditions of Theorem III.5 , follows immediately from Corollary III.1. The important new feature about the above theorem is that the solution $\bar{x}$ can be obtained by an iterative method. This result is due to

Zarantonello [37] . See also Browder and Petryshyn [13].

Remark 2. If $c < k$ in the above theorem, then $k^2\lambda^2 - 2c\lambda + 1$ is positive for any real λ . So inequality (1) can be replaced by the requirement that λ is in the interval $0 < \lambda < 2c/k^2$. In this case, $(c < k)$, the value of λ that gives the best convergence of the sequence $\{x_n\}$ is $\lambda = c/k^2$.

Remark 3. Let us consider the case $c \geqslant k$. Let λ^+ and λ^- be the roots of $k^2\lambda^2 - 2c\lambda + 1$. These roots are real and

$$0 < \lambda^- \leqslant \lambda^+ < 2c/k^2 \quad .$$

So λ satisfies inequality (1) if it is in one of open intervals $(0,\lambda^-)$ and $(\lambda^+, 2c/k^2)$. In this case, $(c \geqslant k)$, there is no value of λ that gives the best convergence of the sequence $\{x_n\}$. The reason being that the infimum of q is 0 for λ varying in the two open intervals above.

The gradient method. We now describe the gradient method for the solution of the functional equation $Tx = f$ in a real Hilbert space. As before we assume that the mapping T from a Hilbert space H into itself is strongly monotone, i.e.,

$$(Tx - Ty, x - y) \geqslant c\,||x - y||^2 , \qquad c > 0$$

and Lipschitzian

$$||Tx - Ty|| \leqslant k\,||x - y|| \quad .$$

For given x_o we define the following sequence of approximations for the solution of $Tx = f$:

(4) $$x_{n+1} = x_n + \alpha_n r_n$$

where

(5) $$r_n = f - Tx_n$$

and α_n is a real number conveniently chosen.

From (4) and (5) it follows that

$$\|r_n\|^2 - \|r_{n+1}\|^2 = \|Tx_n\|^2 - 2(f, Tx_n) - \|Tx_{n+1}\|^2 + 2(f, Tx_{n+1})$$

(7) $$= 2(f - Tx_n, Tx_{n+1} - Tx_n) - \|Tx_n\|^2 - \|Tx_{n+1}\|^2 + 2(Tx_n, Tx_{n+1})$$

$$= \frac{2}{\alpha_n}(x_{n+1} - x_n, Tx_{n+1} - Tx_n) - \|Tx_{n+1} - Tx_n\|^2 \quad .$$

Using the hypotheses made on T we obtain from (7):

(8) $$\|r_n\|^2 - \|r_{n+1}\|^2 \geq (2c\alpha_n - k^2\alpha_n^2)\|r_n\|^2 \quad .$$

Now α_n will be chosen in such a way that there exists a positive number β_n such that $2c\alpha_n - k^2\alpha_n^2 \geq \beta_n$, that is,

(9) $$k^2\alpha_n^2 - 2c\alpha_n + \beta_n \leq 0 \ .$$

So, such a choice of α_n will be possible if β_n satisfies the condition $c^2 - k^2\beta_n \geq 0$. Thus β_n must be in the interval

(10) $$0 < \beta_n \leq \frac{c^2}{k^2} \quad .$$

Once β_n has been chosen in the interval (10) , then α_n can be taken as any number in the interval

(11) $$\alpha_n^- \leq \alpha_n \leq \alpha_n^+ \quad ,$$

where α_n^- and α_n^+ are the roots of $k^2\alpha_n^2 - 2c\alpha_n + \beta_n$.

Now we can prove the following result.

Theorem III.6. Let T be a strongly monotone Lipschitzian mapping in a Hilbert space. Let a be a positive number such that $a < 1$ and $a \leq c^2/k^2$. Suppose that a sequence $\{\beta_n\}$ is chosen in the interval $[a, c^2/k^2]$. Suppose also that a sequence $\{\alpha_n\}$ has been chosen in the interval (11) . Then

(i) the sequence $\{r_n\}$ of residuals defined in (5) converges to 0 ,

(ii) the sequence $\{x_n\}$ defined in (4) converges to the unique solution $\bar{x}$ of $Tx = f$,

(iii) we have the following error estimate

(*) $$\|x_n - \bar{x}\| \leq \frac{q^n}{c} \|Tx_o - f\| \quad ,$$

where $q = \sqrt{1 - a}$.

Proof. 1°) From inequality (8) and the conditions on β_n we obtain

(12) $$\|r_n\|^2 - \|r_{n+1}\|^2 \geq a \|r_n\|^2 \quad .$$

This implies that $(1 - a) \|r_n\|^2 \geq \|r_{n+1}\|^2$, where $1 - a$ is positive by hypothesis. Consequently we have $\|r_{n+1}\| \leq q \|r_n\|$. From this last inequality we conclude that

(13) $$\|r_n\| \leq q^n \|r_o\|$$

Since $q < 1$, Part (i) of the theorem follows.

2°) By monotonicity we have

$$(Tx_{n+1} - Tx_n, x_{n+1} - x_n) \geq c \|x_{n+1} - x_n\|^2$$

This implies $\|x_{n+1} - x_n\| \leq \frac{1}{c} \|Tx_{n+1} - Tx_n\|$. Using (5) it follows that

$$\|x_{n+1} - x_n\| \leq \frac{1}{c} \|r_{n+1} - r_n\| \quad .$$

This implies that the sequence $\{x_n\}$ converges. Let $\bar{x}$ be its limit. From (5) and the fact that $r_n \to 0$ we conclude that $T\bar{x} = f$. So Part (ii) of the theorem is proved.

3°) Using monotonicity again we have

$$(Tx_n - T\bar{x}, x_n - \bar{x}) \geq c \|x_n - \bar{x}\| \quad ,$$

which implies $c \|x_n - \bar{x}\| \leq \|Tx_n - T\bar{x}\|$. Since $T\bar{x} = f$ and

$Tx_n - f = -r_n$ we obtain

$$\|x_n - \bar{x}\| \leq \frac{1}{c} \|r_n\| .$$

Using estimate (13) we obtain the error estimate (*) . The theorem is proved.

Remark 4. Suppose $c < k$. Then the value of a that gives the smallest q is $a = c^2/k^2$, So $\beta_n = c^2/k^2$ for all n and $\alpha_n = c/k^2$ for all n . Thus, in this case $c < k$, the method of Theorem III.5 gives the best convergence.

Remark 6. The gradient method has been used by Vainberg [34] for certain classes of monotone operators. His results are more general than the ones presented here. We also refer to Lagenbach [20] where potential operators are considered.

7. THE DUALITY MAPPING. AN APPLICATION TO FOURIER SERIES.

An important example of a monotone mapping from a Banach space E into its dual space E^* is given by the so-called duality mapping (see definition below). This concept was introduced by Beurling and Livingston [2]. Later it was generalized and extensively studied by Browder [3], [9], [11] . See also Browder and deFigueiredo [12] and Asplund [1] . In this section we establish some of the most important properties of the duality mapping.

<u>Definitions</u>. 1) A <u>gauge function</u> is a real-valued continuous function μ defined in the non-negative half line $R^+ = \{t \in R^1 : t \geq 0\}$ such that (i) $\mu(0) = 0$, (ii) $\lim_{t\to\infty} \mu(t) = +\infty$, (iii) μ is strictly increasing. An example of a gauge function is $\mu(t) = t$.

2) Let E be a Banach space and E^* its dual space. Let $\mu(t)$ be a given gauge function. The <u>duality mapping</u> in E with gauge function μ is a mapping J from E into the set 2^{E^*} of all subsets of E^* such that

$$J(0) = 0$$

$$Jx = \{x^* \in E^* : (x^*,x) = \|x^*\| \, \|x\| , \ \|x^*\| = \mu(\|x\|)\} , \quad x \neq 0 .$$

<u>Remark 1</u>. <u>For</u> $x_o \neq 0$ <u>the set</u> Jx_o <u>is non-empty</u>. Indeed, let L be the one-dimensional subspace generated by x_o. Define a linear functional ℓ in L as follows

$$\ell(x_o) = \mu(\|x_o\|) \, \|x_o\|$$

and

$$\ell(x) = \ell(\lambda x_o) = \lambda \ell(x_o) \quad .$$

It is easy to check that the norm $\|\ell\|$ of the functional ℓ is equal to $\mu(\|x_o\|)$. Now using the Hahn-Banach theorem, we can extend the linear functional ℓ to a linear functional y^* defined in the whole of E which has the same norm, i.e. $\|y^*\| = \|\ell\|$. Such an extension is not necessarily unique. It is clear that Jx_o is the set of all such extensions.

Remark 2. The set Jx is convex. Let x^* and y^* be in Jx. We claim that $z^* = \lambda x^* + (1 - \lambda)y^*$, for $0 < \lambda < 1$, also belongs to Jx. Indeed, first we have

$$(1) \qquad (z^*,x) = \lambda(x^*,x) + (1 - \lambda)(y^*,x) = \mu(\|x\|)\|x\|$$

From this equality it follows that $\|z^*\| \geq \mu(\|x\|)$. On the other hand

$$\|z^*\| \leq \lambda\|x^*\| + (1 - \lambda)\|y^*\| = \mu(\|x\|) .$$

So $\|z_*\| = \mu(\|x\|)$. This together with (1) implies that the linear functional z^* belongs to Jx.

Remark 3. Let μ and μ_1 be two gauge functions. Let J and J_1 be the duality mappings in the Banach space E with gauge functions μ and μ_1, respectively. Then there exists a non-negative real-valued function $\gamma(t)$ such that

$$Jx = \gamma(\|x\|)J_1x .$$

This follows immediately from Remark 1.

Example. Consider the L^p space, $1 < p < \infty$, in the interval $[0,1]$. It is easy to check that the mapping $J_o : L^p \to L^q$, $q = p/(p - 1)$, defined by

$$J_o f = |f|^{p-1} \text{ sgn } f$$

is the duality mapping in E with gauge function $\mu(t) = t^{p-1}$. (Notation: sgn f = signal of f(x)). By Remark 3 above it follows that any other duality mapping J is of the form $Jx = \gamma(\|x\|)J_o x$, where $\gamma(t)$ is some non-negative function defined in the half-line R^+.

The following result is essentially a reformulation of a characterization of reflexivity due to James [18], [19].

Theorem III.7. Let E be a Banach space and E^* its dual space. Let J be the duality mapping in E with a given gauge function μ. Then E is reflexive if and only if the union of all sets Jx, $x \in E$, covers E^*.

Proof. 1°) Let us first assume that E is reflexive. Let $y^* \in E^*$. Let B_1 be the unit ball about the origin. Since B_1 is weakly compact and the linear functional y^* is continuous, it follows that there exists $x_o \in B_1$ such that

$$(y^*, x_o) = \sup_{x \in B_1} (y^*, x) .$$

It is immediate that $\|x_o\| = 1$. So $(y^*, x_o) = \|y^*\| \, \|x_o\|$. Now by the properties of the gauge function μ it follows that there exists $\lambda_o > 0$ such that $\mu(\lambda_o) = \|y^*\|$. Thus $\mu(\|\lambda_o x_o\|) = \|y^*\|$. This together with $(y^*, \lambda_o x_o) = \|y^*\| \, \|\lambda_o x_o\|$ implies that $y^* \in J(\lambda_o x_o)$.

2°) Conversely, suppose that the union of the sets Jx covers E^*. We claim that E^* is reflexive. To prove this we rely on the following characterization of reflexive Banach spaces due to James. "A Banach space E is reflexive if and only if every continuous linear functional on E attains its supremum on the unit ball B_1 about the origin". Let y^* be a continuous linear functional on E. By hypothesis there exists a point x_o in E such that $y^* \in Jx_o$. This implies that

$$(y^*, x_o) = ||y^*|| \, ||x_o|| \quad . \tag{2}$$

now, the point $y_o = x_o / ||x_o||$ is in the unit-ball and from (2) it follows that $(y^*, y_o) = ||y^*|| = \sup_{x \in B_1} (y^*, x)$. So by the characterization of reflexivity stated above it follows that E is reflexive. The proof is complete.

Remark. See Laursen [21] for another characterization of reflexivity using duality mappings.

As we have observed in Remark 1 the duality mapping is multivalued in general. However, if the dual space E^* is strictly convex, then the set Jx consists of exactly one point; see Proposition III.1 immediately below. From now on we restrict ourselves to this case; so the duality mapping $J : E \to E^*$ is a single-valued mapping from E to E^*. For a study of multivalued duality mappings, see Browder [3], [9].

Proposition III.1. Let E be a Banach space with a strictly convex dual space E^*. Let J be the duality mapping in E with gauge function μ. Then the set Jx consists of precisely one point.

Proof. By the definition of duality mapping, the set Jx is on the surface of the ball of radius $\mu(\|x\|)$ about the origin in E^*. By Remark 2 above the set Jx is convex. Thus by the strict convexity of E^* it follows that Jx is a set with only one point.

Corollary III.2. *Let E be a Banach space with a strictly convex dual space E^*. Let $J : X \to X^*$ be the duality mapping with gauge function μ. Then E is reflexive if and only if J is surjective.*

Proof. Use Theorem III.7 and Proposition III.1.

Remark. Let E be a reflexive Banach space with a strictly convex dual space E^*. Then the inverse mapping $J^{-1} : E^* \to E$ is the duality mapping in E^* with gauge function $\mu_1(t) = \mu^{-1}(t)$, where $\mu^{-1}(t)$ is the inverse function of $\mu(t)$.

Proposition III.2. *Let E be a Banach space with a strictly convex dual space E^*. Then the duality mapping J in E with gauge function μ is monotone.*

Proof. By the definition of duality mapping we have

$$(3)\quad (Jx - Jy, x - y) = (Jx,x) + (Jy,y) - (Jx,y) - (Jy,x)$$

$$\geq \mu(\|x\|)\|x\| + \mu(\|y\|)\|y\| - \mu(\|x\|)\|y\| - \mu(\|y\|)\|x\|$$

$$= [\mu(\|x\|) - \mu(\|y\|)][\|x\| - \|y\|] .$$

The last expression is non-negative because μ is a strictly increasing function. So $(Jx - Jy, x - y) \geq 0$, i.e., the mapping J is monotone.

Before stating the next proposition, we prove the following result about strictly convex Banach spaces.

Lemma III.2. A Banach space E is strictly convex if and only if every continuous linear functional on E does not attain its supremum (relatively to the unit ball B_1 about the origin) at more than one point of B_1.

Remark. Observe that the above lemma does not state that every continuous linear functional on E attains its supremum on the unit ball B_1. We have here a uniqueness statement. As we have observed before the James characterization of reflexivity (see proof of Theorem III.7) provides the answer for the existence question.

Proof of Lemma III.2. 1°) Suppose that E is strictly convex. Let y^* be a continuous linear functional on E. Let us assume that there are two points x_1 and x_2 in the ball B_1 such that

$$(y^*, x_1) = (y^*, x_2) = \sup_{x \in B_1} (y^*, x) = ||y^*|| .$$

It is clear that both x_1 and x_2 are in the surface of the ball B_1. Now

$$(y^*, \lambda x_1 + (1 - \lambda) x_2) = \lambda (y^*, x_1) + (1 - \lambda)(y^*, x_2) = ||y^*|| .$$

So $\lambda x_1 + (1 - \lambda) x_2$ is also in the surface of the ball B_1. This however contradicts the assumption that E is strictly convex.

2°) Conversely, let us assume that any given continuous linear functional cannot assume its supremum at more than one point. Let us suppose that E is not strictly convex. That is, there are two points x_o and x_1 on the surface of the ball B_1 such that every point $x_\lambda = \lambda x_o + (1 - \lambda)x_1$, $0 < \lambda < 1$, is also on the surface of B_1 . Let y^* be a continuous linear functional on E such that $(y^*, x_{0.5}) = \|y^*\|$. The existence of such a functional follows from the Hahn-Banach theorem. By our assumption it follows that

$$(y^*, x_o) < \|y^*\| \qquad \text{and} \qquad (y^*, x_1) < \|y^*\| .$$

These two inequalities imply that

$$(y^*, x_{0.5}) = \frac{1}{2}(y^*, x_o) + \frac{1}{2}(y^*, x_1) < \|y^*\| ,$$

which contradicts the fact that $(y^*, x_{0.5}) = \|y^*\|$. So E must be strictly convex. The lemma is proved.

<u>Definition</u>. A mapping $T : E \to E^*$ from a Banach space E into its dual space E^* is said to be <u>strictly monotone</u> if

$$(Tx - Ty, x - y) > 0$$

for all $x = y$ in E .

<u>Proposition III.3</u>. <u>Let E be a strictly convex Banach space with a</u>

strictly convex dual space E^*. Then the duality mapping J with gauge function μ is strictly monotone.

Proof. By Proposition III.2 it follows that J is monotone, i.e., $(Jx - Jy, x - y) \geq 0$ for all x,y in E. Now, let us suppose that there are two points x and y, $x \neq y$ in E such that

$$(Jx - Jy, x - y) = 0 \tag{4}$$

This together with inequality (3) (see proof of Proposition III.2) implies that $||x|| = ||y||$. Now by the definition of duality mapping it follows that Jx attains its supremum on the unit ball B_1 at the point $x/||x||$. Then it follows that $(Jx, y/||y||) < ||Jx||$, i.e.

$$(Jx,y) < ||Jx|| \ ||y|| \ . \tag{5}$$

Similarly

$$(Jy,x) < ||Jy|| \ ||x|| \ . \tag{6}$$

Using estimates (5) and (6) in (3) we obtain

$$(Jx - Jy, x - y) > (||Jx|| - ||Jy||)(||x|| - ||y||) = 0 \ ,$$

which contradicts our assumption (4). So $(Jx - Jy, x - y) > 0$ for $x \neq y$.

Corollary III.3. Let E be a strictly convex Banach space with a *strictly convex dual space E^*. Then the duality mapping J with gauge function μ is injective. If in addition E is reflexive then J is bijective.*

The next result is a statement about "continuity" of a duality mapping.

Proposition III.4. Let E be a Banach space with a strictly convex dual space E^*. Then the duality mapping J with gauge function μ is continuous from the strong topology of E to the weak* topology of E^*.

Proof. Let $\{x_n\}$ be a sequence in E which converges strongly, i.e., $x_n \to x$. We claim that $Jx_n \xrightarrow{*} Jx$, where "$\xrightarrow{*}$" denotes weak* convergence. To prove this we show that every subsequence of $\{Jx_n\}$ contains a further subsequence which converges weakly* to Jx. Since $x_n \to x$, it follows that there exists $M > 0$ such that $||Jx_n|| = \mu(||x_n||) \leq M$ for all n. So, any subsequence of $\{Jx_n\}$ contains a weakly* convergent subsequence. Let us denote this last subsequence by $\{Jy_n\}$ and its weak* limit by w, i.e., $Jy_n \xrightarrow{*} w$. We prove now that $w = Jx$. First we note that

$$(7) \qquad ||x|| \, \mu(||x||) = \lim ||y_n|| \, \mu(||y_n||) = \lim (Jy_n, y_n) = (w, x) .$$

This implies that $\mu(||x||) \leq ||w||$. On the other hand, since $Jy_n \xrightarrow{*} w$, it follows that

$$||w|| \leq \liminf ||Jy_n|| = \lim \mu(||y_n||) = \mu(||x||) .$$

So it follows that $\|w\| = \mu(\|x\|)$. This together with (7) implies that $w = Jx$.

Definition 1. A Banach space E is said to have Property (H) if the following condition is satisfied:

(H) E is strictly convex. Moreover, if x_n converges weakly to x_o and $\lim \|x_n\| = \|x_o\|$, then x_n converges strongly to x_o . This and many other equivalent properties in Banach spaces have been studied by Fan and Glicksberg [15] . Hilbert spaces and uniformly convex Banach spaces are examples of spaces satisfying Property (H) . Moreover, locally uniformly convex Banach spaces (Lovaglia [25]) also satisfy Property (H) , see Lemma III.3 below. A Banach space satisfying Property (H) is not necessarily reflexive, see [15] page 561.

Definition 2. A Banach space E is said to be locally uniformly convex if for each given sequence $\{x_n\}$ and a point $x_o \in E$ such that

(8) $$\|x_n\| \to \|x_o\|$$

(9) $$\|x_o + x_n\| \to 2\|x_o\|$$

it follows that x_n converges strongly to x_o . It is easy to see that every uniformly convex Banach space is locally uniformly convex. Of course the converse is not true, see Lovaglia [25].

Lemma III.3. Let E be a locally uniformly convex Banach space. Then E satisfies Property (H) .

Remark. This result is proved in [15]. We give here a proof using the duality mapping. We make the assumption that E^* is strictly convex in order to have a single-valued mapping. The argument however works without this extraneous assumption.

Proof of Lemma III.3. 1°) E is strictly convex. Indeed, suppose that this is not the case. So there are two distinct points x_o and x_1 such that

$$\|\lambda x_o + (1 - \lambda)x_1\| = \|x_o\| \quad \text{for all} \quad 0 \leq \lambda \leq 1 \ .$$

The sequence $x_n = x_1$, $n = 1, 2, \ldots$ and the point x_o satisfy the conditions (8) and (9) of Definition 2, but x_n does not converge strongly to x_o . This contradicts the assumption that E is locally uniformly convex.

2°) Let $\{x_n\}$ be a sequence in E which converges weakly to x_o and $\|x_n\| \to \|x_o\|$. Let J be the duality mapping in E with gauge function $\mu(t) = t$. Then we have

$$(10) \qquad (x_n + x_o, Jx_o) = (x_n, Jx_o) + (x_o, Jx_o) \to 2\|x_o\|^2 \ .$$

On the other hand

$$(11) \qquad (x_n + x_o, Jx_o) \leq \|x_n + x_o\| \, \|Jx_o\| \leq \|x_n + x_o\| \, \|x_o\| \ .$$

From (10) it follows that

$$\liminf \|x_n + x_o\| \geq 2 \|x_o\|$$

This together with

$$\limsup \|x_n + x_o\| \leq 2 \|x_o\|$$

implies that

$$\lim \|x_n + x_o\| = 2 \|x_o\| .$$

So conditions (8) and (9) of Definition 2 are satisfied. By local uniform convexity it follows that x_n converges strongly to x_o. That is Property (H) is satisfied.

Now we prove the following continuity property of the duality mapping.

Proposition III.5. Let E be a reflexive Banach space and E^* its dual space. Assume that E^* has Property (H). Then the duality mapping J with a given gauge function μ is continuous from the strong topology of E to the strong topology of E^*.

Proof. Using Proposition III.4 we conclude that the duality mapping J is continuous from the strong topology of E to the weak topology of E^*. So if $x_n \to x$ then $Jx_n \rightharpoonup Jx$ (" $\rightharpoonup$" denotes weak convergence). We claim that we actually have $Jx_n \to Jx$. This follows using the fact that E^* satisfies Property (H). Indeed, we already have $Jx_n \rightharpoonup Jx$. Moreover

$$\|Jx_n\| = \mu(\|x_n\|) \to \mu(\|x\|) = \|Jx\| ,$$

because $x_n \to x$. The proof is complete.

Corollary III.4. Let E be a reflexive Banach space which has a locally uniformly convex dual space E^* . Then the duality mapping is continuous from the strong topology of E to the strong topology of E^* .

Proof. Use Proposition III.5 and Lemma III.3.

Corollary III.5. Any duality mapping in L^p , $1 < p < \infty$, is a homeomorphism between L^p and L^q , $q = p/(p - 1)$.

Proof. Use Corollary III.3 and Corollary III.4.

Remark. A more general result than the one stated in the preceding corollary holds true. Namely, let $1 \leq p < \infty$ and $1 \leq r < \infty$; then the mapping $f \to |f|^{\frac{p}{r}-1} f$ is a homeomorphism from L^p onto L^r . See Bourbaki, "Intégration", Chap. IV, §6, Exercise 10.

Now we state a negative result about the weak continuity of the duality mapping. By weak continuity we mean continuity from the weak topology of E into the weak topology of E^* .

Proposition III.6. Consider the Banach space $L^p[0,1]$ for $1 < p < \infty$, $p \neq 2$. Then there exists no weakly continuous duality mapping J in X . See the proof in Browder-deFigueiredo, [13] .

We remark that the duality mapping J in ℓ^p , $1 < p < +\infty$ defined by

$$J(\{\xi_k\}) = \{|\xi_k|^{p-2}\xi_k\}$$

is weakly continuous.

AN APPLICATION TO FOURIER SERIES.

Now we give an application of the concept of duality mapping to a problem in the theory of Fourier series. The result presented here, Theorem III.8, is due to Beurling and Livingston [2].

Let f be an element of some Lebesgue space L^p in the interval $[0, 2\pi]$, $1 < p < \infty$. The Fourier coefficients of f are defined by

$$c_n(f) = \frac{1}{2\pi}\int_0^{2\pi} e^{inx} f(x)dx$$

for $n = 0, \pm 1, \pm 2, \ldots,$ where $f(x)$ is any function in the equivalence class defined by f .

The following result is the well known Riesz-Fisher theorem. See for example, Kolmogorov-Fomin "Elements of the Theory of Functions and Functional Analysis", Vol II.

<u>Theorem A</u>. <u>Let $\{a_n\}$, $n = 0 , \pm 1, \pm 2, \ldots,$ be a sequence of numbers such that</u>

$$\sum_{n=-\infty}^{\infty} |a_n|^2 < \infty .$$

<u>Then there exists a unique element $f \in L^2[0,2\pi]$ such that $c_n(f) = a_n$ for all n</u>.

This result implies the following theorem.

Theorem B. Suppose that the set of all (positive and negative) integers is partitioned in two non-empty distinct classes A and A'. Let $\{a_n\}$, $n = 0, \pm 1, \pm 2, \ldots,$ be a sequence of numbers with the property that there are two elements g and h in $L^2[0,2\pi]$ such that

$$c_n(g) = a_n, \quad n \in A$$

$$c_n(h) = a_n, \quad n \in A' .$$

Then there exists a unique $f \in L^2[0,2\pi]$ such that $c_n(f) = a_n$ for all n.

Theorem B above can be extended to the L^p case. Namely the follow result holds.

Theorem III.8. Suppose that the set of integers is partitioned in two non-empty distinct classes A and A'. Let p be $1 < p < \infty$ and $q = p/(p-1)$. Suppose that there is given a sequence of numbers $\{a_n\}$, $n = 0, \pm 1, \pm 2, \ldots,$ with the property that there are two functions $g_o \in L^p[0,2\pi]$ and $h_o \in L^q[0,2\pi]$ such that

$$c_n(g_o) = a_n \qquad n \in A$$

$$c_n(h_o) = a_n \qquad n \in A' .$$

Then there exists a unique $f_o \in L^p[0,2\pi]$ such that

$$c_n(f_o) = a_n \qquad n \in A$$

$$c_n(Jf_o) = a_n \qquad n \in A' \quad ,$$

where J is the duality mapping in $L^p[0,2\pi]$ with gauge function $\mu(t) = t^{p-1}$.

This theorem will be proved here using the following result, which is due to Browder [3], [11] .

Theorem III.9. Let E be a reflexive strictly convex Banach space, which has a strictly convex dual space E^* . Let V be a closed linear subspace of E, and V^o its annihilator, i.e.

$$V^o = \{x^* \in E^* : (x^*,x) = 0\} , \quad \text{for all} \quad x \in V \quad .$$

Let J be the duality mapping in E with given gauge function μ . Then, for every $x_o \in E$ and every $y_o \in E^*$, the intersection

$$J(x_o + V) \cap (y_o + V^o)$$

consists of precisely one point.

Proof. Let $k : V \to E$ be the inclusion mapping of V into E . Let us consider V with the induced topology. So the mapping k is continuous. Let $k^* : E^* \to V^*$ be the adjoint operator. The mapping k^* is both continuous and weakly continuous. Now let us define the mapping $T : V \to V^*$ as follows

(12) $$Tx = k^*[J(x + x_o) - y_o] \ .$$

Since J is monotone (Proposition III.2) and continuous from the strong topology of E to the weak topology of E^* (Proposition III.4), it follows that T has the same properties. On the other hand, for $x \in V$ we have

$$(Tx,x) = (J(x + x_o) - y_o, x)$$

$$= (J(x + x_o), x + x_o) - (y_o, x) - (J(x + x_o), x_o)$$

$$\geq \|x + x_o\| \mu(\|x + x_o\|) - \|y_o\| \|x\| - \|x_o\| \mu(\|x + x_o\|) \ .$$

From this estimate it follows immediately that $(Tx,x)/\|x\| \to +\infty$ as $\|x\| \to \infty$. So, all the hypotheses of Theorem III.1 are satisfied by the operator T defined in (12). Applying that theorem we conclude that T is surjective. So there exists a point $x \in V$ such that

$$k^*[J(x + x_o) - y_o] = 0 \ ,$$

That is

$$(k^*[J(x + x_o) - y_o], v) = (J(x + x_o) - y_o, v) = 0$$

for all $v \in V$. This means that

$$J(x + x_o) - y_o \in V^o ,$$

i.e. the intersection $J(V + x_o) \cap (V^o + y_o)$ is non-empty. Finally we prove that this intersection consists of exactly one point. Suppose that there are two points x_1 and x_2 in this intersection. Then it follows

$$(13) \qquad (J(x_1 + x_o) - J(x_2 + x_o), x_1 - x_2) = 0$$

Using Proposition III.3, we conclude from (13) that $x_1 = x_2$.

Before proving Theorem III.8 we state a very interesting fact about Fourier series, which is due to Marcel Riesz [30].

Theorem C. Let f be an element in $L^p[0,2\pi]$, $1 < p < \infty$. Then the Fourier series of f converges in the L^p-norm to f , i.e.,

$$||f - \sum_{n=-N}^{N} c_n(f)\, e^{inx}||_{L^p} \to 0$$

as $N \to +\infty$.

Remark. For $p = 2$ this is a classical result which can be found in the standard books on Fourier series. For $p \neq 2$ this theorem is proved [30] using properties of the Hilbert transform. For $p = 1$ the result is false, see Zygmund's book on Tryonometric Series. We thank John Horvath for supplying us with these references.

Now, Theorem C implies the following result, which will be used in the proof of Theorem III.8.

Lemma III.4. Suppose that the set of all integers is partitioned in two distinct non-empty subsets A and A'. Let V be the subset of $L^p[0,2\pi]$, $1 < p < \infty$, consisting of elements f such that $c_n(f) = 0$ for all $n \in A$. Then the closure of the set of all finite linear combinations of e^{inx}, $n \in A'$, is precisely V.

Proof. 1°) First we observe that V is closed. This is a consequence of the following estimate

$$(14) \qquad |c_n(f) - c_n(g)| \leq k \, ||f - g||_p ,$$

where k is a constant depending only on n. Estimate (14) is obtained using Holder's inequality.

2°) Each function e^{inx}, for $n \in A'$, belongs to V. This follows from the fact that $\{e^{inx}\}$, $n = 0, \pm 1, \pm 2, \ldots,$ is an orthonormal set in $L^2[0,2\pi]$.

3°) Given any element f in V, its Fourier series contains only terms of the form $c_n(f)e^{inx}$ for $n \in A'$. Then, using Theorem C we conclude that f is the limit in the L^p-norm of finite linear combinations of e^{inx}, $n \in A'$. So the lemma is proved.

Proof of Theorem III.8. We will apply Theorem III.9 with $E = L^p[0,2\pi]$, $1 < p < \infty$. It is well known that $L^p[0, 2\pi]$, $1 < p < \infty$, is reflexive and strictly convex. Let us denote by V the set

$$V = \{g \in L^p[0,2\pi] : c_n(g) = 0 , \quad n \in A\} .$$

Using Lemma III.4 we conclude that V is a closed subspace and the annihilator is the set

$$V^o = \{h \in L^q[0,2\pi] : c_n(h) = 0 , \quad n \in A'\} \quad .$$

It is immediate that

$$g_o + V = \{g \in L^p[0,2\pi] : c_n(g) = a_n , \ n \in A\}$$

and

$$h_o + V^o = \{h \in L^q[0,2\pi] : c_n(h) = a_n , \ n \in A'\} \quad .$$

Now we use Theorem III.9 to conclude that the intersection

$$J(g_o + V) \cap (h_o + V^o)$$

consists of exactly one point. This means that there exists a unique f_o in L^p such that

$$c_n(f_o) = a_n \qquad n \in A$$

and

$$c_n(Jf_o) = a_n \qquad n \in A'$$

So the theorem is proved.

REFERENCES

[1] Asplund, E. - Positivity of duality mappings. Bull. Amer. Math. Soc. vol. 73(1967) pp. 200-203.

[2] Beurling, A. and Livingston, A.E. - A theorem on duality mappings in Banach spaces. Arkiv. for Mat. 4(1960/3) pp. 405-411.

[3] Browder, F.E. - Problèmes Non-linéaires. Les Presses de l'Université de Montréal (1966).

[4] ———————— - Existence and uniqueness theorems for solutions of nonlinear boundary value problems. Proc. Symp. App. Math. vol XVII (1965) pp. 24-49.

[5] ———————— - The solvability of non-linear functional equations. Duke Math. J. 30(1963) pp. 557-566.

[6] ———————— - Variational boundary value problems for quasi-linear elliptic equations of arbitrary order. Proc. Nat. Acad. Sci. 50(1963) pp. 31-37.

[7] ———————— - Variational boundary value problems for quasi-linear elliptic equations II. Proc. Nat. Acad. Sci. 50(1963) pp. 592-598.

[8] ———————— - Variational boundary value problems for quasi-linear elliptic equations III. Proc. Nat. Acad. Sci. 50(1963) pp. 794-798.

[9] ———————— - Multivalued monotone nonlinear mappings and duality mappings in Banach spaces. Transactions Amer. Math. Soc. 118(1965) pp. 338-351.

[10] ———————— - On functional analysis and partial differential equations I. Math. Ann. 138(1959) pp. 55-79.

[11] ———————— - On a theorem of Beurling and Livingston. Canad. J. Math. 17(1965) pp. 367-372.

[12] Browder, F.E. and deFigueiredo, D.G. - J-monotone nonlinear operators in Banach spaces. Konkl. Nederl. Akad. Wetensch. 69(1966) pp. 412-420.

[13] Browder, F.E. and Petryshyn, W.V. - Construction of fixed points of nonlinear mappings in Hilbert space. (To appear).

[14] Dolph, C.L. and Minty, G.J. - On nonlinear integral equations of the Hammerstein type. Proc. Symp. Nonlinear Integral Equations at Madison. Edited by P.M. Anselone. pp. 99-154.

[15] Fan, K. and Glicksberg, I. - Some geometric properties of the spheres in a normed linear space. Duke Math. J. 25(1958) pp. 553-568.

[16] Friedrichs, K.O. - Spektraltheorie halbbeschränkter Operatoren und Anwendung auf die Spektralzerlegung von Differentialoperatoren. Math. Ann. 109(1934) pp. 465-487.

[17] Hildebrandt, S. and Wienholtz, E. - Constructive proofs of representation theorems in separable Hilbert spaces. Comm. Pure Appl. Math. XVII (1964) pp. 369-373.

[18] James, R.C. - Reflexivity and the supremum of linear functionals. Ann. Math. 66(1957) p. 159-169.

[19] ———————— - Characterizations of reflexivity. Studia Math. XXIII(1964) pp. 205-216.

[20] Lagenbach, A. - Über Gleichungen mit Potential - operatoren und Minimalfolgen nichtquadratischer Funktionale. Math. Nach. (1966).

[21] Laursen, K.B. - A characterization of reflexivity of Banach spaces. Math. Scand. 16(1965) pp. 169-174.

[22] Lax, P.D. and Milgram, A.N. - Parabolic equations. Ann. Math 33(1954) pp. 167-190.

[23] Leray, J. and Lions, J.L. - Quelques résultats de Vishik sur les problèmes elliptiques non linéaires par les méthodes de Minty-Browder. Bull. Soc. Math. France vol. 93 (1965) pp. 97-107.

[24] Lions, J.L. - Problèmes aux limites dans les équations aux derivées partielles. Les Presses de l'Université de Montréal. (1965) 2nd edition.

[25] Lovaglia, A.R. - Locally uniformly convex Banach spaces. Transactions Amer. Math. Soc. vol. 78 (1955) pp. 225-238.

[26] Mikhlin, S.G. - The problem of the minimum of a quadratic functional. Holden-Day. (1965)

[27] Minty, G.J. - Monotone (nonlinear) operators in Hilbert space. Duke Math. J. 29(1962) pp. 341-346.

[28] Nirenberg, L. - Remarks on strongly elliptic partial differential equations. Comm. Pure Appl. Math. 8(1955) pp. 649-675.

[29] Petryshyn, W.V. - Constructional proof of Lax-Milgram lemma and its application to non-K-P.D. abstract and differential operator equations. J. Siam Numer. Anal. 2(1965) pp. 404-420.

[30] Riesz, M. - Sur les fonctions conjuguées. Math. Zeit. vol. 27 (1927) pp. 218-244.

[31] Sauer, N. - Representation theorems in a reflexive B-space. Math Zeit. 93(1966) pp. 202-205

[32] Schwartz, L. - Théorie des distributions, vol. I. Hermann (1950).

[33] Trèves, F. - Topological Vector Spaces, and Theory of Distributions and Kernels. (1967).

[34] Vainberg, M.M. - On the convergence of the method of steepest descent for nonlinear equations. Sibirsky Mat. Jour. 2(1961) pp. 201-220.

[35] Vishik, M.I. - Quasilinear strongly elliptic systems of differential equations having divergence form. Trudi Moscow Math. Soc. vol. 12(1963) pp. 125-184.

[36] Yosida, K. - Functional Analysic. Springer (1965).

[37] Zarantonello, E.H. - Solving functional equations by contractive averaging. Math. Res. Ctr. (Univ. of Wisconsin) Tech. Report No. 160 (1960).

CHAPTER IV

PROJECTIONAL METHODS

INTRODUCTION

This chapter is devoted to fixed point theorems for non-linear mappings $T : C \to E$ defined in a closed convex set C of a Banach space. We use projectional methods in a way very similar to Browder (see, for example, [3]) and Petryshyn [15] . The material presented here has been expounded by us in [7] and [8] . We find it convenient to introduce a class of Banach spaces, see Section 1 , and a class of nonlinear operators, see Section 3 .

1. BANACH SPACES WITH PROPERTY (π_k)

This section is primarily devoted to the study of a certain class of Banach spaces, namely the ones with Property (π_k) . See Definition 1 below. These spaces have been introduced by deFigueiredo [6] in the study of fixed points for weakly continuous non-linear mappings in Banach spaces. (See also Section 4 of this chapter) We learned later that Lindenstrauss had considered such spaces in [11] . The class of Banach spaces with Property (π_k) even for $k = 1$, seems rather large. We show here that most of the standard Banach spaces have such a property. Moreover, we do not know whether there is a Banach space which does not have Property (π_1) .

Definition 1. A Banach space E is said to have Property (π_k) , for some fixed $k \geq 1$, if there exists a collection of finite dimensional subspaces F_α , $\alpha \in \Lambda$, such that :

(i) The collection $\{F_\alpha\}$ is directed by inclusion. That is, given any two elements F_α and F_β in this collection, there exists a third one which contains both.

(ii) The union of all F_α , $\alpha \in \Lambda$, is dense in E .

(iii) Each F_α is the range of a continuous projection P_α of norm $\leq k$ (By projection we mean a linear operator $P : E \to E$ which is idempotent, $P^2 = P$.

<u>Remark 1</u>. In the previous definition, the Banach space E is not necessarily separable. If a separable Banach space has Property (π_k) , then the index family Λ can be taken as the set of all positive integers. So, in this case, the collection $\{F_\alpha\}$ is a linearly ordered set.

<u>Examples</u>. 1) Hilbert spaces have property (π_1) . Indeed, we take for $\{F_\alpha\}$ the collection of <u>all</u> finite dimensional subspaces, and for P_α the orthogonal projection.

2) Banach spaces with a Schauder basis have Property (π_k) . See Proposition A below.

3) Banach spaces with a monotone Schauder basis have Property (π_1) . See Remark 4 .

4) Let $(X,\mathcal{S},\mu)$ be a σ-finite measure space. Let $1 \leq p < \infty$. The space $L^p(X,\mathcal{S},\mu)$ has Property (π_1) . See Proposition IV.2.

5) Let X be a compact metric space. The space $C(X)$ of real-valued continuous functions in X , with the norm of the supremum, has Property (π_1) . See Proposition IV.3.

Definition 2. A sequence $\{x_n\}_{n=1}^{\infty}$ in a Banach space E is said to be a Schauder basis for E if, for each $x \in E$, there exists a unique sequence of numbers a_n, $n = 1, 2, \ldots$, such that $\sum_{j=1}^{n} a_j x_j$ converges strongly to x, as $n \to \infty$.

Remark 1. For each n, the mapping $x \to \alpha_n(x) = a_n$ is a linear functional in E. It is an immediate consequence of Proposition A below that α_n is a continuous functional.

Proposition A. Let E be a Banach space with a Schauder basis $\{x_n\}$. Then, for each n, the mapping $P_n : E \to E$ defined by

$$P_n x = \sum_{j=1}^{n} \alpha_j(x) x_j$$

is a continuous projection. Moreover, there exists a number $k > 0$ such that

$$||P_n|| \leq k$$

for every n.

A proof of this proposition can be found in the book of Goffman and Pedrick, "First course in Functional Analysis", page 102.

Remark 2. For each n, it can be easily seen that the nullspace $N(P_n)$ of the projection P_n coincides with the set of all elements of the form $x = \sum_{j=n+1} \alpha_j(x) x_j$. By Proposition A it follows that $N(P_n)$ is closed. Consequently, the closure of the linear subspace generated by

x_{n+1} , x_{n+2} , ... is also $N(P_n)$. Let us use the following notation: $R(P_n) = F_n$, $N(P_n) = G_n$ and $Q_n = I - P_n$.

Remark 3. A metric question about the projection P_n is as follows. Is $P_n x$ an element of F_n which is closest to x ? Same question for $Q_n x$ and G_n . Before giving an answer to this question, we note that, as a consequence of the finite dimensionality of F_n , it follows that there exists at least one point in F_n which is closest to a given point $x \in E$. That is, given an element x in E , the following set is non-empty.

(1) $$B_{F_n}(x) = \{y_o \in F_n : ||x - y_o|| = \inf_{y \in F_n} ||x - y|| \} .$$

Now the answer to the above question is as follows. In general, the set $B_{F_n}(x)$ does not contain the point $P_n x$. However, Nikolsky [14] has proved that, given a Banach space with a Schauder basis, there exists an equivalent norm such that, with respect to this new norm, we have

$$B_{F_n}(x) = \{P_n x\} \quad \text{and} \quad B_{G_n}(x) = \{Q_n x\} ,$$

where { } denotes a set with a single element and B_{G_n} is defined in a similar way to B_{F_n} . Recently, Rutherford [17] has considered this problem for Banach spaces with an unconditional basis (see definition in Day's book, "Normed Linear Spaces").

Examples. 1) In C_o and ℓ^p , $p \geq 1$, the sequence $\{e_n\} = \{0, \ldots, 0, 1, 0, \ldots\}$ is a Schauder basis. In this case

$\alpha_n(\xi) = \alpha_n(\{\xi_1, \xi_2, \dots\}) = \xi_n$.

2) In $C[0,1]$ a Schauder basis has been constructed by Schauder [20] himself. See also Day's book "Normed Linear Spaces".

3) The Haar system (see the book by Goffman and Pedrick, page 194) is a Schauder basis for $L^p[0,1]$, $1 \leq p < \infty$. See Schauder [21] .

Definition 3. A Schauder basis $\{x_n\}_{n=1}^{\infty}$ for a Banach space E is said to be monotone if, for every $x \in E$, we have

$$\|\textstyle\sum_{j=1}^{n} \alpha_j(x)x_j\| \leq \|x\|$$

for all $n = 1, 2, \dots$.

Remark 4. Let E be a Banach space with a monotone Schauder basis $\{x_n\}_{n=1}^{\infty}$. By Proposition A it follows that the mappings $P_n x = \sum_{j=1}^{n} \alpha_j(x)x_j$ are continuous projections and $\|P_n\| \leq k$. However in view of the monotonicity of the basis we have $\|P_n\| = 1$ for all n . This shows that a Banach space with a monotone Schauder basis has Property (π_1) .

Remark 5. The Schauder bases of Examples 1 and 2 above are monotone.

Remark 6. Let E be a Banach space with a Schauder basis $\{x_n\}_{n=1}^{\infty}$. The following expression defines a new norm in E

$$\|x\|' = \sup_n \|P_n x\| \quad .$$

It can be proved (see the book of Goffman and Pedrick, page 102) that the two norms $\| \|$ and $\| \|'$ are equivalent and that, with respect to this new norm, the basis $\{x_n\}_{n=1}^{\infty}$ is monotone.

The following result, according to Bessaga [24], is due to Mazur. See also Michael and Pelczynsky [13] .

Proposition IV.1. Let E be a Banach space, and P_n , $n = 1, 2, \ldots$, projections in E such that

(i) $\|P_n\| = 1$, $n = 1, 2, \ldots$.

(ii) The range $R(P_n) = E_n$ of P_n has dimension n .

(iii) For every n , $E_n \subset E_{n+1}$.

(iv) The union of all subspaces E_n is dense in E .

Then E has a monotone Schauder basis.

Remark 7. We emphasize the fact that we do not require that the projections of the above proposition be compatible. That is, we do not have necessarily $P_j P_k = P_j$ for $k > j$. If in addition to the hypotheses of Proposition IV.1 , we assume that the projections are compatible, the result could be proved very easily.

The proof of Proposition IV.1 uses the following result of Nikolsky [14] .

Lemma IV.1. Let $\{e_n\}$ be a sequence in a Banach space E , such that the linear subspace generated by the elements of this sequence is dense in E . Assume that, for all positive integers m and n and for all scalars $\lambda_1 , \ldots, \lambda_{m+n}$, we have

$$(*) \qquad \left\|\sum_{j=1}^{m} \lambda_j e_j\right\| \leq \left\|\sum_{j=1}^{m+n} \lambda_j e_j\right\|$$

Then $\{e_n\}$ is a Schauder basis for E.

Proof of Proposition IV.1. We define a sequence $\{e_n\}$ in E as follows. e_1 is a vector in E_1 with unit norm. The nullspace of the projection P_1 restricted to E_2 is a one-dimensional subspace; let e_2 be a vector in this subspace. We proceed by induction. The null-space of the projection P_n restricted to E_{n+1} is a one-dimensional subspace; let e_{n+1} be a vector in this subpsace. We claim that $\{e_n\}$ so constructed is a monotone Schauder basis. Indeed let $\lambda_1, \ldots, \lambda_{n+1}$ be arbitrary scalars. Then

$$\|\textstyle\sum_{j\ 1}^{n+1} \lambda_j e_j\| \geq \|P_n(\sum_{j=1}^{n+1} \lambda_j e_j)\| = \|\sum_{j=1}^{n} \lambda_j e_j\|$$

By induction we have

$$\|\textstyle\sum_{j=1}^{n} \lambda_j e_j\| \leq \|\sum_{j=1}^{n+m} \lambda_j e_j\|$$

for all positive integers m and n. Using Lemma IV.1 the result follows.

The following result is due to R. Beals (see Browder and deFigueiredo [5]).

Proposition IV.2. Let (X,S,μ) be a measure space with a σ-finite measure μ. Then for any p, $1 \leq p < \infty$, the Lebesgue space $L^p(X,S,\mu)$ has Property (π_1).

Proof. 1°) Let $\{S_1, \ldots, S_r\}$ be any finite family of disjoint sets of S with finite measure. Let χ_j be the characteristic function of S_j , $j = 1, \ldots, r$. We consider the finite dimensional subspace F of L^p generated by the functions $\chi_1, \ldots, \chi_r$. Let P be a projection over F defined by

$$Pf = \sum_{j=1}^{r} \frac{1}{\mu(S_j)} \left[\int_{S_j} f d\mu \right] \chi_j \quad .$$

2°) We prove that the projection P has norm 1 . Indeed,

$$\|Pf\|_{L^p}^{p} = \int_X \sum_{j=1}^{r} \frac{1}{\mu(S_j)^p} \left| \int_{S_j} f d\mu \right|^p \chi_j \, d\mu$$

$$= \sum_{j=1}^{r} \mu(S_j)^{1-p} \left| \int_{S_j} f d\mu \right|^p$$

By Hölder's inequality

$$\text{(2)} \qquad \|Pf\|_{L^p}^{p} \leq \sum_{j=1}^{r} \mu(S_j)^{1-p} \left[\int_{S_j} |f|^p \, d\mu \right] \mu(S_j)^{\frac{p}{q}}$$

$$= \sum_{j=1}^{r} \int_{S_j} |f|^p \, d\mu \quad .$$

From (2) it clearly follows that

$$\|Pf\|_{L^p}^{p} \leq \int_X |f|^p \, d\mu = \|f\|_{L^p}^{p} \quad .$$

This implies that $\|P\| = 1$.

3°) Finally we observe that the union of all subsets F (corresponding to all possible choices of disjoint measurable subsets $S_1, \ldots, S_r$, with $r = 1, 2, \ldots,$ is dense in L^p. This is a consequence of the fact that any f in L^p can be approximated in the L^p-norm by simple functions.

The following result is due to Michael and Pelczynsky [12].

Proposition IV.3. Let X be a compact metric space, and $C(X)$ the Banach space of the real-valued continuous functions in X. Then $C(X)$ has Property (π_1).

Remark 8. The proof of this proposition uses the interesting notion of "peaked" partition of unit. In [13] Michael and Pelczynsky show that $C(X)$ has a monotone Schauder basis.

2. DUALITY MAPPING IN BANACH SPACES WITH PROPERTY (π_1).

Let E be a Banach space with Property (π_1), see definition in Section 1 of this Chapter. Let us denote by E^* its dual space. Let $\{F_\alpha\}$ and $\{P_\alpha\}$ be the finite dimensional subspaces and the projections (with norm 1) which are associated with E. For each $\alpha \in \Lambda$, the adjoint operator $P_\alpha^* : E^* \to E^*$ is a projection in E^* also with norm 1. For any subspace M of E, we denote by M^o the annihilator of M, that is, the set

$$M^o = \{y^* \in E^* : (y^*,x) = 0 \quad \text{for all} \quad x \in M\} .$$

It is easy to see that

$$R(P_\alpha^*) = N(P_\alpha)^o \quad \text{and} \quad N(P_\alpha^*) = R(P_\alpha)^o \quad .$$

Let $\mu(r)$ be some given gauge function, see Section III.7. Then there exists a unique multi-valued duality mapping $J : E \to 2^{E^*}$ with gauge function $\mu(r)$, as defined in that section. We now establish some simple results about the way J acts on the subspaces F_α .

Proposition IV.4. Let E be a Banach space with Property (π_1) . Let J be the duality mapping in E with a given gauge function μ . Then, for every $x \in F_\alpha$, the following inclusion holds

$$P_\alpha^*(Jx) \subset Jx \quad .$$

Proof. Let $y' \in Jx$. We prove that $P_\alpha^* y' \in Jx$. In fact, we first have

$$(1) \qquad (P_\alpha^* y', x) = (y', P_\alpha x) = (y', x) = \|y'\| \; \|x\| \quad .$$

From (1) it follows $\|y'\| \leq \|P_\alpha^* y'\|$. Since P_α^* has norm 1 we conclude that $\|P_\alpha^* y'\| = \|y'\|$. This gives $\|P_\alpha^* y'\| = \mu(\|x\|)$. And this together with (1) shows that $P_\alpha^* y'$ is an element of the set Jx .

Corollary IV.1. In addition to the hypotheses of Proposition IV.4 assume that E^* is strictly convex. Then $P^* Jx = Jx$.

Proof. The proof follows immediately from the fact that the additional hypothesis implies that the set Jx consists of exactly one point (Proposition III.1).

Remark. This corollary shows that, when E^* is strictly convex, the duality mapping maps F_α into the finite dimensional subspace $R(P_\alpha^*)$. If in addition the Banach space E is reflexive, it follows from the next proposition that the duality mapping actually maps F_α *onto* $R(P_\alpha^*)$.

Proposition IV.5. Let E be a reflexive Banach space with Property (π_1). Let J be the duality mapping in E with a given gauge function μ. Then, for every $\alpha \in \Lambda$, the following inclusion holds

$$R(P_\alpha^*) \subset J(F_\alpha) .$$

Proof. Let $y' \in R(P_\alpha^*)$. By the generalized Beurling Livingston theorem (see Browder [4]) it follows that the intersection

$$J(F_\alpha) \cap [N(P_\alpha^*) + y'] \neq \phi$$

Thus let $x \in F_\alpha$ and $z' \in N(P_\alpha^*)$ such that $z' + y' \in Jx$. From this it follows that $P_\alpha^*(z' + y') = y' \in Jx$, in view of Proposition IV.4. The proof is complete.

3. FIXED POINT THEOREMS FOR FINITE DIMENSIONAL MAPPINGS

In this section we consider the question of the existence of fixed points for continuous mappings in finite dimensional Banach spaces. Let Ω be a bounded open subset of a finite dimensional Banach space E, and $\bar{\Omega}$ its closure. We consider continuous mappings $T : \bar{\Omega} \to E$, whose range may contain points outside of $\bar{\Omega}$. We shall prove two theorems on the existence of fixed points for such mappings. the first one, Theorem IV.1 , concerns continuous mappings T defined in a ball about the origin. In this case, the proof of the existence of a fixed point uses the Brouwer fixed point theorem. The second result, Theorem IV.2 , concerns continuous mappings T defined in a general bounded closed set $\bar{\Omega}$, and it is an extension of Theorem IV.1 . In this case the tool used in the proof is the Brouwer degree theory.

Theorem A. (Existence and Properties of the Brouwer degree). Let Ω be a bounded open set in the Euclidean n-dimensional space R^n , $\bar{\Omega}$ its closure and $\partial\Omega$ its boundary. Then, for each continuous mapping $T : \bar{\Omega} \to R^n$ and each point z not in the image $T(\partial\Omega)$, there is defined an integer $\deg[T;\Omega,z]$, which is called the Brouwer degree. Moreover, the Brouwer degree has the following properties.

(i) $\deg[I;\Omega,z] = 1$, where I is the identity mapping, $Ix = x$ for all $x \in \bar{\Omega}$.

(ii) If $\deg[T;\Omega,z] \neq 0$, then there exists at least one $x \in \Omega$ such that $Tx = z$.

(iii) Let A be some closed interval in R^1 . Let $F : \bar{\Omega} \times A \to R^n$ be a continuous function, and z some given point in R^n such that $z \neq F(x,t)$ for all $x \in \partial\Omega$ and all $t \in A$. Let us denote by $F_t : \bar{\Omega} \to R^n$ the mapping defined by $F_t(x) = F(x,t)$, for each fixed $T \in A$. Then the degree $\deg[F_t;\Omega, z]$ is constant for $t \in A$.

Remark. There are many nice presentations of the Brouwer degree theory. We think the reader will enjoy the elegant exposition of E. Heinz [9] .

Corollary A. (Brouwer Fixed Point Theorem). Let $T : B_1 \to B_1$ be a continuous mapping of the unit ball B_1 about the origin in R^n into itself. Then T has a fixed point.

Proof. Consider the mapping $F : B_1 \times [0,1] \to R^n$ defined by

$$F(x,t) = x - tTx$$

This function is obviously continuous. We may assume that, for $x \in \partial B_1$ and $t = 1$, we have $F(x,1) \neq 0$. (For otherwise there would exist a fixed point of T and the theorem would be proved). Furthermore for $x \in \partial B_1$ and $0 \leq t < 1$ we have

$$\|F(x,t)\| \geq \|x\| - t\|Tx\| \geq 1 - t > 0 .$$

So the conditions of Theorem A, part (iii), are satisfied. Then $\deg F_1 = \deg F_o$. Since F_o = identity mapping, we obtain, applying Theorem A, parts (i) and (ii), that there exists $x_o \in \text{int } B_o$ such that $x_o - Tx_o = 0$.

Remark 1. It is immediate that Brouwer fixed point theorem holds for any topological space homeomorphic to the unit ball B_1 of R^n. In particular, it holds for the closure of any open convex set in R^n. Consequently, the following result follows.

"Let E be a finite dimensional Banach space, and B a ball the origin in E. Then every continuous mapping $T : B \to B$ has a fixed point in B".

Remark 2. The above result, Corollary A, was proved by Brouwer [Math. Ann. 69(1910), pp. 176-180]. However the result had been previously obtained by Poincaré [Journal de Math. 1886] and Bohl [Journal fur die reine und angewandte Math. 127(1904), pp. 179-276. In fact, the following result is proved by Bohl. "Let K be the n-dimensional rectangle $-a_i \leq x_i \leq a_i$, $i = 1, \ldots n$. Let $T : K \to R^n$ be a continuous function defined in K, such that $Tx \neq 0$ for all $x \in K$. Then there exists a point x_o on the boundary ∂K of K such that $Tx_o = \lambda x_o$ where λ is some negative number." It is easy to see that this result is equivalent to Brouwer fixed point theorem.

Remark 3. A proof of the Brouwer fixed point, without the notion of degree of a mapping, can be found in the book of Dunford-Schwartz, Linear Operators, Part I, pp. 467-470.

Theorem IV.1. Let E be a finite dimensional Banach space, $B = \{x : \|x\| \leq r\}$ and $\partial B = \{x : \|x\| = r\}$. Let $T : B \to E$ be a continuous mapping defined in B, which satisfies the following condition

(*) $Tx - \lambda x \neq 0$ for all $x \in \partial B$ and all $\lambda > 1$.

Then T has a fixed point in B .

Remark. The proof below illustrates the use of the Brouwer fixed point theorem. The next theorem, Theorem IV.2 , could be reduced to Theorem IV.1. We have preferred to give a separate proof of it, so as to illustrate the use of the degree of a mapping.

Proof of Theorem IV.1. We define a new mapping $T_1 : B \to B$ as follows

$$T_1 x = \begin{cases} Tx , & \text{if } \|Tx\| < r \\ rTx/\|Tx\| , & \text{if } \|Tx\| \geq r \end{cases}$$

This new mapping T_1 is continuous, and so, by the Brouwer Fixed Point Theorem, it follows that it has a fixed point $T_1 x_o = x_o$. We claim that x_o is also a fixed point of T . In fact, there are two possibilities.

(i) x_o is in the interior of B . In this case, it follows immediately that $Tx_o = T_1 x_o = x_o$, i.e., x_o is a fixed point of T .

(ii) x_o is in the boundary of B . In this case, it follows from the definition of T_1 that $Tx_o = (\|Tx_o\|/r)x_o$. This implies, in view of hypothesis (*) , that $\|Tx_o\|/r \leq 1$. This together with the fact that $r \leq ||Tx_o||$ gives $r/||Tx_o|| = 1$. Thus $Tx_o = x_o$ also in this second case. The theorem is proved.

Theorem IV.2. *Let Ω be a bounded open subset of a finite dimensional Banach space E . Assume that $0 \in \Omega$. Let $T : \bar{\Omega} \to E$ be a continuous mapping defined in the closure $\bar{\Omega}$ of Ω . Let $\partial\Omega$ denote the boundary of Ω , and suppose that*

$$(*) \qquad Tx - \lambda x \neq 0 , \quad \text{for all} \quad \lambda > 1 \quad \text{and all} \quad x \in \partial\Omega .$$

Then T has a fixed point in $\bar{\Omega}$.

Remark. We give below a proof for the case $E = R^n$. The general case follows immediately from this special case.

Proof. A fixed point of T is a solution of the equation $(I - T)x = 0$. Let us consider the mappings

$$T_\alpha = \alpha(I - T) + (1 - \alpha)I = I - \alpha T \quad ,$$

for $0 \leq \alpha \leq 1$. It is clear that the mapping $F : [0,1] \times \bar{\Omega} \to E$ defined by $F(\alpha,x) = T_\alpha x$ is continuous. We can assume that $T_1 x \neq 0$ for all $x \in \partial\Omega$. For, otherwise, T would have a fixed point in $\bar{\Omega}$, and the theorem would be proved. Furthermore, assumption (*) implies that $T_\alpha x \neq 0$ for all $x \in \partial\Omega$ and all α such that $0 \leq \alpha < 1$. So $T_\alpha x \neq 0$ for all $x \in \partial\Omega$ and all $\alpha \in [0,1]$. By Theorem A it follows that $\deg[T_\alpha;\bar{\Omega},0]$ is constant. Since the degree of $T_o = I$ is 1 , (Theorem A), we obtain that the degree of $T_1 = I - T$ is also 1 . Using Theorem A again we obtain that there exists $x \in \Omega$ such that $(I - T)x = 0$. The theorem is proved.

Remark. Condition (*) will be fulfilled if, for every $x \in \partial\Omega$, there exists a continuous linear functional v_x', $v_x' \in E^*$, such that

$$(Tx, v_x') \leq (x, v_x') \quad \text{and} \quad 0 \leq (x, v_x') \quad .$$

4. GALERKIN APPROXIMABLE OPERATORS.

In this section we introduce a large class of non-linear operators defined in Banach spaces with Property (π_k). We have called [7] them Galerkin approximable operators, or for short G-operators. We show that compact operators, weakly continuous operators and P-compact operators, among others are all G-operators.

Definition 1. Let C be a closed convex subset of a Banach space E with Property (π_k). An operator $T : C \to E$ is said to be Galerkin approximable, (or for short a G-operator) if

(i) $P_\alpha T : C \cap F_\alpha \to F_\alpha$ is continuous for all but a finite number of $\alpha \in \Lambda$.

(ii) T has a fixed point in C whenever there exist $x_\alpha \in F_\alpha$, for all but a finite number of α's, such that $P_\alpha Tx_\alpha = x_\alpha$ and $\|x_\alpha\| \leq R$ for some positive R independent of α.

Now we proceed to establish that the class of G-operators is indeed a large one. In fact, it contains most of the standard non-linear operators. To fix our terminology, let us give some definitions.

Definition 2. A mapping $T : C \to F$ from a closed convex subset C of a Banach space E into another Banach space F is said to be compact if it is continuous and maps bounded sets into relatively compact sets. These operators are called completely continuous by Vainberg [23].

Definition 3. A mapping $T : C \to F$ from a closed convex subset C of a Banach space E into another Banach space F is said to be completely continuous if it takes each weakly convergent sequence into a strongly convergent sequence. These operators are called strongly continuous by Vainberg [23].

Remark. The two classes of mappings just defined are not comparable. That is, neither one is contained in the other. This is shown by the two examples below. However, if the Banach space E is reflexive or the mapping T is linear, then one can prove some relations between these classes, Propositions IV.6 and IV.7.

Example 1. (Example of a compact mapping which is not completely continuous). We learned this example from F.E. Browder. Another example may be found in Vainberg's book [23], page 14. Let ℓ^2 be the Hilbert space of the infinite sequences $\{\xi_1, \xi_2, \ldots\}$ of real numbers such that $\sum_{j=1}^{\infty} |\xi_j|^2 < \infty$. Let $T : \ell^2 \to \ell^2$ be the mapping in ℓ^2 defined as follows,

$$Tx = \{\rho(x), 0, \ldots\}$$

where $\rho : \ell^2 \to R^1$ is defined by

$$\rho(x) = \begin{cases} 1 - \|x\|^2 & \text{if} \quad \|x\| \leq 1 \\ 0 & \text{if} \quad \|x\| > 1 \end{cases} .$$

It is immediate that T is compact. On the other hand, T is not completely continuous. Indeed, if we have $x_n \rightharpoonup x$ and $Tx_n \to Tx$, it will follow $\|x_n\| \to \|x\|$. It is well known that, in Hilbert spaces, the two facts $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ imply $x_n \to x$. Since there are weakly convergent subsequences which do not converge strongly, it follows that T cannot be completely continuous.

Example 2. (Example of a completely continuous mapping which is not compact). (Cf. Vainberg's book [23].) Let $T : C[-\pi,\pi] \to L^2[-\pi,\pi]$ be the identity mapping, $Tf = f$, for all $f \in C[-\pi,\pi]$. The mapping T is not compact. In fact, the sequence $\{\sin jt\}_{j=1}^{\infty}$ is bounded in $C[-\pi,\pi]$, but has no convergent subsequence in $L^2[-\pi,\pi]$. For

$$\|\sin jt - \sin kt\|_{L^2} = 2\pi , \quad \text{for } j \neq k .$$

On the other hand, let $\{f_n(t)\}$ be a sequence in $C[0,1]$ which converges weakly to some continuous function $f(t)$. This implies that the sequence $\{f_n(t)\}$ is uniformly bounded and converges pointwisely to $f(t)$. Applying the Bounded Convergence Theorem (see any book in measure theory), we obtain that

$$\int_{-\pi}^{\pi} |f_n(t) - f(t)|^2 dt \to 0 .$$

This proves that T is completely continuous.

Proposition IV.6. Let E and F be Banach spaces. In addition suppose that E is reflexive. Let C be a closed convex subset of E. Then every completely continuous operator $T : C \to F$ is also compact.

Proof. Let $\{x_n\}$ be a sequence such that $x_n \to x$. Then, a fortiori, $x_n \rightharpoonup x$. ("$\rightharpoonup$" denotes weak convergence). Since T is completely continuous, it follows $Tx_n \to Tx$. So T is a continuous mapping. Now let B be a bounded set in E; we want to prove that every sequence $\{Tx_n\}$, $x_n \in B \cap C$, contains a strongly convergent subsequence. This is however immediate, for the reflexivity of E implies that $\{x_n\}$ contains a weakly convergent subsequence. Since T is completely continuous, this subsequence goes in a strongly convergent subsequence of $\{Tx_n\}$.

Proposition IV.7. Let T be a linear mapping from a Banach space E into another Banach space F. Then, if T is compact, it follows that T is completely continuous.

Proof. Given $x_n \rightharpoonup x$, we claim that $Tx_n \to Tx$. Since T is a linear continuous mapping we have that $Tx_n \rightharpoonup Tx$. So to prove our claim it suffices to show that every subsequence of Tx_n contains a further subsequence which converges. However this is immediate because the sequence x_n is bounded and T is compact.

Corollary IV.2. For linear mappings from a reflexive Banach space into a (not necessarily reflexive) Banach space the two concepts, compactness and complete continuity, coincide.

Theorem IV.3. Let C be a closed convex subset of a Banach space E with Property (π_k). Let $T : C \to E$ be a compact operator defined in C. Then T is a G-operator.

Proof. 1°) It is immediate that $P_\alpha T$ is continuous for all α.

2°) Let B_R be a ball of radius R about the origin. We prove that for every positive integer n there is an index $\alpha(n) \in \Lambda$ with the following property: given $z \in T(C \cap B_R)$ we have an element $z_n \in F_{\alpha(n)}$ such that $||z - z_n|| < 1/n$. Indeed, using the fact that $T(C \cap B_R)$ is relatively compact we can find points $x_1, \ldots, x_m$ in E such that the balls $B(x_j)$ of radius $1/2n$ about these points cover $T(C \cap B_R)$. Since the union of all F_α is dense in E we can find subspaces F_{α_j}, $j = 1, \ldots, m$, such that each F_{α_j} intersects the ball $B(x_j)$. By the fact that the collection $\{F_\alpha\}$ is directed, it follows that there exists a subspace $F_{\alpha(n)}$ in this collection which contains the subspaces $F_{\alpha_1}, \ldots, F_{\alpha_m}$. Now let z be an arbitrary point in $T(C \cap B_R)$. It follows that z belongs to some ball $B(x_j)$, that is

(1) $$||z - x_j|| \leq 1/2n .$$

On the other hand since $F_{\alpha(n)}$ intersects $B(x_j)$ we have for a point $z_n \in F_{\alpha(n)} \cap B(x_j)$

(2) $$||z_n - x_j|| \leq 1/2n .$$

Inequalities (1) and (2) gives $||z - z_n|| < 1/n$, as we have claimed.

3°) Now let us assume that, for all but a finite number of α's, there are $x_\alpha \in F_\alpha$ such that $P_\alpha T x_\alpha = x_\alpha$ and $||x_\alpha|| \leq R$. So there exists an integer n_o such that, for all $n \geq n_o$, we have

(3) $$P_{\alpha(n)} T x_n = x_n , \quad x_n = x_{\alpha(n)} \in F_{\alpha(n)} ,$$

where $F_{\alpha(n)}$ are the subspaces defined in 2°.

Since the sequence $\{x_n\}$ is bounded and T is compact, it follows that Tx_n contains a strongly convergent subsequence. To keep notation simple we assume that $Tx_n \to y$.

4°) Now we claim that $P_{\alpha(n)}Tx_n \to y$. To prove this we first observe that, for all $x \in E$ and all $z \in F_{\alpha(n)}$, we have

$$(4) \qquad \|x - P_{\alpha(n)}x\| \leq (1 + k)\,\|x - z\| \quad .$$

By the triangle inequality and the fact that all projections involved are uniformly bounded by k , we obtain

$$(5) \qquad \|P_{\alpha(n)}Tx_n - y\| \leq k\,\|Tx_n - Tx_m\| + \|P_{\alpha(n)}Tx_m - Tx_m\| + \|Tx_m - y\| \quad .$$

Since $Tx_n \to y$, we have that the first and last terms in the right hand side of (5) can be made small. To estimate the middle term we use Part 2° of this proof and (4). We then conclude that $P_{\alpha(n)}Tx_n \to y$.

5°) Thus, in view of (3), we have that $x_n \to y$. Since T is continuous we have that $Tx_n \to Ty$. This together with the conclusion of 3° gives $Ty = y$. The theorem is proved.

Corollary IV.3. *Let C be a closed convex subset of a reflexive Banach space with Property (π_k) . Let $T : C \to E$ be a completely continuous operator defined in C . Then T is a G-operator.*

Proof. Consequence of Theorem IV.3 and Proposition IV.6.

Definition 4. (Petryshyn [15]) Let E be a separable Banach space with Property (π_k). A mapping $T : E \to E$ is said to be projectionally compact (or for short P-compact) if

(i) $P_n T$ is continuous for all but a finite number of n's.

(ii) Every bounded sequence $\{x_n\}$, $x_n \in F_n$, such that the sequence $\{P_n T x_n - p x_n\}$ (for some $p > 0$) converges strongly, contains a subsequence $\{x_{n(j)}\}$ which converges strongly to some point $x \in E$ and $P_{n(j)} T x_{n(j)} \to Tx$ as $n(j) \to \infty$.

Remark. This notion was introduced by Petryshyn in [15], where he considered the case of bounded mappings, i.e., operators mapping bounded sets in E into bounded sets in E. The general case of a (not necessarily bounded) P-compact mapping was discussed also by Petryshyn in [16]. The concept of P-compact mapping evolved from the notion of a quasi-compact mapping due to Kaniel [10]. In [15] it is proved that every quasi-compact mapping is also P-compact. Furthermore the following two results (Petryshyn [15]) hold.

Proposition IV.8. Let E be a separable Banach space with Property (π_k). Then every compact mapping $T : E \to E$ is also P-compact.

Proof. The continuity of $P_n T$ for all n is immediate. Now let $\{x_n\}$, $x_n \in F_n$, be a bounded sequence and $p > 0$ such that

$$(1) \qquad P_n T x_n - p x_n \to y$$

where y is some point in E. Since T is compact it follows that there exists a subsequence $\{Tx_{n(j)}\}$ which converges strongly. This implies that $P_{n(j)}Tx_{n(j)}$ also converges strongly. From (1) it then follows that there exists a point $x \in E$ such that

(2) $$x_{n(j)} \to x \quad .$$

Since T is continuous, we obtain from (2) that $Tx_{n(j)} \to Tx$. Consequently $P_{n(j)}Tx_{n(j)} \to Tx$. Thus T is a P-compact operator.

Definition. Let E and F be two Banach spaces. A mapping $T : E \to F$ is said to be demicontinuous if it is continuous from the strong topology of E to the weak topology of F .

Proposition IV.9. Let H be a separable Hilbert space, and let H_n be an increasing sequence of finite dimensional subspaces of H , whose union is dense in H . Let P_n be the orthogonal projection over H_n . Let $T : H \to H$ be a bounded demicontinuous monotone mapping in H . Then the mapping $-T$ is P-compact.

Proof. Let $\{x_n\}$ be a bounded sequence in H and $p > 0$ such that $x_n \in H_n$ and

(3) $$P_nTx_n + px_n \to y$$

where y is some point in H . Since H is a Hilbert space we may assume that $x_n \rightharpoonup x$. We claim that $x_n \to x$ and $P_nTx_n \to Tx$, that is, $-T$ is P-compact. We first prove that $y = (T + p)x$. Let z be an arbitrary point in H_{n_o} . Then for $n \geq n_o$ we have that

$$((T + p)x_n - (T + p)z, x_n - z) \geq 0$$

implies

$$(P_n(T + p)x_n - P_n(T + p)z , x_n - z) \geq 0$$

Taking limits in this last inequality we have

$$(4) \qquad (y - (T + p)z, x - z) \geq 0$$

Now we observe that (4) holds for every $z \in \bigcup_{n=1}^{\infty} E_n$. Since this set is dense in E and T is demicontinuous, it follows that (4) holds for every z in E . Take in (4) $z = x + tv$, where v is an arbitrary vector in E and $t > 0$. After simplification, we obtain

$$(y - (T + p)(x + tv), v) \geq 0 .$$

Taking limits and using the fact that T is demicontinuous one gets $(y - (T + p)x, v) \geq 0$ for all $v \in E$. This then implies that $y = (T + p)x$. Now using the monotonicity of T we have

$$p \|x_n - x\|^2 \leq ((T + p)x_n - (T + p)x, x_n - x)$$

$$= (P_n(T + p)x_n, x_n) + ((T + p)x,x) - ((T + p)x, x_n)$$

$$- (P_n(T + p)x_n,x) - ((T + p)x_n,(I - P_n)x)$$

Taking limits and using the fact that T is bounded we conclude that

$$p \overline{\lim} \|x_n - x\|^2 \leq (y,x) + ((T + p)x,x) - ((T + p)x,x) - (y,x) = 0$$

So $x_n \to x$. Then $P_n T x_n \to y - px = (T + p)x - px = Tx$. This proves that $-T$ is P-compact. The proposition is proved.

<u>Theorem IV.4</u>. <u>Let C be a closed convex subset of a separable Banach space E with Property (π_k) . Then every P-compact operator $T : C \to E$ is also a G-operator.</u>

<u>Proof</u>. We assume that, for all but a finite number of n's , there exists $x_n \in F_n$ such that $P_n T x_n = x_n$ and $\|x_n\| \leq R$. We shall prove that there exists $x \in C$ such that $Tx = x$. This together with the fact that $P_n T$ is continuous by hypothesis proves that T is a G-operator. We have $P_n T x_n - x_n = 0$. Using the fact that T is P-compact we obtain a subsequence $\{x_{n(j)}\}$ such that

$$(*) \qquad x_{n(j)} \to x \quad \text{and} \quad P_{n(j)} T x_{n(j)} \to Tx ,$$

where x is some point in C . From (*) it follows that

$$x_{n(j)} \to x \quad \text{and} \quad x_{n(j)} \to Tx ,$$

This implies that $x = Tx$. Thus T is a G-operator.

<u>Definition</u>. Let E and F be two Banach spaces. A mapping $T : E \to F$

is said to be <u>weakly continuous</u> if it is continuous from the weak topology of E to the weak topology of F.

<u>Theorem IV.5. Let C be a closed convex subset of a reflexive Banach space E with Property (π_1). Assume that the dual space E^* is strictly convex. Then every weakly continuous mapping $T : C \to E$ is also a G-operator.</u>

<u>Remark</u>. The above theorem provides an example of a G-operator which is not necessarily P-compact.

We shall need the following result to prove Theorem IV.5.

<u>Lemma IV.2. Let E be a reflexive Banach space with Property (π_k), and E^* its dual space. Let $P_\alpha^* : E^* \to E^*$ be the adjoint of the projection P_α, $\alpha \in \Lambda$. Let us denote by $R(P_\alpha^*)$ the range of P_α^*. Then the union of all $R(P_\alpha^*)$, $\alpha \in \Lambda$, is weakly dense in E^*.</u>

<u>Proof</u>. It suffices to prove that, for each $y' \in E^*$, the set $\{P_\alpha^* y'\}_{\alpha \in \Lambda}$ has y' as one of its weak accumulation points. Indeed, given $\varepsilon > 0$ and $x \in E$, we know that there exists a sequence $\{\alpha(n)\} \subset \Lambda$

$$||P_{\alpha(n)}x - x|| \leq \varepsilon .$$

and then it follows

$$|(P_{\alpha(n)}^* y' - y', x)| = |(y', P_{\alpha(n)}x - x)| \leq \varepsilon ||y'|| .$$

Lemma IV.3. Let E be a reflexive Banach space with Property (π_1), and E^* its dual space. Let J be a duality mapping with given gauge function μ. Then the union of all sets $J(F_\alpha)$, $\alpha \in \Lambda$ is weakly dense in E^*.

Proof. Immediate consequence of Lemma IV.2 and Proposition IV.5.

Remark. The following result can be proved as a consequence of Propositions III.2 and III.4. "Let E be a reflexive Banach space with a strictly convex dual space E^*. Let $\{F_\alpha\}$ be a collection of subspaces whose union is dense in E. Then the union of all sets $J(F_\alpha)$ is weakly dense in E^*."

Proof of Theorem IV.5. For all $\alpha \in \Lambda$, the continuity of $P_\alpha T$ in F_α is immediate. Let us suppose now that, for all but a finite number $\alpha \in \Lambda$, there exists $x_\alpha \in F_\alpha$ such that $P_\alpha T x_\alpha = x_\alpha$ and $\|x_\alpha\| \leq R$. For each $\alpha_o \in \Lambda$ (except, of course, the α's for which $P_\alpha Tx = x$ is not solvable) define

$$V_{\alpha_o} = \text{weak closure of } \bigcup_{F_\alpha \supset F_{\alpha_o}} \{x_\alpha\} .$$

These sets V_{α_o} are weakly compact in virtue of the reflexivity of E. It is immediate that the collection $\{V_\alpha\}$, $\alpha \in \Lambda$, has the finite intersection property. Consequently the intersection $\bigcap_{\alpha \in \Lambda} V_\alpha$ is non-empty. Let x be a point in this intersection. We claim that $Tx = x$. Let z be a point in F_{α_o}. In view of the construction of x, it follows that there exists a sequence $\{x_{\alpha(n)}\}$ in V_{α_o} such that

$$x_{\alpha(n)} \rightharpoonup x .$$

By the weak continuity of T we have

$$Tx_{\alpha(n)} \rightharpoonup Tx .$$

So

(1) $$(Tx_{\alpha(n)} - x_{\alpha(n)}, Jz) \to (Tx - x, Jz) .$$

Since $z \in F_{\alpha(n)}$ for all $\alpha(n)$, we have using Corollary IV.1 that $P^*_{\alpha(n)}Jz = Jz$. Thus

(2) $$(Tx_{\alpha(n)} - x_{\alpha(n)}, Jz) = (P_{\alpha(n)}Tx_{\alpha(n)} - x_{\alpha(n)}, Jz) = 0 .$$

From (1) and (2) follows that

$$(Tx - x, Jz) = 0$$

for all $z \in F_\alpha$. By Lemma IV.3 we conclude that $Tx = x$. So T is a G-operator.

The next example of a G-operator was discussed in a paper by Browder and deFigueiredo [5].

<u>Definition</u>. Let E be a Banach space with a strictly convex dual space E^*. Let $J : E \to E^*$ the duality mapping in E with gauge function $\mu(r)$. (See Section 7 of Chapter III). A mapping $A : E \to E$ is said to

be J-monotone if

$$(Ax - Ay, J(x - y)) \geq 0$$

for all $x, y \in E$.

Example. If $T : E \to E$ is a non-expansive mapping in some Banach space E , then the mapping $I - T$ is J-monotone.

Theorem IV.6. Let E be a reflexive Banach space with Property (π_1) . Assume that the dual space E^* of E is strictly convex. Let $J : E \to E^*$ be a duality mapping in E which is supposed to be continuous and weakly continuous. Let $A : E \to E$ be a J-monotone demicontinuous mapping in E . Then $T = I - A$ is a G-operator.

Proof. 1°) The continuity of $P_\alpha T$ in F_α is immediate. Now suppose that, for all but a finite number of α's , there is $x_\alpha \in F_\alpha$ such that $P_\alpha T x_\alpha = x_\alpha$ and $\|x_\alpha\| \leq R$. For each $\alpha_o \in \Lambda$ define the set

$$V_{\alpha_o} = \text{weak closure of } \bigcup_{F_\alpha \supset F_{\alpha_o}} \{x_\alpha\} .$$

Since $\{x_\alpha\} \subset B_R$ and E is reflexive, it follows that each V_{α_o} is weakly compact. The collection $\{V_\alpha\}$, $\alpha \in \Lambda$, has the finite intersection property. So there exists a point x in the intersection of all V_α . We prove that $Tx = x$, i.e., T is a G-operator.

2°) Let v be an arbitrary point in F_{α_o} . For every α such that $F_\alpha \supset F_{\alpha_o}$ we have by J-monotonicity

(1) $$(Av - Ax_\alpha, J(v - x_\alpha)) \geq 0 .$$

Since $v - x_\alpha \in F_\alpha$ we have using Corollary IV.1 $P_\alpha^* J(v - x_\alpha) = J(v - x_\alpha)$. This implies

(2) $$(Ax_\alpha, J(v - x_\alpha)) = (P_\alpha Ax_\alpha, J(v - x_\alpha)) = 0 .$$

From (1) and (2) it follows

(3) $$(Av, J(v - x_\alpha)) \geq 0$$

for all $x_\alpha \in F_\alpha \supset F_{\alpha_0}$. Since $x \in V_{\alpha_0}$ and J is weakly continuous we obtain from (3)

(4) $$(Av, J(v - x)) \geq 0 .$$

Now observe that (4) holds for every $v \in F_\alpha$. Since the union of all F_α is dense in E, given $u \in E$ there exists a sequence $v_n \in F_{\alpha(n)}$ such that $v_n \to u$. The demicontinuity of A implies $Av_n \rightharpoonup Au$. The continuity of J implies $J(v_n - x) \to J(u - x)$. Since (4) holds for every v_n, it then follows

(5) $$(Au, J(u - x)) \geq 0$$

for all $u \in E$.

3°) Now let w be an arbitrary point in E and $t > 0$. Using (5) with $u = x + tw$ we obtain

$$(A(x + tw), J(tw)) \geq 0 . \tag{6}$$

From results of Section 7 of Chapter III we have that $J(tw) = \beta(t)J(w)$ where $\beta(t)$ is some positive function of t. So it follows from (6) :

$$(A(x + tw), J(w)) \geq 0 .$$

Finally taking $t \to 0$ we have $(Ax,Jw) \geq 0$ for all $w \in E$. Since E is reflexive we have, in virtue of Proposition III.2, that $J(E) = E^*$. So $Ax = 0$, which gives $Tx = x$. The theorem is proved.

Corollary IV.4. Let E be a reflexive Banach space with Property (π_1). Assume that the dual space E^* is strictly convex. Suppose that there exists a duality mapping in E which is both continuous and weakly continuous. Then every nonexpansive mapping $T : E \to E$ is also a G-operator.

5. FIXED POINT THEOREMS FOR GALERKIN APPROXIMABLE OPERATORS

In this section we prove a fixed point theorem, Theorem IV.7, for mappings in the class of Galerkin approximable operators defined in the last section. We then use this theorem to derive most of the known fixed point theorems in Banach spaces.

Theorem IV.7. Let C be a closed convex subset of a Banach space E

with Property (π_k), (see definition in Section IV.1). Let $T : C \to E$ be a G-operator defined in C. Assume that there exists $R > 0$ such that

(i) 0 belongs to the interior of $C \cap B_R \cap F_\alpha$, for all but a finite number of α's. $B_R = \{x \in E : \|x\| \leq R\}$

(ii) For all but a finite number of α's, we have

(A) $Tx - \lambda x \notin N(P_\alpha)$, for all $\lambda > 1$ and all $x \in \partial(B_R \cap C) \cap F_\alpha$

Then T has a fixed point.

Remark 1. The condition (i) above is automatically satisfied in the case when C contains a ball about the origin in E.

Remark 2. The possibility that $\partial(C \cap B_R) = C$ has not been ruled out.

Remark 3. In the case of a Banach space with Property (π_1), Condition (A) of the previous theorem is satisfied if condition (B) below holds. Let J be the duality mapping in E with gauge function $\mu(r)$. Condition (B) is as follows:

(B) $(Tx, Jx) \leq \|x\| \mu(\|x\|)$, for all $x \in \partial(C \cap B_R)$.

To prove that Condition (B) implies Condition (A), we proceed as follows. By Proposition IV.4 there exists $y' \in Jx \cap R(P_\alpha^*)$. For this y' we have

(1) $(P_\alpha(Tx - \lambda x), y') = (Tx - \lambda x, y') \leq \|x\| \mu(\|x\|) - \lambda(x, y')$

where the inequality was obtained using (B) . Now by the definition of duality mapping we have $(x,y') = \|x\| \mu(\|x\|)$. This used in (1) gives $P_\alpha(Tx - \lambda x) \neq 0$, i.e., Condition (A) is satisfied.

<u>Remark 4</u>. Let us consider the case of a bounded P-compact operator in a separable Banach space with Property (π_k) . Then Condition (A) of the previous theorem is satisfied, for all but a finite number of F_n's , if Condition (P) below holds. Condition (P) is as follows:

(P) $\quad Tx - \lambda x \neq 0$, for all $\lambda \geq 1$ and all $x \in \partial B_R$

To prove that (P) implies (A) we proceed as follows. Suppose that this is not the case. So there exist a sequence $\{\lambda_{n(j)}\}$ of numbers greater that 1 and a sequence $\{x_{n(j)}\}$, $x_{n(j)} \in \partial B_R \cap F_{n(j)}$, such that

(2) $$P_{n(j)}(Tx_{n(j)} - \lambda_{n(j)}x_{n(j)}) = 0 .$$

Since T is bounded, it follows that the sequence $\{\lambda_{n(j)}\}$ is bounded. So we may assume that $\lambda_{n(j)} \to \lambda$, $\lambda \geq 1$. From (2) it then follows that

(3) $$P_{n(j)}Tx_{n(j)} - \lambda x_{n(j)} \to 0 .$$

Since T is P-compact we conclude that there exists a subsequence (denote it again by $\{x_{n(j)}\}$) that converges strongly to a point $x \in \partial C$. Moreover $P_{n(j)}Tx_{n(j)} \to Tx$. Thus from (3) we obtain $Tx - \lambda x = 0$ for $\lambda \geq 1$. This contradicts Condition (P) which has been assumed to hold.

Proof of Theorem IV.7. 1°) Since $T : C \to E$ is a G-operator, it follows that, for each α, the mapping $P_\alpha T : C \cap F_\alpha \to F_\alpha$ is continuous.
2°) Now we observe that the boundary (relatively to F_α) $\partial_\alpha(B_R \cap C)$ of the set $B_R \cap C \cap F_\alpha$ is contained in $\partial(B_R \cap C) \cap F_\alpha$. So by assumption (A) it follows that, for each $x \in \partial_\alpha(B_R \cap C)$, we have

$$P_\alpha(Tx - \lambda x) \neq 0 , \quad \text{for all} \quad \lambda > 1 .$$

This implies that hypothesis (*) of Theorem IV.2 is satisfied for the operator $P_\alpha T : B_R \cap C \cap F_\alpha \to F_\alpha$. Applying that theorem it follows that there exists $x_\alpha \in B_R \cap C \cap F_\alpha$ such that $P_\alpha Tx_\alpha = x_\alpha$. Since T is a G-operator it then follows that T has a fixed point in C.

Now we derive, as corollaries to our Theorem IV.7, some of the known fixed point theorems.

Corollary IV.5. (Schauder) Let E be a Banach space with Property (π_1). Let $T : B \to B$ be a compact mapping of a ball B about the origin into itself. Then T has a fixed point in B.

Proof. By Theorem IV.3 it follows that T is a G-operator. It is easy to check that Condition (A) of Theorem IV.7 holds true for $B_R = B$. In fact, $P_\alpha Tx - \lambda x \neq 0$ for all $x \in \partial B$ and all $\lambda > 1$, because $||P_\alpha Tx|| \leq ||Tx|| \leq ||x||$ and $||\lambda x|| = \lambda ||x||$. So, applying Theorem IV.7 the result follows.

Remark 5. This corollary was stated in [7] for the general case of a bounded closed convex set. But there it was given a proof for the case of a ball about the origin.

Remark 6. Corollary IV.5 can be proved by this same method for the case of a separable Banach space with Property (π_k) . (See Petryshyn [15]) . Indeed, one has just to use Proposition IV.8 and Remark 4 above in order to verify Condition (A) of Theorem IV.7.

Corollary IV.6. (Schauder) Let E be a reflexive Banach space with Property (π_1) . Assume that E^* is strictly convex. Let $T : B \to B$ be a weakly continuous mapping of a ball B about the origin into itself. Then T has a fixed point.

Proof. By Theorem IV.5 it follows that T is a G-operator. As in the proof of the previous corollary we have that T satisfies Condition (A) of Theorem IV.7 . Applying that theorem the result follows.

Corollary IV.7 (Rothe). Let E be a Banach space with Property (π_1) . Let $T : B \to E$ be a compact mapping defined in some ball B about the origin. Assume

$$\text{(R)} \qquad T(\partial B) \subset B \ .$$

Then T has a fixed point in B .

Proof. By Theorem IV.3 , T is a G-operator. Similarly to the two previous corollaries, Condition (R) implies Condition (A) of Theorem IV.7 .

Corollary IV.8. (Petryshyn) Let E be a separable Banach space with Property (π_k) . Let $T : E \to E$ be P-compact mapping. Assume that *the following condition holds: there exists $R > 0$ such that*

$$(\hat{P}) \qquad Tx - \lambda x \neq 0 \quad \text{for all} \quad \lambda > 1 \quad \text{and all} \quad x \in \partial B_R \ .$$

Then T has a fixed point in B_R.

Proof. By Theorem IV.4 it follows that the mapping T is a G-operator. Now we have the following alternative: either T has a fixed point in ∂B_R or not. If the first possibility occurs the corollary is proved. The second possibility together with hypothesis $(\hat{P})$ implies Condition (P) of Remark 4 . According to that Remark, Condition (A) of Theorem IV.7 then holds. So it follows, using Theorem IV.7 , that T has a fixed point in B_R .

Remark 7. As remarked by Petryshyn the above result contains a theorem of Altman [2] for compact mappings. The "boundary condition" required by Altman is as follows:

$$(2) \qquad \|x - Tx\|^2 \geq \|Tx\|^2 - \|x\|^2 \quad \text{for all} \quad x \in \partial C \ .$$

It is easy to prove that Condition (2) implies Condition (P) .

Corollary IV.9. *Let E be a reflexive Banach space with Property (π_1) . Assume that E^* is strictly convex. Let $T : C \to E$ be a weakly continuous mapping of a bounded closed convex set C into E . Let J be the duality mapping in E with a given gauge function $\mu(r)$. Assume*

$$(B) \qquad (Tx, Jx) \leq \|x\| \, \mu(\|x\|) \quad \text{for all} \quad x \in \partial C \ .$$

Then T has a fixed point in C .

Proof. By Theorem IV.5 it follows that T is a G-operator. By Theorem IV.7 and Remark 3 the result follows.

Remark. The above result extends previous theorems of Altman [1; Theorem [2] and Shinbrot [22] for the Hilbert space case. This result has proved first by deFigueiredo [6] with the assumption that E^* has property (π_1) .

Corollary IV.10. (Browder-deFigueiredo) Let E be a reflexive Banach space with Property (π_1) . Assume that dual space E^* is strictly convex. Let $J : E \to E^*$ be a duality mapping in E which is assumed to be both continuous and weakly continuous. Let $T : E \to E$ be a demi-continuous mapping such that $I - T$ is J-monotone. Assume that there exists $R > 0$ such that

(B) $$(Tx,Jx) \leq \|x\|\mu(\|x\|) \quad \text{for all} \quad x \in \partial B_R .$$

Then T has a fixed point in B_R .

Proof. Consequence of Theorem IV.7 , Remark 3 and Theorem IV.6 .

Corollary IV.11. Let E be a reflexive Banach space with Property (π_1) . Assume that E^* is strictly convex and that a duality mapping $J : E \to E^*$ is both continuous and weakly continuous. Let T be a nonexpansive mapping in E which maps a ball B about the origin into itself. Then T has a fixed point in B .

<u>Remark</u>: This gives a new proof that a nonexpansive mapping of a ball in a Hilbert space into itself has a fixed point. The above corollary applies also to ℓ^p-spaces $p > 1$, but not to L^p-spaces.

REFERENCES

[1] Altman, M. - A fixed point theorem in Hilbert space. Bull. del'Acad. Polon. des Sciences. Vol. V(1957) pp. 19-22.

[2] ———————— - A fixed point theorem in Banach space. Bull. de l' Acad. Polon. des Sciences. Vol V(1957) pp. 89-92.

[3] Browder, F.E. - Problèmes non-linéaires. Séminaire de Mathématiques Supérieures. Université de Montréal (1965).

[4] ———————— - Multivalued monotone nonlinear mappings and duality mappings in Banach space. Trans. Amer. Math. Soc. 118(1965) pp. 338-351.

[5] Browder, F.E. and deFigueiredo, D.G. - J-monotone nonlinear operators in Banach spaces. Kongl. Nederl. Akad. Wetesch. 69(1966) pp. 412-419.

[6] deFigueiredo, D.G. - Fixed point theorems for weakly continuous mappings. Math. Res. Ctr. (Univ. Wisc.) Tech. Rep. No. 638(1966).

[7] ———————— - Fixed point theorems for non-linear operators and Galerkin approximations. Journal of Diff. Eq. 3(1967) pp. 271-281.

[8] ———————— - Further remarks on fixed points for G-operators in Banach spaces. (To appear).

[9] Heinz, E. - An elementary analytic theory of the degree of mapping in n-dimensional space. J. Math. Mech. 8(1959) pp. 231-247.

[10] Kaniel, S. - Quasi-compact nonlinear operators in Banach space and applications. Arch. Rat. Mech. Anal. 2^{0}(1966) pp. 259-278.

[11] Lindenstrauss, J. - Extension of compact operators. Memoirs of the Amer. Math. Soc. No. 48(1964).

[12] Michael, E.A. and Pelczynsky, A. - Peaked partition subspaces of C(X) . (To appear).

[13] ———————————— - Separable Banach spaces which admit ℓ_n^∞-approximations. (To appear).

[14] Nikolsky, V.N. - Best approximation and basis in Frêchet spaces. Dokl. Akad. Nauk SSSR 59(1948) pp. 639-642.

[15] Petryshyn, W.V. - On a fixed point theorem for nonlinear P-compact operators in Banach space. Bull. Amer. Math. Soc. 72(1966) pp. 329-334.

[16] ———————— - Further remarks on nonlinear P-compact operators in Banach space. Proc. Nat. Acad. Sci. 55(1966) pp. 684-687.

[17] Retherford, J.R. - On Chebyshev subspaces and unconditional bases in Banach spaces. Bull. Amer. Math. Soc. 73(1967) pp. 238-241.

[18] Rothe, E. - Zur Theorie der Topologischen Ordnung und der Vektorfelder in Banachschen Raumen. Compos. Math. 5(1937) pp. 177-197.

[19] Schauder, J. - Der Fixpunktsatz in Funktionalraumen. Studia Math. 2(1930) pp. 171-180.

[20] ———————— - Zur Theorie stetiger Abbildungen in Funktionalraumen. Math. Zeit. 26(1927) pp.

[21] ———————— - Eine Eigenschaft der Haarschen Orthogonal systems. Math. Zeit. 28(1928) pp. 317-320.

[22] Shinbrot, M. - A fixed point theorem, and some applications. Arch. Rat. Mech. Anal. 17(1965) pp. 255-271.

[23] Vainberg, - Variational Methods for the study of nonlinear operators. Holden-Day, (1964).

[24] Bessaga, C. - Basis in certain spaces of continuous functions. Bull. Acad. Polon. Sci. 5(1957) pp. 11-14.