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CAUSAL LOGIC WITH PHYSICAL INTERPRETATION

by

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Abstract.

It is shown that for the orthogonal sets bounded in the time the double orthocomplementation is equal to the causal closure. The family of double orthocomplemented sets generated by orthogonal sets bounded in time is an orthomodular lattice.

1. Introduction.

In the previous papers we investigated the causal logic [2,4] as a family of double orthoclosed sets. But from the physical point of view double orthocomplementation have not simple physical interpretation.

So in the present paper we shall introduce the causal closure as the physical generalization of the property of the light cone in Minkowski space. We shall see, that the family of the causally closed sets forms the complete lattice, which contains the family of double orthoclosed sets as a proper sublattice.

Assuming suitable form for the causal structure the causal logic forms a complete orthomodular lattice as was shown in [2] . So by Foulis-Randall theorem [6] each double orthoclosed set is generated by a maximal orthogonal subset.

Our main result in this paper proves that double orthocomplementation is equal to the causal closure for the orthogonal sets bounded in time. We shall prove also that the causal logic generated by the orthogonal sets bounded in time forms the orthogonal lattice.

2. Causal closure and orthocomplementation generated by causality structure.

By a causal structure /3/ of the set X we shall mean the non-empty family of sets $\mathcal{G}$ covering the set X. Every element f belonging to $\mathcal{G}$ is called a causal path. Let us denote by $\beta(x) := \{f \in \mathcal{G} ; x \in f\}$ the set of all paths containing x. Of course if $x \in f$ then $f \in \beta(x)$. Two points $x, y \in X$ are causally related if there is some path $f \in \mathcal{G}$ passing through both of them. It means that there is a physical signal between them.

But there exists the second aspect of causality when an event x is determined completely by a set A. For a partial differential equation in the Minkowski space the initial data on an achronal set A determines a physical state at each point x in a double light cone generated by A [5,7].

Now we shall examine precisely a causal dependence between a point and a set.

Definition 2.1

A point $x \in X$ is causally controled by a set A iff

$$\forall_{f \in \beta(x)} \quad f \cap A \neq \phi$$

For any set A we shall find the greatest set of points causally controled by the set A .

Definition 2.2

A causal closure of A is the set of all points controlled by A and is denoted by $D(A)$

$$D(A) := \{x \in X \,;\, \forall_{f \in \beta(x)} \; f \cap A \neq \phi\} \tag{2.1}$$

We have a simple physical meaning of this definition. Any element x belongs to $D(A)$ iff every path passing by x intersects the set A .

Lemma 2.1

The map $D: 2^X \to 2^X$ has the following properties:

i/ $A \subset D(A)$

ii/ if $A \subset B$ then $D(A) \subset D(B)$

iii/ $D(A) = D(D(A))$

Proof.

From definition 2.1 i/ and ii/ are obvious. It is enough to prove that $D(D(A)) \subset D(A)$.

Let $x \in D(D(A))$ iff $\forall_{f \in \beta(x)} \; f \cap D(A) \neq \phi$

iff $\forall_{f \in \beta(x)} \; \exists \; z \in f \cap D(A)$ iff $\forall_{f \in \beta(x)} \; \exists_{z \in f} \; \forall_{g \in \beta(z)}$ $g \cap A \neq \phi$ but $z \in f$ so $f \in \beta(z)$. We put $g := f$ then $\forall_{f \in \beta(x)} \; f \cap A \neq \phi$ iff $x \in D(A)$

■

We define the family of causally closed sets

$$\mathcal{C}(X,D) := \{A \subset X;\ A = D(A)\}$$

Lemma 2.2

The family $\mathcal{C}(X,D)$ forms a complete lattice where g.l.b. and l.u.b. are given respectively

$$\bigvee A_i = D(\bigcup A_i) \qquad \bigwedge A_i = \bigcap A_i$$

Proof.

From lemma 2.1 and by virtue of well known theorem [1 p.49] we shall get the proof.

Now we can introduce the orthogonality relation defined by the causal structure [3].

Let $x, y \in X$ $x \neq y$ $x \perp y$ iff x, y is not causally related. Of course it is symmetric and irreflexive relation. The above definition is equivalent to the following one

$$\forall_{f \in \beta(x)} f \cap \{y\} = \phi \quad \text{iff} \quad \forall_{f \in \beta(y)} f \cap \{x\} = \phi \qquad 2.2$$

Using formula 2.2 we are able to rewrite an orthocomplementation in the lenguage of paths.

$$A^{\perp} := \{x \in X;\ x \perp a\ \forall a \in A\} = \{x \in X;\ \forall_{f \in \beta(x)} f \cap A = \phi\}$$

$A^{\perp\perp} := (A^{\perp})^{\perp}$. It is well known [1] that orthogonal map has the following properties:

i/ $A \subset A^{\perp\perp}$

ii/ if $A \subseteq B$ then $B^{\perp} \subseteq A^{\perp}$

iii/ $A \cap A^{\perp} = \phi$

iv/ $A^{\perp} = A^{\perp\perp\perp}$

We have two operations D and $\perp\perp$ and we are interested in the relation between them.

Lemma 2.3

The maps $D : 2^X \to 2^X$ and $\perp : 2^X \to 2^X$ have the following properties:

i/ $D(A^{\perp}) = A^{\perp} = [D(A)]^{\perp}$

ii/ $D(A) \subset A^{\perp\perp}$

Proof.

i/ By lemma 2.1 $A^{\perp} \subset D(A^{\perp})$ and $A \subset D(A)$ from which follows $[D(A)]^{\perp} \subset A^{\perp}$. Therefore it is enough to prove only $D(A^{\perp}) \subset A^{\perp}$ and $A^{\perp} \subset [D(A)]^{\perp}$.

Let $x \in D(A^{\perp})$ iff $\forall_{f \in \beta(x)} f \cap A^{\perp} \neq \phi$ iff $\forall_{f \in \beta(x)} \exists_{z \in f} \forall_{g \in \beta(z)} g \cap A = \phi$ but $z \in f$ so $f \in \beta(z)$. We put $g := f$ and we have $\forall_{f \in \beta(x)} f \cap A = \phi$ iff $x \in A^{\perp}$.

Let $x \in A^{\perp}$ iff $\forall_{f \in \beta(x)} f \cap A = \phi$. From this follows that if $x \in f$ then $\forall_{z \in f}$ if we put $h := f$ we get $h \cap A = \phi$. Hence $\forall_{f \in \beta(x)} \forall_{z \in f} \exists_{h \in \beta(z)} h \cap A = \phi$ iff $x \in [D(A)]^{\perp}$.

ii/ Besause $A \subset A^{\perp\perp}$ then $D(A) \subset D(A^{\perp\perp}) = [D(A^{\perp})]^{\perp} = A^{\perp\perp}$

Corollary 2.1

From the lemmas 2.1 and 2.3 we get the formula

$$A \subset D(A) \subset A^{\perp\perp}$$

The simplest figures ilustrating the above formula in the two dimensional space we shall give as an example in the part 4.

3. Equivalence between the causal closure and the double orthocomplementation for the bounded in time sets.

In this part we shall present the main results. We are interested in the question when $D(A) = A^{\perp\perp}$. From the definitions given in the second part we knew the simple physical meaning of the causal closure. In such situation the causal logic has physical interpretation.

From this place we shall use the causal structure introduced in [2]. The space is identified with $Z = \mathbb{R} \times X$ $\mathbb{R}$ is a real line and X is non-empty set. Minkowski space has of course such a form.

The causal structure is given by the graphs of family $\mathcal{G}$ of functions $f : \mathbb{R} \to X$ which satisfies the following conditions:

1. For any $t_1 \leq t_2 \leq t_3$ and for any $x_1, x_2, x_3 \in X$ if $\beta(t_1, x_1) \cap \beta(t_2, x_2) \neq \phi$ and $\beta(t_2, x_2) \cap \beta(t_3, x_3) \neq \phi$ then

$$\beta(t_1, x_1) \cap \beta(t_3, x_3) \neq \phi$$

2. For any $y \in \mathbb{R} \times X$ and for any $f \in S$ the set $[f, \{y\}] := \{ v \in \mathbb{R} \; ; \; (v, f(v)) \notin \{y\}^{\perp} \}$ is open in $\mathbb{R}$.

The first assumption is a kind of a causal transitivity condition and the second one some kind of continuity condition.

In [2] was shown that one of the equivalent thesis of Foulis-Randall theorem [6] is fulfiled. It means that every double orthoclosed set $A^{\perp\perp} = A$ is generated by any maximal orthogonal set $S \subset A$ such that $S^{\perp\perp} = A^{\perp\perp}$.

From the physical point of view we shall examine the orthogonal set bounded in the time, because only the bounded counter is a good candidate for the measurement apparatus.

<u>Definition 3.1</u>

$A \subset \mathbb{R} \times X$ is <u>bounded in time</u> if exists <u>strip</u> $P(A) := [t_1, t_2] \times X$ containing A.

We denote $P^-(A) := \{t_1\} \times X$, $P^+(A) := \{t_2\} \times X$

Before the theorem let remind the symbols [2]

$$A_+^{\perp} := \{ (t,x) \in \mathbb{R} \times X \; ; \; \forall_{f \in \beta(t,x)} \; f \cap ([t, \infty) \times X) \cap A = \phi \}$$

$$A_-^{\perp} := \{ (t,x) \in \mathbb{R} \times X \; ; \; \forall_{f \in \beta(t,x)} \; f \cap ((-\infty, t] \times X) \cap A = \phi \}$$

$$[f, A]_+ := \{ v \in \mathbb{R} \; ; \; (v, f(v)) \notin A_+^{\perp} \}$$

$$[f, A]_- := \{ v \in \mathbb{R} \; ; \; (v, f(v)) \notin A_-^{\perp} \}$$

Lemma 3.1

If A is an orthogonal set bounded in time and $f \cap A^{\perp} = \phi$ then $f \cap A \neq \phi$.

Proof.

From the boundness in time we see imediately that $[f, A]_+ \neq \phi$ and $[f, A]_- \neq \phi$ so $\sup [f, A]_+$ and $\inf [f, A]_-$ are finite.

We shall consider two cases:

1. $\sup [f, A]_+ > \inf [f, A]_-$
2. $\sup [f, A]_+ \leqslant \inf [f, A]_-$

The first is impossible because of the orthogonality of A

$$[f, A]_+ \cap [f, A]_- = \phi$$

From the second we conclude $f \cap A \neq \phi$ because in the opposite by lemma 2 [2] we get $f \cap A^{\perp} \neq \phi$.

■

Theorem 3.1

If A is orthogonal and bounded in time then

$$A^{\perp\perp} = D(A)$$

Proof.

It is enough to prove $A^{\perp\perp} \subset D(A)$. If $x \in A^{\perp\perp}$ and $x \notin D(A)$ then exists $f \in \beta(x)$ such that $f \cap A^{\perp} = \phi$ and $f \cap A = \phi$. By virtue of lemma 3.1 it is impossible.

■

The above theorem shows that

$$\mathcal{C}_b(X,D) := \{D(A);\ A\text{-orthogonal bounded in time}\} =$$

$$\mathcal{C}_b(X,\perp) := \{A^{\perp\perp};\ A\text{-orthogonal bounded in time}\}$$

Now we shall interest in the structure of the family $\mathcal{C}_b(X,D)$.

__Lemma 3.2__

If $t_1 \leq t \leq t_2$ and $(t_1, f(t_1)),\ (t_2, f(t_2)) \in D(B)$

then $(t, f(t)) \in B^{\perp\perp}$

__Proof.__

Let $g \in \beta(t, f(t))$ and $(t', g(t')) \in g$. If $t' \geq t$ then using the transitivity condition 3.1 exists $h \in \mathcal{G}$ such that $(t', g(t'))$, $(t_1, f(t_1)) \in h$. Because $(t_1, f(t_1)) \in D(B)$ so $h \cap B \neq \phi$. In the opposite case $t' \leq t$ the proof goes in the similar way.

__Lemma 3.3__

Let P be a strip and $B = B^{\perp\perp}$. If S is the maximal orthogonal set in $P \cap B$ and $(P \cap B)^{\perp\perp} = B$ then $S^{\perp\perp} = B$.

__Proof.__

Let $x \in B \setminus (P \cap B)$ .There are two cases:

i/ $x_t \geq P^+$

ii/ $x_t \leq P^-$

We shall examine the case i/.

Because $(P\cap B)^{\perp}=B^{\perp}$ then exists $g\in\beta(x)$ and $y\in g$ such that $y\in P\cap B$. Let $z=g\cap P^{+}$ Because $y_t\leq z_t\leq x_t$ /index t denotes the time coordinate/ and $x,y\in B=D(B)$ then by lemma 3.2 $z\in B^{\perp\perp}=B$.

S is the maximal orthogonal set in $P\cap B$ and $z\in P\cap B$ we conclude that there exists $h\in\beta(z)$ such that $h\cap S\neq\phi$ .By the first condition for the causal structure 3.1 there exists $f\in\beta(x)$ such that $f\cap S=h\cap S\neq\phi$ .So we proved that S is the maximal orthogonal set in B . Using the our version of Randall-Fulis theorem [2] we have $S^{\perp\perp}=B$.

■

Theorem 3.2

$\mathcal{C}_b=\mathcal{C}_b(X,D)=\mathcal{C}_b(X,\perp)$ is an orthomodular lattice.

Proof.

i/ $\mathcal{C}_b$ is closed for l.u.b.

ii/ $\mathcal{C}_b$ is closed for the orthocomplementation

iii/ $\mathcal{C}_b$ is closed for g.l.b.

i/ Let S_1,S_2 be orthogonal bounded in time set such that $S_1^{\perp\perp}=A$, $S_2^{\perp\perp}=B$

Let P be a belt such that $P\supset S_1\cup S_2$. Of course $(P\cap A)^{\perp\perp}=A$, $(P\cap B)^{\perp\perp}=B$ .By virtue of lemma 3.3 it is enough to prove that $(P\cap(A\cup B)^{\perp\perp})^{\perp\perp}=(A\cup B)^{\perp\perp}$

but $A=(P\cap A)^{\perp\perp}\subset(P\cap(A\cup B)^{\perp\perp})^{\perp\perp}$

$B=(P\cap B)^{\perp\perp}\subset(P\cap(A\cup B)^{\perp\perp})^{\perp\perp}$

From this we get $(A\cup B)^{\perp\perp}\subset(P\cap(A\cup B)^{\perp\perp})^{\perp\perp}$

The contrary relation is obvious.

ii/ Let S be the orthogonal bounded in time such that
$$S^{\perp\perp}=A$$
Let P be a strip such that $P\supset S$
By virtue of lemma 3.3 it is enough to prove that
$$(P\cap A^{\perp})^{\perp\perp}=A^{\perp}$$
At first we shall prove that $A^{\perp}\subset D(P\cap A^{\perp})$
Let $x\in A^{\perp}\setminus(P\cap A^{\perp})$. There are two cases:

1. $x_t \geqslant P^{+}$
2. $x_t \leqslant P^{-}$

We shall examine the case1.
Let $f\in\beta(x)$ and $z=f\cap P^{+}$. We shall see that $z\in A^{\perp}$.
If $z\notin A^{\perp}$ then there exists $h\in\beta(z)$, $h\cap A\neq\phi$.
But $D(S)=A$ /theorem 3.1/ then exists $\omega\in h\cap S$
Let notice that $\omega_t\leqslant z_t\leqslant x_t$ so by the transitivity condition for the causal path /3.1/ $x\notin A^{\perp}$. It is easy to see that if $A^{\perp}\subset D(P\cap A^{\perp})$ then $A^{\perp}\subset(P\cap A^{\perp})^{\perp\perp}$.
The contrary relation is obvious.

iii/ It is enough to see that if $A=A^{\perp\perp}$, $B=B^{\perp\perp}$
then $(A\cap B)^{\perp}=(A^{\perp}\cup B^{\perp})^{\perp\perp}$ and to use i/ and ii/.

■

4. Examples.

We shall consider the space $Z = \mathbb{R}_t \times \mathbb{R}$, $\mathbb{R}_t = \mathbb{R}$ /denotes the time/ and the causal structure given by

$$\mathcal{G} = \{ f : \mathbb{R}_t \to \mathbb{R} \ ; \ |f(t_1) - f(t_2)| < |t_1 - t_2| \}$$

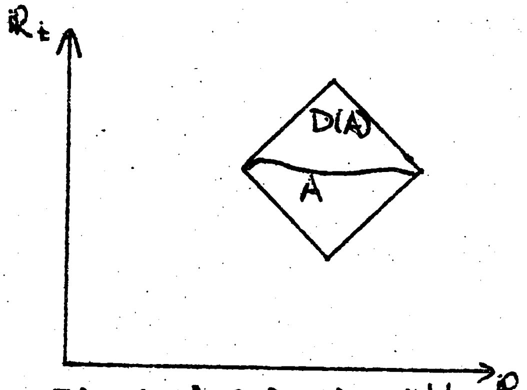

Fig.1 $A \subsetneq D(A) = A^{\perp\perp}$

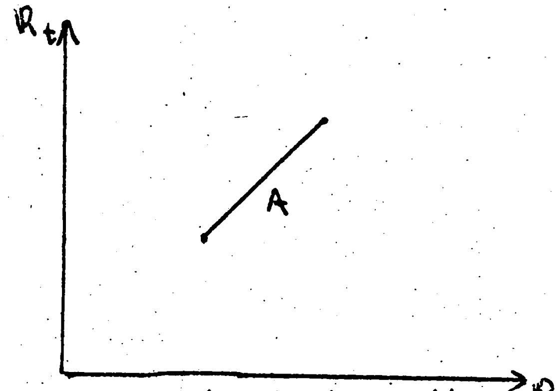

Fig.2 $A = D(A) = A^{\perp\perp}$

A is on the cone

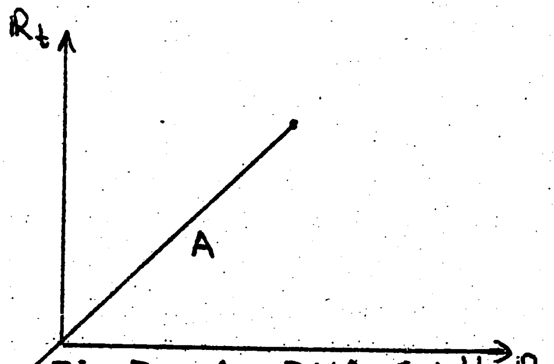

Fig.3 $A = D(A) \subsetneq A^{\perp\perp}$

A is on the cone

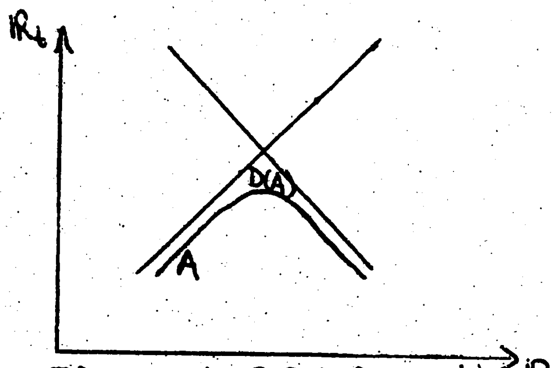

Fig.4 $A \subsetneq D(A) \subsetneq A^{\perp\perp}$

Let us see that orthogonal set A in the figures 3 and 4 are unbounded in time and $D(A) \neq A^{\perp\perp}$.

At last we shall show that in the above causal structure the family $\mathcal{C}_b(X,\perp) = \mathcal{C}_b(X,D)$ is not σ-complete lattice.

Let $A := \{(t,t) \quad t - \text{rational number}\}$. It is not difficult to see that

1. $A = A^{\perp\perp}$
2. if $S \subset A$ S - orthogonal bounded in time
   then $S^{\perp\perp} \neq A$

By virtue of 1 $A = \bigvee_{x_i \in A} \{x_i\}$. Because $\mathcal{C}_b(X,\perp)$ containes the point set then by 2 $\mathcal{C}_b(X,\perp)$ is not σ-complete lattice.

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