INSTYTUT FIZYKI TEORETYCZNEJ
UNIWERSYTETU WROCŁAWSKIEGO

# ORTHOMODULARITY OF CAUSAL LOGICS

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W. Cegła and J. Florek

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Intitute of Theoretical Physics
University of Wrocław
Wrocław, Cybulskiego 36, Poland

Institute of Mathematics
Polish Academy of Sciences
Wrocław, Kopernika 18,Poland

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Abstract:

It is shown that double orthoclosed sets in the space of type $\mathbb{R} \times X$ ; $\mathbb{R}$ real line, $X$ an arbitrary set, form a complete orthomodular lattice.

1. Introduction.

It is well known [3] that in an orthogonality space $(Z,\perp)$ where $Z$ is a non empty set and $\perp$ is a symmetric, nonreflexive binary relation on $Z$, the family of double ortho-closed sets $\mathcal{F}(Z,\perp) = \{A\subset Z\,;\ A = A^{\perp\perp}\}$ forms a complete ortholattice. The family $\mathcal{F}(Z,\perp)$ is partially ordered by the set-theoretic inclucion and equipped with the orthocomplementation $A \longrightarrow A^{\perp}$. For any family of elements of $\mathcal{F}(Z,\perp)$ g.l.b. and l.u.b. are given respectively by the formulas [1]

$$\bigvee A_i = (\bigcup A_i)^{\perp\perp} \qquad \bigwedge A_i = \bigcap A_i$$

In general $\mathcal{F}(Z,\perp)$ need not be orthomodular. This has been discussed in [3] and conditions equivalent to orthomodularity were given there. One of them is the following: if $D$ is an orthogonal subset of $Z$, if $z\notin D^{\perp}$ and $z\notin D^{\perp\perp}$ then $D^{\perp}\cap(z^{\perp}\cap D^{\perp})^{\perp}\neq\emptyset$.

We shall prove this condition in a general case where the space has the form $Z=\mathbb{R}\times X$ ; $\mathbb{R}$ is a real line and $X$ is non empty set. In particular Minkowski space has such form. Orthogonality relation is given by the family of maps satisfying three conditions which have simple physical meaning.

2. Definitions and symbols.

We begin with some definitions. Let $Z = \mathbb{R} \times X$ and $\mathcal{G}$ is a subset of the set of functions $f: S \to X$ where $S$ is the subset of real line $\mathbb{R}$. It will be denoted

$$\mathcal{G} \subset \bigcup_{S \subset \mathbb{R}} \{f;\ f: S \to X\}$$

We define the orthogonality relation as follows: $z_1, z_2 \in Z$, $z_1 \perp z_2$ iff there is no $f \in \mathcal{G}$ such that $z_1 \in f$ and $z_2 \in f$.
We identify the function $f$ with the graph of $f$.
A subset $A \subset Z$ is a partial selector for $\mathcal{G}$ iff for each $f \in \mathcal{G}$ $|f \cap A| \leq 1$.
We define some symbols. For $(t,x) \in \mathbb{R} \times X$

$$\beta(t,x) := \{f \in \mathcal{G}\ ;\ (t,x) \in f\}$$

Let $A \subset \mathbb{R} \times X$

$$A^{\perp} := \{(t,x) \in \mathbb{R} \times X;\ \forall_{f \in \beta(t,x)}\ f \cap A = \emptyset\}$$

$$A^{\perp}_{+} := \{(t,x) \in \mathbb{R} \times X;\ \forall_{f \in \beta(t,x)}\ f \cap ([t,\infty) \times X) \cap A = \emptyset\}$$

$$A^{\perp}_{-} := \{(t,x) \in \mathbb{R} \times X;\ \forall_{f \in \beta(t,x)}\ f \cap ((-\infty,t] \times X) \cap A = \emptyset\}$$

Let $f \in \mathcal{G}$ and $A \subset \mathbb{R} \times X$

$$[f,A]_{+} := \{v \in \mathbb{R}\ ;\ (v, f(v)) \notin A^{\perp}_{+}\}$$

$$[f,A]_{-} := \{v \in \mathbb{R}\ ;\ (v, f(v)) \notin A^{\perp}_{-}\}$$

Let

$$y \in \mathbb{R} \times X,\ f \in \mathcal{G}\ ;\ [f,\{y\}] := \{v \in \mathbb{R}\ ;\ (v, f(v)) \notin \{y\}^{\perp}\}$$

It is easy to see that $A_+^{\perp} \cap A_-^{\perp} = A^{\perp}$ and

$$[f,A]_+ \cup [f,A]_- = \{ v \in \mathbb{R} ;\ (v, f(v)) \notin A^{\perp} \} .$$

Now we shall introduce the restrictions for the family $\mathcal{G}$ .

We shall assume that $\mathcal{G}$ satisfies the following conditions:

1/ For any $t_1 \le t_2 \le t_3$ and for any $x_1, x_2, x_3 \in X$ if $\beta(t_1,x_1) \cap \beta(t_2,x_2) \neq \phi$ and $\beta(t_2,x_2) \cap \beta(t_3,x_3) \neq \phi$ then $\beta(t_1,x_1) \cap \beta(t_3,x_3) \neq \phi$ .

2/ For any $y \in \mathbb{R} \times X$ and for any $f \in \mathcal{G}$ the set $[f, \{y\}]$ is open in $\mathbb{R}$ .

3/ $\mathcal{G} \subset \bigcup_{S \subset \mathbb{R}} \{ f ; f : S \to X \}$ $S$ is a connected subset of $\mathbb{R}$ .

The first assumption is a kind of a causal transitivity condition. The second assumption says that if we can signal from a point $x$ to another point $y$ , then we can signal from point $x$ to the neighbourhood of $y$ .

The last condition is a technical one.

## 3. Basic results.

**Lemma 1.**

Let $\mathcal{G}$ satisfies condition 1/ and $A$ be a partial selector for $\mathcal{G}$ . If $f \in \mathcal{G}$ , $f \cap A^{\perp} \neq \phi$ then $\sup [f,A]_+ \le \inf [f,A]_-$ , and $f \cap A = \phi$ .

Proof:

First of all we shall shaw that $-\infty \leq \sup [f,A]_+ < \infty$ .
Let $(v,f(v)) \in f \cap A^{\perp}$ . Suppose that $\sup [f,A]_+ = \infty$ , then there exists $t \geq v$ and $g \in \beta(t,f(t))$ such that $g \cap ([t,\infty)\times X) \cap A \neq \emptyset$ .From this there exists $(s,x) \in g \cap A$ such that $s \geq t$ .But $v \leq t \leq s$ and $f \in \beta(v,f(v)) \cap \beta(t,f(t))$ , $g \in \beta(t,f(t)) \cap \beta(s,x)$ so by 1/ there is $h \in \mathcal{G}$ such that $(v,f(v)) \in h$ and $(s,x) \in h \cap A$ .
This contradicts to the condition $(v,f(v)) \in A^{\perp}$ .
In a similar way we prove $-\infty < \inf [f,A]_- \leq \infty$ .
If $\sup [f,A]_+ = -\infty$ or $\inf [f,A]_- = \infty$ then the inequality is obvious. We can assume that $\sup [f,A]_+$ and $\inf [f,A]_-$ are finite.
Suppose that $\sup [f,A]_+ > \inf [f,A]_-$ .Then there exists $t_1 > \inf [f,A]_-$ and $t_2 < \sup [f,A]_+$ , $g_1 \in \beta(t_1,f(t_1))$ , $g_2 \in \beta(t_2,f(t_2))$ and $(s_1,x_1) \in g_1 \cap A$ , $(s_2,x_2) \in g_2 \cap A$ such that $s_1 \leq t_1 < t_2 \leq s_2$ .But $g_1 \in \beta(s_1,x_1) \cap \beta(t_1,g(t_1))$ , $f \in \beta(t_1,f(t_1)) \cap \beta(t_2,f(t_2))$ , $g_2 \in \beta(t_2,f(t_2)) \cap \beta(s_2,x_2)$ so by 1/ there is $g_3 \in \beta(s_1,x_1) \cap \beta(s_2,x_2)$ .
But this contradicts the definition of the partial selector because $s_1 \neq s_2$ and $(s_1,x_1)$ , $(s_2,x_2) \in A \cap g_3$ .
If $y \in f \cap A^{\perp}$ then for any $g \in \beta(y)$ $g \cap A = \emptyset$ , especially $f \in \beta(y)$ so $f \cap A = \emptyset$ .

■

Lemma 2.

Let $\mathcal{G}$ satisfies conditions 1/ and 2/ and $A$ be a subset of $\mathbb{R}\times X$ . If $f\in\mathcal{G}$ , $f\cap A=\phi$ , $\sup\,[f,A]_+ = a \le b = \inf\,[f,A]_-$ then

$$[f,A]_+ = (-\infty,a)\cap \operatorname{dom} f\ ,\quad [f,A]_- = (b,\infty)\cap \operatorname{dom} f$$

where dom $f$ is the domain of $f$ .

Proof:

Let us notice that $(-\infty,a)\cap \operatorname{dom} f \subset [f,A]_+$ . Since if $t\in(-\infty,a)\cap \operatorname{dom} f$ then there is $v\in[t,a)$ such that $v\in[f,A]_+$ .But $t\le v$ $t,v\in \operatorname{dom} f$ so by 1/ we have $t\in[f,A]_+$. We shall prove that $[f,A]_+\subset(-\infty,a)\cap \operatorname{dom} f$ If $a=-\infty$ or $a=\infty$ it is obvious because $\sup\{t\in\mathbb{R};\, t\in A\}=-\infty$ iff $A=\phi$ .

It is enough to prove,that if $-\infty<a<\infty$ then $a\notin[f,A]_+$ . If $a\in[f,A]_+$ then exists $g\in\beta(a,f(a))$ and $(t,x)\in g\cap A$ such that $a\le t$ .But $f\cap A=\phi$ so $a<t$ . By virtue of 2/ $[f,\{(t,x)\}]$ is open set in $\mathbb{R}$ therefore exist $\xi\in(a,t)$ and $h\in\beta(\xi,f(\xi))$ such that $(t,x)\in h$, so $(\xi,f(\xi))\notin\{(t,x)\}^{\perp}_+$ . On the other hand $\xi>a$ and $(t,x)\in A$ so that we have contradiction to the definition of $a$ .

In the analogous way we prove $[f,A]_-=(b,\infty)\cap \operatorname{dom} f$ .

■

Now,we are able to prove the main result,the condition equivalent to orthomodularity.

Theorem.

Let $\mathcal{G}$ satisfy 1/,2/,3/ and $A$ is a partial selector for $\mathcal{G}$. If $(t,x)\in\mathbb{R}\times X$, $(t,x)\notin A^{\perp}$, $(t,x)\notin A^{\perp\perp}$ then $A^{\perp}\cap(\{t,x\}^{\perp}\cap A^{\perp})^{\perp}\neq\phi$.

Proof:

We rewrite the thesis in our lenguage. By the property of orthogonality relation we have

$$A^{\perp}\cap(\{t,x\}^{\perp}\cap A^{\perp})^{\perp}\neq\phi \quad \text{iff} \quad A^{\perp}\cap(\{t,x\}\cup A)^{\perp\perp}\neq\phi$$

/i/ $z\in A^{\perp}$ iff $\forall_{f\in\beta(z)}\ f\cap A=\phi$

/ii/ $z\in(\{t,x\}\cup A)^{\perp\perp}$ iff $\forall_{g\in\beta(z)}\ (\{t,x\}\cup A)^{\perp}\cap g=\phi$ iff

$$\forall_{g\in\beta(z)}\ \forall_{(\omega,g(\omega))\in g}\ \exists_{h\in\beta(\omega,g(\omega))}\ [(t,x)\in h \text{ or } h\cap A\neq\phi]$$

From the assumption $(t,x)\notin A^{\perp}$ it follows that $(t,x)\notin A^{\perp}_{+}$ or $(t,x)\notin A^{\perp}_{-}$. We consider only the case $(t,x)\notin A^{\perp}_{+}$. The proof in the second case goes in the similar way.

From the assumption $(t,x)\notin A^{\perp\perp}$ it follows that there exists $f\in\beta(t,x)$ such that $f\cap A^{\perp}\neq\phi$, so there exists $(s,f(s))\in A^{\perp}$. By virtue of lemma 1 and 2 we get $[f,A]_{+}=(-\infty,a)\cap\operatorname{dom} f$ and so $t<a\leq s$. Now by condition 3/ of $\mathcal{G}$ the domain of $f$ is a connected set and so $a\in\operatorname{dom} f$.

From lemma 1 and 2 it also follows that $a\notin[f,A]_{+}\cup[f,A]_{-}$ so $(a,f(a))\in A^{\perp}$.

We shall prove that $(a, f(a)) \in A^{\perp} \cap (\{t,x\} \cup A)^{\perp\perp}$.
Let $g \in \beta(a, f(a))$ and $(\omega, g(\omega)) \in g$. We have two possibilities:

1. $\omega \geqslant a$

   Since $t < a \leq \omega$ then by condition 1/ of $\mathcal{G}$ there exists $h$ such that $h \in \beta(\omega, g(\omega))$, $(t,x) \in h$ and fulfils the first part of condition /ii/.

2. $\omega < a$

   Then by virtue of condition 2/ of $\mathcal{G}$ there exist $\xi \in (\omega, a)$ and $k \in \beta(\xi, f(\xi))$ such that $(\omega, g(\omega)) \in k$. Since $\xi < a$ and $\xi \in \operatorname{dom} f$ and $(-\infty, a) \cap \operatorname{dom} f \subset [f, A]_+$ so $\xi \in [f, A]_+$. Because $\omega < \xi$ and $\xi \in [f,A]_+$ so by condition 1/ of $\mathcal{G}$ exists $h \in \beta(\omega, g(\omega))$ such that $h \cap A \neq \emptyset$. This fulfils the second part of condition /ii/.

$\blacksquare$

4. Example.

Finally we give an example of the space $Z = \mathbb{R} \times X$ and the family $\mathcal{G}$. Let $Z$ be a space-time identified with $\mathbb{R} \times \mathbb{R}^3$ and $\mathcal{G}_\alpha$ be the family of functions satisfying Lipschitz condition with constant $\alpha$ $0 < \alpha \leq \infty$.

$$\mathcal{G}_\alpha = \{ f : \mathbb{R} \to \mathbb{R}^3 \ ; \ \|f(x) - f(y)\| < \alpha |x - y| \}$$

In this case of course the conditions 1/,2/,3/ of $\mathcal{G}$ are satisfied.
For $\alpha = 1$ we get the light cone in Minkowski space considered in [2]. For $\alpha = \infty$, the Galilean logic considered in [2a].

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