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REMARKS ON THE CAUSAL LOGIC

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ABSTRACT

The causal structure of space-time is considered in the quantum logic approach. The properties of corresponding lattices are investigated in nonrelativistic and relativistic examples.

1. Introduction

In the quantum theory there is an old unsolved problem, the problem of a relativistic localization. In the well known Wightman paper [15] a general construction of the position operator for an arbitrary hiperplane has been given. From the physical point of view Wightman's solution has some disadvantages: the main one is the so called non-causal propagation. It means that the scalar product of two localized states (τ_a, τ_b) where a and b are points in Minkowski space is different from zero also for $(a-b)^2<0$ (a and b are space-like separated).

We propose a new approach to the localization problem using the quantum logic. Our scheme is the following: in the space-time we shall construct a family of subsets building a quantum logic and then we shall look for covariant representations of this logic in a Hilbert space.

First of all we need a distinguishability criterion for space-time events weaker than mere difference which has a simple physical connection with the causality. A very useful mathematical model for our investigations is an orthogonality space i.e. a pair $(X,\perp)$ where X is a set and $\perp$ is a symmetric and irreflexive relation on X [12]. The orthogonality relation distinguishes certain families of subsets of X which from the probabilistic point of view should satisfy an orthomodularity condition as a minimum requirement needed for a good definition of the orthogonality additive measure (ortho-states). Of course it imposes the conditions for the orthogonality relation [10,11].

The first part of this paper (section 2 and 3) contains a general investigation of the orthogonality space generated by a causal structure. Physically the orthogonality relation will be interpreted as a causal independence between events. We give the sufficient conditions for orthomodularity and study the problem of structure and physical interpretation of the families of sets distinguished by the orthogonality relation. In section 4 we shall give two examples.

2. The causal structure and the orthogonality relation.

We shall start with some facts about the orthogonality space. An orthogonality space is a pair $(X,\perp)$ where X is a set and $\perp$ an orthogonality relation on X e.a. symmetric and irreflexive. $D\subset X$ is called an orthogonal set ($\perp$-set) iff

$$\forall_{x,y\in D} \quad x\neq y \quad \text{implies} \quad x\perp y \tag{2.1}$$

For $A\subset X$ define $A^{\perp} := \{x\in X;\ x\perp a \quad \forall a\in A\}$ (2.2)

and $A^{\perp\perp} := (A^{\perp})^{\perp}$.

The well known lemma [12] gives $A\subseteq A^{\perp\perp}$, if $A\subset B$ then $B^{\perp}\subset A^{\perp}$, $A\cap A^{\perp} = \phi$, $A^{\perp}=A^{\perp\perp\perp}$, $(\cup A_i)^{\perp}=\cap A_i^{\perp}$.

We introduce causal paths as a method of investigation of orthogonality spaces. Let (X,G) be a pair where X is a nonempty set and G is a distinguished covering of X by nonempty subsets. The pair (X,G) will be called a causal space, the family G a causal structure, an element $f\in G$ a causal path.

Let $x\in X$ we denote by $\beta(x) := \{f\in G;\ x\in f\}$ the set of all causal paths containing x. Of course if $x\in f$ then $f\in\beta(x)$.

In the causal space (X,G) one can introduce an orthogonality relation in a natural way: $x,y\in X$, x is orthogonal to y ($x\perp y$) iff $\beta(x)\cap\beta(y)=\phi$.

Observe that

$$x\perp y \quad \text{iff} \quad \forall_{f\in\beta(x)} \ f\cap\{y\}=\phi \quad \text{iff} \quad \forall_{f\in\beta(y)} \ f\cap\{x\}=\phi \tag{2.3}$$

and

$$A^{\perp} = \{x\in X;\ \forall_{f\in\beta(x)} \ f\cap A=\phi\}. \tag{2.4}$$

If we understand a causal path as a possible physical signal then from (2.4) we see that $A^{\perp}$ is a set of points which are not causally related to any point from the set A.

We say that a point $x \in X$ is <u>causally controlled</u> by a set A iff $\forall_{f\in\beta(x)} f\cap A\neq\phi$.

For each set A we define the <u>causal closure</u>

$$D(A) := \{x\in X;\ \forall_{f\in\beta(x)}\ f\cap A \neq \phi\} \tag{2.5}$$

Observe that D(A) is connected with a concept of a region of causal dependence as in [13.14].

<u>Lemma 2.1.</u>

The map $D : 2^X \Rightarrow 2^X$ has the following properties

i) $A\subset D(A)$

ii) if $A\subset B$ then $D(A)\subset D(B)$

iii) $D(A)=D(D(A))$.

<u>Proof</u>

i) and ii) are obvious from definition. It is enough to prove that $D(D(A)\subset D(A)$.

Let $x\in D(D(A)) \Rightarrow \forall_{f\in\beta(x)}\ f\cap D(A) \neq \phi \Rightarrow \forall_{f\in\beta(x)}\ \exists z\in f\cap D(A) \Rightarrow \forall_{f\in\beta(x)}\ \exists z\in f\ \forall_{g\in\beta(z)}\ g\cap A \neq\phi$.

but $z\in f$ implies $f\in\beta(z)$. We put $g := f$ then $\forall_{g\in\beta(x)}\ f\cap A \neq \phi$ i.e. $x\in D(A)$. ∎

The next lemma gives a relation between the two operations D and $\perp$ in 2^X.

<u>Lemma 2.2.</u>

The maps $D : 2^X \Rightarrow 2^X$ and $\perp: 2^X \Rightarrow 2^X$ have the following properties:

i) $D(A^{\perp}) = A^{\perp} = [D(A)]^{\perp}$

ii) $D(A)\subset A^{\perp\perp}$

<u>Proof</u>

i) From the previous lemma we have $A^{\perp}\subset D(A^{\perp})$ and $A\subset D(A)$ so $[D(A)]^{\perp}\subset A^{\perp}$. Therefore it is enough to prove that $D(A^{\perp})\subset A^{\perp}$ and $A^{\perp}\subset[D(A)]^{\perp}$. This can be done analogously as in lemma 2.1.

ii) Because $A\subset A^{\perp\perp}$ then $D(A)\subset D(A^{\perp\perp})=A^{\perp\perp}$ ∎

As a corollary from the above lemmas we have

$$A\subset D(A)\subset A^{\perp\perp} \tag{2.6}$$

More details one can find in [7].

3. Causal logic in Minkowski's type space.

In this section we restrict our consideration to a special case of a causal space namely $(\mathbb{R}\times X: G)$ where $\mathbb{R}$ the real line, X-any set. The family G consists of graphs of functions $f : S \Rightarrow X$, $S\subset\mathbb{R}$.

$$G \subset \bigcup_{S\subset\mathbb{R}} \{f;\ f:S\Rightarrow X\}.$$

The following symbols and notations have been introduced in [6].

$$(t,x)\in \mathbb{R}\times X, \quad \beta(t,x) = \{f\in G;\ \ (t,x)\in f\}$$

let $A\subset\mathbb{R}\times X$

$$A_+^{\perp} := \{(t,x)\in\mathbb{R}\times X: \ \forall_{f\in\beta(t,x)} \ f\cap([t,\infty)\times X)\cap A=\phi\}$$

$$A_- := \{(t,x)\in\mathbb{R}\times X: \ \forall_{f\in\beta(t,x)} \ f\cap((-\infty,t]\times X)\cap A=\phi\} \tag{3.1}$$

Let $f\in G$ and $A\subset\mathbb{R}\times X$

$$[f,A]_+ := \{v\in\mathbb{R} :\ (v,f(v)) \notin A_+^{\perp}\}$$

$$[f,A]_- := \{v\in\mathbb{R} ;\ (v,f(v)) \notin A_-^{\perp}\} \ . \tag{3.2}$$

Those sets describe points on the path f from which we are able to send a signal to the set A (to the future $[f,A]_+$ or to the past $[f,A]_-$ with an accordance to the order of the real line).
Let us notice that from (3.1) we have $A_+^{\perp} \cap A_-^{\perp} = A^{\perp}$.
From (3.2) we get

$$[f,A]_+ \cup [f,A]_- = \{v \in \mathbb{R} \,;\, (v,f(v)) \notin A_+^{\perp} \cap A_-^{\perp} = A^{\perp}\} =: [f,A]$$

In particular

$$[f,\{y\}] = \{v \in \mathbb{R} \,;\, (v,f(v)) \notin \{y\}^{\perp}\} .$$

In the rest of this section we shall study the structure of the 3 families of sets.

1. $C(X,\perp) := \{A \subset X ; A = A^{\perp\perp}\}$
2. $L(X,\perp) := \{A^{\perp\perp} ; A - \perp \text{set}\}$
3. $C(X,D) := \{A \subset X ; A = D(A)\}$

It is well known [1] that $C(X,\perp)$ form a complete ortholattice (partially ordered by the set theoretic inclusion and equipped with the orthocomplementation $A \Rightarrow A^{\perp}$). The g.l.b. and l.u.b. are given respectively by the formulas

$$\vee A_i = (\cup A_i)^{\perp\perp} \quad , \quad \wedge A_i = \cap A_i$$

But in general $C(X,\perp)$ need not be orthomodular. This has been discussed in [11] and the conditions equivalent to the orthomodularity were given there. One of them is the following: if A is an orthogonal subset of X, if $x \notin A^{\perp}$ and $x \notin A^{\perp\perp}$ then $A^{\perp} \cap (x^{\perp} \cap A^{\perp}) \neq \phi$.

Now we formulate the conditions for the family G such that the orthogonality relation generated by G will satisfy (3.3)

1. $G \subset \bigcup_{S \subset \mathbb{R}} \{f: S \to X\}$ S is a connected subset of $\mathbb{R}$.

2. For any $t_1 \leq t_2 \leq t_3$ and for any $x_1, x_2, x_3 \in X$ if $\beta(t_1,x_1) \cap \beta(t_2,x_2) \neq \phi$ and $\beta(t_2,x_2) \cap \beta(t_3,x_3) \neq \phi$ then $\beta(t_1,x_1) \cap \beta(t_3,x_3) \neq \phi$ (3.4)

3. For any $f \in S$ and for any $y \in \mathbb{R} \times X$ the set $[f,\{y\}]$ is open in $\mathbb{R} \cap \mathrm{dom}\, f$ where $\mathrm{dom}\, f$ is the domain of f.

The second assumption is a causal transitivity condition. The last one is a kind of a continuity (if we are able to send a signal from $x \in f$ to y then we can also signal to y from a neighbourhood of x on f).

Lemma 3.1.

Let G satisfies condition 2 of (3.4) and A be an orthogonal set. If $f \in G$, $f \cap A^{\perp} \neq \phi$ then $\sup[f,A]_+ \leq \inf[f,A]_-$, and $f \cap A = \phi$.

Lemma 3.2.

Let G satisfies condition 2 and 3 of (3.4) and A be a subset of $\mathbb{R} \times X$. If $f \in G$, $f \cap A = \phi$ and $\sup[f,A]_+ \leq \inf[f,A]_-$, then $[f,A]_+ = (-\infty, a) \cap \mathrm{dom}\, f$ and $[f,A]_- = (b, \infty) \cap \mathrm{dom}\, f$.

The proofs can be found in [6]. Using the above lemmas we can prove the orthomodularity condition.

Theorem 3.1.

Let G satisfies conditions 1,2,3 and A be an orthogonal set. If $(t,x) \in \mathbb{R} \times X$, $(t,x) \notin A^{\perp}$, $(t,x) \notin A^{\perp\perp}$ then $A^{\perp} \cap (\{t,x\}^{\perp} \cap A^{\perp})^{\perp} \neq \phi$.

Proof

We rewrite the thesis in the path language.

$$A^{\perp} \cap (\{t,x\}^{\perp} \cap A^{\perp})^{\perp} \neq \phi \quad \text{iff} \quad A^{\perp} \cap (\{t,x\} \cup A)^{\perp\perp} \neq \phi$$

i) $z \in A^{\perp}$ iff $\forall_{f \in \beta(z)} f \cap A = \phi$

ii) $z \in (\{t,x\} \cup A)^{\perp\perp}$ iff $\forall_{g \in \beta(z)} (\{t,x\} \cup A)^{\perp} \cap g = \phi$

iff $\forall_{g \in \beta(z)} \forall_{(\omega,g(\omega)) \in g} \exists_{h \in \beta(\omega,g(\omega))} [(t,x) \in h$ or $h \cap A \neq \phi]$

From the assumption $(t,x) \notin A^{\perp}$ it follows that $(t,x) \notin A_{+}^{\perp}$ or $(t,x) \notin A_{-}^{\perp}$. We consider only the case $(t,x) \notin A_{+}^{\perp}$.

From $(t,x) \notin A^{\perp\perp}$ it follows that there exists $f \in \beta(t,x)$ such that $f \cap A^{\perp} \neq \phi$, so exists $(s,f(s)) \in A^{\perp}$ and by lemma (3.1) and (3.2) and condition 1 of (3.4) one can prove that $a \in \operatorname{dom} f$ and $(a,f(a)) \in A^{\perp}$.

We shall prove that $(a,f(a)) \in A^{\perp} \cap (\{t,x\} \cup A)^{\perp\perp}$. Let $g \in \beta(a,f(a))$ and $(\omega,g(\omega)) \in g$. We have two possibilities

1. $\omega \geq a$

Since $t < a \leq \omega$ so by condition 2 of (3.4) there exists h which satisfies the first part of the thesis $(t,x) \in h$.

2. $\omega < a$

From the 3 condition of (3.4) there exist $\xi \in (\omega,a)$ and $k \in \beta(\xi,f(\xi))$ such that $(\omega,g(\omega)) \in k$. Since $\xi < a$ and $\xi \in \operatorname{dom} f$ and $(-\infty,a) \cap \operatorname{dom} f \subset [f,A]_{+}$ so $\xi \in [f,A]_{+}$. Because $\omega < \xi$ and $\xi \in [f,A]_{+}$ so by condition 2 exists $h \in \beta(\omega,g(\omega))$ such that $h \cap A \neq \phi$ as in the second part of the thesis. ∎

More details one can find in [6]. We also have $C(X,\perp) = L(X,\perp)$ and each $A = A^{\perp\perp}$ is generated by the maximal orthogonal set contained in A (see [11]).

One can prove that $(L(X,\perp), \leq, \perp)$ forms an ortho-complete orthomodular poset also when 3 is replaced by a weaker condition 3′. $\forall\, x \not\perp y\ \exists f \in \beta(x) \cap \beta(y)$, $\forall z \in \mathbb{R} \times X$ $[f,\{z\}]$ is a open set in $\mathbb{R} \cap \operatorname{dom} f$ (see [8,10]).

Consider now the structure of the family $C(X,D)$. Lemma 2.1. gives a sufficient condition for $C(X,D)$ to be a complete lattice [1]. From the corollary 2.6 we have that if $A\subset A^{\perp\perp}$ then $A=D(A)$ and so $L(X,\perp)\subset C(X,D)$ as a subset but not as a sublattice, because usually $A^{\perp\perp}\neq D(A)$. Let us see, that from definition (2.5) $D(A)$ has a simple physical interpretation so it is important to know, when $A^{\perp\perp}=D(A)$. In [7] the conditions has been given for A to satisfy $A^{\perp\perp}=D(A)$. We shall briefly review the results. We assume that each $f\in S$ is a function $f:\mathbb{R}\Rightarrow X$. $A\subset\mathbb{R}\times X$ is said to be <u>bounded in time</u> if there exists a strip $P(A):=[t_1,t_2]\times X$ containg A.

<u>Lemma 3.3.</u>

If A is an orthogonal set bounded in time and $f\cap A^{\perp}=\phi$ then $f\cap A\neq\phi$.

<u>Proof</u>

From the boundness in time we see immediately that $[f,A]_+\neq\phi$ and $[f,A]_-\neq\phi$ so $\sup[f,A]_+$ and $\inf[f,A]_-$ are finite. We consider two cases:

1. $\sup[f,A]_+ > \inf[f,A]_-$

2. $\sup[f,A]_+ \leq \inf[f,A]_-$

The first is impossible because the orthogonality of A. From the second we have $f\cap A\neq\phi$ because otherwise by lemma 3.2 we get $f\cap A^{\perp}\neq\phi$. ∎

<u>Theorem 3.2</u>

If A is an orthogonal set bounded in time then $A^{\perp\perp}=D(A)$.

<u>Proof</u>

It is enough to prove $A^{\perp\perp}\subset D(A)$. If $x\in A^{\perp\perp}$ and $x\notin D(A)$ then exists $f\in\beta(x)$ such that $f\cap A^{\perp}=\phi$ and $f\cap A=\phi$. By virtue of lemma 3.3. this is impossible. ∎

Let $L_b(X,\perp) := \{A^{\perp\perp};$ A-an orthogonal bounded in time$\} = \{D(A);$ A - an orthogonal bounded in time$\}$.

Of course $L_b(X,\perp)\subset L(X,\perp)$ and each element of $L_b(X,\perp)$ can be interpreted as a generalized double cone (diamond) in the space-time. One can prove [7] that $L_b(X.\perp)$ is an orthomodular lattice (but not σ-complete) with $\perp$ as a orthocomplementation and $\perp\perp$ as a closure operation.

4. Examples and final remarks.

Non-relativistic case.

Galilean space-time can be consider as $X=\mathbb{R}\times\mathbb{R}^3$ and G is the family of all functions $f: \mathbb{R}\Rightarrow\mathbb{R}^3$; $\mathbb{R}$ represents the time and $\mathbb{R}^3$ positions. Two points $(t,\underline{x})$, $(s,\underline{y})$ are orthogonal iff $t=s$ and $\underline{x}\neq\underline{y}$. Every constant function $g : \mathbb{R}^3 \Rightarrow \mathbb{R}$ represents a hiperplane S_t of the constant time t, being a maximal orthogonal set, and every maximal orthogonal set has such a form. From the definitions for every orthogonal set E we get $E^{\perp\perp}=E$ and $L(X,\perp)$ in this case is isomorphic to the disjoint sum of Boolean lattices $L_t=2^{S_t}$ indexed by t. We can restrict ourselves to the Borel sets in X and denote by $L^B(X,\perp)$ the corresponding logic. Of course the Borel structure is consistent with the lattice structure, so $L^B(X,\perp)$ is a σ-complete orthomodular lattice.

In the space X we have the action of the Galilean group. It induces authomorphisms of the logic $L^B(X,\perp)$.

To solve the problem of localization we have to find covariant representations, with respect to the Galilean group, of the logic $L^B(X,\perp)$ in a Hilbert space. This problem was discussed in [3] and the general form of the covariant representations has been found there.

Relativistic example.

Minkowski space will be identified with $M=\mathbb{R}\times\mathbb{R}^3$ with the scalar product $x\cdot y = - x_o\cdot y_o + \underline{x}\cdot\underline{y}$. Let G_α be the family of functions $f: \mathbb{R}\Rightarrow \mathbb{R}^3$,

$$\| f(x_o) - f(y_o) \| < \alpha|x_o - y_o| \qquad 0<\alpha\leq\infty .$$

The family G_α satisfies the condition (3.4) and so $C(M,\perp)=L(M,\perp)$ is a complete orthomodular lattice.

Two points $x,y \in M$ are orthogonal iff $|x_o - y_o| \leq \frac{1}{\alpha} \|\underline{x} - \underline{y}\|$ and every maximal orthogonal set is given by a function $\underline{g}: \mathbb{R}^3 \Rightarrow \mathbb{R}$ such that $|g(\underline{x})-g(\underline{y})| \leq \frac{1}{\alpha} \|\underline{x} - \underline{y}\|$.

If α is a velocity of light the orthogonality relation means that x is space-or light-like to y. This case has been considered in [3] where for G has been taken the family of time-like straight lines. For $\alpha = \infty$ we get the Galilean case.

The relativistic logic $L(M,\perp)$ is much less trivial than the non-relativistic one. The illustrating pictures for two dimensional Minkowski space-time one can see in [9].

As in the previous example we can restrict to the Borel sets in Minkowski space and in [4] has been shown that Borel structure is consistent with the lattice one and the corresponding lattice $L^B(M,\perp)$ is a σ-complete orthomodular one. In [4] has been studied also the group of authomorphism of $L^B(M,\perp)$. It has been shown that authomorphism are given by Poincaré transformation and dilation. The first step to the studies of the representations of the logic $L^B(M,\perp)$ in a Hilbert space is the investigation of a generalized ortho-state $\mu : L^B(M,\perp) \Rightarrow \mathbb{R}$ such that $\mu(\vee A_i) = \Sigma\mu(A_i)$, $A_i \perp A_j$.
It has been done in [5] where has been shown that every conserved current generate the state. The problem of finding all covariant representations with respect to the Poincaré group, of the causal logic $L^B(M,\perp)$ is still open. Partial results have been obtained in [2].

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