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MATHEMATICS

On the Representation of Quasi-Boolean Algebras

by

A. BIAŁYNICKI-BIRULA and H. RASIOWA

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The researches on the algebraic treatment of constructive logic with strong negation *), which will be discussed in a separate paper, suggest the examination of a certain kind of lattices here called quasi-Boolean algebras. The aim of this paper is to give the representation theorem of these lattices.

We shall say that an abstract algebra $\langle A, +, \cdot, \sim \rangle$ is a *quasi-Boolean algebra* when:

(i) $\langle A, +, \cdot \rangle$ is a distributive lattice with the zero element 0 and the unit element 1,

(ii) $\sim$ is a unary operation which satisfies the following conditions:

$$\sim\sim a = a, \quad \sim(a \cdot b) = \sim a + \sim b \quad \text{for any} \quad a, b \in A.$$

The operation $\sim$ will be called the *quasi-complement*.

1) *In any quasi-Boolean algebra* $\langle A, +, \cdot, \sim \rangle$

$$\sim(a+b) = \sim a \cdot \sim b \quad \textit{for any} \quad a, b \in A,$$

$$\sim 0 = 1, \quad \sim 1 = 0.$$

This is an immediate consequence of the definition.

Let $\mathfrak{X}$ be a non-empty set and let g be a one-to-one mapping of $\mathfrak{X}$ onto $\mathfrak{X}$ which is an involution, i. e.,

$$g(g(x)) = x \quad \text{for every} \quad x \in \mathfrak{X}.$$

Setting $\sim X = \mathfrak{X} - g(X)$ for every $X \subset \mathfrak{X}$ we find that every family of subsets of the set $\mathfrak{X}$, which is closed under this operation as well under the set-theoretical operations of sum and product, is a quasi-Boolean algebra. Every quasi-Boolean algebra of this kind is said to be a *quasi-*

*) For constructive logic with strong negation cf. [1], [2] and [3].

-field of sets. We shall prove that every quasi-Boolean algebra is isomorphic with a quasi-field of sets, using the method of Stone [4].

Let $\langle A, +, \cdot, \sim\rangle$ be an arbitrary but fixed quasi-Boolean algebra. A non-void subset $\mathfrak{i}\subset A$ is said to be an *ideal* (a *filter*), when $\mathfrak{i}\neq A$ and

(i) if $a, b\in\mathfrak{i}$, then $a+b\in\mathfrak{i}$ $(a\cdot b\in\mathfrak{i})$,

(ii) if $a\in\mathfrak{i}$ and $b\in A$, then $a\cdot b\in\mathfrak{i}$ $(a+b\in\mathfrak{i})$.

An *ideal* $\mathfrak{p}$ (a *filter* $\mathfrak{q}$) is said to be *prime* provided that if $a\cdot b\in\mathfrak{p}$ $(a+b\in\mathfrak{q})$, then either $a\in\mathfrak{p}$ or $b\in\mathfrak{p}$ (either $a\in\mathfrak{q}$ or $b\in\mathfrak{q}$).

Given a subset $A_0\subset A$, let $\widetilde{A_0}$ be the set of all elements $\sim x$ such that $x\in A_0$. It is easy to see that

2) *If $\mathfrak{q}$ is a filter, then $\widetilde{\mathfrak{q}}$ is an ideal. Moreover, if $\mathfrak{q}$ is a prime filter, then $\widetilde{\mathfrak{q}}$ is a prime ideal.*

It is known that

3) *If $\mathfrak{p}$ is a prime ideal, then $A-\mathfrak{p}$ is a prime filter.*

Let $\mathfrak{X}$ be the set of all prime filters of the quasi-Boolean algebra $\langle A, +, \cdot, \sim\rangle$. For every $\mathfrak{q}\in\mathfrak{X}$ let

$$g(\mathfrak{q})=A-\widetilde{\mathfrak{q}}. \tag{*}$$

It follows from the definition of the mapping g and from 2) and 3) that g is a one-to-one mapping of $\mathfrak{X}$ onto $\mathfrak{X}$. Moreover,

$$g(g(\mathfrak{q}))=\mathfrak{q} \quad \text{for every} \quad \mathfrak{q}\in\mathfrak{X}.$$

For every $a\in A$, let $h(a)$ be the class of all prime filters $\mathfrak{q}$ such that $a\in\mathfrak{q}$.

4) *The mapping h is an isomorphism of A into the quasi-field of all subsets of $\mathfrak{X}$, the operation $\sim$ being defined as follows:*

$$\sim X=\mathfrak{X}-g(X) \quad \textit{for every} \quad X\subset\mathfrak{X}.$$

Indeed, it is known from [4] that h is an isomorphism of the distributive lattice $\langle A, +, \cdot\rangle$ into the ring of *all* subsets of $\mathfrak{X}$. To prove 4) it suffices to show that

$$h(\sim a)=\sim h(a), \quad \text{for every} \quad a\in A. \tag{**}$$

To prove (**) let us notice that on account of (*) the condition $\mathfrak{q}\in g(h(a))$ is equivalent to the existence of a prime filter $\mathfrak{q}_1$, belonging to $h(a)$ such that $\mathfrak{q}=g(\mathfrak{q}_1)$, i. e., $\mathfrak{q}=A-\widetilde{\mathfrak{q}}_1$. On account of the equivalences

$$\mathfrak{q}_1\in h(a)\equiv a\in\mathfrak{q}_1\equiv\sim a\in\widetilde{\mathfrak{q}}_1\equiv\sim a\notin A-\widetilde{\mathfrak{q}}_1$$

we obtain that $\mathfrak{q}\in g(h(a))$ if, and only if, there exists a prime filter $\mathfrak{q}_1$ such that $\mathfrak{q}=g(\mathfrak{q}_1)$ and $\sim a\notin\mathfrak{q}$. Since the mapping g transforms $\mathfrak{X}$ onto $\mathfrak{X}$ we infer that for every $\mathfrak{q}\in\mathfrak{X}$ there exists a $\mathfrak{q}_1\in\mathfrak{X}$ such that

$\mathfrak{q} = g(\mathfrak{q}_1)$. In consequence, $\mathfrak{q} \in g(h(a))$ if, and only if, $\sim a \notin \mathfrak{q}$. Hence $\mathfrak{q} \in \sim h(a)$ if, and only if, $\mathfrak{q} \in h(\sim a)$; which completes the proof of 4).

INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES

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