Propagators in curved spacetimes from operator theory

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Abstract

We discuss two distinct operator-theoretic settings useful for describing (or defining) propagators associated with a scalar Klein-Gordon field on a Lorentzian manifold M. Typically, we assume that M is globally hyperbolic, but we will also consider examples where it is not. Here, the term *propagator* refers to any Green function or bisolution of the Klein-Gordon equation pertinent to Classical or Quantum Field Theory. These include the forward, backward, Feynman and anti-Feynman propagtors, the Pauli-Jordan function and 2-point functions of Fock states.

The first operator-theoretic setting is based on the Hilbert space $L^2(M)$. This setting leads to the definition of the operator-theoretic Feynman and anti-Feynman propagators, which often (but not always) coincide with the so-called out-in Feynman and in-out anti-Feynman propagator. On some special spacetimes, the sum of the operator-theoretic Feynman and anti-Feynman propagator equals the sum of the forward and backward propagator. This is always true on static stable spacetimes and, curiously, in some other cases as well. The second setting is the Krein space $\mathcal{W}_{\mathrm{KG}}$ of solutions of the Klein-Gordon equation. Each linear operator on $\mathcal{W}_{\mathrm{KG}}$ corresponds to a bisolution of the Klein-Gordon equation, which we call its *Klein-Gordon kernel*. In particular, the Klein-Gordon kernels of projectors onto maximal uniformly definite subspaces are 2-point functions of Fock states, and the Klein-Gordon kernel of the identity is the Pauli-Jordan function.

After a general discussion, we review a number of examples: static and asymptotically static spacetimes, FLRW spacetimes (reducible by a mode decomposition to 1-dimensional Schrödinger operators), deSitter space and anti-deSitter space, both proper and its universal cover.

Contents

1 Introduction

1.1 Propagators and states

Let M be a globally hyperbolic Lorentzian manifold of dimension d with a *pseudometric tensor* $g_{\mu\nu}$. Let $Y(x)$ be a *scalar potential*, e.g. $Y(x) = m^2$. Consider a field on M satisfying the *Klein-Gordon equation*

$$\big(-\Box + Y(x)\big)\phi(x) = 0, \tag{1.1}$$

where $\Box := |g|^{-\frac{1}{2}}\partial_\mu |g|^{\frac{1}{2}} g^{\mu\nu}\partial_\nu$ is the *d'Alembertian*. If one wants to compute various pertinent quantities related to ϕ, and especially to its quantization $\hat{\phi}$, one needs to know several distributions on $M \times M$, often called "propagators" or "two-point functions".

These distributions fall into two categories: *Green functions* (also called *fundamental solutions*) and *bisolutions* of the Klein-Gordon equation. A *Green function* of the Klein-Gordon equation is a distribution $G^\bullet$ on $M \times M$ satisfying

$$(-\Box_x + Y(x))G^\bullet(x,y) = \delta(x,y) = (-\Box_y + Y(y))G^\bullet(x,y), \tag{1.2}$$

where $\delta(x,y)$ denotes the distributional kernel of the identity. A *bisolution* of the Klein-Gordon equation is a distribution $G^\bullet$ on $M \times M$ satisfying

$$(-\Box_x + Y(x))G^\bullet(x,y) = 0 = (-\Box_y + Y(y))G^\bullet(x,y). \tag{1.3}$$

In our paper we will colloquially use the term "propagator" for various distinguished Green functions and bisolutions of (1.1) motivated by QFT: the advanced and retarded propagators, the Pauli-Jordan propagator, the Feynman and anti-Feynman propagators and the positive/negative frequency bisolutions (often called Wightman two-point functions).[1]

[1]This nomenclature is in accordance with the previous papers [34–36]. Note, however, that the term "propagator" is often reserved for only some of these distributions. Following the usage common in physics we will often also use the term "two-point function" for the positive/negative frequency solutions and (anti-)time-ordered two-point function for the (anti-)Feynman propagator.

The *retarded* (or *forward*) and *advanced* (or *backward*) propagator $G^{\vee}(x,x')$ and $G^{\wedge}(x,x')$ are the unique Green functions supported for x in the causal future resp. causal past of x'. The bisolution defined by

$$G^{\mathrm{PJ}}(x,x') = G^{\vee}(x,x') - G^{\wedge}(x,x') \tag{1.4}$$

is usually called the *Pauli-Jordan propagator* or the *commutator function*. It also possesses a causal support. All three propagators $G^{\vee}$, $G^{\wedge}$ and G^{PJ} are useful in the Cauchy problem of the Klein-Gordon equation. The classical field $\phi(x)$ satisfying (1.1) is equipped with the Poisson bracket

$$\{\phi(x), \phi(y)\} = -G^{\mathrm{PJ}}(x,y).$$

Therefore, following [34–36], $G^{\vee}$ and $G^{\wedge}$, G^{PJ} will be called *classical propagators*. In Quantum Field Theory one uses a few other two-point functions, whose operator-theoretic meaning – especially on curved spacetimes – is the main subject of this article.

Quantization of the classical field $\phi(x)$ is performed in two steps. In the first step we replace it by an operator valued distribution $\hat{\phi}(x)$, which beside the Klein-Gordon equation

$$\big(-\Box + Y(x)\big)\hat{\phi}(x) = 0 \tag{1.5}$$

satisfies the so called *Peierls relation*

$$[\hat{\phi}(x), \hat{\phi}(y)] = -\mathrm{i}G^{\mathrm{PJ}}(x,y)\mathbb{1}.$$

The fields $\hat{\phi}(x)$ generate a $*$-algebra.

In the second step one selects a representation of the fields in a Hilbert space. In practice, this is done by choosing a state ω_α on this algebra, that is, a positive and normalized linear functional. Then ω_α defines the GNS Hilbert space with a distinguished vector Ω_α. One usually considers a Fock state (a pure quasifree state), where the GNS representation has the form of a bosonic Fock space and Ω_α is its vacuum. The expectation values in this state define four important two-point functions:

$$G^{(+)}_{\alpha}(x,y) := \langle \Omega_\alpha \mid \hat{\phi}(x)\hat{\phi}(y) \mid \Omega_\alpha \rangle, \tag{1.6}$$

$$G^{(-)}_{\alpha}(x,y) := \langle \Omega_\alpha \mid \hat{\phi}(y)\hat{\phi}(x) \mid \Omega_\alpha \rangle, \tag{1.7}$$

$$G^{\mathrm{F}}_{\alpha}(x,y) := \mathrm{i}\langle \Omega_\alpha \mid \mathrm{T}\left(\hat{\phi}(x)\hat{\phi}(y)\right) \mid \Omega_\alpha \rangle, \tag{1.8}$$

$$G^{\overline{\mathrm{F}}}_{\alpha}(x,y) := -\mathrm{i}\langle \Omega_\alpha \mid \overline{\mathrm{T}}\big(\hat{\phi}(x)\hat{\phi}(y)\big) \mid \Omega_\alpha \rangle. \tag{1.9}$$

Here, T and $\overline{\mathrm{T}}$ denote the chronological, resp. anti-chronological time ordering. Note that $G^{(+)}_{\alpha}$ and $G^{(-)}_{\alpha}$ are automatically bisolutions; G^{F}_{α} and $G^{\overline{\mathrm{F}}}_{\alpha}$ are Green functions.

It is perhaps less known that it is useful to define mixed propagators corresponding to two *different* states. Suppose that they are given by vectors Ω_α and Ω_β, belonging to the same representation space, with nonzero $\langle\Omega_\alpha|\Omega_\beta\rangle$. Then we set

$$G^{(+)}_{\alpha,\beta}(x,y) := \frac{\langle\Omega_\alpha \mid \hat\phi(x)\hat\phi(y) \mid \Omega_\beta\rangle}{\langle\Omega_\alpha|\Omega_\beta\rangle}, \tag{1.10}$$

$$G^{(-)}_{\alpha,\beta}(x,y) := \frac{\langle\Omega_\alpha \mid \hat\phi(y)\hat\phi(x) \mid \Omega_\beta\rangle}{\langle\Omega_\alpha|\Omega_\beta\rangle}, \tag{1.11}$$

$$G^{\mathrm{F}}_{\alpha,\beta}(x,y) := \mathrm{i}\frac{\langle\Omega_\alpha \mid T\big(\hat\phi(y)\hat\phi(x)\big) \mid \Omega_\beta\rangle}{\langle\Omega_\alpha|\Omega_\beta\rangle}, \tag{1.12}$$

$$G^{\overline{\mathrm{F}}}_{\alpha,\beta}(x,y) := -\mathrm{i}\frac{\langle\Omega_\alpha \mid \overline{T}\big(\hat\phi(y)\hat\phi(x)\big) \mid \Omega_\beta\rangle}{\langle\Omega_\alpha|\Omega_\beta\rangle}. \tag{1.13}$$

Again, $G^{(+)}_{\alpha,\beta}$ and $G^{(-)}_{\alpha,\beta}$ are bisolutions; the Feynman propagator $G^{\mathrm{F}}_{\alpha,\beta}$ and the anti-Feynman propagator $G^{\overline{\mathrm{F}}}_{\alpha,\beta}$ are Green functions.

The functions $G^{(+)}_\alpha(x,y)$ are used to define the GNS representation for the state ω_α and Wick-ordered product of fields. Wick ordering is a first step to renormalization, which is needed to define higher order monomials of fields. The renormalization procedure will not work for an arbitrary state. In practice one assumes that it has the so-called *Hadamard property*, and then renormalization works well. Note that this analysis can be performed on a local level, without considering the whole spacetime.

Let us now describe the application of Feynman propagators. Suppose we perturb the dynamics and we want to compute the scattering operator S_α in the representation given by Ω_α. By a standard argument going back to Dyson, often called the Wick Theorem, S_α can be expressed as a perturbation series with terms labelled by Feynman diagrams. In order to evaluate Feynman diagrams one needs to replace the lines by $G^{\mathrm{F}}_\alpha(x,y)$.

Often it is natural to compute the scattering operator $S_{\alpha,\beta}$, acting from the representation generated by Ω_β to the representation generated by Ω_α. Actually, it is then useful to divide the scattering operator by the overlap between the vacua, and compute

$$\tilde S_{\alpha,\beta} := \frac{S_{\alpha,\beta}}{\langle\Omega_\alpha \mid \Omega_\beta\rangle}. \tag{1.14}$$

The algorithm is similar as above, except that we put $G^{\mathrm{F}}_{\alpha,\beta}$ at each line of a Feynman diagram.

We will see that $G^{\mathrm{F}}_{\alpha,\beta}$ can usually be defined even if $\langle\Omega_\beta \mid \Omega_\alpha\rangle = 0$. Therefore, we can then also compute $\tilde S_{\alpha,\beta}$. In fact, if the theory is linear, $\tilde S_{\alpha,\beta}$ will be usually a well-defined unbounded quadratic form, whose integral kernel $\tilde S_{\alpha\beta}(k_\alpha,k_\beta)$ can be called the "renormalized scattering amplitude". Obviously, the unitarity of $\tilde S_{\alpha\beta}$ is lost, hence renormalized scattering amplitudes will not have a direct probabilistic interpretation. However their ratios

$$\frac{\tilde S_{\alpha\beta}(k_\alpha,k_\beta)}{\tilde S_{\alpha\beta}(k'_\alpha,k'_\beta)} \tag{1.15}$$

have a meaning: they can be used to compute *branching ratios* of various processes.

If we want to compute $\frac{S^*_{\alpha,\beta}}{\langle\Omega_\beta|\Omega_\alpha\rangle}$ we proceed similarly, except that Feynman propagators need to be replaced by anti-Feynman propagators $G^{\overline{\rm F}}_{\beta,\alpha}$

One of important problems of QFT on curved spacetimes is the choice of a state. In Minkowski space and with $Y(x) = m^2 \geq 0$ there is a natural state, described in all textbooks on QFT. More generally, every stationary and stable Klein-Gordon equation possesses a natural state. Stationarity means that one can identify M with $\mathbb{R} \times \Sigma$ so that $g^{\mu\nu}$ and Y are independent of $t \in \mathbb{R}$, Σ is spacelike and ∂_t is timelike. Stability means that the corresponding classical Hamiltonian is bounded from below. Again, requiring that the state is invariant under the time evolution, and in the GNS representation the dynamics is implemented by a positive quantum Hamiltonian fixes the state uniquely. The one-particle Hilbert space is then taken to be the "positive frequency space", that is, the spectral subspace of the generator of the evolution corresponding to the positive part of the spectrum.

On generic spacetimes there are no distinguished states. There is however one class of spacetimes, particularly well adapted to QFT, where there are *two* distinguished states. These are spacetimes with asymptotically stationary and stable future and past. Such spacetimes possess two distinguished states: the "in-state" and the "out-state", given by vectors Ω_- and Ω_+. Obviously, they define two pairs of two-point functions

$$G^{(+)}_\pm(x,x') = \langle\Omega_\pm \mid \hat\phi(x)\hat\phi(x') \mid \Omega_\pm\rangle, \tag{1.16}$$

$$G^{(-)}_\pm(x,x') = \langle\Omega_\pm \mid \hat\phi(x')\hat\phi(x) \mid \Omega_\pm\rangle. \tag{1.17}$$

One can use them to define two GNS representations acting on two Fock spaces.

More interesting are however the following mixed Feynman propagators: the *out-in Feynman propagator* $G^{\rm F}_{+-}$ and the *in-out anti-Feynman propagator* $G^{\overline{\rm F}}_{-+}$,

$$G^{\rm F}_{+-}(x,x') = \mathrm{i}\frac{\langle\Omega_+ \mid T\hat\phi(x)\hat\phi(x') \mid \Omega_-\rangle}{\langle\Omega_+ \mid \Omega_-\rangle}, \tag{1.18}$$

$$G^{\overline{\rm F}}_{-+}(x,x') = -\mathrm{i}\frac{\langle\Omega_- \mid \overline{T}\hat\phi(x)\hat\phi(x') \mid \Omega_+\rangle}{\langle\Omega_- \mid \Omega_+\rangle}. \tag{1.19}$$

We will see below that $G^{\rm F}_{+-}$ and $G^{\overline{\rm F}}_{-+}$ play an important role in applications.

In a generic situation, (1.18), (1.19) and (1.20) may be ill defined because the overlap $\langle\Omega_+ \mid \Omega_-\rangle$ is zero. Fortunately, as we will see, one can define $G^{\rm F}_{+-}$ and $G^{\overline{\rm F}}_{-+}$ independently via operator theory, without a division by zero.

On an asymptotically stationary and stable spacetime it is natural to use for the initial, resp. final representation the Hilbert space generated by Ω_-, resp. Ω_+. Thus the main objects of interest are

$$\frac{S_{+-}}{\langle\Omega_+ \mid \Omega_-\rangle}, \quad \frac{S^*_{+-}}{\langle\Omega_- \mid \Omega_+\rangle}. \tag{1.20}$$

They can be evaluated using $G^{\rm F}_{+-}$ and $G^{\overline{\rm F}}_{-+}$, even if $\langle\Omega_+ \mid \Omega_-\rangle = 0$.

The main topic of the present article is how to define various propagators using tools of operator theory. We will see in particular that one does not need to worry about dividing by the overlap $\langle\Omega_\alpha|\Omega_\beta\rangle$. It is possible to give a purely operator theoretic definition of (1.10), (1.11), (1.12), (1.13), which works also if $\langle\Omega_\alpha \mid \Omega_\beta\rangle = 0$.

1.2 Operator-theoretic interpretations of propagators

There are two distinct operator-theoretic settings related to the Klein-Gordon equation, which are useful in defining and computing propagators: the space of solutions to (1.1), which we denote $\mathcal{W}_{\mathrm{KG}}$, and the Hilbert space $L^2(M, |g|^{\frac{1}{2}})$. Let us first outline the first setting.

To define $\mathcal{W}_{\mathrm{KG}}$ one usually starts from the space of complex space-compact solutions to (1.1). This space is endowed with the so-called *Klein-Gordon charge form*—an indefinite sesquilinear form obtained by integrating the natural current over an arbitrary Cauchy surface. In the generic case, this space does not have a distinguished positive scalar product. Nevertheless, it often possesses a natural family of equivalent positive scalar products. Then, for technical reasons, it is useful to use them to define its completion, as is described in [36]. One obtains a Krein space $\mathcal{W}_{\mathrm{KG}}$: a space of solutions with a Hilbertian topology equipped with a distinguished indefinite Klein-Gordon charge form. Using elements of the theory of Krein spaces one is able to give meaning to the quantities (1.10), (1.11), (1.12), (1.13), avoiding expressions of the type $\frac{0}{0}$. This is a big advantage of the operator-theoretic viewpoint.

In practice, it is convenient to represent the space $\mathcal{W}_{\mathrm{KG}}$ in terms of Cauchy data. More precisely, we first identify $M = \mathbb{R} \times \Sigma$, where Σ has a spatial signature and ∂_t a temporal signature. Each element of $\mathcal{W}_{\mathrm{KG}}$ is uniquely determined by its value at $\{t\} \times \Sigma$ and its temporal derivative. This allows us to describe elements of $\mathcal{W}_{\mathrm{KG}}$ as pairs of functions on Σ.

The space $\mathcal{W}_{\mathrm{KG}}$ is not the only operator-theoretic setting for propagators. There is another one, provided by the Hilbert space is $L^2(M, |g|^{\frac{1}{2}})$. At first many readers may protest – this space does not describe physically relevant states. However, as we will see it is very useful for the computation of propagators.

It can be easily shown that on Minkowski space the usual Feynman and anti-Feynman propagator are the boundary values of the resolvent kernel of the Klein-Gordon operator on $L^2(\mathbb{R}^{1,d-1})$:

$$G^{\mathrm{F}}(x,y) := \lim_{\epsilon \searrow 0} \frac{1}{(-\Box + m^2 + \mathrm{i}\epsilon)}(x,y), \tag{1.21}$$

$$G^{\overline{\mathrm{F}}}(x,y) := \lim_{\epsilon \searrow 0} \frac{1}{(-\Box + m^2 - \mathrm{i}\epsilon)}(x,y), \tag{1.22}$$

It is not difficult to see that an analogous statement is true on stationary stable spacetimes.

More generally, suppose we use the path integral formalism to define perturbative QFT. The usual prescription says that one should split the action in a quadratic part and the interaction, and then derive Feynman diagrams from the path integral. It is easy to see that this prescription formally yields (1.21) and (1.22) as the expressions corresponding to the lines in Feynman diagrams. This suggests an alternative definition of Feynman and anti-Feynman propagator, which we describe below.

It is clear that $-\Box + Y(x)$ is a Hermitian operator on $L^2(M, |g|^{\frac{1}{2}})$. Suppose that it is essentially self-adjoint. Then its spectrum is contained in $\mathbb{R}$ and the following definition makes sense:

$$G^{\mathrm{F}}_{\mathrm{op}}(x,y) := \lim_{\epsilon\searrow 0} \frac{1}{(-\Box + Y(x) + \mathrm{i}\epsilon)}(x,y), \tag{1.23}$$

$$G^{\overline{\mathrm{F}}}_{\mathrm{op}}(x,y) := \lim_{\epsilon\searrow 0} \frac{1}{(-\Box + Y(x) - \mathrm{i}\epsilon)}(x,y), \tag{1.24}$$

where we use the distributional limit. We call $G^{\mathrm{F}}_{\mathrm{op}}(x,y)$ and $G^{\overline{\mathrm{F}}}_{\mathrm{op}}(x,y)$ the *operator-theoretic Feynman and anti-Feynman propagator*.

One can heuristically derive [34, 35] that on asymptotically stationary and stable spacetimes the out-in Feynman and the in-out anti-Feynman propagator coincide with the operator-theoretic Feynman propagators:

$$G^{\mathrm{F}}_{\mathrm{op}}(x,y) = G^{\mathrm{F}}_{+-}(x,y), \tag{1.25}$$

$$G^{\overline{\mathrm{F}}}_{\mathrm{op}}(x,y) = G^{\overline{\mathrm{F}}}_{-+}(x,y). \tag{1.26}$$

Indeed, these identities can be viewed as a justification of the path-integral approach to QFT.

The definitions (1.23) and (1.24) raise difficult mathematical questions. First, the essential self-adjointness for generic spacetimes is a nontrivial problem. For asymptotically Minkowskian spacetimes satisfying some non-trapping conditions it has been proven in [62, 63, 74]. Under similar conditions one can show that (1.25) and (1.26) are true.

Propagators may satisfy various identities. We already mentioned (1.4), which defines the Pauli-Jordan propagator. Another identity universally true is

$$G^{\mathrm{PJ}}(x,x') = \mathrm{i}G^{(+)}_{\alpha,\beta}(x,x') - \mathrm{i}G^{(-)}_{\alpha,\beta}(x,x'), \tag{1.27}$$

valid for any pair of Fock states $\omega_\alpha, \omega_\beta$.

On Minkowski space with $Y(x) = m^2 \geq 0$, and more generally for a stationary stable Klein-Gordon equation, we have the identity

$$G^{\mathrm{F}}_{\mathrm{op}} + G^{\overline{\mathrm{F}}}_{\mathrm{op}} = G^{\vee} + G^{\wedge}. \tag{1.28}$$

In particular, the support of $G^{\mathrm{F}}_{\mathrm{op}} + G^{\overline{\mathrm{F}}}_{\mathrm{op}}$ is causal.

Definition 1.1. We will say that the Klein-Gordon equation is *special* if one can define $G^{\mathrm{F}}_{\mathrm{op}}$ and $G^{\overline{\mathrm{F}}}_{\mathrm{op}}$ (which we expect to be true in typical situations) and the support of $G^{\mathrm{F}}_{\mathrm{op}} + G^{\overline{\mathrm{F}}}_{\mathrm{op}}$ is causal. We will then also say that the *specialty condition* is satisfied.

Special Klein-Gordon equations have the following advantage. One may expect that that it is in many situations comparably simple to compute the distributions $G^{\mathrm{F}}_{\mathrm{op}}$ and $G^{\overline{\mathrm{F}}}_{\mathrm{op}}$ using operator-theoretic tools. Then, splitting $G^{\mathrm{F}}_{\mathrm{op}} + G^{\overline{\mathrm{F}}}_{\mathrm{op}}$ into two distributions, one supported in the causal future and the other supported in the causal past, we may determine $G^{\vee}$ and $G^{\wedge}$.

Remark 1.2. Note that if the spacetime is not globally hyperbolic, then $G^{\vee}$ and $G^{\wedge}$ may not be uniquely defined. In this case, the splitting of $G^{\mathrm{F}}_{\mathrm{op}} + G^{\overline{\mathrm{F}}}_{\mathrm{op}}$ yields one possible pair of forward and backward propagators.

The specialty condition is generically violated. It is however very useful if it holds. We will discuss some interesting cases when it is true.

1.3 Outline of the paper

Section 2 is a didactic introduction containing a discussion of propagators on Minkowski space. Here, all arguments are simple and well-known. In particular, we describe both basic operator-theoretic settings: the space of solutions to the Klein-Gordon equation and the Hilbert space $L^2(\mathbb{R}^{1,d-1})$.

We then describe in Section 3 various kinds of propagators in a generic spacetime. Again, we have two settings: the Krein space of solutions to the Klein-Gordon equation $\mathcal{W}_{\mathrm{KG}}$ and the Hilbert space $L^2(M, \sqrt{|g|})$.

The remaining sections are dedicated to the discussion of various examples of spacetimes with largely different properties:

1. First we discuss stationary spacetimes. Here one can give fairly explicit formulas for all four basic Green functions and the Pauli-Jordan function. If in addition the Klein-Gordon equation is stable, then one can also define the positive/negative frequency bisolution, and the specialty condition is fulfilled.

   In the tachyonic case, that is, if the Hamiltonian is not positive, the special property is violated, and we cannot define positive/negative frequency bisolutions. This includes the Minkowski space with imaginary mass, that is, $m^2 < 0$. Of course, this case is not very physical, however, all four basic Green functions are usually well defined.

2. Spacetimes asymptotically stationary and stable in the past and future form a class well suited for the formalism of QFT. After identifying M with $\mathbb{R} \times \Sigma$, where $\mathbb{R}$ describes time and Σ is a Cauchy surface with a time dependent Riemannian metric, one can give a fairly explicit description of all propagators using the time evolution of solutions, as described in [36]. Remarkably, the out-in Feynman and in-out anti-Feynman propagator are well defined—this is a non-trivial statement proven in [36]. As we mentioned above, the specialty condition is rarely fulfilled.

3. The Klein-Gordon equation on $1+0$-dimensional spacetimes essentially reduces to a one-dimensional Schrödinger operator. The corresponding propagators are well-known objects from the theory of such operators. The speciality condition is fulfilled if and only if the scattering operator is reflectionless. Obviously, it is satisfied if the potential is a constant. But curiously, as is well known, there exist potentials which are reflectionless at all energies. The best known such potential is

$$-\frac{\mu^2 - \frac{1}{4}}{\cosh^2 x} \tag{1.29}$$

   for half-integer μ.

4. Spacetimes, whose pseudometric depends on time only through a conformal factor, are usually called Friedmann-Lemaître-Robertson-Walker (FLRW) spacetimes. In such spacetimes, after diagonalization of the spatial Laplacian, or in other words, after decomposing it into "modes", the Klein-Gordon equation can be reduced to the $1+0$-dimensional setting. Thus in principle one can write all propagators as the direct sum

or integral of propagators for each mode. In particular, the Klein-Gordon equation is special if each mode is reflectionless.

5. The theory of propagators on the d-dimensional de Sitter space dS_d is especially rich and surprising.

   The deSitter space can be interpreted as the "Wick rotated" d-dimensional sphere. Analytically continuing the Green function of the sphere in the usual spherical coordinates we obtain a certain Feynman and anti-Feynman propagator. For $m^2 \geq (\frac{d-1}{2})^2$, they can be used to write down the Wightman two-point functions of a state as well as the classical propagators. This state is usually called the Euclidean (or Bunch-Davies) state and is believed to be the physical choice on the deSitter space, because it is Hadamard. In other words, the Euclidean state satisfies the Hadamard condition.

   The d'Alembertian on deSitter space is essentially self-adjoint on smooth compactly supported functions. This is a special case of a general mathematical theorem saying that invariant differential operators on maximally symmetric pseudoRiemannian manifolds are essentially self-adjoint. One can compute the resolvent of the d'Alembertian on dS_d. Taking its boundary values yields the operator-theoretic Feynman and anti-Feynman propagator. Curiously, they are different from the Euclidean Feynman and anti-Feynman propagator. The specialty condition is satisfied in odd dimensions; it is not true in even dimensions.

   It is well-known that all deSitter invariant states can be described and expressed in terms of Gegenbauer functions. They are usually called α-vacua, where α is a complex parameter that can be used to parametrize them. $\alpha = 0$ corresponds to the Euclidean vacuum. All other α-vacua are not Hadamard.

   The deSitter space is not asymptotically stationary. However, it possesses two distinguished states, which can be called the in state and the out state. The former has an incoming behavior in the past, the latter is outgoing in the future. The operator theoretic Feynman and anti-Feynman propagators satisfy the identitities (1.25) and (1.26). In odd dimensions the in-state coincides with the out-state. In even dimensions this is not the case. In all dimensions, the in-state and out-state are distinct from Euclidean state.

   dS_d is a FLRW spacetime (with a conformal factor that blows up exponentially). Therefore, it is possible to decompose the Klein-Gordon equation into modes. In each mode one obtains the 1-dimensional Schrödinger operator with the potential (1.29), where μ depends on the dimension and the degree of spherical harmonics. μ is a half-integer for odd dimensions and an integer for even dimensions. This is another way to see that the Klein-Gordon equation in odd dimensions is special and in even dimension is not.

   One can define retarded and advanced propagators for all values of $m^2 \in \mathbb{R}$. However, the case $m^2 < (\frac{d-1}{2})^2$ seems not physical. In fact, below $(\frac{d-1}{2})^2$ the spectrum of the d'Alembertian is discrete. Operator-theoretic Feynman and anti-Feynman propagators are well defined (and identical) outside of this spectrum. As can be expected, the specialty condition is then violated.

6. The universal cover of anti-deSitter space $\widetilde{\mathrm{AdS}}_d$ is another maximally symmetric spacetime, where one can compute all propagators. It is a stationary spacetime, which is not globally hyperbolic: it possesses geodesics that escape to the spatial boundary in a finite proper time. One can apply two approaches to define the propagators on the universal cover of anti-deSitter space.

   The first approach uses $L^2(\widetilde{\mathrm{AdS}}_d)$. The d'Alembertian is essentially self-adjoint—there is no need to fix boundary conditions. One can define the operator-theoretic Feynman and anti-Feynman propagators as the limits of its resolvent. If $m^2 > -(\frac{d-1}{2})^2$, then their sum has a causal support, so one can define the retarded and advanced propagator by splitting this sum. In particular, the specialty condition is satisfied.

   Alternatively, one can use the evolution of the Cauchy data. For $m^2 \geq -(\frac{d-1}{2})^2 + 1$ this evolution is uniquely defined–one does not need to specify boundary conditions. For $m^2 < -(\frac{d-1}{2})^2 + 1$ boundary conditions are needed. For $-(\frac{d-1}{2})^2 \leq m^2 < -(\frac{d-1}{2})^2 + 1$ there exists a distinguished boundary condition (corresonding to the Friedrichs extension), which agrees with the propagators obtained from the operator-theoretic Feynman propagator. In particular, we have distinguished retarded and advanced propagators. For $m^2 < -(\frac{d-1}{2})^2$ there are no distinguished boundary conditions at spatial infinity. Thus retarded and advanced propagators are non-unique and none is distinguished.

Pertinent elements of the theory of Krein spaces are discussed in Appendix A. Propagators on deSitter and anti-deSitter space can be described explicitly in terms of special functions (Gegenbauer functions). We introduce their relevant properties in Appendix B.

Remark 1.3. We restrict our considerations to a real scalar field $\hat{\phi}(x)$, but they can be generalized to a complex scalar field in a fairly straightforward manner. One needs then two pairs of creation and annihilation operators. Both the real and the complex formalism are treated in [36].

Our analysis can be also easily extended to cover the Klein-Gordon equation with electromagnetic potential. Then we will be forced to use complex fields, and not real fields.

1.4 Literature about the subject

Quantum Field Theory on curved spacetimes is one of the most discussed and developed areas of theoretical physics. It has enormous literature, including numerous standard textbooks [7, 10, 45, 65]. Our paper has many features of a review article, describing various facts and concepts known from the literature. However, we think that the paper also provides new insights and that quite a number of ideas are here stated clearly for the first time. Let us in particular mention the description of the operator-theoretic setups in Section 3 and Appendix A, which is a continuation of the works [34–36] of D. Siemssen and one of the authors (JD), the study of four different approaches to the Klein-Gordon equation on deSitter space from Section 6 and the discussions of the "speciality condition" throughout all sections.

We start our review of the literature with the "classical propagators", that is, the retarded and advanced propagator, and the Pauli-Jordan function. They belong to standard knowledge and are well-studied in standard references. In the massless case on the flat $\mathbb{R}^{1,3}$ the retarded and advanced propagators are well known from classical electrodynamics, and are sometimes called the *Lienard-Wiechert potentials*. In the massive flat case their expressions in terms of Hankel functions are contained in many textbooks. The Cauchy problem of the wave equation on curved spacetimes was studied already by Hadamard [49], at least locally. A recent reference to this subject on arbitrary globally hyperbolic manifolds is the book by Bär, Ginoux and Pfäffle [8]. In the introduction to this book one reads: "Tracing back the references [on the uniqueness and existence of linear wave equation on lorentzian manifolds] one typically ends at unpublished lecture notes of Leray [58] or their exposition by Choquet-Bruhat [26]."

In the literature the Pauli-Jordan function is often called the commutator function or (recently, in mathematics oriented literature) the causal propagator, [7, 45]. Note, however, that the latter name can lead to confusion: in [14] the Feynman propagator is called the causal Green function.

Propagators on the Minkowski space, including "non-classical" ones, are well-known from various textbooks on Quatum Field Theory (especially the old-fashioned ones). For instance, Appendix 2 of Bogoliubov–Shirkov [14] and Appendix C of Bjorken–Drell [11] contain expressions for these functions in the position space in the physical case of $\mathbb{R}^{1,3}$, and discuss conventions used by various authors.

"Non-classical" propagators are expectation values of products of two fields. Those without time-ordering, sometimes called Wightman functions, are ubiquitous in the mathematical literature, since they are needed to define the GNS representation and multiplication in appropriately defined local algebras. One of major questions, which is asked in various papers is whether they satisfy the Hadamard condition.

Expectation values of time-ordered fields, that is, Feynman propagators, are needed when we want to find scattering amplitudes. They often appear in the physics literature as mixed two-point functions, typically with the out-vacuum on the left and in-vacuum on the right. For instance, in Birrel-Davies [10] in (9.13) one finds the following definition of Green functions:

$$\tau(x_1, x_2 ... x_m) = \frac{\langle \mathrm{out}, 0 | T(\phi(x_1)\phi(x_2)...\phi(x_m)|0, \mathrm{in}\rangle}{\langle \mathrm{out}, 0|0, \mathrm{in}\rangle}. \tag{1.30}$$

Then the authors write: "...unlike the case of Minkowski space where $|0, \mathrm{out}\rangle = |0, \mathrm{in}\rangle$ (up to a phase factor), the vacuum $|0, \mathrm{in}\rangle$ in curved spacetime will not in general be stable: $\langle \mathrm{out}, 0|0, \mathrm{in}\rangle \neq 1$." In particular the relationship (1.25), which says that the "out-in Feynman propagator"G^{F}_{+-} coincides with the Feynman propagator formally computed in the path-integral approach (which can be interpreted as $G^{\mathrm{F}}_{\mathrm{op}}$) is implicitly contained in [10] (and in general in the physics literature). Elements of this philosophy are also found in [69, 70].

In the more recent rigorous literature, mixed (two-state) propagators are almost absent. The majority of recent works, for example the seminal papers [21, 54], emphasize the local point of view. Their usual goal is to construct a net of local algebras, for which it is enough to fix a single state, preferably Hadamard, which can be done locally.

A systematic rigorous study of various natural propagators on curved spacetimes was undertaken in the series of papers by one of the authors (JD) with a coauthor [34–36]. In particular, the construction of the distinguished Feynman propagator by methods of Krein spaces on an asymptotically stationary stable spacetimes is contained in [36]. A construction of the same Feynman propagator on a (more narrow) class of asymptotically Minkowskian spaces by methods of pseudodifferential calculus was given by Gérard and Wrochna [45, 47].

There exist many works, especially in the PDE literature, about *parametrices* of the Klein-Gordon equations, that is, inverses modulo a *smoothing operator*. A celebrated paper with this philosophy is the work by Duistermaat and Hörmander [39], which describes four natural parametrices: retarded, advanced, Feynman and anti-Feynman. Such parametrices are enough in the study of propagation of singularities, and they do not require a global knowlege of the spacetime. Similarly, it is often argued in mathematical physics papers that it is enough to know a two-point function only up to a *smooth term*. This is indeed sufficient if we want to prove the existence of renormalized powers of fields [21].

In our paper we are interested only in *exact* Green functions and bisolutions. Clearly, they are needed if we want to be able to compute scattering amplitudes exactly.

We will discuss the literature about the examples that we present in Sections 4, 5, 6 and 7 in the respective sections.

2 Minkowski space

In this section, we provide a didactic and partially heuristic introduction to the operator-theoretic approach to propagators using the example of Minkowski space. Readers who are interested only in the more general setup on curved spacetimes may skip this section.

2.1 Propagators in Minkowski space

Consider a real scalar field in d-dimensional Minkowski spacetime. For non-negative squared mass $m^2 \geq 0$, the Klein-Gordon equation for $\phi(x)$ (in natural units) is[2]

$$(-\Box + m^2)\phi(x) := \left(\partial_{x^0}^2 - \sum_{k=1}^{d-1} \partial_{x^k}^2 + m^2\right)\phi(x) = 0. \tag{2.1}$$

There are four fundamental solutions invariant with respect to the restricted Poincare group: the *Feynman propagator* $G^{\mathrm{F}}(x,y)$, the *anti-Feynman propagator* $G^{\overline{\mathrm{F}}}(x,y)$, the *forward (retarded) propagator* $G^{\vee}(x,y)$ and the *backward (advanced) propagator* $G^{\wedge}(x,y)$, given by

$$\begin{aligned} G^{\mathrm{F}/\overline{\mathrm{F}}}(x,y) &= \frac{1}{(2\pi)^d}\int \frac{\mathrm{e}^{\mathrm{i}p(x-y)}}{p^2+m^2\mp \mathrm{i}0}\mathrm{d}^d p, \\ G^{\vee/\wedge}(x,y) &= \frac{1}{(2\pi)^d}\int \frac{\mathrm{e}^{\mathrm{i}p(x-y)}}{p^2+m^2\mp \mathrm{i}0\,\mathrm{sgn}(p^0)}\mathrm{d}^d p. \end{aligned} \tag{2.2}$$

[2]We adapt the signature $(-,+,+,\cdots,+)$ for the metric tensor throughout this paper.

The retarded and advanced propagators are distinguished by the fact that they have support inside the closed forward/backward lightcone $(V^{\pm})^{\mathrm{cl}}$, where

$$V^{\pm} := \{(x,y) \mid (x-y)^2 < 0,\ x^0 - y^0 \gtrless 0\}, \tag{2.3}$$

and cl denotes the closure. The Feynman and anti-Feynman propagators are supported everywhere.

Three bisolutions invariant with respect to the restricted Poincare group play an important role in QFT: the *positive* and *negative frequency solutions*

$$\begin{aligned} G^{(\pm)}(x,y) &= \frac{1}{(2\pi)^{d-1}} \int \theta(\pm p^0)\, \delta(p^2+m^2)\, \mathrm{e}^{\mathrm{i}p(x-y)} \mathrm{d}^d p, \\ &= \frac{1}{(2\pi)^{d-1}} \int \frac{\mathrm{e}^{\mp \mathrm{i} E_{\boldsymbol{p}}(x^0-y^0) + \mathrm{i}\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{y})}}{2E_{\boldsymbol{p}}} \mathrm{d}^{d-1}\boldsymbol{p}, \end{aligned} \tag{2.4}$$

and the *Pauli-Jordan function*

$$\begin{aligned} G^{\mathrm{PJ}}(x,y) &= \frac{\mathrm{i}}{(2\pi)^{d-1}} \int \mathrm{sgn}(p^0)\, \delta(p^2+m^2)\, \mathrm{e}^{\mathrm{i}p(x-y)} \mathrm{d}^d p \\ &= \frac{1}{(2\pi)^{d-1}} \int \frac{\sin\big(E_{\boldsymbol{p}}(x^0-y^0)\big)}{E_{\boldsymbol{p}}} \mathrm{e}^{\mathrm{i}\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{y})} \mathrm{d}^{d-1}\boldsymbol{p}, \end{aligned} \tag{2.5}$$

where the bold variables indicate the $d-1$-component spatial part of the full vectors and where $E_{\boldsymbol{p}} := \sqrt{\boldsymbol{p}^2 + m^2}$. Sometimes one also introduces the *symmetric two-point function*

$$G^{\mathrm{sym}}(x,y) := G^{(+)}(x,y) + G^{(-)}(x,y). \tag{2.6}$$

The propagators satisfy the following identities:

$$G^{\mathrm{F}} + G^{\overline{\mathrm{F}}} = G^{\vee} + G^{\wedge}, \tag{2.7a}$$

$$G^{\mathrm{F}} - G^{\overline{\mathrm{F}}} = \mathrm{i}\big(G^{(+)} + G^{(-)}\big), \tag{2.7b}$$

$$G^{\mathrm{PJ}} = G^{\vee} - G^{\wedge} = \mathrm{i}\big(G^{(+)} - G^{(-)}\big), \tag{2.7c}$$

$$G^{\mathrm{F}} = \mathrm{i}G^{(+)} + G^{\wedge} = \mathrm{i}G^{(-)} + G^{\vee}, \tag{2.7d}$$

$$G^{\overline{\mathrm{F}}} = -\mathrm{i}G^{(+)} + G^{\vee} = -\mathrm{i}G^{(-)} + G^{\wedge}. \tag{2.7e}$$

The classical field theory has a natural canonical structure with the following Poisson bracket:

$$\{\phi(x), \phi(y)\} = -G^{\mathrm{PJ}}(x-y). \tag{2.8}$$

2.2 Quantum fields on Minkowski space

Suppose $\hat{\phi}(x)$ is a real scalar quantum field on $\mathbb{R}^{1,d-1}$ that satisfies the same equation as the classical one:

$$(-\Box + m^2)\hat{\phi}(x) = 0. \tag{2.9}$$

It is an operator valued distribution with commutation relations that go under the name of the Peierls bracket and are obtained by the obvious quantization of (2.8):

$$[\hat{\phi}(x), \hat{\phi}(y)] = -\mathrm{i}G^{\mathrm{PJ}}(x, y). \tag{2.10}$$

The fields $\hat{\phi}(x)$ act in the Fock space with the vacuum Ω. $G^{(\pm)}$ and $G^{\mathrm{F}/\overline{\mathrm{F}}}$ are vacuum expectation values of products of fields, as described in (1.6), (1.7), (1.8) resp. (1.9).

2.3 Operator-theoretic interpretation of the Feyman propagator

The Minkowski space is equipped with a natural Poincaré-invariant Lebesgue measure. Therefore one can invariantly define the Hilbert space $L^2(\mathbb{R}^{1,d-1})$. Clearly, it is not a "state space" of a quantum system. However, it is useful in the theory of the Feynman propagator.

The d'Alembertian $-\Box$ is a self-adjoint operator. Its spectrum is the whole real line $\mathbb{R}$. Therefore, if $m^2 \in \mathbb{C} \setminus \mathbb{R}$, we can define its resolvent, which we denote

$$G(-m^2) := (-\Box + m^2)^{-1}. \tag{2.11}$$

Its integral kernel will be denoted $G(-m^2; x, y)$. Clearly,

$$G(-m^2; x, y) = \frac{1}{(2\pi)^d} \int \frac{\mathrm{e}^{\mathrm{i}p(x-y)}}{p^2 + m^2} \mathrm{d}^d p. \tag{2.12}$$

It is easy to see that

$$G^{\mathrm{F}}(x, y) := \lim_{\epsilon \searrow 0} G(-m^2 - \mathrm{i}\epsilon; x, y), \tag{2.13}$$

$$G^{\overline{\mathrm{F}}}(x, y) := \lim_{\epsilon \searrow 0} G(-m^2 + \mathrm{i}\epsilon; x, y), \tag{2.14}$$

where the convergence is understood in the sense of distributions.

2.4 The space of solutions

Bisolutions of the Klein-Gordon equation also have an operator-theoretic interpretation, which is, however, based on a different functional-analytic setting.

Definition 2.1. A function ζ on Minkowski space is called *space-compact* if there exists a compact set $K \subset \mathbb{R}^{1,d-1}$ such that

$$\operatorname{supp} \zeta \subset J(K), \quad \text{where} \quad J(K) := J^+(K) \cup J^-(K), \quad J^{\pm}(K) := K + (V^{\pm})^{\mathrm{cl}}. \tag{2.15}$$

Let us denote by $\mathcal{W}_{\mathrm{sc}}$ the space of smooth and space-compact solutions of the Klein-Gordon equation. It is equipped with the indefinite *Klein-Gordon charge form*, sometimes called the *Klein-Gordon inner product:*

$$(\xi|\zeta)_{\mathrm{KG}} := \mathrm{i} \int_{\Sigma} \overline{\xi(x)} \overleftrightarrow{\partial}_{\mu} \zeta(x) \mathrm{d}\Sigma^{\mu}(x), \tag{2.16}$$

where Σ is any Cauchy surface, $\mathrm{d}\Sigma^\mu$ is the standard measure on Σ times the future-directed unit-vector normal to Σ and

$$X\overleftrightarrow{\partial}_\mu Y := X\partial_\mu Y - (\partial_\mu Y)X. \tag{2.17}$$

Since ξ and ζ satisfy the Klein-Gordon equation, the definition (2.16) is independent of the choice of the Cauchy surface Σ. $(\cdot,\cdot)_{\mathrm{KG}}$ is not positive definite but it can be decomposed (in many ways) into maximal uniformly positive and maximal negative subspaces.[3]

For Minkowski space, we can without loss of generality choose $\Sigma = \{x^0 = const.\}$, so that (2.16) becomes

$$(\xi,\zeta)_{\mathrm{KG}} = \mathrm{i}\int_{\{x^0=const.\}} \overline{\xi(x)}\overleftrightarrow{\partial}_0\zeta(x)\mathrm{d}^{d-1}\boldsymbol{x}. \tag{2.18}$$

Definition 2.2. Let $A(x,y)$ be a bisolution of the Klein-Gordon equation and $\zeta \in \mathcal{W}_{\mathrm{sc}}$. Then

$$(A\zeta)(x) = \big(\overline{A(x,\cdot)}|\zeta(\cdot)\big)_{\mathrm{KG}} = \mathrm{i}\int_\Sigma A(x,y)\overleftrightarrow{\partial}_{y^\mu}\zeta(y)\mathrm{d}\Sigma^\mu(y) \tag{2.19}$$

does not depend on the choice of the Cauchy surface Σ and defines a linear map on $\mathcal{W}_{\mathrm{sc}}$. We call $A(x,y)$ the *Klein-Gordon kernel of the operator* A.

If A and B are two operators with the Klein-Gordon kernels $A(x,y)$ and $B(x,y)$, then the Klein-Gordon kernel of their composition is

$$AB(x,y) = \big(\overline{A(x,\cdot)}|B(\cdot,y)\big)_{\mathrm{KG}}. \tag{2.20}$$

From (2.5), we can derive $G^{\mathrm{PJ}}(x,y)\Big|_{x^0=y^0} = 0$ and $\partial_{y^0}G^{\mathrm{PJ}}(x,y)\Big|_{x^0=y^0} = -\delta(\boldsymbol{x}-\boldsymbol{y})$. Using these identities, we obtain

$$\Big(\overline{-\mathrm{i}G^{\mathrm{PJ}}(x,\cdot)}|\zeta(\cdot)\Big)_{\mathrm{KG}} = \zeta(x). \tag{2.21}$$

Therefore, $-\mathrm{i}G^{\mathrm{PJ}}(x,y)$ is the Klein-Gordon kernel of the identity operator on $\mathcal{W}_{\mathrm{sc}}$. By (2.7c), we have $-\mathrm{i}G^{\mathrm{PJ}}(x,y) = G^{(+)}(x,y) - G^{(-)}(x,y)$. Also the positive/negative frequency solutions have an interpretation as integral kernels of operators.

We formally have

$$\begin{aligned}\Big(\overline{G^{(\pm)}(x,\cdot)}|G^{(\pm)}(\cdot,y)\Big)_{\mathrm{KG}} &= \pm G^{(\pm)}(x,y), \\ \Big(\overline{G^{(\pm)}(x,y)}|G^{(\mp)}(y,z)\Big)_{\mathrm{KG}} &= 0.\end{aligned} \tag{2.22}$$

[3]Cf. Def. A.23 for the precise definition of maximal uniformly postive/negative spaces. Note also that the necessary uniform definiteness is stronger than definiteness used in [36]. However, the application to QFT remains untouched by this since we are anyway dealing with uniformly definite subspaces.

Therefore, $\Pi^{(\pm)}(x,y) := \pm G^{(\pm)}(x,y)$ are the Klein-Gordon kernels of projections $\Pi^{(\pm)}$ on $\mathcal{W}_{\rm sc}$.[4] By (2.4), these projections are orthogonal:

$$\big(\Pi^{(\pm)}\xi,\zeta\big)_{\rm KG} = \big(\xi,\Pi^{(\pm)}\zeta\big)_{\rm KG}. \tag{2.23}$$

From the relation of $G^{(\pm)}$ and $G^{\rm PJ}$, it follows that $\Pi^{(+)} + \Pi^{(-)} = \mathbb{1}_{\mathcal{W}_{\rm sc}}$.

We may write any $\zeta \in \mathcal{W}_{\rm sc}$ as

$$\zeta(x) = \frac{1}{(2\pi)^{d-1}} \int \frac{1}{\sqrt{2E_{\boldsymbol{p}}}} \Big(\mathrm{e}^{-\mathrm{i}E_{\boldsymbol{p}}x^0+\mathrm{i}\boldsymbol{p}\boldsymbol{x}}\zeta^{(+)}(\boldsymbol{p}) + \mathrm{e}^{\mathrm{i}E_{\boldsymbol{p}}x^0-\mathrm{i}\boldsymbol{p}\boldsymbol{x}}\zeta^{(-)}(\boldsymbol{p})\Big)\mathrm{d}^{d-1}\boldsymbol{p}. \tag{2.24}$$

Making use of (2.4) one last time, we find

$$\Pi^{(\pm)}\zeta(x) = \frac{1}{(2\pi)^{d-1}} \int \frac{\mathrm{e}^{\mp\mathrm{i}(E_{\boldsymbol{p}}x^0-\mathrm{i}\boldsymbol{p}\boldsymbol{x})}}{\sqrt{2E_{\boldsymbol{p}}}}\zeta^{(\pm)}(\boldsymbol{p})\mathrm{d}^{d-1}\boldsymbol{p}. \tag{2.25}$$

Therefore,

$$(\Pi^{(\pm)}\zeta,\Pi^{(\pm)}\zeta)_{\rm KG} = \pm\frac{1}{(2\pi)^{d-1}}\int \big|\zeta^{(\pm)}(\boldsymbol{p})\big|^2\mathrm{d}^{d-1}\boldsymbol{p}, \tag{2.26}$$

and the integral on the right-hand side is non-negative. That is, $\Pi^{(\pm)}$ are projections onto maximal uniformly positive resp. maximal uniformly negative subspaces.

$\Pi^{(+)}$ and $\Pi^{(-)}$ define an involution

$$S := \Pi^{(+)} - \Pi^{(-)}, \quad S^2 = \mathbb{1}_{\mathcal{W}_{\rm sc}}, \tag{2.27}$$

whose Klein-Gordon kernel is the symmetric two-point function $G^{\rm sym} = G^{(+)} + G^{(-)}$.

The charge form $(\cdot|\cdot)_{\rm KG}$ on $\mathcal{W}_{\rm sc}$ is indefinite. It is useful to consider a positive scalar product on $\mathcal{W}_{\rm sc}$, defined in terms of the involution S:

$$(\xi,\zeta)_0 := (\xi,\Pi^{(+)}\zeta)_{\rm KG} - (\xi,\Pi^{(-)}\zeta)_{\rm KG} = (\xi,S\zeta)_{\rm KG}. \tag{2.28}$$

One can take the completion of $\mathcal{W}_{\rm sc}$ with respect to $(\cdot,\cdot)_0$. We denote this completion by $\mathcal{W}_{\rm KG}$. It is simultaneously a Hilbert space with the positive scalar product (2.28) and a Krein space with the indefinite product (2.16). The operators $\Pi^{(+)}$ and $\Pi^{(-)}$ extend to orthogonal projections on $\mathcal{W}_{\rm KG}$. The space $\Pi^{(+)}\mathcal{W}_{\rm KG}$ can be identified with the one particle space of the Fock representation of the quantum field $\hat{\phi}(x)$.

3 Propagators in curved spacetimes

3.1 Klein-Gordon equation

Consider a Lorentzian manifold M of dimension d with *pseudometric tensor* $g_{\mu\nu}$. Define the *d'Alembertian*

$$\Box := |g|^{-\frac{1}{2}}\partial_\mu|g|^{\frac{1}{2}}g^{\mu\nu}\partial_\nu \tag{3.1}$$

[4] To be precise: on its completion described below.

and the *Klein-Gordon operator* $-\Box + Y(x)$, where $Y(x)$ is an x-dependent *scalar potential*. Most of the time we will assume that $Y(x) = m^2$, so that the Klein-Gordon operator is $-\Box + m^2$. How to generalize the well-known propagators from $\mathbb{R}^{1,d-1}$ to generic spacetimes?

Note that the d'Alembertian (3.1) acts on scalar functions. It is sometimes more convenient to replace it by the d'Alembertian in the half-density formalism, that is

$$\Box_{\frac{1}{2}} := |g|^{\frac{1}{4}}\Box|g|^{-\frac{1}{4}} = |g|^{-\frac{1}{4}}\partial_\mu|g|^{\frac{1}{2}}g^{\mu\nu}\partial_\nu|g|^{-\frac{1}{4}}. \tag{3.2}$$

In the half-density formalism the space $L^2(M, |g|^{\frac{1}{2}})$ is replaced by $L^2(M)$, where we just take the Lebesgue measure with respect to given coordinates. We will write $\Box$ for $\Box_{\frac{1}{2}}$ when it is clear from the context that we use the half-density formalism. See e.g. [36].

3.2 Classical propagators

Suppose that M is globally hyperbolic. It is well-known that there exist unique fundamental solutions $G^\vee(x,y)$ and $G^\wedge(x,y)$ of the Klein-Gordon equation which have future- respectively past-directed causal support:

$$\begin{aligned} (x,y) \in \operatorname{supp} G^\vee \quad &\Rightarrow \quad \exists \text{ causal curve connecting } x \text{ and } y \text{ and } x^0 - y^0 \geq 0, \\ (x,y) \in \operatorname{supp} G^\wedge \quad &\Rightarrow \quad \exists \text{ causal curve connecting } x \text{ and } y \text{ and } x^0 - y^0 \leq 0. \end{aligned} \tag{3.3}$$

$G^\vee(x,y)$ is called the *forward* (or *retarded*) *propagator*, $G^\wedge(x,y)$ is called the *backward* (or *advanced*) *propagator*. Their difference, which obviously is a bisolution of the Klein-Gordon equation, is called the *Pauli-Jordan propagator* (or *commutator function)*

$$G^{\mathrm{PJ}}(x,y) := G^\vee(x,y) - G^\wedge(x,y). \tag{3.4}$$

These three propagators are sometimes called jointly *classical propagators*.

3.3 Quantum fields and non-classical propagators

We still assume that M is globally hyperbolic. Consider a real scalar quantum field $\hat\phi(x)$ on M satisfying

$$\begin{aligned} \big(-\Box + Y(x)\big)\hat\phi(x) &= 0, \\ [\hat\phi(x), \hat\phi(y)] &= -\mathrm{i}G^{\mathrm{PJ}}(x,y)\mathbb{1}. \end{aligned} \tag{3.5}$$

Identify M with $\mathbb{R} \times \Sigma$, where $x^0 \in \mathbb{R}$ can be interpreted as time. Then we introduce the chronological and antichronological time ordering by

$$\mathrm{T}\left(\hat\phi(x)\hat\phi(y)\right) := \begin{cases} \hat\phi(x)\hat\phi(y), & x^0 > y^0, \\ \hat\phi(y)\hat\phi(x), & y^0 > x^0, \end{cases} \tag{3.6}$$

$$\overline{\mathrm{T}}\big(\hat\phi(x)\hat\phi(y)\big) := \begin{cases} \hat\phi(y)\hat\phi(x), & x^0 > y^0, \\ \hat\phi(x)\hat\phi(y), & y^0 > x^0. \end{cases} \tag{3.7}$$

Note that this definition does not depend on the choice of coordinates, because of the Einstein causality of the field $\hat{\phi}(x)$.

Suppose that we have a Fock representation of the fields with the vacuum Ω_α. We define the 2-point functions $G_\alpha^{(+)}$, $G_\alpha^{(-)}$, G_α^{F} and $G_\alpha^{\overline{\mathrm{F}}}$ as in (1.6), (1.7), (1.12) and (1.9) in the introduction. Note that

$$G_\alpha^{\mathrm{F}}(x,y) := \mathrm{i}\Big(\theta(x^0-y^0)G_\alpha^{(+)}(x,y) + \theta(y^0-x^0)G_\alpha^{(-)}(x,y)\Big), \tag{3.8}$$

$$G_\alpha^{\overline{\mathrm{F}}}(x,y) := -\mathrm{i}\Big(\theta(x^0-y^0)G_\alpha^{(-)}(x,y) + \theta(y^0-x^0)G_\alpha^{(+)}(x,y)\Big), \tag{3.9}$$

provided that one can properly interpret the products of distributions on the right-hand side.

Sometimes, it is also useful to consider the symmetric two-point function

$$G_\alpha^{\mathrm{sym}}(x,x') := G_\alpha^{(+)}(x,x') + G_\alpha^{(-)}(x,x'). \tag{3.10}$$

so that $G_\alpha^{(\pm)} = \frac{1}{2}\big(G_\alpha^{\mathrm{sym}} \mp \mathrm{i}G^{\mathrm{PJ}}\big)$.

More generally, suppose we have two Fock represention with the vacua Ω_α and Ω_β. We tacitly assume that these two representations are unitarily equivalent, and $\langle\Omega_\alpha|\Omega_\beta\rangle \neq 0$. We define the 2-point functions $G_{\alpha,\beta}^{(+)}$, $G_{\alpha,\beta}^{(-)}$, the Feynman propagator $G_{\alpha,\beta}^{\mathrm{F}}$ and the anti-Feynman propagator $G_{\alpha,\beta}^{\overline{\mathrm{F}}}$ as in (1.10), (1.11), (1.12) and (1.9) from the introduction. Similarly, we have

$$G_{\alpha,\beta}^{\mathrm{F}}(x,y) := \mathrm{i}\Big(\theta(x^0-y^0)G_{\alpha,\beta}^{(+)}(x,y) + \theta(y^0-x^0)G_{\alpha,\beta}^{(-)}(x,y)\Big), \tag{3.11}$$

$$G_{\alpha,\beta}^{\overline{\mathrm{F}}}(x,y) := -\mathrm{i}\Big(\theta(x^0-y^0)G_{\alpha,\beta}^{(-)}(x,y) + \theta(y^0-x^0)G_{\alpha,\beta}^{(+)}(x,y)\Big), \tag{3.12}$$

again provided that one can properly interpret the products of distributions on the right-hand side.

Note that all these definitions do not work if $\langle\Omega_\alpha \mid \Omega_\beta\rangle = 0$ (which is actually quite common). Later we will see how to avoid this problem.

Proposition 3.1. *$G_{\alpha,\beta}^{(+)}$ and $G_{\alpha,\beta}^{(-)}$ are bisolutions of the Klein-Gordon equation. The Feynman propagator $G_{\alpha,\beta}^{\mathrm{F}}$ and the anti-Feynman propagator $G_{\alpha,\beta}^{\overline{\mathrm{F}}}$ are Green functions of the Klein-Gordon equation. They satisfy the following identities:*

$$G_{\alpha,\beta}^{\mathrm{F}} + G_{\alpha,\beta}^{\overline{\mathrm{F}}} = G^{\vee} + G^{\wedge}, \tag{3.13a}$$

$$G_{\alpha,\beta}^{\mathrm{F}} - G_{\alpha,\beta}^{\overline{\mathrm{F}}} = \mathrm{i}\Big(G_{\alpha,\beta}^{(+)} + G_{\alpha,\beta}^{(-)}\Big), \tag{3.13b}$$

$$G^{\mathrm{PJ}} = G^{\vee} - G^{\wedge} = \mathrm{i}\Big(G_{\alpha,\beta}^{(+)} - G_{\alpha,\beta}^{(-)}\Big), \tag{3.13c}$$

$$G_{\alpha,\beta}^{\mathrm{F}} = \mathrm{i}G_{\alpha,\beta}^{(+)} + G^{\wedge} = \mathrm{i}G_{\alpha,\beta}^{(-)} + G^{\vee}, \tag{3.13d}$$

$$G_{\alpha,\beta}^{\overline{\mathrm{F}}} = -\mathrm{i}G_{\alpha,\beta}^{(+)} + G^{\vee} = -\mathrm{i}G_{\alpha,\beta}^{(-)} + G^{\wedge}. \tag{3.13e}$$

Proof. The idendities (3.13c) follow from the definition of the involved propagators. (3.13a) and (3.13b) follows from the definition of the Feynman and anti-Feynman propagator via (anti-)time-ordering as in (3.11) and (3.12), from (3.13c) and the support properties of the advanced and retarded propagators. (3.13d) and (3.13e) are obtained by inserting the other identities into each other in various ways. □

3.4 Propagators as Klein-Gordon kernels

Here is an alternative, more satisfactory definition of non-classical propagators, which is based entirely on operator theory, without going through quantum fields.

For $\zeta, \xi \in C^\infty(M)$ set

$$\overline{\zeta(x)}\overleftrightarrow{\nabla}_\mu \xi(x) := \big(\nabla_\mu \overline{\zeta(x)}\big)\xi(x) - \overline{\zeta(x)}\nabla_\mu \xi(x). \tag{3.14}$$

Let $\mathcal{W}_{\rm sc}$ denote the space of smooth, space-compact solutions to the Klein-Gordon equation

$$\big(-\Box + Y(x)\big)\zeta(x) = 0. \tag{3.15}$$

We have $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$, where the left ∇ is the covariant derivative on covectors, and the right ∇ on scalars.[5] Therefore, if $\zeta, \xi \in \mathcal{W}_{\rm sc}$, then

$$J_\mu[\zeta,\xi](x) := \overline{\zeta(x)}\overleftrightarrow{\nabla}_\mu \xi(x) = \big(\nabla_\mu \overline{\zeta(x)}\big)\xi(x) - \overline{\zeta(x)}\nabla_\mu \xi(x) \tag{3.16}$$

is a covariantly conserved current, which means

$$|g|^{-\frac{1}{2}}\partial_\mu |g|^{\frac{1}{2}} J_\mu = \nabla^\mu J_\mu = 0, \tag{3.17}$$

Therefore,

$$(\zeta, \xi)_{\rm KG} := \mathrm{i}\int_\Sigma \overline{\zeta(x)}\overleftrightarrow{\nabla}_\mu \xi(x) \mathrm{d}\Sigma^\mu(x), \tag{3.18}$$

does not depend on the choice of the Cauchy surface Σ, where $\mathrm{d}\Sigma^\mu(x)$ is the natural measure on Σ times the future-directed normal vector. (3.18) is called the Klein-Gordon charge form.

The Klein-Gordon charge form is not positive definite. We can, however, usually extend the space $\mathcal{W}_{\rm sc}$ to a larger space, denoted $\mathcal{W}_{\rm KG}$, which admits a direct sum decomposition

$$\mathcal{W}_{\rm KG} = \mathcal{Z}_\alpha^{(+)} \oplus \mathcal{Z}_\alpha^{(-)}, \quad \mathcal{Z}_\alpha^{(+)} = \overline{\mathcal{Z}_\alpha^{(-)}}, \tag{3.19}$$

where the components are orthogonal with respect to $(\cdot|\cdot)_{\rm KG}$, and where $\mathcal{Z}_\alpha^{(\pm)}$ is maximal uniformly positive/negative. Every $\zeta \in \mathcal{W}_{\rm sc}$, decomposed according to (3.19) as

$$\zeta = \zeta_\alpha^{(+)} + \zeta_\alpha^{(-)}, \quad \zeta_\alpha^{(+)} = \overline{\zeta_\alpha^{(-)}}, \tag{3.20}$$

[5]Note that when acting on scalars, we may replace ∇ by ∂. However, the used notation is standard.

satisfies

$$\pm(\zeta_\alpha^{(\pm)}|\zeta_\alpha^{(\pm)})_{\rm KG}\geq 0,\quad (\zeta_\alpha^{(\pm)}|\zeta_\alpha^{(\mp)})_{\rm KG}=0. \tag{3.21}$$

Thus

$$(\zeta|\xi)_{\rm KG}=(\zeta_\alpha^{(+)}|\xi_\alpha^{(+)})_{\rm KG}+(\zeta_\alpha^{(-)}|\xi_\alpha^{(-)})_{\rm KG}. \tag{3.22}$$

The index α indicates the decomposition (3.19). We also have a positive definite scalar product

$$(\zeta|\xi)_\alpha=(\zeta_\alpha^{(+)}|\xi_\alpha^{(+)})_{\rm KG}-(\zeta_\alpha^{(-)}|\xi_\alpha^{(-)})_{\rm KG}, \tag{3.23}$$

which is however less canonical than the Klein-Gordon charge form because it depends on the decomposition (3.19). It is natural to assume that $\mathcal{W}_{\rm KG}$ is complete in the topology given by (3.23), and that $\mathcal{W}_{\rm sc}$ is dense in $\mathcal{W}_{\rm KG}$.

It is important to note that there are many decompositions of the form (3.23) with properties as above leading to the same space $\mathcal{W}_{\rm KG}$. In fact, probably in typical spacetimes all physically reasonable decompositions lead to the same $\mathcal{W}_{\rm KG}$. Clearly, not all elements of $\mathcal{W}_{\rm KG}$ are space-compact, but they decay at an appropriate rate in spatial directions.

Mathematically, $\mathcal{W}_{\rm KG}$ has the structure of a *Krein space*. A decomposition (3.19) is an example of a *fundamental decomposition* of a Krein space. Let $\Pi_\alpha^{(\pm)}$ be the orthogonal projections onto $\mathcal{Z}_\alpha^{(\pm)}$. Denoting by $\mathcal{N}$ the nullspace and by $\mathcal{R}$ the range, we thus have

$$\begin{aligned}\mathcal{N}(\Pi_\alpha^{(\pm)})=\mathcal{Z}_\alpha^{(\mp)},\quad \mathcal{R}(\Pi_\alpha^{(\pm)})=\mathcal{Z}_\alpha^{(\pm)},\\ \big(\Pi_\alpha^{(\pm)}\big)^2=\Pi_\alpha^{(\pm)},\quad \Pi_\alpha^{(\pm)}\Pi_\alpha^{(\mp)}=0,\\ \big(\Pi_\alpha^{(\pm)}\zeta,\Pi_\alpha^{(\pm)}\zeta\big)\gtrless 0,\quad \zeta\in\mathcal{W}_{\rm KG}\\ \big(\Pi_\alpha^{(\pm)}\zeta,\xi\big)=\big(\zeta,\Pi_\alpha^{(\pm)}\xi\big),\quad \zeta,\xi\in\mathcal{W}_{\rm KG}.\end{aligned} \tag{3.24}$$

Definition 3.2. Let $B(x,y)$ be a bisolution of the Klein-Gordon equation. Then it defines a linear operator B on $\mathcal{W}_{\rm KG}$ via

$$B\zeta(x):={\rm i}\int_\Sigma B(x,y)\overset{\leftrightarrow}{\nabla}_{y^\mu}\zeta(y){\rm d}\Sigma^\mu(y). \tag{3.25}$$

The function or distribution $B(x,y)$ will be called the *Klein-Gordon kernel* of B.

A decomposition of the form (3.19) has also a physical meaning. Let $\pm G_\alpha^{(\pm)}(x,y)$ be the Klein-Gordon kernel of the projection $\Pi_\alpha^{(\pm)}$, so that the sum $G_\alpha^{\rm sym}(x,y):=G_\alpha^{(+)}(x,y)+G_\alpha^{(-)}(x,y)$ is the Klein-Gordon kernel of the involution $S_\alpha:=\Pi_\alpha^{(+)}-\Pi_\alpha^{(-)}$. Then there exists a Fock representation with the Fock vacuum Ω_α such that $G_\alpha^{(\pm)}(x,y)$ are the corresponding two-point functions

$$G_\alpha^{(+)}(x,y):=\langle\Omega_\alpha\mid\hat\phi(x)\hat\phi(y)\mid\Omega_\alpha\rangle, \tag{3.26}$$

$$G_\alpha^{(-)}(x,y):=\langle\Omega_\alpha\mid\hat\phi(y)\hat\phi(x)\mid\Omega_\alpha\rangle. \tag{3.27}$$

Let $\mathcal{W}_{\rm KG} = \mathcal{Z}_\beta^{(+)} \oplus \mathcal{Z}_\beta^{(-)}$ be another orthogonal decomposition of the Krein space $\mathcal{W}_{\rm KG}$ into a maximal uniformly positive and maximal uniformly negative subspace, defining the vacuum Ω_β. One can show [36] (see also Appendix A) that the spaces $\mathcal{Z}_\beta^{(+)}$ and $\mathcal{Z}_\alpha^{(-)}$ are complementary, so that we have a (non-orthogonal) direct sum decomposition

$$\mathcal{W}_{\rm KG} = \mathcal{Z}_\beta^{(+)} \oplus \mathcal{Z}_\alpha^{(-)}. \tag{3.28}$$

Therefore, we can define projections $\Pi_{\alpha,\beta}^{(+)}$, $\Pi_{\alpha,\beta}^{(-)}$ corresponding to this decomposition satisfying

$$\begin{aligned} \mathcal{N}(\Pi_{\alpha,\beta}^{(+)}) = \mathcal{R}(\Pi_{\alpha,\beta}^{(-)}) &= \mathcal{Z}_\beta^{(-)}, \\ \mathcal{R}(\Pi_{\alpha,\beta}^{(+)}) = \mathcal{N}(\Pi_{\alpha,\beta}^{(-)}) &= \mathcal{Z}_\alpha^{(+)}, \\ \Pi_{\alpha,\beta}^{(+)} + \Pi_{\alpha,\beta}^{(-)} &= \mathbb{1}. \end{aligned} \tag{3.29}$$

We may also decompose $\mathcal{W}_{\rm KG}$ the other way around,

$$\mathcal{W}_{\rm KG} = \mathcal{Z}_\alpha^{(+)} \oplus \mathcal{Z}_\beta^{(-)}. \tag{3.30}$$

The corresponding projections are denoted $\Pi_{\beta,\alpha}^{(+)}$, $\Pi_{\beta,\alpha}^{(-)}$. Then one finds

$$\big(\Pi_{\alpha,\beta}^{(\pm)}\zeta|\xi\big)_{\rm KG} = \big(\zeta|\Pi_{\beta,\alpha}^{(\pm)}\xi\big)_{\rm KG}. \tag{3.31}$$

Thus, these projections are orthogonal if and only if $\mathcal{Z}_\alpha^{(+)} = \mathcal{Z}_\beta^{(+)}$.

Let R be a bounded linear transformation on $\mathcal{W}_{\rm KG}$. The Klein-Gordon Hermitian conjugate $R^{*{\rm KG}}$ of R is defined by

$$(R^{*{\rm KG}}\zeta|\xi)_{\rm KG} := (\zeta|R\xi)_{\rm KG}, \tag{3.32}$$

and the complex conjugate $\overline{R}$ by

$$\overline{R}\,\overline{\zeta} := \overline{R\zeta}. \tag{3.33}$$

Linear transformations that preserve the structure of $\mathcal{W}_{\rm KG}$ are called *symplectic*, or (especially in the physics literature) *Bogoliubov transformations*. Here, R preserving the structure of $\mathcal{W}_{\rm KG}$ means that R is *pseudounitary* and *real*, i.e.,

$$(R\zeta|R\xi)_{\rm KG} = (\zeta|\xi)_{\rm KG} \quad \text{and} \quad R\overline{\zeta} = \overline{R\zeta}, \tag{3.34}$$

or in other words $R^{*{\rm KG}} = R^{-1}$ and $\overline{R} = R$.

Proposition 3.3. *Let R be a pseudounitary transformation. If K is a linear operator on $\mathcal{W}_{\rm KG}$ with Klein-Gordon kernel $G(x,y)$, then the Klein-Gordon kernel of the operator RKR^{-1} is obtained by applying $R \otimes \overline{R}$ to $G(\cdot,\cdot)$.*

Proof. We will drop the subscript KG in the proof, since we will use only the Klein-Gordon form. It is enough to consider K of the form $|f)(g|$, that is

$$(\zeta|K\xi) = (\zeta|f)(g|\xi), \tag{3.35}$$

with the Klein-Gordon kernel $f \otimes \overline{g}$. It is easy to compute that

$$RKR^{-1} = |Rf)(R^{-1*{\rm KG}}g| = |Rf)(Rg|. \tag{3.36}$$

Thus, the Klein-Gordon kernel of RKR^{-1} is given by $Rf \otimes \overline{Rg} = (R \otimes \overline{R})\,(f \otimes \overline{g})$. □

3.5 Mode expansions

In the physics literature, the construction described in the previous subsection is often implemented as follows.

One starts by assuming that the classical field can be written as

$$\phi(x) = \int \big(\overline{\varphi_{\alpha,k}(x)} a_{\alpha,k} + \varphi_{\alpha,k}(x) a_{\alpha,k}^*\big) \mathrm{d}k, \tag{3.37}$$

where $\varphi_{\alpha,k}$ are mode functions spanning $\mathcal{Z}_\alpha^{(+)}$. They satisfy

$$\begin{aligned} \big(\varphi_{\alpha,k}, \varphi_{\alpha,k'}\big)_{\mathrm{KG}} = -\big(\overline{\varphi_{\alpha,k}}, \overline{\varphi_{\alpha,k'}}\big)_{\mathrm{KG}} &= -\delta(k-k'), \\ \big(\varphi_{\alpha,k}, \overline{\varphi_{\alpha,k'}}\big)_{\mathrm{KG}} &= 0, \\ (-\Box_g + m^2)\varphi_{\alpha,k}(x) &= 0. \end{aligned} \tag{3.38}$$

The variable k is here merely a parametrization of “generalized eigenfunctions” of the d’Alembertian. In Minkowski space, it usually coincides with the $d-1$-momentum, which is not available on a generic spacetime. If the Cauchy surfaces of the spacetime are compact, the set of k is discrete, and the Dirac delta must be replaced by the Kronecker delta and mode integrals by mode sums.

Remark 3.4. Many papers about QFT on curved spacetimes do not mention the word “Krein space”. Instead they introduce a decomposition of solutions to the Klein-Gordon equation into a “positive frequency part” and a “negative frequency part”. This is usually done through modes, as in (3.37). However, if it is possible to represent the fields in terms of orthogonal modes as above, then we automatically fix an admissible involution (cf. Appendix A, in particular Definition A.15), and the space consisting of square integrable “wave packets” of these modes is automatically a Krein space. Thus the idea of a Krein spaces is introduced in many papers “through the back door”.

After quantization we obtain

$$\hat{\phi}(x) = \int \big(\overline{\varphi_{\alpha,k}(x)} \hat{a}_{\alpha,k} + \varphi_{\alpha,k}(x) \hat{a}_{\alpha,k}^*\big) \mathrm{d}k, \tag{3.39}$$

$$[\hat{a}_{\alpha,k}, \hat{a}_{\alpha,k'}^*] = \delta(k-k'), \quad [\hat{a}_{\alpha,k}, \hat{a}_{\alpha,k'}] = 0. \tag{3.40}$$

Then

$$\begin{aligned} G_\alpha^{(+)}(x,y) &= \int \overline{\varphi_{\alpha,k}(x)} \varphi_{\alpha,k}(y) \mathrm{d}k, \\ G_\alpha^{(-)}(x,y) &= \int \varphi_{\alpha,k}(x) \overline{\varphi_{\alpha,k}(y)} \mathrm{d}k. \end{aligned} \tag{3.41}$$

Now assume that we have a second state Ω_β and the corresponding decomposition

$$\hat{\phi}(x) = \int \big(\overline{\varphi_{\beta,k}(x)} \hat{a}_{\beta,k} + \varphi_{\beta,k}(x) \hat{a}_{\beta,k}^*\big) \mathrm{d}k, \tag{3.42}$$

$$[\hat{a}_{\beta,k}, \hat{a}_{\beta,k'}^*] = \delta(k-k'), \quad [\hat{a}_{\beta,k}, \hat{a}_{\beta,k'}] = 0. \tag{3.43}$$

Let us assume that the two decompositions are related by a Bogoliubov transformation

$$\begin{aligned}\varphi_{\beta,k}(x) &= N_{\alpha\beta}(k)\varphi_{\alpha,k}(x) + \int \Lambda_{\alpha,\beta}(k,k')\overline{\varphi_{\alpha,k'}(x)}\mathrm{d}k', \\ \hat{a}_{\beta,k} &= N_{\alpha\beta}(k)\hat{a}_{\alpha,k} - \int \Lambda_{\alpha,\beta}(k,k')\hat{a}^*_{\alpha,k'}\mathrm{d}k',\end{aligned} \tag{3.44}$$

and $\overline{\varphi_{\beta,k}(x)}$ resp. $a^*_{\beta,k}$ are obtained by taking the complex resp. hermitian conjugate of the right-hand side of (3.44). Here, $N_{\alpha\beta}(k) = \overline{N_{\beta\alpha}(k)}$ and the functions (or distributions) $\Lambda_{\alpha,\beta}(k,k')$ satisfy

$$\begin{aligned}\Lambda_{\alpha,\beta}(k,k') = -\Lambda_{\beta,\alpha}(k,k'), \quad \Lambda_{\alpha,\beta}(k,k') = \Lambda_{\alpha,\beta}(k',k), \\ \int \Lambda_{\alpha,\beta}(k,p)\overline{\Lambda_{\alpha,\beta}(k',p)}\mathrm{d}p = \big(|N_{\alpha\beta}(k)|^2 - 1\big)\delta(k-k').\end{aligned} \tag{3.45}$$

The conditions (3.45) ensure that the transformed mode functions satisfy the same pseudo-orthogonality relations and the transformed creation and annihilation operators satisfy the same commutation relations as their counterparts before the transformation, that the field expansion looks the same for any α, and that applying two transformations $\alpha \to \beta \to \alpha$ yields the identity.

Now the mixed propagators expressed in terms of modes are

$$\begin{aligned}G^{(+)}_{\alpha\beta}(x,y) &= \int \frac{1}{N_{\alpha\beta}(k)}\overline{\varphi_{\alpha,k}(x)}\varphi_{\beta,k}(y)\mathrm{d}k, \\ G^{(-)}_{\alpha\beta}(x,y) &= \int \frac{1}{N_{\alpha\beta}(k)}\overline{\varphi_{\alpha,k}(y)}\varphi_{\beta,k}(x)\mathrm{d}k.\end{aligned} \tag{3.46}$$

Let us check that the ranges of $\Pi^{(\pm)}_{\alpha\beta}$ remain maximal uniformly positive resp. maximal uniformly negative subspaces of $\mathcal{W}_{\mathrm{sc}}$. Expanding $\zeta \in \mathcal{W}_{\mathrm{sc}}$ into modes,

$$\zeta(x) = \int \varphi_{\alpha,k}(x)\zeta^{(-)}_{\alpha,k} + \overline{\varphi_{\alpha,k}(x)}\zeta^{(+)}_{\alpha,k}\mathrm{d}k, \tag{3.47}$$

and using (3.46), we find

$$\Pi^{(+)}_{\alpha\beta}\zeta(x) = \int \frac{\zeta^{(+)}_{\beta,p}}{N_{\alpha\beta}(p)}\,\overline{\varphi_{\alpha,p}(x)}\mathrm{d}p, \quad \Pi^{(-)}_{\alpha\beta}\zeta(x) = \int \frac{\zeta^{(-)}_{\alpha,p}}{N_{\alpha\beta}(p)}\,\varphi_{\beta,p}(x)\mathrm{d}p \tag{3.48}$$

Therefore,

$$\big(\Pi^{(+)}_{\alpha\beta}\zeta, \Pi^{(+)}_{\alpha\beta}\zeta\big) = \int \left|\frac{\zeta^{(+)}_{\alpha,p}}{N_{\alpha\beta}(p)}\right|^2 \mathrm{d}p, \quad \big(\Pi^{(-)}_{\alpha\beta}\zeta, \Pi^{(-)}_{\alpha\beta}\zeta\big) = -\int \left|\frac{\zeta^{(-)}_{\beta,p}}{N_{\alpha\beta}(p)}\right|^2 \mathrm{d}p. \tag{3.49}$$

3.6 Operator-theoretic (anti-)Feynman propagator

The d'Alembertian on a globally hyperbolic spacetime M with pseudometric g, in the half-density formalism given by

$$-\Box = -|\det g|^{-\frac{1}{4}}\partial_\mu g^{\mu\nu}|\det g|^{\frac{1}{2}}\partial_\nu|\det g|^{-\frac{1}{4}}, \tag{3.50}$$

is Hermitian (symmetric) in the sense of $L^2\big(M\big)$. The same is true for the Klein-Gordon operator $-\Box + Y(x)$ with a real potential Y. Assume that $-\Box + Y(x)$ is *essentially self-adjoint* (if not, choose a self-adjoint extension).

Then its resolvent $G(z) := (-\Box+Y(x)-z)^{-1}$ is well-defined for $z \in \mathbb{C}\setminus\mathbb{R}$. It possesses a distributional kernel $G(z; x, y)$. Suppose that there exists

$$G^{\mathrm{F}}_{\mathrm{op}}(x, y) := \lim_{\epsilon\searrow 0} G(+\mathrm{i}\epsilon; x, y) = (-\Box + Y(x) - \mathrm{i}\epsilon)^{-1}(x, y), \tag{3.51}$$

$$G^{\overline{\mathrm{F}}}_{\mathrm{op}}(x, y) := \lim_{\epsilon\searrow 0} G(-\mathrm{i}\epsilon; x, y) = (-\Box + Y(x) + \mathrm{i}\epsilon)^{-1}(x, y), \tag{3.52}$$

where we use the distributional limit. The distributions $G^{\mathrm{F}}_{\mathrm{op}}(x, x')$ and $G^{\overline{\mathrm{F}}}_{\mathrm{op}}(x, x')$ will be called the *operator-theoretic Feynman and anti-Feynman propagator*.

We expect that the limits (3.51) and (3.52) exist in most physically interesting situations. They will not exist at the point spectrum of $-\Box + Y(x)$ (which is probably quite rare).

We believe that the following argument justifies this definition. Here is an elementary fact about *Fresnel integrals*. Let c be a real symmetric $n \times n$ matrix, u a variable in $\mathbb{R}^n$ and $J \in \mathbb{R}^n$. Then

$$\frac{\int \mathrm{e}^{\pm\mathrm{i}(u^{\mathrm{T}}\frac{c}{2}u+J^{\mathrm{T}}u)}\mathrm{d}u}{\int \mathrm{e}^{\pm\mathrm{i}u^{\mathrm{T}}\frac{c}{2}u}\mathrm{d}u} = \exp\Big(\mp\frac{\mathrm{i}}{2}J^{\mathrm{T}}(c\pm\mathrm{i}0)^{-1}J\Big). \tag{3.53}$$

If we use *path integrals* to construct a quantum field theory, we usually start from defining formally the generating function as

$$Z(J) := \frac{\int \mathrm{e}^{\mathrm{i}S(\phi)+\mathrm{i}\int\phi(x)J(x)\mathrm{d}x}\mathcal{D}\phi}{\int \mathrm{e}^{\mathrm{i}S(\phi)}\mathcal{D}\phi}.$$

If the action is *quadratic*

$$\begin{aligned} S(\phi) &= -\frac{1}{2}\int \big(\partial_\mu\phi(x)\partial^\mu\phi(x) + m^2\phi(x)^2\big)\sqrt{|g|}(x)\mathrm{d}x \\ &= -\frac{1}{2}\big(\phi|(-\Box+m^2)\phi\big), \end{aligned}$$

then the path integral by analogy to (3.53) can be *rigorously defined* as

$$\begin{aligned} Z(J) &= \exp\frac{\mathrm{i}}{2}\big(J|(-\Box+m^2-\mathrm{i}0)^{-1}J\big) \\ &= \exp\Big(\frac{\mathrm{i}}{2}\int\int J(x)G^{\mathrm{F}}_{\mathrm{op}}(x,y)J(y)\sqrt{|g|}(x)\sqrt{|g|}(y)\mathrm{d}x\mathrm{d}y\Big). \end{aligned}$$

Spacetimes where the d'Alembertian is essentially self-adjoint include stationary spacetimes, Friedmann-Lemaître-Robertson-Walker (FLRW) spacetimes, $1+0$-dimensional spacetimes, de Sitter and the universal cover of the anti-deSitter space. Essential self-adjointness was also recently proven on a class of asymptotically Minkowski spacetimes [62, 74]. However, even on well-behaved spacetimes, essential self-adjointness is not always true [56].

Remark 3.5. Essential self-adjointness is destroyed if there are spacelike or timelike boundaries. The problem with spacelike boundries can be sometimes cured by imposing boundary conditions—we will see this in Section 7 about the universal cover of the anti-deSitter space. Time-like boundaries are different. In particular, if the time is confined to an interval $]a, b[$ instead of $\mathbb{R}$, then it is not appropriate to consider self-adjoint realizations of the Klein-Gordon operators to define Feynman propagators. Instead, one can consider other types of non-self-adjoint boundary conditions, as explained in [36].

3.7 Special Klein-Gordon equations

Definition 3.6. Suppose that the Klein-Gordon operator $-\Box+Y(x)$ on a Lorentzian manifold M is essentially self-adjoint. We say that $-\Box+Y(x)$ is *special* if the sum

$$G^{\mathrm{F}}_{\mathrm{op}}(x,x')+G^{\overline{\mathrm{F}}}_{\mathrm{op}}(x,x') \tag{3.54}$$

has causal support.

Definition 3.7. Suppose that the Klein-Gordon operator $-\Box+Y(x)$ is essentially self-adjoint and M is globally hyperbolic. We say that it is *strongly special* if

$$G^{\mathrm{F}}_{\mathrm{op}}(x,x')+G^{\overline{\mathrm{F}}}_{\mathrm{op}}(x,x')=G^{\vee}(x,x')+G^{\wedge}(x,x'). \tag{3.55}$$

Clearly, strong specialty implies specialty. We expect that under broad conditions the converse is also true.

Special Klein-Gordon operators are interesting because the associated propagators can be determined in an easy way. Indeed, it is often not very difficult to compute $G^{\mathrm{F}}_{\mathrm{op}}(x,x')$ and $G^{\overline{\mathrm{F}}}_{\mathrm{op}}(x,x')$. After all, there are various techniques to compute the kernel of the resolvent of a differential operator. From these, one can determine the retarded and advanced propagators by

$$G^{\vee/\wedge}(x,x')=\theta\big(\pm(x^0-x'^0)\big)\Big(G^{\mathrm{F}}_{\mathrm{op}}(x,x')+G^{\overline{\mathrm{F}}}_{\mathrm{op}}(x,x')\Big) \tag{3.56}$$

as well as the Pauli-Jordan function $G^{\mathrm{PJ}}=G^{\vee}-G^{\wedge}$.

Strictly speaking, (3.56) is not fully legal, because it involves multiplying a distribution by a discontinuous function $\theta\big(\pm(x^0-x'^0)\big)$. In practice, we expect that this obstacle can be overcome, see [37]. In particular, there is no problem with the multiplication with the theta function when we can apply the method of evolution equations, see Subsect. 4.2.

More interestingly, there is a natural candidate for the positive and negative frequency bisolutions of a distinguished state:

$$G^{(\pm)}:=-\mathrm{i}\big(G^{\mathrm{F}}_{\mathrm{op}}-G^{\wedge/\vee}\big)=\mathrm{i}\big(G^{\overline{\mathrm{F}}}_{\mathrm{op}}-G^{\vee/\wedge}\big). \tag{3.57}$$

Remark 3.8. Actually, we do not know if $G^{(\pm)}$ defined by (3.57) in the case when $-\Box+Y(x)$ is special always satisfy the positivity requirement — in all cases that we worked out they do.

4 Stationary and asymptotically stationary spaces

4.1 Propagators on stationary spacetimes

Assume that $M=\mathbb{R}\times\Sigma$, with the variables typically denoted by $(t,\mathbf{x})$, and sometimes $(s,\mathbf{y})$. Suppose that neither $g_{\alpha\beta}$ nor Y depends on time t, the time slices $\{t\}\times\Sigma$ are spacelike and ∂_t is timelike. Such spacetimes are called *stationary*.

In addition, we will assume that the spacetime is *static*, i.e. there are no time-position cross-terms. This is not a necessary condition for the present analysis, however for static spacetimes many formulas are more explicit. In other words, we assume that the metric is

$$-\alpha^2(\mathbf{x})\mathrm{d}t^2+h_{ij}(\mathbf{x})\mathrm{d}\mathbf{x}^i\mathrm{d}\mathbf{x}^j. \tag{4.1}$$

Here and in the following, Latin indices run over the spatial directions. We write $|h|$ for $\det h$. The Klein-Gordon operator in the half-density formalism is

$$-\Box+Y(\mathbf{x})=\frac{1}{\alpha^2}\partial_t^2-\alpha^{-\frac12}\,|h|^{-\frac14}\partial_i\alpha\,|h|^{\frac12}h^{ij}\partial_j\,\alpha^{-\frac12}\,|h|^{-\frac14}+Y. \tag{4.2}$$

It is convenient to replace (4.2) by

$$-\tilde{\Box}+\tilde{Y}:=\alpha\big(-\Box+Y\big)\alpha=\partial_t^2+L, \tag{4.3}$$

where

$$L:=-\Delta_{\tilde h}+\tilde Y,\quad \Delta_{\tilde h}:=\gamma^{-\frac12}\partial_i\gamma\tilde h^{ij}\partial_j\gamma^{-\frac12}, \tag{4.4}$$

$$\tilde h_{ij}:=\frac{1}{\alpha^2}h_{ij},\quad [\tilde h^{ij}]=[\tilde h_{ij}]^{-1},\quad \gamma:=\frac{|h|^{\frac12}}{\alpha},\quad \tilde Y=\alpha^2 Y.$$

Note that if $\tilde u$ solves

$$(-\tilde\Box+\tilde Y)\tilde u=0, \tag{4.5}$$

then $u:=\alpha^{-1}\tilde u$ solves

$$(-\Box+Y)u=0. \tag{4.6}$$

Let us first describe the approach based on the evolution of Cauchy data, which is particularly simple for static spacetimes. The equation (4.5) for $u(t)=u(t,\mathbf{x})$ can be rewritten as a 1st order uquation for the Cauchy data

$$\big(\partial_t+\mathrm{i}B\big)w=0, \tag{4.7}$$

$$B:=\begin{bmatrix}0 & \mathbb{1}\\ L & 0\end{bmatrix},\qquad w=\begin{bmatrix}w_1\\ w_2\end{bmatrix}:=\begin{bmatrix}u\\ \mathrm{i}\partial_t u\end{bmatrix}. \tag{4.8}$$

Assume that L is positive and self-adjoint.

We can compute the evolution operator and the spectral projection of B onto the positive and negative part of the spectrum:

$$\mathrm{e}^{-\mathrm{i}tB} := \begin{bmatrix} \cos t\sqrt{L} & -\mathrm{i}\frac{\sin t\sqrt{L}}{\sqrt{L}} \\ -\mathrm{i}\sqrt{L}\sin t\sqrt{L} & \cos t\sqrt{L} \end{bmatrix}. \tag{4.9}$$

$$\Pi^{(\pm)} := 1\!\!1_{\mathbb{R}_+}(\pm B) = \frac{1}{2}\begin{bmatrix} 1\!\!1 & \pm\frac{1\!\!1}{\sqrt{L}} \\ \pm\sqrt{L} & 1\!\!1 \end{bmatrix}. \tag{4.10}$$

We assume that 0 is not an eigenvalue of L and endow the space of Cauchy data with the (positive) scalar product

$$(w|v)_0 := (w_1|\sqrt{L}v_1) + \big(w_2\big|\tfrac{1}{\sqrt{L}}v_2\big). \tag{4.11}$$

The completion of $\mathcal{W}_{\mathrm{sc}}$ with respect to this scalar product will be denoted $\mathcal{W}_0$. Note that B is Hermitian with respect to this scalar product:

$$(Bw|v)_0 = (w|Bv)_0 = (w_2|\sqrt{L}v_1) + (w_1|\sqrt{L}v_2). \tag{4.12}$$

The space $\mathcal{W}_0$ is also endowed with the (indefinite) Klein-Gordon charge form

$$(w|v)_{\mathrm{KG}} = (w|Qv) := (w_1|v_2) + (w_2|v_1), \quad Q = \begin{bmatrix} 0 & 1\!\!1 \\ 1\!\!1 & 0 \end{bmatrix}. \tag{4.13}$$

Note that the evolution $\mathrm{e}^{-\mathrm{i}tB}$ preserves the Klein-Gordon charge form (4.13). Therefore, $\mathcal{R}(\Pi^{(\pm)})$ are maximal uniformly positive/negative subspaces with respect to the Klein-Gordon charge form. Then we can define the propagators on the level of the Cauchy data as follows:

$$\begin{aligned} E^{\mathrm{PJ}}(t,s) &:= \mathrm{e}^{-\mathrm{i}(t-s)B}, \\ E^{\vee/\wedge}(t,s) &:= \pm\theta\big(\pm(t-s)\big)\mathrm{e}^{-\mathrm{i}(t-s)B}, \\ E^{(\pm)}(t,s) &:= \mathrm{e}^{-\mathrm{i}(t-s)B}\Pi^{(\pm)}, \\ E^{\mathrm{F}/\overline{\mathrm{F}}}(t,s) &:= \mathrm{e}^{-\mathrm{i}(t-s)B}\big(\theta(t-s)\Pi^{(\pm)} - \theta(s-t)\Pi^{(\mp)}\big). \end{aligned}$$

At least formally, $E^{\vee}$, $E^{\wedge}$, E^{F}, $E^{\overline{\mathrm{F}}}$ are inverses and E^{PJ}, $E^{(+)}$, $E^{(-)}$ are bisolutions of $\partial_t + \mathrm{i}B$. They are 2×2 matrices:

$$E^{\bullet}(t,s) = \begin{bmatrix} E^{\bullet}_{11}(t,s) & E^{\bullet}_{12}(t,s) \\ E^{\bullet}_{21}(t,s) & E^{\bullet}_{22}(t,s) \end{bmatrix}.$$

We set

$$\begin{aligned} G^{\bullet} &:= \mathrm{i}\alpha^{-1}E^{\bullet}_{12}\alpha^{-1}, \quad \bullet = \mathrm{PJ}, \vee, \wedge, \mathrm{F}, \overline{\mathrm{F}}, \\ G^{(\pm)} &:= \pm\alpha^{-1}E^{(\pm)}_{12}\alpha^{-1}, \end{aligned} \tag{4.14}$$

obtaining propagators for a general stationary stable case. Thus

$$G^{\mathrm{PJ}}(t,\mathbf{x};s,\mathbf{y}) = \frac{1}{\alpha(\mathbf{x})}\frac{\sin(t-s)\sqrt{L}}{\sqrt{L}}(\mathbf{x},\mathbf{y})\frac{1}{\alpha(\mathbf{y})}, \tag{4.15}$$

$$G^{\vee/\wedge}(t,\mathbf{x};s,\mathbf{y}) = \pm\theta\big(\pm(t-s)\big)\frac{1}{\alpha(\mathbf{x})}\frac{\sin(t-s)\sqrt{L}}{\sqrt{L}}(\mathbf{x},\mathbf{y})\frac{1}{\alpha(\mathbf{y})}, \tag{4.16}$$

$$G^{(\pm)}(t,\mathbf{x};s,\mathbf{y}) = \frac{1}{\alpha(\mathbf{x})}\frac{\mathrm{e}^{\mp\mathrm{i}(t-s)\sqrt{L}}}{2\sqrt{L}}(\mathbf{x},\mathbf{y})\frac{1}{\alpha(\mathbf{y})}, \tag{4.17}$$

$$G^{\mathrm{F}/\overline{\mathrm{F}}}(t,\mathbf{x};s,\mathbf{y}) = \pm\mathrm{i}\frac{1}{\alpha(\mathbf{x})}\left(\theta(t-s)\frac{\mathrm{e}^{\mp\mathrm{i}(t-s)\sqrt{L}}}{2\sqrt{L}} + \theta(s-t)\frac{\mathrm{e}^{\pm\mathrm{i}(t-s)\sqrt{L}}}{2\sqrt{L}}\right)(\mathbf{x},\mathbf{y})\frac{1}{\alpha(\mathbf{y})}. \tag{4.18}$$

Note that all the identities (3.13) still hold. In particular, the specialty condition is true:

$$G^{\mathrm{F}} + G^{\overline{\mathrm{F}}} = G^{\vee} + G^{\wedge}, \tag{4.19}$$

Let us describe now the approach based on the Hilbert space $L^2(M)$. We assume that L is essentially self-adjoint on $C_{\mathrm{c}}^{\infty}(\Sigma)$ in the sense of $L^2(\Sigma)$ Then it is easy to see $\partial_t^2 + L$ is essentially self-adjoint on $C_{\mathrm{c}}^{\infty}(M)$ in the sense of $L^2(M)$. Assume that for some $0 < c, C$ we have $c \leq \alpha \leq C$. Then $\alpha(\mathbf{x})$ is a bounded invertible on $L^2(M)$, and using this we can show that $-\Box + Y(\mathbf{x})$ is essentially self-adjoint on $\alpha(\mathbf{x})C_{\mathrm{c}}^{\infty}(M)$. As proven in [34], under some minor additional technical conditions we can then define $G_{\mathrm{op}}^{\mathrm{F}}$ and $G_{\mathrm{op}}^{\overline{\mathrm{F}}}$ and they coincide with G^{F} and $G^{\overline{\mathrm{F}}}$ computed from the evolution equation.

Note that the stability condition $L \geq 0$ was an important condition of the analysis based on the evolution equation. Suppose now that L is not positive, but only self-adjoint, which can be called the *tachyonic case*. In the tachyonic case, the formulas (4.15) and (4.16) for the classical propagators $G^{\mathrm{PJ}}, G^{\vee}, G^{\wedge}$ are still true. However, the evolution approach does not allow us to define $G^{(\pm)}$, G^{F} or $G^{\overline{\mathrm{F}}}$. We can then define the operator-theoretic $G_{\mathrm{op}}^{\mathrm{F}}$ or $G_{\mathrm{op}}^{\overline{\mathrm{F}}}$, which however are not given by the formula (4.18). Note that the specialty condition (4.19) is no longer true in the tachyonic case.

For instance, in the Minkowski space, with $Y(x) = m^2 < 0$, $G_{\mathrm{op}}^{\mathrm{F}}$ and $G_{\mathrm{op}}^{\overline{\mathrm{F}}}$ are well-behaved tempered distributions while the forward and backward propagators have exponential growth as $t \to \pm\infty$ inside $V^{\pm}$. A detailed discussion can for example be found in [37].

4.2 Classical propagators from evolution equations

Let us now consider a generic (not necessarily stationary) globally hyperbolic spacetime M. In order to compute non-classical (actually, also classical) propagators, it is useful to convert the Klein–Gordon equation into a 1st order evolution equation on the phase space describing Cauchy data. To this end, we fix a decomposition $M = \mathbb{R} \times \Sigma$. We assume that $\{t\} \times \Sigma$ is Riemannian for all $t \in \mathbb{R}$ and ∂_t is always timelike. We will use Latin letters for spatial indices. We introduce

$$
\begin{aligned}
h = [h_{ij}] = [g_{ij}], \quad h^{-1} = [h^{ij}],\\
\beta^j := g_{0i} h^{ij}, \quad \alpha^2 := g_{0i} h^{ij} g_{j0} - g_{00},\\
|h| = |\det h| = \det h, \quad |g| = |\det g|.
\end{aligned}
$$

$\alpha^2 > 0$. In coordinates, the metric can be written as

$$
g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu = -\alpha^2 \mathrm{d}t^2 + h_{ij}(\mathrm{d}x^i + \beta^i \mathrm{d}t)(\mathrm{d}x^j + \beta^j \mathrm{d}t),
$$

for some $\alpha(x) > 0$ and $[\beta^i(x)]$. We have $|g| = \alpha^2 |h|$. The Klein–Gordon operator in the half-density formalism can now be written

$$
\begin{aligned}
-\Box + Y(x) = {} & |g|^{-\frac{1}{4}} (\partial_t - \partial_i \beta^i) \frac{|g|^{\frac{1}{2}}}{\alpha^2} (\partial_t - \beta^j \partial_j) |g|^{-\frac{1}{4}} \\
& - |g|^{-\frac{1}{4}} \partial_i |g|^{\frac{1}{2}} h^{ij} \partial_j |g|^{-\frac{1}{4}} + Y.
\end{aligned}
$$

Instead of the operator $\Box$ on $L^2(M)$, it is more convenient to work with the operator

$$
\tilde{\Box} := \alpha \Box \alpha.
$$

It can be written as

$$
\begin{aligned}
-\tilde{\Box} + \alpha^2 Y &= \gamma^{-\frac{1}{2}} (\partial_t - \partial_i \beta^i) \gamma (\partial_t - \beta^j \partial_j) \gamma^{-\frac{1}{2}} \\
&\quad - \gamma^{-\frac{1}{2}} \partial_i \alpha^2 \gamma h^{ij} \partial_j \gamma^{-\frac{1}{2}} + \alpha^2 Y \\
&= (\partial_t + \mathrm{i} W^*)(\partial_t + \mathrm{i} W) + L,
\end{aligned}
$$

where we introduced

$$
\begin{aligned}
\gamma &:= \alpha^{-2} |g|^{\frac{1}{2}} = \alpha^{-1} |h|^{\frac{1}{2}}, \\
W &:= \frac{\mathrm{i}}{2} \gamma^{-1} \gamma_{,t} + \mathrm{i} \gamma^{\frac{1}{2}} \beta^i \partial_i \gamma^{-\frac{1}{2}}, \\
L &:= -\partial_i^{\gamma\,*} \tilde{h}^{ij} \partial_j^\gamma + \tilde{Y},
\end{aligned}
$$

and we use the shorthands

$$
\tilde{h}^{ij} := \alpha^2 h^{ij}, \quad \tilde{Y} := \alpha^2 Y, \quad \partial_i^\gamma := \gamma^{\frac{1}{2}} \partial_i \gamma^{-\frac{1}{2}}, \quad \gamma_{,t} := \partial_t \gamma.
$$

Clearly, propagators for $\tilde{\Box}$ induce corresponding propagators for $\Box$.

For each $t \in \mathbb{R}$, we (formally) define

$$
B(t) = \begin{bmatrix} B_{11}(t) & B_{12}(t) \\ B_{21}(t) & B_{22}(t) \end{bmatrix} := \begin{bmatrix} W(t) & \mathbb{1} \\ L(t) & W(t)^* \end{bmatrix}.
$$

Setting $u_1(t) = u(t)$ and $u_2(t) = (\mathrm{i}\partial_t - W(t))u(t)$, we find that

$$\big(\partial_t + \mathrm{i}B(t)\big)\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = 0$$

if and only if u is a (weak) solution of the Klein–Gordon equation $\tilde{\Box} u = 0$. Therefore we occasionally call $\partial_t + \mathrm{i}B(t)$ the *first-order Klein–Gordon operator*. The half-densities $u_1(t)$ and $u_2(t)$ may be called the *Cauchy data* for u at time t. The operator $L(t)$ is a Hermitian operator on $C_{\mathrm{c}}^\infty(\Sigma)$ in the sense of the Hilbert space $L^2(\Sigma)$. The current preserved by the dynamics is given by the matrix

$$Q = \begin{bmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{bmatrix}.$$

It is natural to introduce the *classical Hamiltonian*

$$H(t) = QB(t) = \begin{bmatrix} L(t) & W^*(t) \\ W(t) & \mathbb{1} \end{bmatrix}$$

We typically assume that $H(t) \geq 0$.

Under some mild conditions [35, 36] the evolution equation leads to a dynamics $R(t,s)$ satisfying

$$R(t,t) = \mathbb{1}, \tag{4.20}$$

$$\big(\partial_t + \mathrm{i}B(t)\big)R(t,s) = 0, \tag{4.21}$$

$$\partial_s R(t,s) - \mathrm{i}R(t,s)B(t) = 0. \tag{4.22}$$

The dynamics is a 2×2 matrix of operators acting on functions on Σ:

$$R(t,s) = \begin{bmatrix} R_{11}(t,s) & R_{12}(t,s) \\ R_{21}(t,s) & R_{22}(t,s) \end{bmatrix} \tag{4.23}$$

with distributional kernels $R_{ij}(t,\mathbf{x};s,\mathbf{y})$. The classical propagators in the Cauchy data formalism are:

$$E^{\mathrm{PJ}}(t,\mathbf{x};s,\mathbf{y}) = \mathrm{i}R(t,\mathbf{x};s,\mathbf{y}), \tag{4.24}$$

$$E^{\vee}(t,\mathbf{x};s,\mathbf{y}) = \mathrm{i}\theta(t-s)R(t,\mathbf{x};s,\mathbf{y}), \tag{4.25}$$

$$E^{\wedge}(t,\mathbf{x};s,\mathbf{y}) = -\mathrm{i}\theta(s-t)R(t,\mathbf{x};s,\mathbf{y}). \tag{4.26}$$

Then we set

$$G^{\bullet}(t,\mathbf{x};s,\mathbf{y}) = \mathrm{i}\frac{1}{\alpha(t,\mathbf{x})}E^{\bullet}_{12}(t,\mathbf{x};s,\mathbf{y})\frac{1}{\alpha(s,\mathbf{y})}, \qquad \bullet = \mathrm{PJ}, \vee, \wedge. \tag{4.27}$$

4.3 Non-classical propagators on asymptotically stationary spacetimes

Assume now that the Klein-Gordon equation is

$$\text{asymptotically stationary: } \lim_{t\pm\infty} B(t) =: B_\pm \quad \text{exists;} \tag{4.28}$$

$$\text{and asymptotically stable: } H_\pm := QB_\pm \geq 0. \tag{4.29}$$

Assume that 0 is not an eigenvalue of B_+ and B_-. Define the "out/in particle/antiparticle projections":

$$\Pi_\pm^{(+)} := \mathbb{1}_{]0,\infty[}(B_\pm), \tag{4.30}$$

$$\Pi_\pm^{(-)} := \mathbb{1}_{]0,\infty[}(-B_\pm). \tag{4.31}$$

We can transport them by the evolution to any time t:

$$\Pi_\pm^{(+)}(t) := \lim_{s\to\pm\infty} R(t,s)\Pi_\pm^{(+)}R(s,t), \tag{4.32}$$

$$\Pi_\pm^{(-)}(t) := \lim_{s\to\pm\infty} R(t,s)\Pi_\pm^{(-)}R(s,t). \tag{4.33}$$

We can now define the "out/in positive/negative frequency bisolutions in the Cauchy data formalism":

$$E_\pm^{(+)}(t,s) = \Pi_\pm^{(+)}(t)R(t,s), \tag{4.34}$$

$$E_\pm^{(-)}(t,s) = \Pi_\pm^{(-)}(t)R(t,s). \tag{4.35}$$

Note that $\mathcal{R}(\Pi^{(\pm)})$ and $\mathcal{R}(\Pi^{(\pm)}(t))$ are maximal uniformly positive/negative subspaces of the Krein space $\mathcal{W}_{\rm KG}$.

We will need also projections $\Pi_{+-}^{(\pm)}(t)$ and $\Pi_{-+}^{(\pm)}(t)$ defined by specifying their range and nullspace:[6]

$$\begin{aligned}
\mathcal{R}\big(\Pi_{+-}^{(+)}(t)\big) &= \mathcal{N}\big(\Pi_{+-}^{(-)}(t)\big) = \mathcal{R}\big(\Pi_{+}^{(+)}(t)\big),\\
\mathcal{R}\big(\Pi_{+-}^{(-)}(t)\big) &= \mathcal{N}\big(\Pi_{+-}^{(+)}(t)\big) = \mathcal{R}\big(\Pi_{-}^{(-)}(t)\big),\\
\mathcal{R}\big(\Pi_{-+}^{(+)}(t)\big) &= \mathcal{N}\big(\Pi_{-+}^{(-)}(t)\big) = \mathcal{R}\big(\Pi_{-}^{(+)}(t)\big),\\
\mathcal{R}\big(\Pi_{-+}^{(-)}(t)\big) &= \mathcal{N}\big(\Pi_{-+}^{(+)}(t)\big) = \mathcal{R}\big(\Pi_{+}^{(-)}(t)\big).
\end{aligned} \tag{4.36}$$

Note that

$$\Pi_{+-}^{(\pm)} + \Pi_{+-}^{(\mp)} = \mathbb{1}. \tag{4.37}$$

Now we can define the Feynman and anti-Feynman Green functions in the Cauchy data formalism:

$$E_{+-}^{\rm F}(t,s) = \theta(t-s)\Pi_{+-}^{(+)}(t)R(t,s) - \theta(s-t)\Pi_{+-}^{(-)}(t)R(t,s), \tag{4.38}$$

$$E_{-+}^{\overline{\rm F}}(t,s) = \theta(t-s)\Pi_{-+}^{(-)}(t)R(t,s) - \theta(s-t)\Pi_{-+}^{(+)}(t)R(t,s). \tag{4.39}$$

[6] Note that our notation is different from the convention in [35, 36].

Next we set

$$G^{(+)}_{\pm}(t,\mathbf{x};s,\mathbf{y}) = \frac{1}{\alpha(t,\mathbf{x})}E^{(+)}_{\pm,12}(t,\mathbf{x};s,\mathbf{y})\frac{1}{\alpha(s,\mathbf{y})}, \tag{4.40}$$

$$G^{(-)}_{\pm}(t,\mathbf{x};s,\mathbf{y}) = -\frac{1}{\alpha(t,\mathbf{x})}E^{(-)}_{\pm,12}(t,\mathbf{x};s,\mathbf{y})\frac{1}{\alpha(s,\mathbf{y})}, \tag{4.41}$$

$$G^{\mathrm{F}}_{+-}(t,\mathbf{x};s,\mathbf{y}) = \mathrm{i}\frac{1}{\alpha(t,\mathbf{x})}E^{\mathrm{F}}_{+-,12}(t,\mathbf{x};s,\mathbf{y})\frac{1}{\alpha(s,\mathbf{y})}, \tag{4.42}$$

$$G^{\overline{\mathrm{F}}}_{-+}(t,\mathbf{x};s,\mathbf{y}) = \mathrm{i}\frac{1}{\alpha(t,\mathbf{x})}E^{\overline{\mathrm{F}}}_{-+,12}(t,\mathbf{x};s,\mathbf{y})\frac{1}{\alpha(s,\mathbf{y})}. \tag{4.43}$$

$G^{(\pm)}_{-}$ are two-point functions of the “in-vacuum” Ω_- and $G^{(\pm)}_{+}$ are two-point functions of the “out-vacuum” Ω_+. Both are Hadamard states [46].

The *out-in Feynman propagator* $G^{\mathrm{F}}_{+-}(x,x')$ and the *in-out anti-Feynman propagator* $G^{\overline{\mathrm{F}}}_{-+}(x,x')$ are the “mixed Feynman propagators” corresponding to those states. In fact it is easy to see that if $\langle\Omega_+ \mid \Omega_-\rangle \neq 0$ then

$$G^{\mathrm{F}}_{+-}(x,y) = \mathrm{i}\frac{\langle\Omega_+ \mid \mathrm{T}\,\hat\phi(x)\hat\phi(y) \mid \Omega_-\rangle}{\langle\Omega_+ \mid \Omega_-\rangle}, \tag{4.44}$$

$$G^{\overline{\mathrm{F}}}_{-+}(x,y) = -\mathrm{i}\frac{\langle\Omega_- \mid \overline{\mathrm{T}}\hat\phi(x)\hat\phi(y) \mid \Omega_+\rangle}{\langle\Omega_- \mid \Omega_+\rangle}. \tag{4.45}$$

Assume in addition that $\alpha(x)$ and $\alpha^{-1}(x)$ are bounded on M. One can then heuristically derive [34, 35], and under some technical assumptions rigorously prove [62, 74], that they coincide with the operator-theoretic propagators:

$$G^{\mathrm{F}}_{\mathrm{op}}(x,y) = G^{\mathrm{F}}_{+-}(x,y), \tag{4.46}$$

$$G^{\overline{\mathrm{F}}}_{\mathrm{op}}(x,y) = G^{\overline{\mathrm{F}}}_{-+}(x,y). \tag{4.47}$$

Recall that the definition of Feynman and antiFeynman propagators includes multiplications with step functions, which strictly speaking needs an additional justification. On the other hand, the multiplication with step functions in the evolution equation approach, see (4.38) and (4.39), is unproblematic.

5 FLRW spacetimes

5.1 1+0-dimensional spacetimes

$1+0$-dimensional spacetimes form an important class of spacetimes for which we can understand various propagators rather completely.

The Klein-Gordon operator can be written as a one-dimensional Schrödinger operator (with the wrong sign in front of the second derivative):

$$K := -\Box + Y(t) = \partial_t^2 + Y(t). \tag{5.1}$$

We will assume that

$$Y(t) = -V(t) + m^2, \quad \lim_{t\to\pm\infty} V(t) = 0, \tag{5.2}$$

so that we can write

$$H := -\partial_t^2 + V(t), \quad K = -H + m^2. \tag{5.3}$$

Thus to discuss propagators on $1+0$-dimensional spacetimes one needs to understand the theory of Green functions of the one-dimensional Schrödinger operator H. A standard reference for the subject is [76]. In the following subsection, we present this well-known theory following [33] in a style adjusted to the QFT applications that we have in mind.

5.2 Green functions of one-dimensional Schrödinger operators

Suppose that we are given two solutions ψ_1, ψ_2 of the equation

$$(H - m^2)\psi(t) = 0. \tag{5.4}$$

Their Wronskian

$$\mathcal{W}(\psi_1, \psi_2) := \psi_1(t)\psi_2'(t) - \psi_1'(t)\psi_2(t) \tag{5.5}$$

does not depend on t. We may then define a bisolution of (5.4) by

$$G^{\leftrightarrow}(-m^2; t, s) := \frac{\psi_1(t)\psi_2(s) - \psi_2(t)\psi_1(s)}{\mathcal{W}(\psi_1, \psi_2)}. \tag{5.6}$$

Note that

$$G^{\leftrightarrow}(-m^2; t, t) = 0 \quad \text{and} \quad \partial_s G^{\leftrightarrow}(-m^2; t, s)\Big|_{s=t} = -\partial_t G^{\leftrightarrow}(-m^2; t, s)\Big|_{t=s} = 1. \tag{5.7}$$

It is easy to see that $G^{\leftrightarrow}(-m^2; t, s)$ is independent of the choice of ψ_1 and ψ_2. We call $G^{\leftrightarrow}(-m^2; t, s)$ the *canonical bisolution*. It is the analog of the Pauli-Jordan propagator.

We then can define the *forward and backward Green functions* via

$$G^{\rightarrow}(-m^2; t, s) := \theta(t-s) G^{\leftrightarrow}(-m^2; t, s), \tag{5.8}$$

$$G^{\leftarrow}(-m^2; t, s) := -\theta(s-t) G^{\leftrightarrow}(-m^2; t, s). \tag{5.9}$$

Using (5.7), one readily verifies that $G^{\rightarrow}$ and $G^{\leftarrow}$ are indeed Green functions. Needless to say, they are the analogs of the retarded and advanced propagators.

Now let $\mathrm{Re}(m) > 0$. The *Jost solutions* $\psi_{\pm}(m, t)$ are the unique solutions of (5.4) with the asymptotic behavior

$$\psi_{\pm}(m, t) \sim \mathrm{e}^{\mp m t} \quad \text{as} \quad t \to \pm\infty. \tag{5.10}$$

The *Jost function* is

$$\omega(m) := \mathcal{W}\big(\psi_+(m,\cdot), \psi_-(m,\cdot)\big). \tag{5.11}$$

Then, the unique fundamental solution with appropriate decay behavior as $|t| \to \infty$, that is, the integral kernel of the resolvent $G(-m^2)$ of H, is

$$G(-m^2; t, s) := \frac{1}{\omega(m)}\Big(\theta(t-s)\psi_+(m,t)\psi_-(m,s) + \theta(s-t)\psi_-(m,t)\psi_+(m,s)\Big). \tag{5.12}$$

To verify that $G(-m^2; t, s)$ is indeed a fundamental solution, one may use integration by parts and the time-independence of the Wronskian to show that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{g(t)} G(-m^2; t, s)(-H + m^2) f(s)\mathrm{d}t\mathrm{d}s = \int_{-\infty}^{\infty} \overline{g(s)} f(s)\mathrm{d}s.$$

Now let $m > 0$. The distributional boundary values $G(m^2 \mp \mathrm{i}0; t, s)$ of $G(-m^2; t, s)$ on the spectrum are then given by

$$G(m^2 \mp \mathrm{i}0; t, s) = \frac{\theta(t-s)\psi_+(\pm\mathrm{i}m, t)\psi_-(\pm\mathrm{i}m, s) + \theta(s-t)\psi_-(\pm\mathrm{i}m, t)\psi_+(\pm\mathrm{i}m, s)}{\omega(\pm\mathrm{i}m)}. \tag{5.13}$$

Thus we computed all four basic Green functions of the Klein-Gordon equation given by (5.3):

$$\begin{aligned}
&\text{retarded propagator:} && G^{\to}(m^2; t, s), && (5.14)\\
&\text{advanced propagator:} && G^{\leftarrow}(m^2; t, s), && (5.15)\\
&\text{Feynman propagator:} && G(m^2 - \mathrm{i}0; t, s), && (5.16)\\
&\text{anti-Feynman propagator:} && G(m^2 + \mathrm{i}0; t, s). && (5.17)
\end{aligned}$$

One can now ask when the Klein-Gordon equation given by the operator (5.3) on a $1+0$-dimensional spacetime is special, i.e., when the following identity holds:

$$G(m^2 - \mathrm{i}0) + G(m^2 + \mathrm{i}0) = G^{\to}(m^2) + G^{\leftarrow}(m^2)? \tag{5.18}$$

To answer this question, it is useful to introduce the concept of *reflectionlessness*.

Definition 5.1. Let $A(\pm\mathrm{i}m)$ and $B(\pm\mathrm{i}m)$ denote the coefficients of the scattering matrix, i.e.,

$$\psi_+(\pm\mathrm{i}m, t) = A(\pm\mathrm{i}m)\psi_-(\mp\mathrm{i}m, t) + B(\pm\mathrm{i}m)\psi_+(\mp\mathrm{i}m, t). \tag{5.19}$$

The potential $Y(t)$ is called *reflectionless at energy* m^2 if $B(\pm\mathrm{i}m) = 0$.

We have the following theorem.

Theorem 5.2. *The potential $Y(t)$ is reflectionless if and only if the spacetime is special, i.e., if and only if* (5.18) *is true.*

Proof. We have

$$
\begin{aligned}
&G(m^2 - \mathrm{i}0) + G(m^2 + \mathrm{i}0) \\
&= \theta(t-s)\Bigg(\frac{\psi_+(\mathrm{i}m,t)\psi_-(\mathrm{i}m,s)}{\omega(\mathrm{i}m)} + \frac{\psi_+(-\mathrm{i}m,t)\psi_-(-\mathrm{i}m,s)}{\omega(-\mathrm{i}m)}\Bigg) \qquad (5.20)\\
&\quad + \theta(s-t)\Bigg(\frac{\psi_-(\mathrm{i}m,t)\psi_+(\mathrm{i}m,s)}{\omega(\mathrm{i}m)} + \frac{\psi_-(-\mathrm{i}m,t)\psi_+(-\mathrm{i}m,s)}{\omega(-\mathrm{i}m)}\Bigg). \qquad (5.21)
\end{aligned}
$$

Moreover,

$$\omega(\pm\mathrm{i}m) = \pm A(\pm\mathrm{i}m)\mathcal{W}(\psi_-(-\mathrm{i}m), \psi_-(\mathrm{i}m)) + B(\pm\mathrm{i}m)\mathcal{W}(\psi_+(\mp\mathrm{i}m), \psi_-(\pm\mathrm{i}m)). \quad (5.22)$$

Then the part (5.20) becomes

$$
\begin{aligned}
\theta(t-s)\Bigg(&\frac{A(\mathrm{i}m)\psi_-(-\mathrm{i}m,t)\psi_-(\mathrm{i}m,s) + B(\mathrm{i}m)\psi_+(-\mathrm{i}m,t)\psi_-(\mathrm{i}m,s)}{A(\mathrm{i}m)\mathcal{W}(\psi_-(-\mathrm{i}m),\psi_-(\mathrm{i}m)) + B(\mathrm{i}m)\mathcal{W}(\psi_+(-\mathrm{i}m),\psi_-(\mathrm{i}m))} \qquad (5.23)\\
&- \frac{A(-\mathrm{i}m)\psi_-(\mathrm{i}m,t)\psi_-(-\mathrm{i}m,s) + B(-\mathrm{i}m)\psi_+(\mathrm{i}m,t)\psi_-(-\mathrm{i}m,s)}{A(-\mathrm{i}m)\mathcal{W}(\psi_-(-\mathrm{i}m),\psi_-(\mathrm{i}m)) - B(-\mathrm{i}m)\mathcal{W}(\psi_+(\mathrm{i}m),\psi_-(-\mathrm{i}m))}\Bigg)
\end{aligned}
$$

Since $A(\pm\mathrm{i}m) \neq 0$, this is $G^{\to}$ if and only if $B(\pm\mathrm{i}m) = 0$. Similar for (5.21). □

5.3 Mode decomposition of FLRW spacetimes

Consider a Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime, which has the line element

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2\mathrm{d}\Sigma^2, \quad (5.24)$$

where $\mathrm{d}\Sigma^2$ is the line element of a fixed $d-1$-dimensional complete Riemannian manifold, e.g. elliptic, Euclidean or hyperbolic space. The Klein-Gordon operator is

$$-\Box_g + m^2 = \partial_t^2 + (d-1)\frac{\dot{a}(t)}{a(t)}\partial_t - \frac{\Delta_\Sigma}{a(t)^2} + m^2, \quad (5.25)$$

where the dot indicates a derivative with respect to t. Then

$$a^{\frac{d-1}{2}}\big(-\Box_g + m^2\big)a^{-\frac{d-1}{2}} = \partial_t^2 - \frac{d-1}{2}\Big(\frac{\ddot{a}}{a} + \frac{d-3}{2}\Big(\frac{\dot{a}}{a}\Big)^2\Big) - \frac{\Delta_\Sigma}{a(t)^2} + m^2. \quad (5.26)$$

We can use the spectral theorem to diagonalize $-\Delta_\Sigma$, and then to restrict (5.26) to a (generalized) eigenfunction (a "mode") of $-\Delta_\Sigma$ with eigenvalue λ. Thus, for each such mode, (5.26) becomes $-H_\lambda + m^2$, where

$$H_\lambda := -\partial_t^2 + V_\lambda(t) \quad (5.27)$$

is the one-dimensional Schrödinger operator with potential

$$V_\lambda(t) = \frac{d-1}{2}\Big(\frac{\ddot{a}}{a} + \frac{d-3}{2}\Big(\frac{\dot{a}}{a}\Big)^2\Big) - \frac{\lambda}{a(t)^2}. \tag{5.28}$$

We can then write all propagators as the integral over all modes.

Thus the Klein-Gordon equation is "special" if and only if (5.27) is reflectionless at energy m^2 for all λ in the spectrum of $-\Delta_\Sigma$.

6 DeSitter space

Our next example is the d-dimensional deSitter space dS_d. DeSitter space is an important example of a non-stationary spacetime and one of the simplest examples to model a universe with an accelerated expansion. It exhibits a particularly rich structure and, being a symmetric space, all its invariant propagators can be given explicitly in terms of special functions.

We will describe four different approaches to investigate propagators on dS_d. The first approach is based on Wick rotation (analytic continuation) from the sphere $\mathbb{S}^d$. The second approach is the operator-theoretic one based on the resolvent of the d'Alembertian on $L^2(\mathrm{dS}_d)$. Somewhat surprisingly, the operator-theoretic approach leads to non-physical two-point functions. The third approach is the one based on the Krein space of solutions of the Klein-Gordon equation. It leads to the well-known family of deSitter invariant two-point functions corresponding to the so-called α-*vacua*. In fact, the general setup leads to invariant correlation functions between *two different* α-vacua. Finally, we may interpret dS_d as a special case of a FLRW spacetime and apply the methods from Section 5. This last approach breaks manifest deSitter invariance. To obtain the full invariant propagators, one needs to sum over all modes using rather complicated addition formulas for special functions. On the other hand, the first three approaches directly lead to the invariant propagators.

There is a very large literature about propagators on deSitter space. Particularly useful for our considerations were [2, 4, 5, 12, 15, 17, 19, 20, 25, 28, 42, 43, 51–53, 59, 60, 69, 71, 72]. In these references, one finds different approaches to investigate propagators on deSitter space.

Many of them use mode sums to construct propagators – sometimes explicitly like in [2, 15, 43, 61], sometimes abstractly like in [4]. The papers [17, 19, 20] have an axiomatic approach much in the spirit of Gårding and Wightman. Only the reference [69] uses the operator-theoretic approach to define the Feynman propagator in $d = 4$ dimensions.

6.1 Geometry of deSitter space

The d-dimensional deSitter space dS_d is defined by an embedding into $d+1$-dimensional Minkowski space $\mathbb{R}^{1,d}$. Let $[\cdot|\cdot]$ denote the pseudo-scalar product on $\mathbb{R}^{1,d}$ defined by

$$[x|x'] = -x^0 x'^0 + \sum_{i=1}^{d} x^i x'^i. \tag{6.1}$$

(That is, the Minkowski pseudo-metric is mostly positive.) Then the d-dimensional deSitter space is the one-sheeted hyperboloid

$$\mathrm{dS}_d := \{x \in \mathbb{R}^{1,d} \mid [x|x] = 1\}. \tag{6.2}$$

Let us introduce some notation that will frequently appear throughout this section. For $x, x' \in \mathrm{dS}_d \hookrightarrow \mathbb{R}^{1,d}$, we define

the invariant quantity	$Z \equiv Z(x, x') := [x\|x'],$	(6.3)
the antipodal point to x:	$x^A := -x,$	
the time variable	$t \equiv t(x, x') := x^0 - x'^0,$	
the "antipodal time" variable	$t^A := t(x^A, x') := -(x^0 + x'^0).$	

While t and t^A are two independent variables, we have $Z(x^A, x') = -Z(x, x') = -Z$.

DeSitter space has various regions:

$Z > 1:$	x and x' are timelike sepearated,	(6.4)
$Z = 1:$	x and x' are sepearated by a null-geodesic,	
$Z < 1:$	x and x' are not connected by a causal curve.	

The last region includes the subregions

$Z = -1:$	x^A and x' are sepearated by a null-geodesic,	(6.5)
$Z < -1:$	x^A and x' are timelike sepearated.	

One may further divide the regions $Z > 1$ and $Z < -1$ into future and past dependent on whether t resp. t^A are positive or negative. Thus, if we fix a point $x' \in \mathrm{dS}_d$, then we can partition dS_d into 5 regions:

$$\mathrm{dS}_d = V^+ \cup V^- \cup A^+ \cup A^- \cup S \tag{6.6}$$

as depicted in Figure 1.

The deSitter space possesses a global system of coordinates

$$x^0 = \sinh\tau, \quad x^i = \cosh\tau\,\Omega^i,\ i = 1, \dots, d, \quad \text{where} \quad \tau \in \mathbb{R},\ \Omega \in \mathbb{S}^{d-1} \hookrightarrow \mathbb{R}^d. \tag{6.7}$$

In these coordinates we have $\mathrm{d}s^2 = -\mathrm{d}\tau^2 + \cosh^2(\tau)\mathrm{d}\Omega^2$ and

$$Z = -\sinh\tau\sinh\tau' + \cosh\tau\cosh\tau'\cos\theta, \tag{6.8}$$

where θ is the angle between Ω and Ω'. If $x = (0, 1, 0 \dots)$, then $Z = \cosh\tau'\cos\theta$.

Both the *(full) deSitter group* $O(1, d)$ and the *restricted deSitter group* $SO_0(1, d)$, that is, the connected component of the identity in $O(1, d)$, act on dS_d. The Klein-Gordon equation restricted to invariant solutions and written in terms of Z reduces to the Gegenbauer equation,

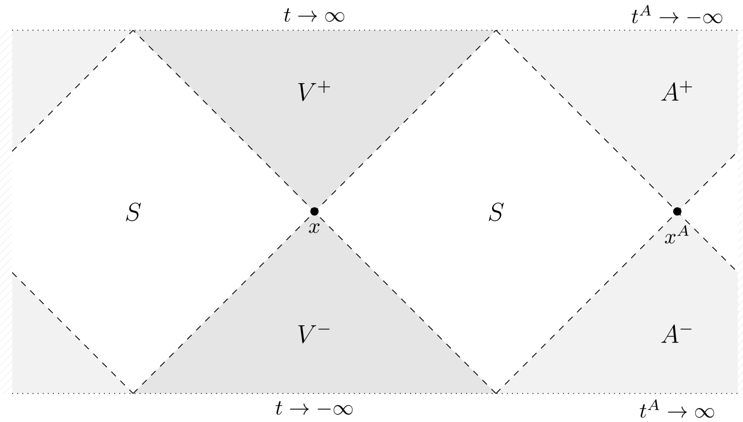


Figure 1: Conformal diagram of deSitter space with the reference point x and the regions $V^{\pm} := \{Z(x,x') > 1 \mid t(x,x') \gtrless 0\}$, $A^{\pm} := \{Z(x,x') < -1 \mid t(x^A,x') \lesseqgtr 0\}$ and $S := \{|Z(x,x')| < 1\}$. The left and right side of the diagram are glued together and each point represents a $d-2$-sphere.

a form of the hypergeometric equation [4, 10, 25, 44, 61, 71] whose properties we discuss in Appendix B.

In the literature one often restricts analysis to subsets of dS_d, such as the *Poincare patch* or the *static patch*, which allow for coordinate systems with special properties. In our paper we consider only the "*global patch*", that is the full deSitter space. Otherwise, we would have to consider boundary conditions for the d'Alembertian at the boundary of our patch (which would break the deSitter invariance and presumably be non-physical).

For more information about deSitter space, consult the overviews [60, 72] .

6.2 The sphere

On generic spacetimes the notion of a Wick rotation is not uniquely defined. However the deSitter space can be viewed as a Wick-rotated sphere. Therefore, in this subsection we recall some facts about the sphere and the Green function of the spherical Laplacian.

Consider the $d+1$ dimensional Euclidean space equipped with the scalar product

$$(x|x') = \sum_{i=1}^{d+1} x^i x'^i. \tag{6.9}$$

The d-dimensional (unit) sphere is defined as

$$\mathbb{S}^d := \{x \in \mathbb{R}^{d+1} \mid (x|x) = 1\}. \tag{6.10}$$

For $\mathrm{Re}(\nu) > 0$ or $\nu \in \mathrm{i}\mathbb{R}_{\geq 0} \setminus \mathrm{i}\big(\frac{d-1}{2} + \mathbb{N}_0\big)$, let us consider the resolvent of the spherical Laplacian $G^{\mathrm{s}}(-\nu^2) := (-\Delta^{\mathrm{s}} + (\frac{d-1}{2})^2 + \nu^2)^{-1}$. Its integral kernel $G^{\mathrm{s}}(-\nu^2; x, x')$ can be expressed in terms of the invariant quantity $(x|x')$ (see e.g. [31, 32]) as:

$$G^{\mathrm{s}}(-\nu^2; x, x') = C_{d,\nu} \mathbf{S}_{\frac{d-2}{2}, \mathrm{i}\nu}\big(-(x|x')\big), \tag{6.11}$$

where

$$C_{d,\nu} := \frac{\Gamma\left(\frac{d-1}{2}+\mathrm{i}\nu\right)\Gamma\left(\frac{d-1}{2}-\mathrm{i}\nu\right)}{(4\pi)^{\frac{d}{2}}}, \tag{6.12}$$

and where $\mathrm{S}_{\alpha,\lambda}(z)$ is the Gegenbauer function described in Appendix B.

In the literature [27, 73], the spherical Green function is often expressed in terms of associated Legendre functions. However, the equivalent representation in terms of Gegenbauer functions is arguably simpler.

6.3 Propagators related to the Euclidean state

We now turn to the d-dimensional deSitter space for $d \geq 2$. We will analyze bi- and fundamental solutions of the Klein-Gordon equation

$$(-\Box + m^2)\phi(x) = 0 \tag{6.13}$$

in deSitter space, which are invariant under the full or restricted deSitter group. Note that m might contain a coupling to the scalar curvature. Hence it is sometimes called *effective mass*. Anyway, we prefer to use the parameter ν defined by

$$\nu := \sqrt{m^2 - \left(\tfrac{d-1}{2}\right)^2} \in \mathbb{C} \cap \{\mathrm{Re}(\nu) > 0 \text{ or } \nu \in \mathrm{i}\mathbb{R}_{\geq 0}\}. \tag{6.14}$$

Thus (6.13) is replaced with

$$\Big(-\Box + \left(\tfrac{d-1}{2}\right)^2 + \nu^2\Big)\phi(x) = 0. \tag{6.15}$$

The case of positive ν^2 has analogous properties to that of positive m^2 in Minkowski space, while the case of negative ν^2 has analogous properties to the tachyonic case in Minkowski space. We will also allow for complex ν^2, choosing the principal sheet of the square root.

The case $\nu^2 < 0$ is more intricate than the case $\mathrm{Re}(\nu) > 0$ and contains various subcases with different exotic properties. For example, it contains the case of small effective masses $0 < m < \frac{d-1}{2}$ and the (singular) massless case $m = 0$.

On the deSitter space embedded in $\mathbb{R}^{1,d}$ there is a natural kind of a Wick rotation, which we will use: the replacement of x^{d+1} with $\pm\mathrm{i}x^0$. We note first that

$$(x|x') = 1 - \frac{(x-x'|x-x')}{2} \quad \text{for} \quad x, x' \in \mathbb{S}^d. \tag{6.16}$$

The replacement of $x^{d+1} - x'^{d+1}$ with $(x^0 - x'^0)\mathrm{e}^{\pm\mathrm{i}\phi}$, $\phi \in [0, \frac{\pi}{2}]$, yields

$$\begin{aligned}&(x^{d+1} - x'^{d+1})^2 \to (x^0 - x'^0)^2\mathrm{e}^{\pm 2\mathrm{i}\phi} \xrightarrow{\phi \to \frac{\pi}{2}} -(x^0 - x'^0)^2 \pm \mathrm{i}0 \\ \Rightarrow\ &(x|x') \to [x|x'] \mp \mathrm{i}0.\end{aligned} \tag{6.17}$$

Moreover, we need to insert a prefactor $\pm\mathrm{i}$ coming from the change of the integral measure.

Let $\mathrm{Re}(\nu) > 0$ or $\nu \in \mathrm{i}\mathbb{R}_{\geq 0} \setminus \mathrm{i}\big(\frac{d-1}{2} + \mathbb{N}_0\big)$. The Feynman and anti-Feynman propagators in the d-dimensional deSitter space obtained by Wick rotation of the Green function (6.11) on the sphere are given by

$$G_0^{\mathrm{F}/\overline{\mathrm{F}}}(x, x') = \pm\mathrm{i}C_{d,\nu}\, \mathbf{S}_{\frac{d}{2}-1,\mathrm{i}\nu}\big(- Z \pm \mathrm{i}0\big), \tag{6.18}$$

where $C_{d,\nu}$ is given by (6.12) and $Z := [x|x']$. We easily check that (6.18) are Green functions of the Klein-Gordon equation on dS_d.

The sum of the Euclidean Feynman and anti-Feynman propagator has a causal support, for $\mathbf{S}_{\alpha,\lambda}(z)$ is holomorphic on $\mathbb{C}\backslash] - \infty, -1]$, and therefore

$$G_0^{\mathrm{F}} + G_0^{\overline{\mathrm{F}}} = \mathrm{i}C_{d,\nu} \Big(\mathbf{S}_{\frac{d}{2}-1,\mathrm{i}\nu}\big(- Z + \mathrm{i}0\big) - \mathbf{S}_{\frac{d}{2}-1,\mathrm{i}\nu}\big(- Z - \mathrm{i}0\big)\Big) \tag{6.19}$$

vanishes for $Z < 1$.

As we will see later, G_0^{F} and $G_0^{\overline{\mathrm{F}}}$ are not the operator-theoretic Feynman and anti-Feynman propagators. However, we can still apply to them the procedure described in Subsection 3.7. This leads to the classical propagators

$$G^{\vee/\wedge}(x, x') = \mathrm{i}\theta\big(\pm (x^0 - x'^0)\big)C_{d,\nu} \Big(\mathbf{S}_{\frac{d}{2}-1,\mathrm{i}\nu}\big(- Z + \mathrm{i}0\big) - \mathbf{S}_{\frac{d}{2}-1,\mathrm{i}\nu}\big(- Z - \mathrm{i}0\big)\Big), \tag{6.20}$$

$$G^{\mathrm{PJ}}(x, x') = \mathrm{i}\,\mathrm{sgn}\left(x^0 - x'^0\right)C_{d,\nu}\Big(\mathbf{S}_{\frac{d}{2}-1,\mathrm{i}\nu}\big(- Z + \mathrm{i}0\big) - \mathbf{S}_{\frac{d}{2}-1,\mathrm{i}\nu}\big(- Z - \mathrm{i}0\big)\Big), \tag{6.21}$$

as well as to the positive/negative frequency solutions

$$G_0^{(\pm)}(x, x') = C_{d,\nu}\, \mathbf{S}_{\frac{d}{2}-1,\mathrm{i}\nu}\Big(- Z \pm \mathrm{i}0\,\mathrm{sgn}\left(x^0 - x'^0\right)\Big). \tag{6.22}$$

$G_0^{(\pm)}$ have the Hadamard property and are two-point functions of a state called the *Euclidean state* Ω_0 (sometimes also called the *Bunch-Davies state*) [4, 23, 25, 44, 61, 71]. Thus

$$\begin{aligned} -\mathrm{i}G^{\mathrm{PJ}}(x, x') &= [\hat\phi(x), \hat\phi(x')], \\ G_0^{(+)}(x, x') &= \langle\Omega_0 \mid \hat\phi(x)\hat\phi(x') \mid \Omega_0\rangle, \quad G_0^{(-)}(x, x') = \langle\Omega_0 \mid \hat\phi(x')\hat\phi(x) \mid \Omega_0\rangle \\ G_0^{\mathrm{F}}(x, x') &= \mathrm{i}\langle\Omega_0 \mid \mathrm{T}\,\hat\phi(x)\hat\phi(x') \mid \Omega_0\rangle, \quad G_0^{\overline{\mathrm{F}}}(x, x') = -\mathrm{i}\langle\Omega_0 \mid \overline{\mathrm{T}}\hat\phi(x)\hat\phi(x') \mid \Omega_0\rangle. \end{aligned} \tag{6.23}$$

Note that the propagators associated to the Euclidean vacuum satisfy all relations (2.7). The classical propagators (6.20) and (6.21) are universal: they do not depend on the Euclidean vacuum, therefore we do not decorate them with the subscript 0.

6.4 Bisolutions and Green functions

The family of invariant propagators on the deSitter space is quite rich and is not limited to those related to the Euclidean state, discussed in the previous subsection. In order to prepare for their analysis, in this subsection we will descibe invariant solutions of the Klein-Gordon equation on deSitter space.

From the analysis of previous subsection we easily see that the following functions are bisolutions invariant with respect to the restricted deSitter group:

$$
\begin{aligned}
G_0^{\mathrm{sym}}(x,x') &:= G_0^{(+)}(x,x') + G_0^{(-)}(x,x') &\quad (6.24)\\
&= C_{d,\nu}\Big(\mathbf{S}_{\frac{d}{2}-1,\mathrm{i}\nu}\big(-Z+\mathrm{i}0\big) + \mathbf{S}_{\frac{d}{2}-1,\mathrm{i}\nu}\big(-Z-\mathrm{i}0\big)\Big),\\
G_0^{\mathrm{sym},A}(x,x') &:= G_0^{\mathrm{sym}}(x^A,x') = G_0^{\mathrm{sym}}(x,x'^A) &\quad (6.25)\\
&= C_{d,\nu}\Big(\mathbf{S}_{\frac{d}{2}-1,\mathrm{i}\nu}\big(Z+\mathrm{i}0\big) + \mathbf{S}_{\frac{d}{2}-1,\mathrm{i}\nu}\big(Z-\mathrm{i}0\big)\Big),\\
G^{\mathrm{PJ}}(x,x') &:= \mathrm{i}\big(G_0^{(+)}(x,x') - G_0^{(-)}(x,x')\big) &\quad (6.26)\\
&= \mathrm{i}\,\mathrm{sgn}\,(t) C_{d,\nu}\Big(\mathbf{S}_{\frac{d}{2}-1,\mathrm{i}\nu}\big(-Z+\mathrm{i}0\big) - \mathbf{S}_{\frac{d}{2}-1,\mathrm{i}\nu}\big(-Z-\mathrm{i}0\big)\Big),\\
G^{\mathrm{PJ},A}(x,x') &:= G^{\mathrm{PJ}}(x^A,x') = -G^{\mathrm{PJ}}(x,x'^A) &\quad (6.27)\\
&= \mathrm{i}\,\mathrm{sgn}\,(t^A) C_{d,\nu}\Big(\mathbf{S}_{\frac{d}{2}-1,\mathrm{i}\nu}\big(Z+\mathrm{i}0\big) - \mathbf{S}_{\frac{d}{2}-1,\mathrm{i}\nu}\big(Z-\mathrm{i}0\big)\Big).
\end{aligned}
$$

Indeed, we already know that $G_0^{(\pm)}$ are bisolutions, hence so are (6.24) and (6.26). It is also clear that replacing x with x^A, used in (6.25) and (6.27) leads to invariant bisolutions. More precisely, we have the following theorem:

Theorem 6.1. *For generic ν, $\{G_0^{\mathrm{sym}}, G_0^{\mathrm{sym},A}\}$ is a basis of the space of fully deSitter invariant bisolutions, and $\{G_0^{\mathrm{sym}}, G_0^{\mathrm{sym},A}, G^{\mathrm{PJ}}, G^{\mathrm{PJ},A}\}$ is a basis of the space of bisolutions invariant under the restricted deSitter group.*

Proof. The integral kernel of an operator invariant with respect to the full deSitter group can always be written in terms of the invariant quantity Z alone. If we only demand invariance under the restricted deSitter group, the regions V^+ and V^- as well as A^+ and A^- need to be treated as independent. Hence for $|Z| > 1$, solutions invariant under the restricted deSitter group may depend on $\mathrm{sgn}(t)$ resp. $\mathrm{sgn}(t^A)$.

As explained above, the Klein-Gordon equation restricted to invariant solutions and written in terms of Z reduces to the Gegenbauer equation. As a 2nd order differential equation, its solution space on an interval away from the singularities at $Z = 1, -1$ is 2-dimensional. In particular, for $Z \in]-1, 1[$, bisolutions are generically spanned by $\mathbf{S}_{\frac{d}{2}-1,\mathrm{i}\nu}(Z)$ and $\mathbf{S}_{\frac{d}{2}-1,\mathrm{i}\nu}(-Z)$.

Let us describe the continuation to the interval $Z \in]-1, \infty[$. The extension to the interval $Z \in]-\infty, 1[$ works analogously.

The first bisolution, being regular for $Z \in]-1, \infty[$, has an obvious unique continuation to that interval. The second, having a singularity at $Z = 1$, must be continued to a distribution on $]-1, \infty[$. There is a one-parameter family of such continuations that yield a bisolution on $]-1, \infty[$ given by

$$
a\mathbf{S}_{\frac{d}{2}-1,\mathrm{i}\nu}\big(-Z+\mathrm{i}0\,\mathrm{sgn}(t)\big) + (1-a)\mathbf{S}_{\frac{d}{2}-1,\mathrm{i}\nu}\big(-Z-\mathrm{i}0\,\mathrm{sgn}(t)\big). \quad (6.28)
$$

Finally, one can also continue the 0-function from $]-1, 1[$ to a nontrivial bisolution on

$]-1,\infty[$. It is easy to see that the only such continuation is

$$\begin{aligned}&\mathbf{S}_{\frac{d}{2}-1,\mathrm{i}\nu}\big(-Z+\mathrm{i}0\,\mathrm{sgn}(t)\big)-\mathbf{S}_{\frac{d}{2}-1,\mathrm{i}\nu}\big(-Z-\mathrm{i}0\,\mathrm{sgn}(t)\big)\\&=\mathrm{sgn}\,t\Big(\mathbf{S}_{\frac{d}{2}-1,\mathrm{i}\nu}(-Z+\mathrm{i}0)-\mathbf{S}_{\frac{d}{2}-1,\mathrm{i}\nu}(-Z-\mathrm{i}0)\Big).\end{aligned}\tag{6.29}$$

This is precisely the a-dependent part in (6.28). Therefore, the distributions $\mathbf{S}_{\frac{d}{2}-1,\mathrm{i}\nu}\big(Z\big)$, $\Big(\mathbf{S}_{\frac{d}{2}-1,\mathrm{i}\nu}\big(-Z+\mathrm{i}0)+\mathbf{S}_{\frac{d}{2}-1,\mathrm{i}\nu}\big(-Z-\mathrm{i}0)\Big)$ and $\mathrm{sgn}\,t\Big(\mathbf{S}_{\frac{d}{2}-1,\mathrm{i}\nu}\big(-Z+\mathrm{i}0)-\mathbf{S}_{\frac{d}{2}-1,\mathrm{i}\nu}\big(-Z-\mathrm{i}0)\Big)$ span the space of bisolutions in the region $Z>-1$. □

Consequently, the explicit form of the general bisolution is

$$\begin{aligned}G_{\underline{a}}^{\mathrm{bisol}}=&:=\mathrm{i}a_1G_0^{\mathrm{sym}}+a_2G^{\mathrm{PJ}}+\mathrm{i}a_3G_0^{\mathrm{sym},A}+a_4G^{\mathrm{PJ}A}\\&=\mathrm{i}C_{d,\nu}\Big(\big(a_1+a_2\,\mathrm{sgn}(t)\big)\mathbf{S}_{\frac{d-2}{2},\mathrm{i}\nu}(-Z+\mathrm{i}0)\\&\qquad+\big(a_1-a_2\,\mathrm{sgn}(t)\big)\mathbf{S}_{\frac{d-2}{2},\mathrm{i}\nu}(-Z-\mathrm{i}0)\\&\qquad+\big(a_3-a_4\,\mathrm{sgn}(t^A)\big)\mathbf{S}_{\frac{d-2}{2},\mathrm{i}\nu}(Z+\mathrm{i}0)\\&\qquad+\big(a_3+a_4\,\mathrm{sgn}(t^A)\big)\mathbf{S}_{\frac{d-2}{2},\mathrm{i}\nu}(Z-\mathrm{i}0)\Big)\end{aligned}\tag{6.30}$$

and the explicit form of the general fundamental solution is

$$\begin{aligned}G_{\underline{a}}&:=G_0^{\mathrm{F}}+G_{\underline{a}}^{\mathrm{bisol}}\\&=\mathrm{i}C_{d,\nu}\Big(\big(1+a_1+a_2\,\mathrm{sgn}(t)\big)\mathbf{S}_{\frac{d-2}{2},\mathrm{i}\nu}(-Z+\mathrm{i}0)\\&\qquad+\big(a_1-a_2\,\mathrm{sgn}(t)\big)\mathbf{S}_{\frac{d-2}{2},\mathrm{i}\nu}(-Z-\mathrm{i}0)\\&\qquad+\big(a_3-a_4\,\mathrm{sgn}(t^A)\big)\mathbf{S}_{\frac{d-2}{2},\mathrm{i}\nu}(Z+\mathrm{i}0)\\&\qquad+\big(a_3+a_4\,\mathrm{sgn}(t^A)\big)\mathbf{S}_{\frac{d-2}{2},\mathrm{i}\nu}(Z-\mathrm{i}0)\Big).\end{aligned}\tag{6.31}$$

6.5 Resolvent of the d'Alembertian and operator-theoretic propagators

The d'Alembertian $-\Box$ is an essentially self-adjoint operator on $C_{\mathrm{c}}^\infty(\mathrm{dS}_d)$. This follows from a general theory of invariant differential operators on symmetric spaces [9, 68] and the fact that deSitter space can be seen as the quotient of Lie groups $O(1,d)/O(1,d-1)$. In this subsection we will compute its resolvent and operator-theoretic Feynman and anti-Feynman propagators. In the four-dimensional case, this has been studied [69].

Outside of the spectrum of $-\Box+\big(\frac{d-1}{2}\big)^2$ we set

$$G(-\nu^2):=\Big(-\Box+\big(\tfrac{d-1}{2}\big)^2+\nu^2\Big)^{-1}.\tag{6.32}$$

Its integral kernel will be denoted $G(-\nu^2;x,x')$.

Theorem 6.2. *Let* $\operatorname{Re}\nu > 0$.
Odd d**.** *The resolvent is given by*

$$\begin{aligned}
&G(-\nu^2;x,x') \\
&= \frac{\Gamma\big(\frac{d-1}{2}+\mathrm{i}\nu\big)}{2^{2+\mathrm{i}\nu}(2\pi)^{\frac{d-1}{2}}\sinh\pi\nu}\Big(\mathbf{Z}_{\frac{d-2}{2},\mathrm{i}\nu}(-Z-\mathrm{i}0)-\mathbf{Z}_{\frac{d-2}{2},\mathrm{i}\nu}(-Z+\mathrm{i}0)\Big), \quad \operatorname{Im}\nu<0; \qquad (6.33)\\
&= \frac{\Gamma\big(\frac{d-1}{2}-\mathrm{i}\nu\big)}{2^{2-\mathrm{i}\nu}(2\pi)^{\frac{d-1}{2}}\sinh\pi\nu}\Big(\mathbf{Z}_{\frac{d-2}{2},-\mathrm{i}\nu}(-Z+\mathrm{i}0)-\mathbf{Z}_{\frac{d-2}{2},-\mathrm{i}\nu}(-Z-\mathrm{i}0)\Big), \quad \operatorname{Im}\nu>0. \qquad (6.34)
\end{aligned}$$

Therefore, for $\nu > 0$ *the Feynman and anti-Feynman propagators are*

$$G_{\mathrm{op}}^{\mathrm{F}}(x,x') = \frac{\Gamma\big(\frac{d-1}{2}+\mathrm{i}\nu\big)}{2^{2+\mathrm{i}\nu}(2\pi)^{\frac{d-1}{2}}\sinh\pi\nu}\Big(\mathbf{Z}_{\frac{d-2}{2},\mathrm{i}\nu}(-Z-\mathrm{i}0)-\mathbf{Z}_{\frac{d-2}{2},\mathrm{i}\nu}(-Z+\mathrm{i}0)\Big), \qquad (6.35)$$

$$G_{\mathrm{op}}^{\overline{\mathrm{F}}}(x,x') = \frac{\Gamma\big(\frac{d-1}{2}-\mathrm{i}\nu\big)}{2^{2-\mathrm{i}\nu}(2\pi)^{\frac{d-1}{2}}\sinh\pi\nu}\Big(\mathbf{Z}_{\frac{d-2}{2},-\mathrm{i}\nu}(-Z+\mathrm{i}0)-\mathbf{Z}_{\frac{d-2}{2},-\mathrm{i}\nu}(-Z-\mathrm{i}0)\Big), \qquad (6.36)$$

Even d**.** *The resolvents are*

$$\begin{aligned}
&G(-\nu^2;x,x') \\
&= -\frac{\Gamma\big(\frac{d-1}{2}+\mathrm{i}\nu\big)}{2^{2+\mathrm{i}\nu}(2\pi)^{\frac{d-1}{2}}\cosh\pi\nu}\Big(\mathbf{Z}_{\frac{d-2}{2},\mathrm{i}\nu}(-Z+\mathrm{i}0)+\mathbf{Z}_{\frac{d-2}{2},\mathrm{i}\nu}(-Z-\mathrm{i}0)\Big), \quad \operatorname{Im}\nu<0; \qquad (6.37)\\
&= -\frac{\Gamma\big(\frac{d-1}{2}-\mathrm{i}\nu\big)}{2^{2-\mathrm{i}\nu}(2\pi)^{\frac{d-1}{2}}\cosh\pi\nu}\Big(\mathbf{Z}_{\frac{d-2}{2},-\mathrm{i}\nu}(-Z+\mathrm{i}0)+\mathbf{Z}_{\frac{d-2}{2},-\mathrm{i}\nu}(-Z-\mathrm{i}0)\Big), \ \operatorname{Im}\nu>0. \qquad (6.38)
\end{aligned}$$

Therefore, for $\nu > 0$ *the operator-theoretic Feynman and anti-Feynman propagators are*

$$G_{\mathrm{op}}^{\mathrm{F}}(x,x') = -\frac{\Gamma\big(\frac{d-1}{2}+\mathrm{i}\nu\big)}{2^{2+\mathrm{i}\nu}(2\pi)^{\frac{d-1}{2}}\cosh\pi\nu}\Big(\mathbf{Z}_{\frac{d-2}{2},\mathrm{i}\nu}(-Z+\mathrm{i}0)+\mathbf{Z}_{\frac{d-2}{2},\mathrm{i}\nu}(-Z-\mathrm{i}0)\Big), \qquad (6.39)$$

$$G_{\mathrm{op}}^{\overline{\mathrm{F}}}(x,x') = -\frac{\Gamma\big(\frac{d-1}{2}-\mathrm{i}\nu\big)}{2^{2-\mathrm{i}\nu}(2\pi)^{\frac{d-1}{2}}\cosh\pi\nu}\Big(\mathbf{Z}_{\frac{d-2}{2},-\mathrm{i}\nu}(-Z+\mathrm{i}0)+\mathbf{Z}_{\frac{d-2}{2},-\mathrm{i}\nu}(-Z-\mathrm{i}0)\Big). \qquad (6.40)$$

Note that to obtain $G_{\mathrm{op}}^{\overline{\mathrm{F}}}$ from $G_{\mathrm{op}}^{\mathrm{F}}$ we need to replace ν with $-\nu$.

Proof. We first need to compute the Green operator $G_{\mathrm{op}}(-\nu^2)$ for $\nu^2 \in \mathbb{C}\setminus\mathbb{R}$. Clearly, it should be invariant under the *full* deSitter group. Its integral kernel (as the integral kernel of a bounded operator) must not grow too fast as $Z \to \pm\infty$. To start, we thus use the connection formula (B.14) to write the general fundamental solution (6.31) in terms of the Gegenbauer functions $\mathbf{Z}_{\alpha,\pm\lambda}(-Z \pm \mathrm{i}0)$, which have a determined behavior as $|Z| \to \infty$. Since we require invariance under the full deSitter group, we must have $a_2 = a_4 = 0$.

This yields

$$\frac{\sinh\pi\nu}{2^{\frac{d-3}{2}}\sqrt{\pi}C_{d,\nu}}G_{\underline{a}} = -\frac{2^{-\mathrm{i}\nu}}{\Gamma\big(\frac{d-1}{2}-\mathrm{i}\nu\big)}\mathbf{Z}_{\frac{d-2}{2},\mathrm{i}\nu}(-Z+\mathrm{i}0)\Big(1+a_1+a_3\mathrm{e}^{\mathrm{i}\pi\left(\frac{d-1}{2}+\mathrm{i}\nu\right)}\Big) \tag{6.41}$$
$$+\frac{2^{\mathrm{i}\nu}}{\Gamma\big(\frac{d-1}{2}+\mathrm{i}\nu\big)}\mathbf{Z}_{\frac{d-2}{2},-\mathrm{i}\nu}(-Z+\mathrm{i}0)\Big(1+a_1+a_3\mathrm{e}^{\mathrm{i}\pi\left(\frac{d-1}{2}-\mathrm{i}\nu\right)}\Big)$$
$$-\frac{2^{-\mathrm{i}\nu}}{\Gamma\big(\frac{d-1}{2}-\mathrm{i}\nu\big)}\mathbf{Z}_{\frac{d-2}{2},\mathrm{i}\nu}(-Z-\mathrm{i}0)\Big(a_1+a_3\mathrm{e}^{-\mathrm{i}\pi\left(\frac{d-1}{2}+\mathrm{i}\nu\right)}\Big)$$
$$+\frac{2^{\mathrm{i}\nu}}{\Gamma\big(\frac{d-1}{2}+\mathrm{i}\nu\big)}\mathbf{Z}_{\frac{d-2}{2},-\mathrm{i}\nu}(-Z-\mathrm{i}0)\Big(a_1+a_3\mathrm{e}^{-\mathrm{i}\pi\left(\frac{d-1}{2}-\mathrm{i}\nu\right)}\Big).$$

We have $\mathbf{Z}_{\frac{d-2}{2},\pm\mathrm{i}\nu}(Z) \sim cZ^{-\frac{d-1}{2}\mp\mathrm{i}\nu}$ as $|Z|\to\infty$, while the measure on $L^2(\mathrm{dS}_d,\sqrt{|g|})$ behaves as cZ^{d-2} as $|Z|\to\infty$.[7]

Thus, the resolvent should, for $|Z|>1$, only contain

$$\mathbf{Z}_{\frac{d-2}{2},\mathrm{i}\nu}(|Z|) \quad \text{if} \quad \mathrm{Im}(\nu)<0 \quad \text{and} \quad \mathbf{Z}_{\frac{d-2}{2},-\mathrm{i}\nu}(|Z|) \quad \text{if} \quad \mathrm{Im}(\nu)>0, \tag{6.42}$$

for otherwise it could not be the integral kernel of a bounded operator on $L^2(\mathrm{dS}_d,\sqrt{|g|})$. The parameters that correspond to such a decay behavior are different in even and odd dimensions:

Odd dimensions. In odd dimensions, $\frac{d-1}{2}$ is an integer, and we obtain

$$\text{Solution } \sim \mathbf{Z}_{\frac{d-2}{2},\pm\mathrm{i}\nu}(Z) \text{ for } |Z|>1: \quad a_1 = \pm\frac{\mathrm{e}^{\mp\pi\nu}}{2\sinh\pi\nu}, \quad a_3 = \pm\frac{(-1)^{\frac{d+1}{2}}}{2\sinh\pi\nu}. \tag{6.43}$$

Even dimensions. In even dimensions, $\frac{d-1}{2}$ is a half-integer but not an integer. We obtain

$$\text{Solution } \sim \mathbf{Z}_{\frac{d-2}{2},\pm\mathrm{i}\nu}(Z) \text{ for } |Z|>1: \quad a_1 = -\frac{\mathrm{e}^{\mp\pi\nu}}{2\cosh\pi\nu}, \quad a_3 = -\mathrm{i}\frac{(-1)^{\frac{d}{2}}}{2\cosh\pi\nu}. \tag{6.44}$$

These values of a_1 and a_3 yield the formulas for the resolvents. The operator-theoretic Feynman and anti-Feynman propagators are the limits of the resolvents on the spectrum from below resp. above. □

We will give an interpretation of the operator-theoretic (anti-)Feynman propagators in terms of time-ordered two-point functions between two states in Section 6.7. However, from their formulas, we can already see the surprising fact that they are different from the propagators in the Euclidean state Ω_0, which is the only deSitter-invariant Hadamard state.

One can ask when the Klein-Gordon operator on deSitter space is special. The situation is quite remarkable:

[7] This can be verified using the global coordinates (6.7), in which Z is given by (6.8).

Theorem 6.3. *Let $\nu > 0$. Then*

$$G_{\mathrm{op}}^{\mathrm{F}} + G_{\mathrm{op}}^{\overline{\mathrm{F}}} = G^{\vee} + G^{\wedge}, \quad \textbf{for odd } d; \tag{6.45}$$

$$\textit{but} \quad G_{\mathrm{op}}^{\mathrm{F}} + G_{\mathrm{op}}^{\overline{\mathrm{F}}} \neq G^{\vee} + G^{\wedge}, \quad \textbf{for even } d. \tag{6.46}$$

Proof. We use the connection formula (B.13) to rewrite $G_{\mathrm{op}}^{\mathrm{F}}$ and $G_{\mathrm{op}}^{\overline{\mathrm{F}}}$ in terms of $\mathbf{S}_{\frac{d-2}{2},\pm\mathrm{i}\nu}(\cdot)$ and compare to the formulas (6.20). Actually, in odd dimensions, the result follows immediately if one uses (B.14) instead of (B.13). □

Let us finally consider the "tachyonic" region of parameters in the deSitter space.

Theorem 6.4. 1. **Odd** d. *The spectrum of* $-\Box + \left(\frac{d-1}{2}\right)^2$ *equals*

$$\big\{ -\mu^2 \mid \mu \in \mathbb{N}_0 \big\} \cup [0, \infty[. \tag{6.47}$$

Setting $\mu := -\mathrm{i}\nu$*, the resolvent for* $\mu \in [0, \infty[\backslash \mathbb{N}_0$ *is given by*

$$\begin{aligned} &G_{\mathrm{op}}(\mu^2; x, x') \qquad (6.48) \\ &= -\mathrm{i}\frac{\Gamma\big(\frac{d-1}{2} + \mu\big)}{2^{2+\mu}(2\pi)^{\frac{d-1}{2}} \sin \pi\mu} \Big(\mathbf{Z}_{\frac{d-2}{2},\mu}(-Z + \mathrm{i}0) - \mathbf{Z}_{\frac{d-2}{2},\mu}(-Z - \mathrm{i}0) \Big). \end{aligned}$$

2. **Even** d. *The spectrum of* $-\Box + \left(\frac{d-1}{2}\right)^2$ *equals*

$$\big\{ -\mu^2 \mid \mu \in \mathbb{N}_0 + \tfrac{1}{2} \big\} \cup [0, \infty[. \tag{6.49}$$

Setting $\mu := -\mathrm{i}\nu$*, the resolvent for* $\mu \in [0, \infty[\backslash \big(\mathbb{N}_0 + \frac{1}{2}\big)$ *is given by*

$$\begin{aligned} &G_{\mathrm{op}}(\mu^2; x, x') \qquad (6.50) \\ &= -\frac{\Gamma\big(\frac{d-1}{2} + \mu\big)}{2^{2+\mu}(2\pi)^{\frac{d-1}{2}} \cos \pi\mu} \Big(\mathbf{Z}_{\frac{d-2}{2},\mu}(-Z + \mathrm{i}0) + \mathbf{Z}_{\frac{d-2}{2},\mu}(-Z - \mathrm{i}0) \Big). \end{aligned}$$

Proof. Let $\mu > 0$. If the limits of (6.33) and (6.34) resp. of (6.37) and (6.38) as ν approaches the imaginary line exist, they coincide:

$$\lim_{\epsilon \to 0} G_{\mathrm{op}}((-(\mathrm{i}\mu + \epsilon)^2; x, x') = \lim_{\epsilon \to 0} G_{\mathrm{op}}((-(-\mathrm{i}\mu + \epsilon)^2; x, x'). \tag{6.51}$$

The results of these limits are the integral kernels of the resolvents in the "tachyonic" case (6.48) resp. (6.50).

For even d, the limit diverges for $\mu \in \mathbb{N}_0 + \frac{1}{2}$ due to the presence of $\cos \pi\mu$ in the denominator of (6.50). This is not a removable singularity. For $Z < -1$, we have

$$\mathbf{Z}_{\frac{d-2}{2},\mu}(-Z + \mathrm{i}0) = \mathbf{Z}_{\frac{d-2}{2},\mu}(-Z - \mathrm{i}0) = \mathbf{Z}_{\frac{d-2}{2},\mu}(|Z|), \tag{6.52}$$

and this does not vanish identically.

For odd d, the limit diverges for $\mu \in \mathbb{N}_0$ due to the presence of $\sin \pi\mu$ in the denominator of (6.48). Although less obvious than in the even-dimensional case, this is also not a removable singularity. Due to (B.15), we have

$$\mathbf{Z}_{\frac{d-2}{2},\mu}(-Z+\mathrm{i}0) - \mathbf{Z}_{\frac{d-2}{2},\mu}(-Z-\mathrm{i}0) = 0, \quad |Z|>1,\ \mu \in \mathbb{N}_0. \tag{6.53}$$

But using the connection formula (B.13), we find for $|Z|<1$ and $\mu \in \mathbb{N}_0$,

$$\mathbf{Z}_{\frac{d-2}{2},\mu}(-Z+\mathrm{i}0) - \mathbf{Z}_{\frac{d-2}{2},\mu}(-Z-\mathrm{i}0) = \frac{-\mathrm{i}\,\mathrm{sgn}(Z)2^{\mu+\frac{d+1}{2}}}{\Gamma\big(\frac{d-1}{2}+\mu\big)(1-Z^2)^{\frac{d-2}{2}}}\mathbf{S}_{\frac{d-2}{2},\mu}(-Z). \tag{6.54}$$

This does not vanish identically. □

6.6 Alpha vacua

For the rest of the section on deSitter space, we restrict ourselves to the case of real and positive $\nu > 0$.

The Euclidean vacuum is not the only deSitter invariant state on deSitter space. There exists a whole family of such states, called *alpha vacua* [4, 15, 61]. We describe these states using the Krein space language introduced in Section 3 and then explain the relation to the approach based on mode expansions, which is commonly used in the physics literature [4, 15].

6.6.1 Alpha vacua in the Krein space picture

As usual, let $\mathcal{W}_{\mathrm{KG}}$ be the Krein space of solutions of the Klein-Gordon equation and let $\Pi_0^{(\pm)}$ be the Klein-Gordon kernels associated with the bisolutions $\pm G_0^{(\pm)}$. It is easy to verify (e.g. using mode expansions) that they correspond to a fundamental decomposition of $\mathcal{W}_{\mathrm{KG}}$. We will use this fundamental decomposition as a reference point. Then the ranges $\mathcal{Z}_0^{(\pm)} := \mathcal{R}(\Pi_0^{(\pm)})$ define a $(\cdot|\cdot)_{\mathrm{KG}}$-orthogonal direct sum decomposition of $\mathcal{W}_{\mathrm{KG}}$ via

$$\mathcal{W}_{\mathrm{KG}} = \mathcal{Z}_0^{(+)} \oplus \mathcal{Z}_0^{(-)}, \quad \mathcal{Z}_0^{(-)} = \overline{\mathcal{Z}_0^{(+)}}. \tag{6.55}$$

Using the explicit representations (6.22), it is easy to see that

$$G_0^{(+)}(x^A, x'^A) = \overline{G_0^{(+)}(x,x')} = G_0^{(-)}(x,x'). \tag{6.56}$$

Introducing the map $(J^A\varphi)(x) := \varphi(x^A)$, (6.56) implies

$$J^A\Pi_0^{(+)}J^A = \Pi_0^{(-)}, \quad J^A(\mathcal{Z}_0^{(\pm)}) = \mathcal{Z}_0^{(\mp)}. \tag{6.57}$$

Now let $\alpha \in \mathbb{C}$ with $|\alpha| < 1$. We define a Bogoliubov transformation R_α on $\mathcal{W}_{\mathrm{KG}}$ (i.e., a real pseudounitary map) via

$$(R_\alpha\varphi)(x) = \frac{1}{\sqrt{1-|\alpha|^2}}\varphi(x) + \frac{\overline{\alpha}}{\sqrt{1-|\alpha|^2}}\varphi(x^A), \quad \varphi \in \mathcal{Z}_0^{(+)}; \tag{6.58}$$

$$(R_\alpha\overline{\varphi})(x) = \frac{1}{\sqrt{1-|\alpha|^2}}\overline{\varphi(x)} + \frac{\alpha}{\sqrt{1-|\alpha|^2}}\overline{\varphi(x^A)}, \quad \overline{\varphi} \in \mathcal{Z}_0^{(-)}. \tag{6.59}$$

The projections $R_\alpha \Pi_0^{(\pm)} R_\alpha^{-1}$ define another fundamental decomposition of $\mathcal{W}_{\mathrm{KG}}$, hence another Fock vacuum, called the α-vacuum. Their two-point functions are given by the Klein-Gordon kernels of $\pm R_\alpha \Pi_0^{(\pm)} R_\alpha^{-1}$, that is $(R_\alpha \otimes R_\alpha) G_0^{(\pm)}(\cdot,\cdot)$ (see Prop. 3.3). After inserting (6.58) and (6.59), we obtain

$$\begin{aligned} &G_\alpha^{(+)}(x,x') \\ &= \frac{1}{1-|\alpha|^2}\Big(G_0^{(+)}(x,x') + \alpha G_0^{(+)}(x^A,x') + \overline{\alpha} G_0^{(+)}(x,x'^A) + |\alpha|^2 G_0^{(+)}(x^A,x'^A)\Big); \end{aligned} \tag{6.60}$$

$$\begin{aligned} &G_\alpha^{(-)}(x,x') \\ &= \frac{1}{1-|\alpha|^2}\Big(G_0^{(-)}(x,x') + \overline{\alpha} G_0^{(-)}(x^A,x') + \alpha G_0^{(-)}(x,x'^A) + |\alpha|^2 G_0^{(-)}(x^A,x'^A)\Big). \end{aligned} \tag{6.61}$$

Using $G_0^{(\pm)}(x^A,x') = G_0^{(\mp)}(x,x'^A)$ we can rewrite this as

$$\begin{aligned} &G_\alpha^{(\pm)}(x,x') \\ &= \frac{1}{1-|\alpha|^2}\Big(\tfrac{1+|\alpha|^2}{2} G_0^{\mathrm{sym}}(x,x') \mp \mathrm{i}\tfrac{1-|\alpha|^2}{2} G^{\mathrm{PJ}}(x,x') + \tfrac{\alpha+\overline{\alpha}}{2} G_0^{\mathrm{sym},A}(x,x') - \mathrm{i}\tfrac{\alpha-\overline{\alpha}}{2} G_0^{\mathrm{PJ},A}(x,x')\Big). \end{aligned} \tag{6.62}$$

From (6.62), we obtain the well-known expressions for the Feynman and anti-Feynman propagator [4, 15]:[8]

$$\begin{aligned} &G_\alpha^{\mathrm{F}/\overline{\mathrm{F}}}(x,x') \\ &= G_0^{\mathrm{F}/\overline{\mathrm{F}}}(x,x') \pm \frac{\mathrm{i}}{1-|\alpha|^2}\Big(|\alpha|^2 G_0^{\mathrm{sym}}(x,x') + \mathrm{i}\tfrac{\alpha+\overline{\alpha}}{2} G_0^{\mathrm{sym},A}(x,x') - \mathrm{i}\tfrac{\alpha-\overline{\alpha}}{2} G_0^{\mathrm{PJ},A}(x,x')\Big). \end{aligned} \tag{6.63}$$

It is known that only the α-vacuum satisfying the Hadamard condition is the Euclidean vacuum, that is, corresponding to $\alpha = 0$ (see [4] and references therein). This can also be read off the expansion of the Gegenbauer function around the singularity.

From the point of view of perturbative QFT, the usefulness of alpha vacua for $\alpha \neq 0$ is therefore questionable. It is not clear how one can renormalize quantities that are local and non-linear in the fields [22]. However, they are reasonable objects in *linear* QFT and possibly also in an effective field-theory. We shall see that the operator-theoretic propagators correspond to field expectation values in specific alpha vacua.

6.6.2 Alpha vacua and mode expansions

In the literature α-vacua are often introduced as follows [4, 15]. First one expands the real scalar Klein-Gordon field $\hat{\phi}(x)$ into modes with respect to the Euclidean vacuum,

$$\hat{\phi}(x) = \sum_n \varphi_n(x)\hat{a}_n^* + \overline{\varphi_n(x)}\hat{a}_n. \tag{6.64}$$

[8] Note that the two references have different conventions for the parameter α, and in addition, both conventions are different from ours. In particular, [4] uses two real labels α, β that are both described by a single $\alpha \in \mathbb{C}$ in our notation.

Here, $\hat{a}_n$ and $\hat{a}_n^*$ are annihilation and creation operators and $\varphi_n(x)$ are mode functions that satisfy the orthogonality relations (3.38) with the Dirac delta replaced by the Kronecker delta. This is essentially a choice of an orthonormal basis of the space $\mathcal{Z}_0^{(+)}$. The positive frequency solution can then be written as a mode sum,

$$G_0^{(+)}(x, x') = \sum_n \overline{\varphi_n(x)} \varphi_n(x'). \tag{6.65}$$

Next, using the explicit form of the modes, one shows [4, 15] that the modes associated to the Euclidean vacuum can be chosen to satisfy

$$\varphi_n(x) = \overline{\varphi_n(x^A)}. \tag{6.66}$$

Then one defines the Bogoliubov transformation (6.58) by its action on the modes,

$$\varphi_{\alpha,n}(x) := \frac{1}{\sqrt{1-|\alpha|^2}} \varphi_n(x) + \frac{\overline{\alpha}}{\sqrt{1-|\alpha|^2}} \overline{\varphi_n(x)}, \tag{6.67}$$

and the positive frequency solution associated to the alpha vacuum with parameter α is given by

$$G_\alpha^{(+)}(x, x') = \sum_n \overline{\varphi_{\alpha,n}(x)} \varphi_{\alpha,n}(x'). \tag{6.68}$$

Needless to say, the construction using the mode expansion and the construction based on the Krein space $\mathcal{W}_{\mathrm{KG}}$ are equivalent. In particular, $\varphi_{\alpha,n} = R_\alpha \varphi_n$.

6.6.3 Correlation functions between two different alpha vacua

Suppose now that α, β be two complex parameter with $|\alpha|, |\beta| < 1$ and consider a pair of Bogoliubov transformations R_α, R_β and a pair of Fock vacua Ω_α, Ω_β. Using modes, we can write

$$\begin{aligned} \varphi_{\beta,n}(x) &:= N_{\alpha\beta} \varphi_{\alpha,n}(x) + M_{\alpha,\beta} \overline{\varphi_{\alpha,n}(x)}, \\ N_{\alpha\beta} &= \frac{1 - \overline{\beta}\alpha}{\sqrt{(1-|\alpha|^2)(1-|\beta|^2)}}, \\ M_{\alpha,\beta} &= \frac{\overline{\beta} - \overline{\alpha}}{\sqrt{(1-|\alpha|^2)(1-|\beta|^2)}}. \end{aligned} \tag{6.69}$$

Note that this definition is a special case of the more general form (3.44). It relates to the latter equation via

$$N_{\alpha,\beta} = N_{\alpha,\beta}(n), \quad M_{\alpha,\beta} \delta_{n,m} = \Lambda_{\alpha,\beta}(n, m). \tag{6.70}$$

Therefore, we may use (3.46) to obtain the mixed two-point functions

$$\begin{aligned} &G^{(+)}_{\alpha,\beta}(x,x') &(6.71)\\ &= \frac{1}{1-\overline{\beta}\alpha}\Big(G_0^{(+)}(x,x') + \alpha G_0^{(+)}(x^A,x') + \overline{\beta} G_0^{(+)}(x,x'^A) + \alpha\overline{\beta} G_0^{(+)}(x^A,x'^A)\Big), \\ &G^{(-)}_{\alpha,\beta}(x,x') &(6.72)\\ &= \frac{1}{1-\overline{\beta}\alpha}\Big(G_0^{(-)}(x,x') + \overline{\alpha} G_0^{(-)}(x^A,x') + \beta G_0^{(-)}(x,x'^A) + \alpha\overline{\beta} G_0^{(-)}(x^A,x'^A)\Big). \end{aligned}$$

This can be rewritten as

$$\begin{aligned} &G^{(\pm)}_{\alpha,\beta}(x,x') &(6.73)\\ &= \frac{1}{1-\overline{\beta}\alpha}\Big(\tfrac{1+\alpha\overline{\beta}}{2} G_0^{\mathrm{sym}}(x,x') \mp \mathrm{i}\tfrac{1-\alpha\overline{\beta}}{2} G^{\mathrm{PJ}}(x,x') + \tfrac{\alpha+\overline{\beta}}{2} G_0^{\mathrm{sym},A}(x,x') - \mathrm{i}\tfrac{\alpha-\overline{\beta}}{2} G_0^{\mathrm{PJ},A}(x,x')\Big). \end{aligned}$$

The corresponding Feynman and anti-Feynman propagator are

$$\begin{aligned} &G^{\mathrm{F}/\overline{\mathrm{F}}}_{\alpha,\beta}(x,x') &(6.74)\\ &= G_0^{\mathrm{F}/\overline{\mathrm{F}}}(x,x') \pm \frac{\mathrm{i}}{1-\overline{\beta}\alpha}\Big(\alpha\overline{\beta} G_0^{\mathrm{sym}}(x,x') + \mathrm{i}\tfrac{\alpha+\overline{\beta}}{2} G_0^{\mathrm{sym},A}(x,x') - \mathrm{i}\tfrac{\alpha-\overline{\beta}}{2} G_0^{\mathrm{PJ},A}(x,x')\Big). \end{aligned}$$

6.7 "In" and "out" vacua

The deSitter space is not asymptotically stationary. Therefore, the usual definition of "in" and "out" vacua is not applicable. Nevertheless, one can define a pair of deSitter invariant states that deserve to be called the "in" and "out" vacuum. In this subsection we will compute the corresponding propagators.

Every bisolution of the Klein-Gordon equation is a linear combination of appropriately regularized functions $\mathbf{Z}_{\frac{d-2}{2},\mathrm{i}\nu}(Z)$ and $\mathbf{Z}_{\frac{d-2}{2},-\mathrm{i}\nu}(Z)$. They behave for large Z proportionally to $Z^{-\frac{d-1}{2}-\mathrm{i}\nu}$, resp. $Z^{-\frac{d-1}{2}+\mathrm{i}\nu}$. We are looking for two-point functions, which in the "causal asymptotic region", that is for $Z\to\infty$ and $t\to\pm\infty$, have a *definite behavior*, that is, they behave either as $cZ^{-\frac{d-1}{2}-\mathrm{i}\nu}$, or as $cZ^{-\frac{d-1}{2}+\mathrm{i}\nu}$.

Note that the propagators have also the "antipodal asymptotic region": $Z\to-\infty$, $t^A\to\pm\infty$. It will be interesting to determine their behavior of that region as well.

The following theorem describes all deSitter invariant two-point functions with a definite behavior in the causal asymptotic region.

Theorem 6.5. *1.* **Odd dimensions.** *There exists a unique α-vacuum with the propagators behaving as*

$$G^{(\pm)}_{\alpha} \sim cZ^{-\frac{d-1}{2}\pm\mathrm{i}\nu},\quad Z\to+\infty,\quad t\to-\infty; \tag{6.75}$$

$$\textit{and}\quad G^{(\pm)}_{\alpha} \sim cZ^{-\frac{d-1}{2}\mp\mathrm{i}\nu},\quad Z\to+\infty,\quad t\to+\infty. \tag{6.76}$$

These functions vanish for $Z < -1$ and their parameter α is

$$\alpha_- = \alpha_+ = \alpha_{\rm as} := (-1)^{\frac{d+1}{2}} \mathrm{e}^{-\pi\nu} = \mathrm{e}^{-\pi\nu \pm \mathrm{i}\pi\frac{d+1}{2}}. \tag{6.77}$$

This vacuum could be called the "in" vacuum or the "out" vacuum. We will call it the asymptotic vacuum. *We will write* as *instead of $\alpha_{\rm as}$ in the subscripts of propagators and two-point functions. The two point functions of these states are*

$$\begin{aligned} &\frac{\mathrm{i}\sinh\pi\nu}{2^{\frac{d-3}{2}}\sqrt{\pi}C_{d,\nu}} G_{\rm as}^{(\pm)}(x,x') \\ =& \frac{2^{-\mathrm{i}\nu}\theta(\pm t)}{\Gamma\big(\frac{d-1}{2}-\mathrm{i}\nu\big)}\Big(\mathbf{Z}_{\frac{d-2}{2},\mathrm{i}\nu}(-Z-\mathrm{i}0) - \mathbf{Z}_{\frac{d-2}{2},\mathrm{i}\nu}(-Z+\mathrm{i}0)\Big) \\ &+ \frac{2^{\mathrm{i}\nu}\theta(\mp t)}{\Gamma\big(\frac{d-1}{2}+\mathrm{i}\nu\big)}\Big(\mathbf{Z}_{\frac{d-2}{2},-\mathrm{i}\nu}(-Z-\mathrm{i}0) - \mathbf{Z}_{\frac{d-2}{2},-\mathrm{i}\nu}(-Z+\mathrm{i}0)\Big), \end{aligned} \tag{6.78}$$

and their Feynman and anti-Feynman propagators coincide with the operator-theoretic ones from (6.33) *and* (6.34)*:*

$$G_{\rm as}^{\rm F}(x,x') = G_{\rm op}^{\rm F}(x,x'), \quad G_{\rm as}^{\overline{\rm F}}(x,x') = G_{\rm op}^{\overline{\rm F}}(x,x'). \tag{6.79}$$

2. **Even dimensions.** *There exist two α-vacua that satisfy* (6.75) *and* (6.76)*. One of the two values is*

$$\alpha_- = \mathrm{i}\mathrm{e}^{-\pi\nu}(-1)^{\frac{d}{2}} = \mathrm{e}^{-\pi\nu + \mathrm{i}\pi\frac{d+1}{2}} \tag{6.80}$$

and its positive/negative frequency solutions vanish for $Z < -1$, $t^A < 0$. It will be called the "in" vacuum.

The other value is

$$\alpha_+ = -\mathrm{i}\mathrm{e}^{-\pi\nu}(-1)^{\frac{d}{2}} = \mathrm{e}^{-\pi\nu - \mathrm{i}\pi\frac{d+1}{2}} = -\alpha_- \tag{6.81}$$

and its positive/negative frequency solutions vanish for $Z < -1$, $t^A > 0$. It will be called the "out" vacuum.

We will write $-$, resp. $+$ instead of α_- and α_+ in subscripts. The two-point functions of these states are

$$\begin{aligned} &\frac{\mathrm{i}\sinh\pi\nu}{2^{\frac{d-3}{2}}\sqrt{\pi}C_{d,\nu}} G_{-}^{(\pm)}(x,x') \\ =& \frac{2^{-\mathrm{i}\nu}\theta(\pm t)}{\Gamma\big(\frac{d-1}{2}-\mathrm{i}\nu\big)}\Big(\mathbf{Z}_{\frac{d-2}{2},\mathrm{i}\nu}(-Z-\mathrm{i}0) - \mathbf{Z}_{\frac{d-2}{2},\mathrm{i}\nu}(-Z+\mathrm{i}0)\Big) \\ &+ \frac{2^{\mathrm{i}\nu}\theta(\mp t)}{\Gamma\big(\frac{d-1}{2}+\mathrm{i}\nu\big)}\Big(\mathbf{Z}_{\frac{d-2}{2},-\mathrm{i}\nu}(-Z-\mathrm{i}0) - \mathbf{Z}_{\frac{d-2}{2},-\mathrm{i}\nu}(-Z+\mathrm{i}0)\Big) \\ &+ \frac{(-1)^{\frac{d}{2}}\theta(t^A)}{2^{\frac{d-3}{2}}\sqrt{\pi}}\Big(\mathbf{S}_{\frac{d-2}{2},\mathrm{i}\nu}(Z+\mathrm{i}0) - \mathbf{S}_{\frac{d-2}{2},\mathrm{i}\nu}(Z-\mathrm{i}0)\Big), \end{aligned} \tag{6.82}$$

and

$$\frac{\mathrm{i}\sinh\pi\nu}{2^{\frac{d-3}{2}}\sqrt{\pi}C_{d,\nu}}G_{+}^{(\pm)}(x,x') \tag{6.83}$$
$$= \frac{2^{-\mathrm{i}\nu}\theta(\pm t)}{\Gamma\big(\frac{d-1}{2}-\mathrm{i}\nu\big)}\Big(\mathbf{Z}_{\frac{d-2}{2},\mathrm{i}\nu}(-Z-\mathrm{i}0)-\mathbf{Z}_{\frac{d-2}{2},\mathrm{i}\nu}(-Z+\mathrm{i}0)\Big)$$
$$+\frac{2^{\mathrm{i}\nu}\theta(\mp t)}{\Gamma\big(\frac{d-1}{2}+\mathrm{i}\nu\big)}\Big(\mathbf{Z}_{\frac{d-2}{2},-\mathrm{i}\nu}(-Z-\mathrm{i}0)-\mathbf{Z}_{\frac{d-2}{2},-\mathrm{i}\nu}(-Z+\mathrm{i}0)\Big)$$
$$+\frac{(-1)^{\frac{d}{2}}\theta(-t^{A})}{2^{\frac{d-3}{2}}\sqrt{\pi}}\Big(\mathbf{S}_{\frac{d-2}{2},\mathrm{i}\nu}(Z+\mathrm{i}0)-\mathbf{S}_{\frac{d-2}{2},\mathrm{i}\nu}(Z-\mathrm{i}0)\Big).$$

The out-in Feynman and the in-out anti-Feynman propagator coincide with the operator-theoretic Feynman, resp. anti-Feynman propagator, (6.39)*, resp.* (6.40)*:*

$$G^{\mathrm{F}}_{+-}=G^{\mathrm{F}}_{\mathrm{op}};\quad G^{\overline{\mathrm{F}}}_{-+}=G^{\overline{\mathrm{F}}}_{\mathrm{op}}. \tag{6.84}$$

Remark 6.6. The concrete values for α corresponding to "in" and "out" states are well-known [15, 61] but typically derived by asymptotic properties of the modes. We derive them in the following using a "global picture".

Proof of Thm 6.5. Due to our preparations in Section 6.5, it is easy to write the general bisolution $G^{\mathrm{bisol}}_{\underline{a}}(x,x')$ in terms of $\mathbf{Z}_{\frac{d-2}{2},\mathrm{i}\nu}$ and $\mathbf{Z}_{\frac{d-2}{2},-\mathrm{i}\nu}$. Using (6.62) to express the values $a_1,\dots,a_4$ of $G^{\mathrm{bisol}}_{\underline{a}}$ in terms of α, we obtain

$$2\mathrm{i}(1-|\alpha|^2)\frac{\sinh\pi\nu}{2^{\frac{d-3}{2}}\sqrt{\pi}C_{d,\nu}}G^{(\pm)}_{\alpha} \tag{6.85}$$
$$=-\frac{2^{-\mathrm{i}\nu}}{\Gamma\big(\frac{d-1}{2}-\mathrm{i}\nu\big)}\mathbf{Z}_{\frac{d-2}{2},\mathrm{i}\nu}(-Z+\mathrm{i}0)\Big(\big(1+|\alpha|^2\pm(1-|\alpha|^2)\operatorname{sgn}(t)\big)$$
$$+\big(2\operatorname{Re}(\alpha)+2\mathrm{i}\operatorname{Im}(\alpha)\operatorname{sgn}(t^{A})\big)\mathrm{e}^{\mathrm{i}\pi\left(\frac{d-1}{2}+\mathrm{i}\nu\right)}\Big)$$
$$-\frac{2^{-\mathrm{i}\nu}}{\Gamma\big(\frac{d-1}{2}-\mathrm{i}\nu\big)}\mathbf{Z}_{\frac{d-2}{2},\mathrm{i}\nu}(-Z-\mathrm{i}0)\Big(\big(1+|\alpha|^2\mp(1-|\alpha|^2)\operatorname{sgn}(t)\big)$$
$$+\big(2\operatorname{Re}(\alpha)-2\mathrm{i}\operatorname{Im}(\alpha)\operatorname{sgn}(t^{A})\big)\mathrm{e}^{-\mathrm{i}\pi\left(\frac{d-1}{2}+\mathrm{i}\nu\right)}\Big)$$
$$-(\nu\leftrightarrow-\nu).$$

The analysis of the asymptotic behavior of the latter function differs in odd and even dimensions.

Odd dimensions In this case, $\frac{d-1}{2}$ is an integer and we obtain

$$\text{Solution } G_\alpha^{(\pm)} \sim \mathbf{Z}_{\frac{d-2}{2},\mathrm{i}\nu}(Z) \text{ for } Z > 1: \tag{6.86a}$$
$$(-1)^{\frac{d+1}{2}} 2\,\mathrm{Re}(\alpha) = \mathrm{e}^{\mp\,\mathrm{sgn}(t)\pi\nu} + |\alpha|^2 \mathrm{e}^{\pm\,\mathrm{sgn}(t)\pi\nu},$$
$$\text{Solution } G_\alpha^{(\pm)} \sim \mathbf{Z}_{\frac{d-2}{2},-\mathrm{i}\nu}(Z) \text{ for } Z > 1: \tag{6.86b}$$
$$(-1)^{\frac{d+1}{2}} 2\,\mathrm{Re}(\alpha) = \mathrm{e}^{\pm\,\mathrm{sgn}(t)\pi\nu} + |\alpha|^2 \mathrm{e}^{\mp\,\mathrm{sgn}(t)\pi\nu},$$
$$\text{Solution } G_\alpha^{(\pm)} \sim \mathbf{Z}_{\frac{d-2}{2},\mathrm{i}\nu}(-Z) \text{ for } Z < -1: \tag{6.86c}$$
$$(-1)^{\frac{d+1}{2}} (1 + |\alpha|^2) = 2\,\mathrm{Re}(\alpha)\cosh\pi\nu + 2\mathrm{i}\,\mathrm{Im}(\alpha)\,\mathrm{sgn}(t^A)\sinh\pi\nu,$$
$$\text{Solution } G_\alpha^{(\pm)} \sim \mathbf{Z}_{\frac{d-2}{2},-\mathrm{i}\nu}(-Z) \text{ for } Z < -1: \tag{6.86d}$$
$$(-1)^{\frac{d+1}{2}} (1 + |\alpha|^2) = 2\,\mathrm{Re}(\alpha)\cosh\pi\nu - 2\mathrm{i}\,\mathrm{Im}(\alpha)\,\mathrm{sgn}(t^A)\sinh\pi\nu.$$

Now equations (6.86c) and (6.86d) imply $\mathrm{Im}(\alpha) = 0$ and

$$\frac{1}{2}\left(\alpha + \frac{1}{\alpha}\right) = (-1)^{\frac{d+1}{2}} \cosh\pi\nu \quad \Rightarrow \quad \alpha = (-1)^{\frac{d+1}{2}} \mathrm{e}^{-\pi\nu}. \tag{6.87}$$

Furthermore, $G_\alpha^{(\pm)}|_{Z<-1} = 0$ if α solves (6.86c) and (6.86d) with $\mathrm{Im}(\alpha) = 0$. Finally, inserting (6.87) into (6.86a) and (6.86b) implies

$$\begin{aligned} G_\alpha^{(\pm)} &\sim \mathbf{Z}_{\frac{d-2}{2},\pm\mathrm{i}\nu}(Z), \quad Z > 1,\ t > 0; \\ G_\alpha^{(\pm)} &\sim \mathbf{Z}_{\frac{d-2}{2},\mp\mathrm{i}\nu}(Z), \quad Z > 1,\ t < 0. \end{aligned} \tag{6.88}$$

Inserting the obtained value for α into (6.85) yields the explicit formulas for $G_{\mathrm{as}}^{(\pm)}$. The (anti-)Feynman propagator is $\pm\mathrm{i}$ times the (anti-)time-ordered two-point function and it is easy to see that they coincide with $G_{\mathrm{op}}^{\mathrm{F}/\bar{\mathrm{F}}}$.

Even dimensions The situation is more divers in even dimensions. The conditions become:

$$\text{Solution } G^{(\pm)}_\alpha \sim \mathbf{Z}_{\frac{d-2}{2},\mathrm{i}\nu}(Z) \text{ for } Z>1: \tag{6.89a}$$
$$\mp(-1)^{\frac{d-2}{2}} 2\mathrm{i}\,\mathrm{sgn}(t)\,\mathrm{Re}(\alpha) = \mathrm{e}^{\mp\,\mathrm{sgn}(t)\pi\nu} - |\alpha|^2 \mathrm{e}^{\pm\,\mathrm{sgn}(t)\pi\nu},$$
$$\text{Solution } G^{(\pm)}_\alpha \sim \mathbf{Z}_{\frac{d-2}{2},-\mathrm{i}\nu}(Z) \text{ for } Z>1: \tag{6.89b}$$
$$\mp(-1)^{\frac{d-2}{2}} 2\mathrm{i}\,\mathrm{sgn}(t)\,\mathrm{Re}(\alpha) = \mathrm{e}^{\pm\,\mathrm{sgn}(t)\pi\nu} - |\alpha|^2 \mathrm{e}^{\mp\,\mathrm{sgn}(t)\pi\nu},$$
$$\text{Solution } G^{(\pm)}_\alpha \sim \mathbf{Z}_{\frac{d-2}{2},\mathrm{i}\nu}(-Z) \text{ for } Z<-1: \tag{6.89c}$$
$$(-1)^{\frac{d-2}{2}}(1+|\alpha|^2) = -2\mathrm{i}\,\mathrm{Re}(\alpha)\sinh\pi\nu + 2\,\mathrm{Im}(\alpha)\,\mathrm{sgn}(t^A)\cosh\pi\nu,$$
$$\text{Solution } G^{(\pm)}_\alpha \sim \mathbf{Z}_{\frac{d-2}{2},-\mathrm{i}\nu}(-Z) \text{ for } Z<-1: \tag{6.89d}$$
$$(-1)^{\frac{d-2}{2}}(1+|\alpha|^2) = 2\mathrm{i}\,\mathrm{Re}(\alpha)\sinh\pi\nu + 2\,\mathrm{Im}(\alpha)\,\mathrm{sgn}(t^A)\cosh\pi\nu.$$

We immediately read off $\mathrm{Re}(\alpha)=0$. Then, by (6.89a) and (6.89b), the existence of a definite behavior in the region $Z>1$ implies $|\alpha| = \mathrm{e}^{-\pi\nu}$. Hence $\alpha = \mathrm{e}^{\mathrm{i}\pi(n+\frac{1}{2})-\pi\nu}$ with $n\in\mathbb{Z}$. Then (6.89c) and (6.89d) simplify to

$$(-1)^{\frac{d-2}{2}-n} = \mathrm{sgn}(t^A). \tag{6.90}$$

$n=\frac{d-2}{2}$ yields a solution that vanishes for $Z<-1$ and $t^A>0$ but has indeterminate behavior as $Z<-1$ and $t^A\to-\infty$, while $n=\frac{d}{2}$ yields a solution that vanishes for $Z<-1$ and $t^A<0$ but has indeterminate behavior as $Z<-1$ and $t^A\to+\infty$. We obtain the values for α_+ and α_-.

Inserting the obtained values for α into (6.85) yields the explicit formulas for $G^{(\pm)}_\pm$: this rather cumbersome computation involves the connection formula (B.14), the identity (B.15) and repeated use of identities of the type $1\pm\mathrm{sgn}(\cdot)=2\theta(\pm\cdot)$. The (anti-)Feynman propagators are obtained from (6.74) and also using the connection formulas. □

Parts of the literature, cf. for example [43], also sometimes consider the "in-in" or "out-out" Feynman propagators. In odd dimensions, all these objects agree with the operator-theoretic Feynman propagator. In even dimension, there is a zoo of various propagators. Inserting the respective values for α_+ and α_- into (6.74) and making use of the connection formulas (B.12), (B.13) and (B.14) as well as identities of the type $1\pm\mathrm{sgn}(\cdot)=2\theta(\pm\cdot)$, one finds

$$G^{\mathrm{F}}_{--}(x,x') = \frac{\Gamma\left(\frac{d-1}{2}+\mathrm{i}\nu\right)}{2^{2+\mathrm{i}\nu}(2\pi)^{\frac{d-1}{2}}\sinh\pi\nu}\Big(\mathbf{Z}_{\frac{d-2}{2},\mathrm{i}\nu}(-Z-\mathrm{i}0) - \mathbf{Z}_{\frac{d-2}{2},\mathrm{i}\nu}(-Z+\mathrm{i}0)\Big) \tag{6.91}$$
$$+\frac{(-1)^{\frac{d}{2}}\theta(t^A)C_{d,\nu}}{\sinh\pi\nu}\Big(\mathbf{S}_{\frac{d-2}{2},\mathrm{i}\nu}(Z+\mathrm{i}0) - \mathbf{S}_{\frac{d-2}{2},\mathrm{i}\nu}(Z-\mathrm{i}0)\Big),\quad d\text{ even},$$

and

$$G^{\rm F}_{++}(x,x') = \frac{\Gamma\big(\frac{d-1}{2}+{\rm i}\nu\big)}{2^{2+{\rm i}\nu}(2\pi)^{\frac{d-1}{2}}\sinh\pi\nu}\Big({\bf Z}_{\frac{d-2}{2},{\rm i}\nu}(-Z-{\rm i}0) - {\bf Z}_{\frac{d-2}{2},{\rm i}\nu}(-Z+{\rm i}0)\Big) \tag{6.92}$$
$$+\frac{(-1)^{\frac{d}{2}}\theta(-t^A)C_{d,\nu}}{\sinh\pi\nu}\Big({\bf S}_{\frac{d-2}{2},{\rm i}\nu}(Z+{\rm i}0) - {\bf S}_{\frac{d-2}{2},{\rm i}\nu}(Z-{\rm i}0)\Big), \quad d \text{ even}.$$

The formulas for $G^{\overline{\rm F}}_{--}(x,x')$ resp. $G^{\overline{\rm F}}_{++}(x,x')$ are obtained by replacing ν with $-\nu$ in the formulas for $G^{\rm F}_{--}(x,x')$ resp. $G^{\rm F}_{++}(x,x')$. Note that the second lines of (6.91) and (6.92) are supported in the region $Z < -1$, $t^A \gtrless 0$.

6.8 Symmetric Scarf Hamiltonian

We will discuss in the next subsection another approach to the Klein-Gordon equation on the deSitter space. In this approach we will use the one-dimensional Schrödinger Hamiltonian on $L^2(\mathbb{R})$ of the form

$$H^{\rm S}_\alpha := -\partial_\tau^2 - \frac{\alpha^2-\frac14}{\cosh(\tau)^2}. \tag{6.93}$$

It is sometimes called *symmetric Scarf Hamiltonian* [38]. It is well-known that this Hamiltonian for some values of parameters is reflectionless. For competeness, let us verify this.

First we check that $H^{\rm S}_\alpha + \lambda^2$ is equivalent to the Gegenbauer equation after the consecutive change of variables $\sinh\tau = w$, ${\rm i}w = v$:

$$\begin{aligned} &\cosh(\tau)^{-\alpha-\frac12}\big(H^{\rm S}_\alpha+\lambda^2\big)\cosh(\tau)^{\alpha+\frac12} \\ &= -\partial_\tau^2 - (2\alpha+1)\tanh(\tau)\partial_\tau - \big(\alpha+\tfrac12\big)^2 + \lambda^2 \\ &= -(1+w^2)\partial_w^2 - 2(\alpha+1)w\partial_w - \big(\alpha+\tfrac12\big)^2+\lambda^2 \\ &= (1-v^2)\partial_v^2 - 2(\alpha+1)v\partial_v - \big(\alpha+\tfrac12\big)^2+\lambda^2. \end{aligned} \tag{6.94}$$

For ${\rm Re}(\lambda) > 0$, the Jost solutions can thus be expressed in terms of the Gegenbauer Z-function:

$$\psi_\pm(\lambda,t) = 2^{\mp\lambda}\Gamma(1\pm\lambda){\rm e}^{{\rm i}\frac{\pi}{2}(\frac12+\alpha\pm\lambda)}\cosh(\tau)^{\alpha+\frac12}Z_{\alpha,\pm\lambda}(\pm{\rm i}\sinh\tau), \tag{6.95}$$

such that

$$\psi_\pm(\lambda,\tau) \sim {\rm e}^{\mp\lambda\tau}, \quad \pm\tau\to\infty. \tag{6.96}$$

The Gegenbauer functions on the righthand-side of (6.95) have purely imaginary arguments. They are to be interpreted as living on the cut plane $\mathbb{C}\setminus\big(]-\infty,-1]\cup[1,\infty[\big)$ instead of the usual $\mathbb{C}\backslash]-\infty,1]$. $\psi_+(\lambda,\cdot)$ is expressed in terms of the analytic continuation of $Z_{\alpha,\lambda}(w)$ defined on the standard sheet $\mathbb{C}\backslash]-\infty,1]$ *to the upper half-plane*, while $\psi_-(\lambda,\cdot)$ is expressed in terms of the analytic continuation of $Z_{\alpha,-\lambda}(w)$ defined on the standard sheet $\mathbb{C}\backslash]-\infty,1]$

to the lower half plane. Using the connection formulas (B.13) and (B.14), and the fact that $\mathbf{S}_{\alpha,\lambda}$ is holomorphic on $]-1,1[$, one can derive a connection formula for the two holomorphic continuations:

$$\begin{aligned}&\mathbf{Z}_{\alpha,\lambda}(w+\mathrm{i}0) \\ =\;&\frac{\mathrm{i}\cos\pi\alpha\,\mathrm{e}^{-\mathrm{i}\pi(\alpha+\lambda)}\mathbf{Z}_{\alpha,\lambda}(w-\mathrm{i}0)}{\sin\pi\lambda}-\frac{\mathrm{i}2^{2\lambda}\mathrm{e}^{-\mathrm{i}\pi\alpha}\pi\mathbf{Z}_{\alpha,-\lambda}(w-\mathrm{i}0)}{\Gamma\big(\frac12+\alpha+\lambda\big)\Gamma\big(\frac12-\alpha+\lambda\big)\sin\pi\lambda},\quad w\in]-1,1[.\end{aligned}\tag{6.97}$$

In particular, $\mathbf{Z}_{\alpha,\lambda}(w+\mathrm{i}0)$ is proportional to $\mathbf{Z}_{\alpha,-\lambda}(w-\mathrm{i}0)$ if and only if $\cos\pi\alpha=0$, i.e., if and only if $\alpha\in\mathbb{Z}+\frac12$.

Consequently, the symmetric Scarf Hamiltonian is reflectionless for all energies ν^2 iff $\alpha\in\mathbb{Z}+\frac12$.

6.9 Partial wave decomposition

Using the global system of coordinates (6.7), the deSitter space can be viewed as a FLRW space, and can be identified with $\mathbb{R}\times\mathbb{S}^{d-1}$. In these coordinates, the (gauged) Klein-Gordon operator takes the form

$$\begin{aligned}&\cosh(\tau)^{\frac{d-1}{2}}\big(-\Box_g+m^2\big)\cosh(\tau)^{-\frac{d-1}{2}} \\ =\;&\partial_\tau^2-\frac{d-1}{2}\Big(1+\frac{(d-3)\sinh(\tau)^2}{2\cosh(\tau)^2}\Big)-\frac{\Delta_{\mathbb{S}^{d-1}}}{\cosh(\tau)^2}+m^2 \\ =\;&\partial_\tau^2+\frac{\big(\frac{d-2}{2}\big)^2-\frac14-\Delta_{\mathbb{S}^{d-1}}}{\cosh(\tau)^2}+\nu^2\end{aligned}\tag{6.98}$$

The spectrum of $-\Delta_{\mathbb{S}^{d-1}}$ is $\{l(l+d-2)\mid l\in\mathbb{N}_0\}$. Hence, restricted to eigenfunctions with eigenvalue $l(l+d-2)$, the above operator becomes $-H_\alpha^{\mathrm{S}}+\nu^2$, where H_α^{S} is the *symmetric Scarf Hamiltonian* with $\alpha=l+\frac{d-2}{2}$. The symmetric Scarf potential is reflectionless for all energies $\nu^2\in\mathbb{R}$ and $\alpha\in\frac12+\mathbb{Z}$. This corresponds to odd dimensions. Thus for each mode the in state coincides with the out state. In even dimensions $\alpha\in\mathbb{Z}$, and then for each mode the in state is different from the out state.

7 Anti-deSitter space and its universal cover

Our final example of Lorentzian manifolds are the d-dimensional anti-deSitter space AdS_d and its universal covering $\widetilde{\mathrm{AdS}}_d$.

AdS_d is pathological from several points of view. First of all, it has time loops, which makes it unsuitable as a model of a spacetime. It does not make much sense to speak about propagators on AdS_d. However, it is instructive to consider its d'Alembertian as a self-adjoint operator and to compute its spectrum and the kernel of the resolvent (Green function), which we will do in this section.

The cyclicity of time can be cured by replacing the proper anti-deSitter space by its universal cover $\widetilde{\mathrm{AdS}}_d$. It is still not globally hyperbolic, because of a timelike boundary at spacelike infinity. However the latter problem is not very serious, and various propagators can be defined on $\widetilde{\mathrm{AdS}}_d$.

Therefore, most of this section will be devoted to $\widetilde{\mathrm{AdS}}_d$. We will apply two methods to define propagators: through the resolvent of the d'Alembertian on $L^2(\widetilde{\mathrm{AdS}}_d)$, and by considering the evolution of the Cauchy data. The latter approach is facilitated by the fact that $\widetilde{\mathrm{AdS}}_d$ is static. The absence of global hyperbolicity is not a problem for the first approach. For the second approach it manifests itself by the need to set boundary conditions at the spatial infinity for m^2 below a certain value.

One can view anti-deStter space as a Wick rotated hyperbolic space $\mathbb{H}^d$. This is not very useful for AdS_d, but works for $\widetilde{\mathrm{AdS}}_d$.

Various propagators of massive scalar fields on AdS_d and $\widetilde{\mathrm{AdS}}_d$ have been intensively studied. Among the vast literature, we mention the references [1, 3, 5, 16, 18, 24, 29, 30, 40, 55, 69], which are particularly useful for understanding the analytic structure. Similar to the deSitter example, the only of these references using the operator-theoretic view on the Feynman propagator is [69] (here in two dimensions). The references [16, 18] have an axiomatic approach. Appendix A of [3] is particularly helpful to understand the analytic structure of propagators on the universal cover. Subsection 7.3.5, where we present the approach based on the evolution of Cauchy data, is based on the seminal work [55].

7.1 Geometry of anti-deSitter space and its universal cover

The d-dimensional anti-deSitter space AdS_d can be defined as an embedded submanifold of $\mathbb{R}^{2,d-1}$:

$$\mathrm{AdS}_d = \{x \in \mathbb{R}^{2,d-1} \mid \langle x|x\rangle = -1\}, \tag{7.1}$$

where

$$\langle x|x'\rangle := -x^0 x'^0 - x^d x'^d + \sum_{i=1}^{d-1} x^i x'^i =: Z(x,x') \equiv Z. \tag{7.2}$$

A coordinate system covering all of AdS_d is given by

$$x^0 = \cosh\rho\cos\tau, \quad x^i = \sinh\rho\,\Omega^i, \quad x^d = \cosh\rho\sin\tau, \tag{7.3}$$
$$\text{where} \quad \tau \in [-\pi,\pi[,\ \rho\in\mathbb{R}_{\geq 0},\ \Omega\in\mathbb{S}_{d-2}\hookrightarrow\mathbb{R}^{d-1} \quad \text{and} \quad i=1,\dots,d-1.$$

In these coordinates, the line element reads

$$\mathrm{d}s^2 = -\cosh(\rho)^2\mathrm{d}\tau^2 + \mathrm{d}\rho^2 + \sinh(\rho)^2\mathrm{d}\Omega^2. \tag{7.4}$$

Note the famous cyclicity of time, $x(\tau+2\pi k,\rho,\Omega) = x(\tau,\rho,\Omega)$ for all $k\in\mathbb{Z}$. Therefore, AdS_d has closed timelike curves and is not globally hyperbolic.

AdS_d is equipped with an involution $x \mapsto -x$. This involution maps the coordinates (τ,ρ,Ω) to $(\tau+\pi,\rho,-\Omega)$.

Another system of coordinates is obtained by replacing ρ with $u \in [0, \frac{\pi}{2}[$, where $\sinh\rho = \tan u$. In these coordinates, the line element (7.4) becomes

$$\mathrm{d}s^2 = \frac{-\mathrm{d}\tau^2 + \mathrm{d}u^2 + \sin(u)^2\,\mathrm{d}\Omega^2}{\cos(u)^2}. \tag{7.5}$$

In the coordinates (7.3) and (7.5), we find

$$Z = -\cosh\rho\cosh\rho'\cos(\tau-\tau') + \sinh\rho\sinh\rho'\cos\theta \tag{7.6}$$

$$= -\frac{\cos(\tau-\tau')}{\cos u\cos u'} + \frac{\sin u\sin u'\cos\theta}{\cos u\cos u'}. \tag{7.7}$$

where θ is the angle between Ω and Ω'.

Let us fix the vector $x' = (1, 0\ldots, 0)$. Then $-\langle x|x'\rangle = \frac{\cos\tau}{\cos u}$ and we can partition AdS_d into the following regions:

$$V_0 := \{|\tau| < u\}, \tag{7.8a}$$

$$V_2 := \{\pi - \tau < u\} \cup \{\pi + \tau < u\}, \tag{7.8b}$$

$$V_1 := \{\min(\tau, \pi - \tau) > u\}, \tag{7.8c}$$

$$V_{-1} := \{\min(-\tau, \pi + \tau) > u\}. \tag{7.8d}$$

Note that

$$-Z > 1, \quad \tau \in [-\tfrac{\pi}{2}, \tfrac{\pi}{2}] \qquad \text{on } V_0, \tag{7.9a}$$

$$Z > 1, \quad \tau \in [-\pi, -\tfrac{\pi}{2}] \cup [\tfrac{\pi}{2}, \pi] \qquad \text{on } V_2, \tag{7.9b}$$

$$|Z| < 1, \quad \tau \in [0, \pi] \qquad \text{on } V_1, \tag{7.9c}$$

$$|Z| < 1, \quad \tau \in [-\pi, 0] \qquad \text{on } V_{-1}. \tag{7.9d}$$

AdS_d has the topology of $\mathbb{S}_1 \times \mathbb{R}^{d-1}$. Therefore it has a universal covering space

$$\widetilde{\mathrm{AdS}}_d \to \mathrm{AdS}_d. \tag{7.10}$$

In the literature, this universal cover is sometimes called anti-deSitter space instead [50]. We will, however, use the name anti-deSitter space for the embedded submanifold (7.1), adding the adjective “proper” whenever we think it is necessary to avoid confusion.

It is easy to describe $\widetilde{\mathrm{AdS}}_d$ in coordinates: we just assume that $\tau \in \mathbb{R}$, and keep the line element (7.4) or (7.5). $\widetilde{\mathrm{AdS}}_d$ is a static Lorentzian manifold. It is still not globally hyperbolic, since there are geodesics, which in finite time escape to its boundary.

Let us fix the vector $x' = (1, 0\ldots, 0)$. Then we can partition $\widetilde{\mathrm{AdS}}_d$ into the following regions:

$$V_{2n} := \{|\tau - n\pi| < u\}, \tag{7.11}$$

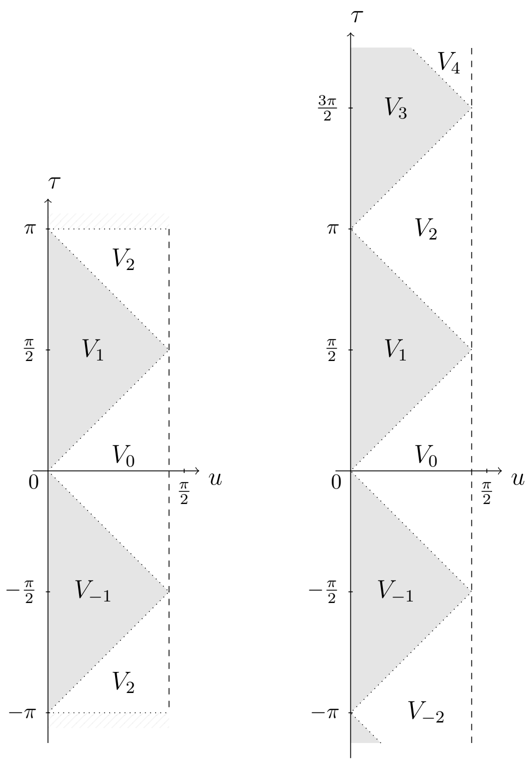
$$V_{2n+1} := \{\min(\tau - n\pi, (n+1)\pi - \tau) > u\}. \tag{7.12}$$

Note that

$$-(-1)^n Z > 1, \quad \tau \in [(n - \tfrac{1}{2})\pi, (n + \tfrac{1}{2})\pi] \qquad \text{on } V_{2n} \tag{7.13}$$

$$|Z| < 1, \quad \tau \in [n\pi, (n+1)\pi] \qquad \text{on } V_{2n+1}. \tag{7.14}$$

The spaces AdS_d and $\widetilde{\mathrm{AdS}}_d$ with their various regions are depicted in Figure 2.



(a) (b)

Figure 2: (a) Anti-deSitter space in the coordinates $u \in [0, \frac{\pi}{2}[$ and $\tau \in]-\pi, \pi]$ from (7.5) and its partition into the regions V_0, V_2, V_1 and V_{-1}. Each point represents a $d-2$-sphere of the coordinates Ω. The lines $\tau = \pi$ and $\tau = -\pi$ are glued together, reflecting the cyclicity of time. An observer can reach spatial infinity ($u = \frac{\pi}{2}$, indicated by the dashed line) in finite time, which makes it necessary to impose boundary conditions when solving the Cauchy problem for certain masses, see Section 7.3.5. (b) The universal cover of anti-deSitter space in the same coordinates, where however τ ranges over all of $\mathbb{R}$, removing the cylicity of time. The time-like boundary at $u = \frac{\pi}{2}$ is still present.

7.2 (Proper) anti-deSitter space

7.2.1 Invariant Green functions on anti-deSitter space

Consider first the Klein-Gordon equation on the (proper) anti-deSitter space AdS_d:

$$(-\Box + m^2)\phi(x) = 0. \tag{7.15}$$

Instead of m we will use the parameter ν

$$\nu := \sqrt{m^2 + \left(\tfrac{d-1}{2}\right)^2}, \tag{7.16}$$

where as usual we use the principal branch of the square root. Thus (7.15) is replaced with

$$\Big(-\Box - \left(\tfrac{d-1}{2}\right)^2 + \nu^2 \Big)\phi(x) = 0. \tag{7.17}$$

Set

$$C_{d,\mathrm{i}\nu} = \frac{\Gamma\left(\frac{d-1}{2} + \nu\right)\Gamma\left(\frac{d-1}{2} - \nu\right)}{(4\pi)^{\frac{d}{2}}}, \tag{7.18}$$

$$G^{\mathrm{sym}}(x, x') := \mathrm{i}C_{d,\mathrm{i}\nu}\big(\mathbf{S}^{\mathrm{sym}}_{\frac{d-2}{2},\nu}(Z + \mathrm{i}0) + \mathbf{S}_{\frac{d-2}{2},\nu}(Z + \mathrm{i}0)\big), \tag{7.19}$$

$$G^{\mathrm{sym},A}(x, x') := G^{\mathrm{sym}}(x^A, x') = \mathrm{i}C_{d,\mathrm{i}\nu}\big(\mathbf{S}_{\frac{d-2}{2},\nu}(-Z + \mathrm{i}0) + \mathbf{S}_{\frac{d-2}{2},\nu}(-Z - \mathrm{i}0)\big). \tag{7.20}$$

Lemma 7.1. *A general form of a fundamental solution invariant with respect to the full anti-deSitter group is*

$$\begin{aligned} G_{ab}(x,x') =& \mathrm{i}C_{d,\mathrm{i}\nu}\mathbf{S}_{\frac{d-2}{2},\nu}(Z+\mathrm{i}0) \\ &+ aG^{\mathrm{sym}}(x,x') + bG^{\mathrm{sym},A}(x,x'), \quad a,b\in\mathbb{C}. \end{aligned} \tag{7.21}$$

Proof. Consider

$$G_{\pm}(x,x') := \pm\mathrm{i}C_{d,\mathrm{i}\nu}\mathbf{S}_{\frac{d-2}{2},\nu}(Z+\mathrm{i}0). \tag{7.22}$$

Away from the diagonal they clearly satisfy the Klein-Gordon equation. We have

$$Z := \langle x|x'\rangle = -1 - \frac{\langle x-x'|x-x'\rangle}{2}, \quad x,x'\in \mathrm{AdS}_d. \tag{7.23}$$

Therefore, we easily see that at the diagonal they have the same singularity as the usual Feynman and anti-Feynman propagator on the Minkowski space. Therefore, they satisfy the equation for the fundamental solution

$$\Big(-\Box_x - \big(\tfrac{d-1}{2}\big)^2 + \nu^2\Big)G_{\pm}(x,x') = \delta(x,x'). \tag{7.24}$$

Their difference $G^{\mathrm{sym}} = \mathrm{i}(G_+ - G_-)$ is clearly a bisolution, which is invariant under the full anti-deSitter group, because it only depends on Z. The second, linearly independent invariant bisolution is obtained by replacing $Z\to -Z$ in the first bisolution.

We then show that the space of bisolutions is 2-dimensional, following the arguments used for the deSitter space. □

7.2.2 Resolvent of the d'Alembertian

The d'Alembertian is an essentially self-adjoint operator on $C_{\mathrm{c}}^{\infty}(\mathrm{AdS}_d)$. As for deSitter space, this follows from a general theory of invariant differential operators on symmetric spaces [9,68]. Indeed, the anti-deSitter space is a symmetric space $SO(2,d-2)/SO(1,d-2)$.

For $-\nu^2$ outside of the spectrum of $-\Box - \big(\frac{d-1}{2}\big)^2$ on AdS_d we set

$$G(-\nu^2) := \Big(-\Box + \nu^2 - \big(\tfrac{d-1}{2}\big)^2\Big)^{-1}. \tag{7.25}$$

Its integral kernel will be denoted $G(-\nu^2;x,x')$.

Theorem 7.2. *1.* **Odd dimensions.** *The spectrum of* $-\Box - \big(\frac{d-1}{2}\big)^2$ *is*

$$\{-\nu^2 \mid \nu\in\mathbb{N}\}\cup[0,\infty[. \tag{7.26}$$

Away from the spectrum, we have

$$G(-\nu^2;x,x') = \mathrm{i}\frac{\sqrt{\frac{\pi}{2}}\,\Gamma\big(\frac{d-1}{2}+\nu\big)}{2^{\nu+1}(2\pi)^{\frac{d}{2}}\sin(\pi\nu)}\Big(\mathbf{Z}_{\frac{d-2}{2},\nu}(Z-\mathrm{i}0) - \mathbf{Z}_{\frac{d-2}{2},\nu}(Z+\mathrm{i}0)\Big). \tag{7.27}$$

2. **Even dimensions.** *The spectrum of* $-\Box - \left(\frac{d-1}{2}\right)^2$ *is*

$$\{-\nu^2 \mid \nu \in \mathbb{N} - \tfrac{1}{2}\} \cup [0, \infty[. \tag{7.28}$$

Away from the spectrum, we have

$$G(-\nu^2; x, x') = -\frac{\sqrt{\frac{\pi}{2}}\,\Gamma\left(\frac{d-1}{2} + \nu\right)}{2^{\nu+1}(2\pi)^{\frac{d}{2}} \cos(\pi\nu)} \Big(\mathbf{Z}_{\frac{d-2}{2},\nu}(Z - \mathrm{i}0) + \mathbf{Z}_{\frac{d-2}{2},\nu}(Z + \mathrm{i}0)\Big). \tag{7.29}$$

Proof. Let $\mathrm{Re}(\nu) > 0$ and $\mathrm{Im}(\nu) \neq 0$. We repeat the analysis that we did on deSitter space. That is, using the connection formula (B.14), we first expand the Gegenbauer functions $\mathbf{S}_{\frac{d-2}{2},\nu}$ in the general fundamental solution (7.21) in terms of the Gegenbauer functions $\mathbf{Z}_{\frac{d-2}{2},\nu}$ and $\mathbf{Z}_{\frac{d-2}{2},-\nu}$. They have a definite behavior as $|Z| \to \infty$. Note, however, that the parameter is now $\pm\nu$, while on deSitter space it was $\pm\mathrm{i}\nu$. Therefore, for any sign of $\mathrm{Im}(\nu)$, solutions with a sufficiently fast decay behavior must only contain $\mathbf{Z}_{\frac{d-2}{2},\nu}$. This fixes the parameters a and b uniquely. The corresponding Green function are (7.27) and (7.29). The poles of the prefactor give the point spectrum for $\nu \in]0, \infty[$. The limits as $\mathrm{Re}(\nu) \searrow 0$ are different for $\mathrm{Im}(\nu) \gtrless 0$, giving the continuous spectrum. $\square$

7.3 Universal cover of anti-deSitter space

7.3.1 Wick rotation

Anti-deSitter space is closely related to the hyperbolic space

$$\mathbb{H}^d := \{x \in \mathbb{R}^{1,d} \mid [x|x] = -1\}, \tag{7.30}$$

where

$$[x|x'] = -x^0 x'^0 + \sum_{i=1}^{d} x^i x'^i, \tag{7.31}$$

as in (6.1). Let Δ be the Laplace-Beltrami operator on $\mathbb{H}^d$. Set

$$G^{\mathrm{h}}(-\nu^2) := \Big(- \Delta^{\mathrm{h}} - (\tfrac{d-1}{2})^2 + \nu^2\Big)^{-1}. \tag{7.32}$$

For $\mathrm{Re}(\nu) > 0$, the integral kernel of $G^{\mathrm{h}}(-\nu^2)$ can be expressed in terms of the invariant quantity $[x|x']$ and the Gegenbauer function $\mathbf{Z}_{\alpha,\lambda}(w)$ as (see e.g. [31,32], and for an equivalent expression in terms of associated Legendre functions [27])

$$G^{\mathrm{h}}\big(- \nu^2; x, x'\big) = \frac{\sqrt{\pi}\Gamma(\frac{d-1}{2} + \nu)}{\sqrt{2}(2\pi)^{\frac{d}{2}} 2^{\nu}} \mathbf{Z}_{\frac{d}{2}-1,\nu}\big(- [x|x']\big). \tag{7.33}$$

Let us try to introduce a kind of a Wick rotation from $\mathbb{H}^d$ to anti-deSitter space by replacing x^d with $\pm\mathrm{i}x^d$. We have

$$[x|x'] = -1 - \frac{[x-x'|x-x']}{2}, \quad x,x' \in \mathbb{H}^d, \tag{7.34}$$
$$Z := \langle x|x'\rangle = -1 - \frac{\langle x-x'|x-x'\rangle}{2}, \quad x,x' \in \mathrm{AdS}_d.$$

Thus, similar to the case of de Sitter space, we have to replace $-[x|x']$ in the argument of the Gegenbauer function in (7.33) by $-\big(\langle x|x'\rangle \mp \mathrm{i}0\big) = -\langle x|x'\rangle \pm \mathrm{i}0$ and insert a prefactor $\pm\mathrm{i}$ coming from the change of the integral measure. In this way, we obtain

$$\pm\mathrm{i}\frac{\sqrt{\pi}\Gamma(\frac{d-1}{2}+\nu)}{\sqrt{2}(2\pi)^{\frac{d}{2}}2^{\nu}}\mathbf{Z}_{\frac{d}{2}-1,\nu}\big(-Z\pm\mathrm{i}0\big) \tag{7.35}$$

as candidates for (anti-)Feynman propagators on AdS_d.

On the proper anti-deSitter space AdS_d the latter expression cannot be a Green function. In fact, due to the identity (B.15), the application of the Klein-Gordon operator to (7.35) yields a nonzero distribution supported at $\{Z=-1\}\cup\{Z=1\}$ (the diagonal and the antipode of the diagonal).

This problem dissapears on the universal cover $\widetilde{\mathrm{AdS}}_d$ of anti-deSitter space. The expression (7.35), properly continued to further regions, yields a Green function of the Klein-Gordon operator, as we shall see in the next subsection.

7.3.2 Resolvent of the d'Alembertian

The essential self-adjointness of the d'Alembertian $-\Box$ on $C_{\mathrm{c}}^\infty(\widetilde{\mathrm{AdS}}_d)$ is not covered by the references [9, 68]. However, we expect that the methods of above references can be extended to $\widetilde{\mathrm{AdS}}_d$, so that one can show that the d'Alembertian is indeed essentially self-adjoint on $C_{\mathrm{c}}^\infty(\widetilde{\mathrm{AdS}}_d)$. If this is the case, then in this subsection we will find its resolvent $(-\Box+m^2)^{-1}$ for $m^2\in\mathbb{C}\setminus\mathbb{R}$. Indeed, we will see that there is a unique invariant Green function with appropriate decay behavior.

As before, it is convenient to set

$$G(-\nu^2) := \Big(-\Box - \big(\tfrac{d-1}{2}\big)^2 + \nu^2\Big)^{-1}, \tag{7.36}$$

and denote the integral kernel of $G(-\nu^2)$ by $G(-\nu^2;x,x')$.

To describe $G(-\nu^2;x,x')$ explicitly, it is convenient for $n\in\mathbb{Z}$ to introduce open regions

$$W_n := \Big(\big(V_{2n-1}\cup V_{2n}\cup V_{2n+1}\big)^{\mathrm{cl}}\Big)^{\circ}, \quad n\in\mathbb{Z}, \tag{7.37}$$

with V_n as defined in Subsection 7.1 and with cl denoting the closure and $\circ$ the interior. We have $W_n\cap W_{n+1}=V_{2n+1}$ and

$$\widetilde{\mathrm{AdS}}_d = \bigcup_{n\in\mathbb{Z}} W_n. \tag{7.38}$$

Theorem 7.3. *For $\nu^2 \in \mathbb{C} \setminus \mathbb{R}$ and $\mathrm{Re}(\nu) > 0$, the integral kernel of the resolvent (7.36) is given on W_n by the formula*

$$G(-\nu^2; x, x') = \frac{\sqrt{\pi}\Gamma(\frac{d-1}{2}+\nu)}{\sqrt{2}(2\pi)^{\frac{d}{2}}2^{\nu}} \tag{7.39}$$
$$\times \begin{cases} \mathrm{i}\mathrm{e}^{-\mathrm{i}|n|(\frac{d-1}{2}+\nu)\pi}\mathbf{Z}_{\frac{d}{2}-1,\nu}\Big(-(-1)^n Z + (-1)^n \mathrm{i}0s\Big), & \mathrm{Im}\,\nu < 0; \\ -\mathrm{i}\mathrm{e}^{\mathrm{i}|n|(\frac{d-1}{2}+\nu)\pi}\mathbf{Z}_{\frac{d}{2}-1,\nu}\Big(-(-1)^n Z - (-1)^n \mathrm{i}0s\Big), & \mathrm{Im}\,\nu > 0, \end{cases}$$

Here s can be represented by $s = \mathrm{sgn}(\sin(|\tau - \tau'|))$, or

$$\begin{aligned} (x, x') \in V_{2n-1} &\Rightarrow s = (-1)^n \,\mathrm{sgn}(2n-1), \\ (x, x') \in V_{2n} &\Rightarrow s = 0, \\ (x, x') \in V_{2n+1} &\Rightarrow s = (-1)^{n+1} \,\mathrm{sgn}(2n+1). \end{aligned} \tag{7.40}$$

(Note that in V_{2n} we may set $s = -1$ or $s = 1$ because the function is univalent).

Proof. We start from the Wick-rotated expressions (7.35). As discussed in Section 7.3.1, they are no fundamental solutions of the Klein-Gordon equation on AdS_d — but restricted to W_0, they become fundamental solutions. Any fundamental solution on W_0 must differ from (7.35) by a linear combination of the four bisolutions

$$\sim \mathbf{Z}_{\frac{d}{2}-1,\nu}\big(-Z \pm \mathrm{i}0\,\mathrm{sgn}(\tau)\big) \quad \text{and} \quad \sim \mathbf{Z}_{\frac{d}{2}-1,-\nu}\big(-Z \pm \mathrm{i}0\,\mathrm{sgn}(\tau)\big). \tag{7.41}$$

However, to find $G(-\nu^2)$, we may not add any terms proportional to $\mathbf{Z}_{\frac{d}{2}-1,-\nu}$ because this would spoil the decay behavior as $|Z| \to \infty$.

We split the proof of (7.39) in two steps. First, we show that (7.39) is a fundamental solution with appropriate decay behavior as $|Z| \to \infty$ *and* $|\tau| \to \infty$. Second, we argue that adding any bisolution, which is a (non-zero) linear combination of (7.41) has exponential growth as $\tau \to +\infty$ or $\tau \to -\infty$.

On W_0, consider (7.35). On the overlap $V_1 = W_0 \cap W_2$, we have

$$\mathbf{Z}_{\frac{d}{2}-1,\nu}\big(-Z \pm \mathrm{i}0\big) = \mathrm{e}^{\mp\mathrm{i}\pi\left(\frac{d-1}{2}+\nu\right)}\mathbf{Z}_{\frac{d}{2}-1,\nu}\big(-(-Z) \pm (-1)\mathrm{i}0\big). \tag{7.42}$$

On the chart W_2, the integral kernel of the resolvent must be a bisolution and it must on V_1 agree with (7.42). Therefore, the i0 should switch the sign from V_1 to V_3. On V_1, we have $\tau \in]0, \pi[$. Hence

$$(7.42) = \mathrm{e}^{\mp\mathrm{i}\pi\left(\frac{d-1}{2}+\nu\right)}\mathbf{Z}_{\frac{d}{2}-1,\nu}\big(-(-Z) \pm (-1)\mathrm{i}0\,\mathrm{sgn}(\sin(|\tau-\tau'|))\big) \quad \text{on} \quad V_1 \tag{7.43}$$

and (7.43) is the appropriate continuation of (7.42) to W_2.

Now notice that $\mathrm{sgn}(\sin(|\tau - \tau'|)) = -1$ on V_3. Therefore, in this region,

$$\begin{aligned} &\mathrm{e}^{\mp\mathrm{i}\pi\left(\frac{d-1}{2}+\nu\right)}\mathbf{Z}_{\frac{d}{2}-1,\nu}\big(-(-Z) \pm (-1)\mathrm{i}0\,\mathrm{sgn}(\sin(|\tau-\tau'|))\big) \\ &= \mathrm{e}^{\mp\mathrm{i}\pi\left(\frac{d-1}{2}+\nu\right)}\mathbf{Z}_{\frac{d}{2}-1,\nu}\big(-(-Z) \pm \mathrm{i}0\big) \\ &= \mathrm{e}^{\mp2\pi\mathrm{i}\left(\frac{d-1}{2}+\nu\right)}\mathbf{Z}_{\frac{d}{2}-1,\nu}\big(-(-1)^2 Z \pm (-1)^2\mathrm{i}0\,\mathrm{sgn}(\sin(|\tau-\tau'|))\big). \end{aligned} \tag{7.44}$$

Inductively, we obtain (7.39) for $n \geq 0$. The continuation to negative n works analogously. Since only $\mathbf{Z}_{\frac{d}{2}-1,\nu}$ appears, both formulas have an appropriate decay behavior as $|Z| \to \infty$ for any sign of $\mathrm{Im}(\nu)$. However, the exponential prefactor

$$\mathrm{e}^{\mp|n|\pi\mathrm{i}\left(\frac{d-1}{2}+\nu\right)} \tag{7.45}$$

decays only for $\mathrm{Im}(\nu) \lessgtr 0$ as $|n| \to \infty$ (or equivalently, as $|\tau| \to \infty$).

To see that these are the only fundamental solutions with appropriate decay behavior, notice that a basis of bisolutions that decay as $|Z| \to \infty$ is on W_0 given by

$$\begin{aligned} &\mathbf{Z}_{\frac{d}{2}-1,\nu}\big(-Z+\mathrm{i}0)\big) + \mathbf{Z}_{\frac{d}{2}-1,\nu}\big(-Z-\mathrm{i}0\big) \\ \text{and} \quad &\mathrm{sgn}(\tau)\Big(\mathbf{Z}_{\frac{d}{2}-1,\nu}\big(-Z+\mathrm{i}0)\big) - \mathbf{Z}_{\frac{d}{2}-1,\nu}\big(-Z-\mathrm{i}0\big)\Big). \end{aligned} \tag{7.46}$$

Both choices contain $+\mathrm{i}0$ and $-\mathrm{i}0$, and it is easy to see that their continuation to the higher W_n contains terms that exponentially increase with time at least in one of the directions $\tau > 0$ resp. $\tau < 0$. □

Remark 7.4. Note in particular that the resolvent (7.27) resp. (7.29) on AdS_d cannot be continued to the higher charts W_n in a similar way. For any sign of $\mathrm{Im}(\nu)$, it will have a term that exponentially increases with n.

7.3.3 Propagators from the resolvent

From the formula for the resolvent we can immediately determine the operator-theoretic Feynman and anti-Feynman propagators for for $n \in \mathbb{Z}$ in the regions W_n. We have

$$G^{\mathrm{F}/\overline{\mathrm{F}}}_{\mathrm{op}}(x,x') = \pm\,\mathrm{i}\frac{\sqrt{\pi}\Gamma(\frac{d-1}{2}+\nu)}{\sqrt{2}(2\pi)^{\frac{d}{2}}2^{\nu}}\mathrm{e}^{\mp\mathrm{i}|n|(\frac{d-1}{2}+\nu)\pi}\mathbf{Z}_{\frac{d}{2}-1,\nu}\Big(-(-1)^n Z \pm (-1)^n\mathrm{i}0s\Big), \tag{7.47}$$

where s is as in Theorem 7.3.

The sum $G^{\mathrm{F}}_{\mathrm{op}} + G^{\overline{\mathrm{F}}}_{\mathrm{op}}$ has a causal support (or in the terminology of Def. 3.6 the specialty condition holds):

$$\begin{aligned} G^{\mathrm{F}}_{\mathrm{op}}(x,x') + G^{\overline{\mathrm{F}}}_{\mathrm{op}}(x,x') = \mathrm{i}\frac{\sqrt{\pi}\Gamma(\frac{d-1}{2}+\nu)}{\sqrt{2}(2\pi)^{\frac{d}{2}}2^{\nu}}\Bigg(&\mathrm{e}^{-\mathrm{i}|n|(\frac{d-1}{2}+\nu)\pi}\mathbf{Z}_{\frac{d}{2}-1,\nu}\Big(-(-1)^n Z + (-1)^n\mathrm{i}0s\Big) \\ -\,\mathrm{e}^{\mathrm{i}|n|(\frac{d-1}{2}+\nu)\pi}\mathbf{Z}_{\frac{d}{2}-1,\nu}\Big(-(-1)^n Z - (-1)^n\mathrm{i}0s\Big)\Bigg). \end{aligned} \tag{7.48}$$

In fact, (7.48) vanishes for $x \in V_0$. We obtain the retarded and advanced propagator by multiplying it with $\theta\big(\pm(\tau-\tau')\big)$. The "Pauli-Jordan function" is then the difference of the retarded and advanced propagator. We use (2.7d) to define $G^{(\pm)}$ obtaining on the chart W_n:

$$\begin{aligned} G^{(\pm)}(x,x') = &\frac{\sqrt{\pi}\Gamma(\frac{d-1}{2}+\nu)}{\sqrt{2}(2\pi)^{\frac{d}{2}}2^{\nu}}\mathrm{e}^{\mp\mathrm{i}n(\frac{d-1}{2}+\nu)\pi} \\ &\times\mathbf{Z}_{\frac{d}{2}-1,\nu}\Big(-(-1)^n Z \pm (-1)^n\mathrm{i}0\tilde{s}\Big). \end{aligned} \tag{7.49}$$

Here $\tilde{s}$ can be represented by $\tilde{s} = \mathrm{sgn}(\sin(\tau - \tau'))$, or

$$(x, x') \in V_{2n-1} \Rightarrow \tilde{s} = (-1)^n, \tag{7.50}$$

$$(x, x') \in V_{2n} \Rightarrow \tilde{s} = 0, \tag{7.51}$$

$$(x, x') \in V_{2n+1} \Rightarrow \tilde{s} = (-1)^{n+1}. \tag{7.52}$$

Note that for $\nu^2 < 0$ the specialty condition is no longer true. Therefore, although we can define $G^{\mathrm{F}}_{\mathrm{op}}$ and $G^{\overline{\mathrm{F}}}_{\mathrm{op}}$, we are not able to obtain other propagators from them.

7.3.4 Trigonometric Pöschl-Teller Hamiltonian

In our further analysis of the anti-deSitter space we will need properties of the following 1-dimensional Schrödinger operator on $L^2[0, \frac{\pi}{2}]$:

$$H^{\mathrm{PT}}_{\alpha,\nu} := -\partial_u^2 + \frac{\alpha^2 - \frac{1}{4}}{\sin(u)^2} + \frac{\nu^2 - \frac{1}{4}}{\cos(u)^2}. \tag{7.53}$$

It is called the *trigonometric Pöschl-Teller Hamiltonian* [66] and is one of the 1-dimensional Schrödinger operators exactly solvable in terms of hypergeometric functions.

By an extension of standard arguments (cf. [67, Chapter X]), one finds that $H^{\mathrm{PT}}_{\alpha,\nu}$, viewed as an operator on $L^2[0, \frac{\pi}{2}]$, is essentially self-adjoint if both $\nu^2 \geq 1$ and $\alpha^2 \geq 1$, it has a positive Friedrichs extension if $\nu^2 \geq 0$ and $\alpha^2 \geq 0$, and all self-adjoint extensions are unbounded from below if $\nu^2 < 0$ or $\alpha^2 < 0$.

7.3.5 Propagators from the evolution of Cauchy data

In this subsection we present an approach to propagators on $\widetilde{\mathrm{AdS}}_d$ different from that of Subsection 7.3.3 It is based on the evolution of the Cauchy data. We will use the stationarity of $\widetilde{\mathrm{AdS}}_d$.

The Klein-Gordon operator with effective mass m in the coordinates (7.5) is given by

$$\begin{aligned} -\Box_g + m^2 &= -\frac{1}{\sqrt{|\det g|}} \partial_\mu g^{\mu\nu} \sqrt{|\det g|} \partial_\nu + m^2 \qquad (7.54) \\ &= \cos(u)^2 \Big(\partial_\tau^2 - \frac{\Delta_{\mathbb{S}^{d-2}}}{\sin(u)^2} - \tan(u)^{2-d} \partial_u \tan(u)^{d-2} \partial_u + \frac{m^2}{\cos(u)^2} \Big) \end{aligned}$$

with $\Delta_{\mathbb{S}^{d-2}}$ being the Laplace-Beltrami operator on the $d-2$-dimensional sphere parametrized by the coordinates Ω. Gauging (7.54) we obtain

$$\tan(u)^{\frac{d-2}{2}} \big(-\Box_g + m^2 \big) \tan(u)^{\frac{2-d}{2}} \tag{7.55}$$

$$= \cos(u)^2 \left(\partial_\tau^2 - \partial_u^2 + \frac{-\Delta_{\mathbb{S}^{d-2}} + \big(\frac{d-3}{2}\big)^2 - \frac{1}{4}}{\sin(u)^2} + \frac{\nu^2 - \frac{1}{4}}{\cos(u)^2} \right) \tag{7.56}$$

with ν^2 as in (7.16). For $d \geq 3$,the spectrum of $-\Delta_{\mathbb{S}^{d-2}}$ is $\{l(l+d-3) \mid l \in \mathbb{N}_0\}$. For $d = 2$, the term proportional to $\sin(u)^{-2}$ vanishes.

Hence, restricted to eigenfunctions of $-\Delta_{\mathbb{S}^{d-2}}$, (7.55) becomes, up to the prefactor $\cos(u)^2$, the trigonometric Pöschl-Teller Hamiltonian (7.53) with $\alpha := l + \frac{d-3}{2}$ if $d \geq 3$ and $\alpha^2 = \frac{1}{4}$ if $d = 2$.

To define dynamics in anti-deSitter space, one needs to fix a self-adjoint extension of $H^{\mathrm{PT}}_{\alpha,\nu}$, i.e., boundary conditions at spacelike infinity. A comprehensive analysis of boundary conditions for H_{PT} and their application to anti-deSitter QFT has been carried out by Ishibashi and Wald [55].

Notice first that $\alpha^2 < 1$ if and only if $d = 2$ or $d \in \{3, 4\}$ and $l = 0$. Hence, one might expect that boundary conditions at the origin need to be fixed in these cases. But one can show that this is merely an artifact of the choice of coordinates and that no boundary conditions at $u = 0$ are required [55]. The important part is fixing the boundary conditions (i.e., a self-adjoint extension of $H^{\mathrm{PT}}_{\alpha,\nu}$) at spatial infinity $u = \frac{\pi}{2}$.

Now for $\nu^2 \geq 0$ the operator $H^{\mathrm{PT}}_{\alpha,\nu}$ is essentially self-adjoint, so the dynamics is uniquely determined. We can compute all propagators—they agree with those obtained from the operator-theoretic Feynman propagator. In particular, the specialty condition is true.

For $0 \leq \nu^2 < 1$ we have a one-parameter family of self-adjoint extensions, depending on the boundary condition at spatial infinity. All of them can be used to define the propagators. Among them there is a distinguished boundary condition given by the Friedrichs extension, or equivalently, by the analytic continuation in the parameter ν. By the uniqueness of analytic continuation, this leads to propagators that agree with those obtained from the operator-theoretic Feynman propagator.

Finally, for $\nu^2 < 0$ there is a one-parameter family of realizations of $H^{\mathrm{PT}}_{\alpha,\nu}$, and all are unbounded from below. Each of them can be used to define an evolution of Cauchy data, and hence the retarded and advanced propagator. However we do not have a distinguished state.

A Projections and Krein spaces

The main goal of this appendix is a short presentation of basic facts about Krein spaces, which provide a natural functional-analytic setting for the Klein-Gordon equation. There exist comprehensive textbook treatments of spaces with indefinite inner products [6, 13]. Our treatment is perhaps more concise, concentrating on the concepts directly needed in our paper. To a large extent we follow [36], with some simplifications and improvements.

We start with some useful but not well-known lemmas about projections, involutions and complementary subspaces, presenting constructions related to pairs of complementary subspaces, which go back to Kato [57]. Then we describes elements of the theory of Krein spaces. The main result that we prove is the proposition saying that every pair consisting of a maximal uniformly positive and maximal uniformly negative subspace is complementary, which is crucial in the construction of the out-in Feynman propagator.

A.1 Involutions

Let $\mathcal{W}$ be a vector space. We do not need topology on $\mathcal{W}$ for the moment. We use the term "invertible" as the synonym of "bijective".

Definition A.1. We say that a pair $(\mathcal{Z}_\bullet^{(+)}, \mathcal{Z}_\bullet^{(-)})$ of subspaces of $\mathcal{W}$ is *complementary* if

$$\mathcal{Z}_\bullet^{(+)} \cap \mathcal{Z}_\bullet^{(-)} = \{0\}, \quad \mathcal{Z}_\bullet^{(+)} + \mathcal{Z}_\bullet^{(-)} = \mathcal{W}.$$

Definition A.2. We say that a pair of operators $(\Pi_\bullet^{(+)}, \Pi_\bullet^{(-)})$ on $\mathcal{W}$ is a pair of complementary projections if

$$\left(\Pi_\bullet^{(\pm)}\right)^2 = \Pi_\bullet^{(\pm)}, \quad \Pi_\bullet^{(+)} + \Pi_\bullet^{(-)} = \mathbb{1}.$$

Definition A.3. An operator $S_\bullet$ on $\mathcal{W}$ is called an *involution*, if $S_\bullet^2 = \mathbb{1}$.

Note that there is a 1-1 correspondence between involutions, pairs of complementary projections and pairs of complementary subspaces:

$$\Pi_\bullet^{(\pm)} := \frac{1}{2}(\mathbb{1} \pm S_\bullet), \quad \mathcal{Z}_\bullet^{(\pm)} := \mathcal{R}(\Pi_\bullet^{(\pm)}). \tag{A.1}$$

A.2 Pair of involutions I

In this subsection we give a criterion for complementarity of two subspaces, and then we construct the corresponding projections following Kato [57].

Suppose that S_1 and S_2 are two involutions on $\mathcal{W}$. Let

$$\Pi_i^{(\pm)} := \frac{1}{2}(\mathbb{1} \pm S_i), \quad \mathcal{Z}_i^{(\pm)} := \mathcal{R}(\Pi_i^{(\pm)}), \quad i = 1, 2,$$

be the corresponding pairs of complementary projections and subspaces. Define

$$\Upsilon \ = \frac{1}{4}(S_1 + S_2)^2. \tag{A.2}$$

Observe that Υ commutes with $\Pi_1^{(+)}$, $\Pi_1^{(-)}$, $\Pi_2^{(+)}$ and $\Pi_2^{(-)}$.

Proposition A.4. *The following conditions are equivalent:*

(i) Υ *is invertible.*

(ii) $\Pi_1^{(+)} + \Pi_2^{(-)}$ *and* $\Pi_2^{(+)} + \Pi_1^{(-)}$ *are invertible.*

Moreover, if one of the above holds, then the pairs $(\mathcal{Z}_1^{(+)}, \mathcal{Z}_2^{(-)})$ *as well as* $(\mathcal{Z}_2^{(+)}, \mathcal{Z}_1^{(-)})$ *are complementary.*

Proof. (i) $\iff$ (ii) and (i) $\iff$ (iii) follow from

$$\Upsilon = (\Pi_1^{(+)} + \Pi_2^{(-)})(\Pi_2^{(+)} + \Pi_1^{(-)}) \tag{A.3}$$

by the following easy fact: If R, S, T are maps such that $R = ST = TS$, then R is bijective if and only if both T and S are bijective.

The last implication follows from the next proposition. □

In the setting of the above proposition we can use Υ to construct two pairs of complementary projections:

Proposition A.5. *Suppose that Υ is invertible. Then*

$$\begin{aligned}
\Lambda_{12}^{(+)} &:= \Pi_1^{(+)}\Upsilon^{-1}\Pi_2^{(+)} && \textit{is the projection onto } \mathcal{Z}_1^{(+)} \textit{ along } \mathcal{Z}_2^{(-)},\\
\Lambda_{12}^{(-)} &:= \Pi_2^{(-)}\Upsilon^{-1}\Pi_1^{(-)} && \textit{is the projection onto } \mathcal{Z}_2^{(-)} \textit{ along } \mathcal{Z}_1^{(+)},\\
\Lambda_{21}^{(+)} &:= \Pi_2^{(+)}\Upsilon^{-1}\Pi_1^{(+)} && \textit{is the projection onto } \mathcal{Z}_2^{(+)} \textit{ along } \mathcal{Z}_1^{(-)},\\
\Lambda_{21}^{(-)} &:= \Pi_1^{(-)}\Upsilon^{-1}\Pi_2^{(-)} && \textit{is the projection onto } \mathcal{Z}_1^{(-)} \textit{ along } \mathcal{Z}_2^{(+)}.
\end{aligned}$$

In particular,

$$\Lambda_{12}^{(+)} + \Lambda_{12}^{(-)} = \mathbb{1}, \quad \Lambda_{21}^{(+)} + \Lambda_{21}^{(-)} = \mathbb{1}.$$

Proof. First we check that $\Lambda_{12}^{(+)}$ is a projection:

$$\begin{aligned}
\big(\Lambda_{12}^{(+)}\big)^2 &= \Pi_1^{(+)}\Upsilon^{-1}\Pi_2^{(+)}\Pi_1^{(+)}\Upsilon^{-1}\Pi_2^{(+)}\\
&= \Pi_1^{(+)}\Upsilon^{-1}(\Pi_2^{(+)}\Pi_1^{(+)} + \Pi_1^{(-)}\Pi_2^{(-)})\Upsilon^{-1}\Pi_2^{(+)} = \Lambda_{12}^{(+)}.
\end{aligned}$$

Moreover,

$$\Lambda_{12}^{(+)} = \Pi_1^{(+)}(\Pi_2^{(+)} + \Pi_1^{(-)})\Upsilon^{-1} = \Upsilon^{-1}(\Pi_1^{(+)} + \Pi_2^{(-)})\Pi_2^{(+)}.$$

But $(\Pi_2^{(+)} + \Pi_1^{(-)})\Upsilon^{-1}$ and $\Upsilon^{-1}(\Pi_1^{(+)} + \Pi_2^{(-)})$ are invertible. Hence $\mathcal{R}(\Lambda_{12}^{(+)}) = \mathcal{R}(\Pi_1^{(+)})$ and $\mathcal{N}(\Lambda_{12}^{(+)}) = \mathcal{N}(\Pi_2^{(+)}) = \mathcal{R}(\Pi_2^{(-)})$. This proves the statement of the proposition about $\Lambda_{12}^{(+)}$. The remaining statements are proven analogously. □

Remark A.6. Note that the notation for projections $\Lambda_{12}^{(\pm)}$ and $\Lambda_{21}^{(\pm)}$ is different than in [36].

A.3 Pair of involutions II

Let S_i, $(\Pi_i^{(+)}, \Pi_i^{(-)})$, $(\mathcal{Z}_i^{(+)}, \mathcal{Z}_i^{(-)})$, $i = 1, 2$, be as in the previous subsection. Set

$$K := S_2 S_1. \tag{A.4}$$

Proposition A.7. *K is invertible and*

$$S_1 K S_1 = S_2 K S_2 = K^{-1}. \tag{A.5}$$

In what follows we will use the decomposition $\mathcal{W} = \mathcal{Z}_1^{(+)} \oplus \mathcal{Z}_1^{(-)}$. Under the assumption that $\mathbb{1} + K$ is invertible, we define

$$c := \Pi_1^{(+)} \frac{\mathbb{1} - K}{\mathbb{1} + K} \Pi_1^{(-)}, \quad d := \Pi_1^{(-)} \frac{\mathbb{1} - K}{\mathbb{1} + K} \Pi_1^{(+)}. \tag{A.6}$$

where c, resp. d are interpreted as operators from $\mathcal{Z}_1^{(-)}$ to $\mathcal{Z}_1^{(+)}$, resp. from $\mathcal{Z}_1^{(+)}$ to $\mathcal{Z}_1^{(-)}$.

Proposition A.8. *The following conditions are equivalent:*

(i) Υ *is invertible (or Condition (ii) of Proposition A.4 is true).*

(ii) $\mathbb{1} + K$ *is invertible.*

Suppose that the above conditions are true. As we know from Prop. A.4, the pairs of subspaces $(\mathcal{Z}_1^{(+)}, \mathcal{Z}_2^{(-)})$ *and* $(\mathcal{Z}_2^{(+)}, \mathcal{Z}_1^{(-)})$ *are then complementary. Here are new formulas for the corresponding projections:*

$$\Lambda_{12}^{(+)} = \begin{bmatrix} \mathbb{1} & c \\ 0 & 0 \end{bmatrix} \qquad \textit{projects onto } \mathcal{Z}_1^{(+)} \textit{ along } \mathcal{Z}_2^{(-)},$$
$$\Lambda_{12}^{(-)} = \begin{bmatrix} 0 & -c \\ 0 & \mathbb{1} \end{bmatrix} \qquad \textit{projects onto } \mathcal{Z}_2^{(-)} \textit{ along } \mathcal{Z}_1^{(+)},$$
$$\Lambda_{21}^{(+)} = \begin{bmatrix} \mathbb{1} & 0 \\ -d & 0 \end{bmatrix} \qquad \textit{projects onto } \mathcal{Z}_2^{(+)} \textit{ along } \mathcal{Z}_1^{(-)},$$
$$\Lambda_{21}^{(-)} = \begin{bmatrix} 0 & 0 \\ d & \mathbb{1} \end{bmatrix} \qquad \textit{projects onto } \mathcal{Z}_1^{(-)} \textit{ along } \mathcal{Z}_2^{(+)}.$$

Besides, $\mathbb{1} - dc$ *and* $\mathbb{1} - cd$ *are invertible, and we have the following formulas:*

$$\Upsilon = \frac{1}{4}(\mathbb{1} + K)(\mathbb{1} + K^{-1}) = \begin{bmatrix} (\mathbb{1} - cd)^{-1} & 0 \\ 0 & (\mathbb{1} - dc)^{-1} \end{bmatrix}, \tag{A.7a}$$

$$K = \begin{bmatrix} (\mathbb{1} + cd)(\mathbb{1} - cd)^{-1} & -2c(\mathbb{1} - dc)^{-1} \\ -2d(\mathbb{1} - cd)^{-1} & (\mathbb{1} + dc)(\mathbb{1} - dc)^{-1} \end{bmatrix}, \tag{A.7b}$$

$$\Pi_1^{(+)} = \begin{bmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{bmatrix}, \quad \Pi_2^{(+)} = \begin{bmatrix} (\mathbb{1} - cd)^{-1} & c(\mathbb{1} - dc)^{-1} \\ -d(\mathbb{1} - cd)^{-1} & -dc(\mathbb{1} - dc)^{-1} \end{bmatrix}, \tag{A.7c}$$

$$\Pi_1^{(-)} = \begin{bmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{bmatrix}, \quad \Pi_2^{(-)} = \begin{bmatrix} -cd(\mathbb{1} - cd)^{-1} & -c(\mathbb{1} - dc)^{-1} \\ d(\mathbb{1} - cd)^{-1} & (\mathbb{1} - dc)^{-1} \end{bmatrix}, \tag{A.7d}$$

$$S_1 = \begin{bmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{bmatrix}, \quad S_2 = \begin{bmatrix} (\mathbb{1} + cd)(\mathbb{1} - cd)^{-1} & 2c(\mathbb{1} - dc)^{-1} \\ -2d(\mathbb{1} - cd)^{-1} & -(\mathbb{1} + dc)(\mathbb{1} - dc)^{-1} \end{bmatrix}. \tag{A.7e}$$

Proof. We have

$$\Upsilon = \frac{1}{4}(S_1 + S_2)^2 = \frac{1}{4}(\mathbb{1} + K)(\mathbb{1} + K^{-1}). \tag{A.8}$$

But $(\mathbb{1} + K^{-1}) = K^{-1}(\mathbb{1} + K)$. Hence $\mathbb{1} + K$ is invertible iff $\mathbb{1} + K^{-1}$ is. Therefore, (i)⇔(ii).

For the remainder of the proof we assume that $\mathbb{1}+K$ is invertible. We have

$$S_1\frac{\mathbb{1}-K}{\mathbb{1}+K}S_1=-\frac{\mathbb{1}-K}{\mathbb{1}+K}. \tag{A.9}$$

Therefore

$$\Pi_1^{(+)}\frac{\mathbb{1}-K}{\mathbb{1}+K}\Pi_1^{(+)}=\Pi_1^{(-)}\frac{\mathbb{1}-K}{\mathbb{1}+K}\Pi_1^{(-)}=0. \tag{A.10}$$

Hence,

$$\frac{\mathbb{1}-K}{\mathbb{1}+K}=\begin{bmatrix}0 & c\\ d & 0\end{bmatrix}. \tag{A.11}$$

This implies

$$\frac{1}{\mathbb{1}+K}=\frac{1}{2}\begin{bmatrix}\mathbb{1} & c\\ d & \mathbb{1}\end{bmatrix},\qquad \frac{1}{\mathbb{1}+K^{-1}}=\frac{1}{2}\begin{bmatrix}\mathbb{1} & -c\\ -d & \mathbb{1}\end{bmatrix}. \tag{A.12}$$

Multiplying the two expressions of (A.12) yields

$$\Upsilon^{-1}=\begin{bmatrix}\mathbb{1}-cd & 0\\ 0 & \mathbb{1}-dc\end{bmatrix}. \tag{A.13}$$

Hence we proved both identities of (A.7a), as well as invertibility of $\mathbb{1}-cd$ and $\mathbb{1}-dc$.

We check that

$$\begin{bmatrix}\mathbb{1} & c\\ d & \mathbb{1}\end{bmatrix}^{-1}=\begin{bmatrix}(\mathbb{1}-cd)^{-1} & -c(\mathbb{1}-dc)^{-1}\\ -d(\mathbb{1}-cd)^{-1} & (\mathbb{1}-dc)^{-1}\end{bmatrix}. \tag{A.14}$$

Now

$$K=2\begin{bmatrix}\mathbb{1} & c\\ d & \mathbb{1}\end{bmatrix}^{-1}-\begin{bmatrix}\mathbb{1} & 0\\ 0 & \mathbb{1}\end{bmatrix} \tag{A.15}$$

yields (A.7b).

The formulas for $\Pi_1^{(\pm)}$ and S_1 are obvious. We obtain S_2 from $S_2=KS_1$. From S_2 we get $\Pi_2^{(\pm)}$.

Now $\Lambda_{12}^{(+)}=\Pi_1^{(+)}\Upsilon^{-1}\Pi_2^{(+)}$ yields (A.7c), etc. □

The operators c, d are sometimes called *angular operators*.

A.4 Pair of self-adjoint involutions in a Hilbert space

Suppose now that $\mathcal{W}$ is a Hilbert space and S_i, $i=1,2$, is a pair of self-adjoint involutions. Obviously, the corresponding projections $\Pi_i^{(+)}$, $\Pi_i^{(-)}$ are then orthogonal.

We will use the orthogonal decomposition $\mathcal{W}=\mathcal{Z}_1^{(+)}\oplus\mathcal{Z}_1^{(-)}$. In this decomposition we can wrtie

$$\Pi_2^{(+)}=\begin{bmatrix}A & B\\ B^* & C\end{bmatrix},\qquad \text{where}\quad 0\le A\le\mathbb{1},\quad 0\le C\le\mathbb{1}. \tag{A.16}$$

Using $(\Pi_2^{(+)})^2 = \Pi_2^{(+)}$ we obtain

$$\left(A - \tfrac{1}{2}\right)^2 = \tfrac{1}{4} - BB^*, \qquad \left(C - \tfrac{1}{2}\right)^2 = \tfrac{1}{4} - B^*B. \tag{A.17}$$

For an operator K, $\sigma(K)$ will denote its spectrum. If K is self-adjoint we will write

$$\inf K = \inf \sigma(K), \quad \sup K = \sup \sigma(K). \tag{A.18}$$

It follows from (A.17) that $\frac{1}{4} \geq \sup BB^* = \sup B^*B = \|B\|^2$, and hence

$$0 \leq \inf \left(\tfrac{1}{4} - BB^*\right) = \inf \left(\tfrac{1}{4} - B^*B\right). \tag{A.19}$$

The following proposition describes the situation where the angle between the projections $\Pi_1^{(+)}$ and $\Pi_2^{(+)}$ is not more than $\frac{\pi}{4}$:

Proposition A.9. *The following conditions are equivalent:*

1. $A \geq \frac{1}{2}$ *and* $C \leq \frac{1}{2}$.
2. $A = \frac{1}{2} + \sqrt{\frac{1}{4} - BB^*}$ *and* $C = \frac{1}{2} - \sqrt{\frac{1}{4} - B^*B}$.

Proof. 1.$\Leftarrow$ 2. is obvious.

1.$\Rightarrow$ 2. follows from (A.17), where by 1. we need to take the positive square root. ☐

The following consequence of Prop. A.9 will be useful in the theory of Krein spaces:

Lemma A.10. *Let P be an orthogonal projection and S a self-adjoint involution. Let $\alpha > 0$ and*

$$PSP \geq \alpha P, \tag{A.20}$$

$$(\mathbb{1} - P)S(\mathbb{1} - P) \leq 0. \tag{A.21}$$

Then

$$(\mathbb{1} - P)S(\mathbb{1} - P) \leq -\alpha(\mathbb{1} - P). \tag{A.22}$$

Proof. We set $S_1 := 2P - \mathbb{1}$, so that $P = \Pi_1^{(+)}$, and $S_2 := S$. Thus we are in the setting of this subsection. We write $\Pi_2^{(+)} = \frac{S_2 + \mathbb{1}}{2}$ as in (A.16), and then

$$PSP \geq 0 \;\Leftrightarrow\; A \geq \frac{1}{2}, \tag{A.23}$$

$$(\mathbb{1} - P)S(\mathbb{1} - P) \leq 0 \Leftrightarrow\; C \leq \frac{1}{2}. \tag{A.24}$$

Hence (A.20) and (A.21) imply the conditions of Proposition A.9. Therefore,

$$PSP = \sqrt{1 - 4BB^*}, \tag{A.25}$$

$$(\mathbb{1} - P)S(\mathbb{1} - P) = -\sqrt{1 - 4B^*B}. \tag{A.26}$$

By (A.20), $\sqrt{1 - 4BB^*} \geq \alpha$. So $-\sqrt{1 - 4B^*B} \leq -\alpha$, which proves (A.22). ☐

A.5 Hilbertizable spaces

Definition A.11. Let $\mathcal{W}$ be a complex[9] topological vector space. We say that it is *Hilbertizable* if it has the topology of a Hilbert space for some scalar product $(\cdot|\cdot)_\bullet$ on $\mathcal{W}$. We will then say that $(\cdot|\cdot)_\bullet$ is *compatible with (the Hilbertizable structure of)* $\mathcal{W}$. The Hilbert space $\big(\mathcal{W}, (\cdot|\cdot)_\bullet\big)$ will be occasionally denoted $\mathcal{W}_\bullet$. We denote the corresponding norm by $\|\cdot\|_\bullet$, the orthogonal complement of $\mathcal{Z} \subset \mathcal{W}$ by $\mathcal{Z}^{\perp\bullet}$ and the Hermitian adjoint of an operator A by $A^{*\bullet}$.

In what follows $\mathcal{W}$ is a Hilbertizable space. Let $(\cdot|\cdot)_1$, $(\cdot|\cdot)_2$ be two scalar products compatible with $\mathcal{W}$. Then there exist constants $0 < c \leq C$ such that

$$c(w|w)_1 \leq (w|w)_2 \leq C(w|w)_1.$$

Let R be a linear operator on $\mathcal{W}$. We say that it is *bounded* if for some (hence for all) compatible scalar products $(\cdot|\cdot)_\bullet$ there exists a constant $C_\bullet$ such that

$$\|Rw\|_\bullet \leq C_\bullet \|w\|_\bullet.$$

Let Q be a sesquilinear form on $\mathcal{W}$. We say that it is *bounded* if for some (hence for all) compatible scalar products $(\cdot|\cdot)_\bullet$ there exists $C_\bullet$ such that

$$|(v|Qw)| \leq C_\bullet \|v\|_\bullet \|w\|_\bullet, \quad v, w \in \mathcal{W}.$$

A.6 Pseudounitary spaces

Let $(\mathcal{W}, Q)$ be a Hilbertizable space equipped with a bounded Hermitian form,

$$(v|Qw) = \overline{(w|Qv)}, \quad v, w \in \mathcal{W}. \tag{A.27}$$

Definition A.12. Let $\mathcal{Z} \subset \mathcal{W}$. We define its *Q-orthogonal companion* as follows:

$$\mathcal{Z}^{\perp Q} := \{w \in \mathcal{W} \mid (w|Qv) = 0,\ v \in \mathcal{Z}\}.$$

Clearly, $\mathcal{Z}^{\perp Q}$ is a closed subspace of $\mathcal{W}$.

Definition A.13. Let $w \in \mathcal{W}$. We say that w is *positive, negative, resp. neutral* if

$$(w|Qw) \geq 0, \quad (w|Qw) \leq 0, \quad \text{resp.} \quad (w|Qw) = 0. \tag{A.28}$$

We say that a subspace $\mathcal{Z} \subset \mathcal{W}$ is *positive, negative, resp. neutral* if all its elements are positive, negative, resp. neutral elements.

Definition A.14. We say that $(\mathcal{W}, Q)$ is a *pseudounitary space* if $\mathcal{W}^{\perp Q} = \{0\}$.

[9] Analogous definitions and results are valid for *real* Hilbertizable spaces.

A.7 Krein spaces

Let $(\mathcal{W}, Q)$ be a Hilbertizable space equipped with a bounded Hermitian form.

Definition A.15. A bounded involution $S_\bullet$ on $\mathcal{W}$ will be called *admissible* if it preserves Q, that is,

$$(S_\bullet v \,|\, Q S_\bullet w) = (v \,|\, Qw), \tag{A.29}$$

and

$$(v \,|\, w)_\bullet := (v \,|\, Q S_\bullet w) = (S_\bullet v \,|\, Qw) \tag{A.30}$$

is a scalar product compatible with the Hilbertizable structure of $\mathcal{W}$.

Definition A.16. A space $(\mathcal{W}, Q)$ is called a *Krein space* if it possesses an admissible involution.

Clearly, a Krein space is a pseudounitary space.

Remark A.17. In the literature sometimes instead of the term "admissible involution" one finds "fundamental symmetry".

For any admissible involution $S_\bullet$, we define the corresponding *particle projection* $\Pi_\bullet^{(+)}$ and *particle space* $\mathcal{Z}_\bullet^{(+)}$, as well as the *antiparticle projection* $\Pi_\bullet^{(-)}$ and *antiparticle space* $\mathcal{Z}_\bullet^{(-)}$, as in (A.1). The decomposition $\mathcal{W} \simeq \mathcal{Z}_\bullet^{(+)} \oplus \mathcal{Z}_\bullet^{(-)}$ is often called a *fundamental decomposition*. Note the following relations:

$$\begin{aligned}(v \,|\, w)_\bullet &= (\Pi_\bullet^{(+)} v \,|\, \Pi_\bullet^{(+)} w)_\bullet + (\Pi_\bullet^{(-)} v \,|\, \Pi_\bullet^{(-)} w)_\bullet, \\ (v \,|\, Qw) &= (\Pi_\bullet^{(+)} v \,|\, \Pi_\bullet^{(+)} w)_\bullet - (\Pi_\bullet^{(-)} v \,|\, \Pi_\bullet^{(-)} w)_\bullet.\end{aligned}$$

Definition A.18. Let A be a bounded operator on $\mathcal{W}$. Then there exists a unique operator A^{*Q} called the *Q-adjoint of* A such that

$$(A^{*Q} v \,|\, Qw) = (v \,|\, QAw), \quad v, w \in \mathcal{W}. \tag{A.31}$$

Let $\mathcal{Z} \subset \mathcal{W}$ and let A be an operator on $\mathcal{W}$. We have the identities:

$$\mathcal{Z}^{\perp Q} = S_\bullet \mathcal{Z}^{\perp \bullet}, \tag{A.32}$$

$$A^{*Q} = S_\bullet A^{*\bullet} S_\bullet. \tag{A.33}$$

With the help of these identities it is easy to show various properties of $\perp Q$ and $*\bullet$:

Proposition A.19. *1. If $\mathcal{Z}$ is a closed subspace, then $(\mathcal{Z}^{\perp Q})^{\perp Q} = \mathcal{Z}$.*

2. If $\mathcal{Z}_1, \mathcal{Z}_2$ are complementary subspaces in $\mathcal{W}$, then so are $\mathcal{Z}_1^{\perp Q}, \mathcal{Z}_2^{\perp Q}$.

3. Suppose that $(\Pi^{(+)}, \Pi^{(-)})$ is a pair of complementary projections. Then $(\Pi^{(+)*Q}, \Pi^{(-)*Q})$ is also a pair of complementary projections and

$$\mathcal{R}(\Pi^{(\pm)*Q}) = \mathcal{N}(\Pi^{(\mp)*Q}) = \mathcal{R}(\Pi^{(\mp)})^{\perp Q} = \mathcal{N}(\Pi^{(\pm)})^{\perp Q}. \tag{A.34}$$

Definition A.20. Let R be a bounded invertible operator on $(\mathcal{W}, Q)$. We say that R is *pseudo-unitary* if

$$(Rv \,|\, QRw) = (v \,|\, Qw). \tag{A.35}$$

A.8 Krein spaces with conjugation

Definition A.21. An antilinear involution $v \mapsto \varepsilon v$ on a Krein space $(\mathcal{W}, Q)$ will be called a *conjugation* if it antipreserves Q, that is

$$(v|Qw) = -\overline{(\varepsilon v|Q\varepsilon w)} \tag{A.36}$$

and there exists an admissible involution $S_\bullet$ such that $\varepsilon S_\bullet \varepsilon = -S_\bullet$.

Note that then

$$(\varepsilon v\,|\,\varepsilon w)_\bullet = \overline{(v\,|\,w)_\bullet}.$$

Definition A.22. We say that an operator R is *real* if $\overline{R} := \varepsilon R \varepsilon = R$. We say that R is *anti-real* if $\overline{R} = -R$, that is, if $\mathrm{i}R$ is real.

Krein spaces with conjugations are especially important: Suppose that $(\mathcal{W}, Q)$ is a Krein space with conjugation. Clearly, if $S_\bullet$ is an admissible anti-real involution, then

$$\overline{\Pi_\bullet^{(+)}} = \Pi_\bullet^{(-)}, \quad \overline{\mathcal{Z}_\bullet^{(+)}} = \mathcal{Z}_\bullet^{(-)},$$

so that $\mathcal{W} = \mathcal{Z}_\bullet^{(+)} \oplus \overline{\mathcal{Z}_\bullet^{(+)}}$.

A.9 Maximal uniformly positive/negative subspaces

Let $(\mathcal{W}, Q)$ be a Krein space. We want to characterize definite subspaces with good properties. Following [13] we make the following definition.

Definition A.23. Let $\mathcal{Z}$ be a subspace of $\mathcal{W}$.

1. We say that it is uniformly positive/negative if for some scalar product $(\cdot|\cdot)_\bullet$ compatible with the Hilbertizable structure of $\mathcal{W}$ there exists $\alpha_\bullet > 0$ such that

$$v \in \mathcal{Z} \Rightarrow (v\,|\,Qv) \geq \alpha_\bullet (v|v)_\bullet, \quad \text{resp.} \quad v \in \mathcal{Z} \Rightarrow (v\,|\,Qv) \leq -\alpha_\bullet (v|v)_\bullet. \tag{A.37}$$

2. We say that $\mathcal{Z}$ is maximal uniformly positive/negative if it is a maximal subspace with the property of uniform positivity/negativity.

The following proposition, whose statement partially overlaps with Thm. V.5.2. and Cor. V. 7.4. in [13], relates maximal uniformly positive/negative spaces to fundamental decompositions and admissible involutions.

Proposition A.24. *Let $\mathcal{Z}_\bullet^{(+)}$ be a subspace of $\mathcal{W}$. Set $\mathcal{Z}_\bullet^{(-)} := \mathcal{Z}_\bullet^{(+)\perp Q}$. The following conditions are equivalent:*

1. *$\mathcal{Z}_\bullet^{(+)}$ is maximal uniformly positive.*
2. *$\mathcal{Z}_\bullet^{(+)}$ is maximal uniformly positive and $\mathcal{Z}_\bullet^{(-)}$ is maximal uniformly negative.*

3. *The spaces* $\mathcal{Z}_\bullet^{(+)}$ *and* $\mathcal{Z}_\bullet^{(-)}$ *are complementary, and if* $(\Pi_\bullet^{(+)}, \Pi_\bullet^{(-)})$ *is the corresponding pair of projections, then* $S_\bullet := \Pi_\bullet^{(+)} - \Pi_\bullet^{(-)}$ *is an admissible involution.*

Proof of Prop. A.24. Assume 3). Then $(\cdot|\cdot)_\bullet := (\cdot|QS_\bullet\cdot)$ is compatible and

$$(v|Qv)_\bullet = \pm(v|v)_\bullet, \quad v \in \mathcal{Z}_\bullet^{(\pm)}. \tag{A.38}$$

Hence $\mathcal{Z}_\bullet^{(\pm)}$ are maximal uniformly positive/negative. This proves 3)⇒2).

2)⇒1) is obvious.

Now assume 1). Let S_0 be an arbitrary admissible involution with the corresponding scalar product $(\cdot|\cdot)_0$. First note that $\mathcal{Z}_\bullet^{(-)}$ is negative. Indeed, suppose that $v_1 \in \mathcal{Z}_\bullet^{(-)}$ is strictly positive. Then for some α_1

$$(v_1|Qv_1) \geq \alpha_1(v|v)_0. \tag{A.39}$$

Hence $\mathrm{Span}(\mathcal{Z}_\bullet^{(+)}, v_1)$ is uniformly positive, which contradicts the maximality of $\mathcal{Z}_\bullet^{(+)}$.

Let P by the orthogonal projection (in the sense of $(\cdot|\cdot)_0$) onto $\mathcal{Z}_\bullet^{(+)}$. Then an arbitrary element of $\mathcal{Z}_\bullet^{(+)}$ has the form Pv and of $\mathcal{Z}_\bullet^{(-)}$ the form $S_0(\mathbb{1}-P)v$ for some $v \in \mathcal{W}$.

By the uniform positivity of $\mathcal{Z}_\bullet^{(+)}$, resp. by negativity of $\mathcal{Z}_\bullet^{(-)}$, we have

$$(v|PS_0Pv)_0 = (Pv|S_0Pv)_0 = (Pv|QPv) \geq \alpha(Pv|Pv)_0 \tag{A.40}$$

and

$$\begin{aligned}(v|(\mathbb{1}-P)S_0(\mathbb{1}-P)v)_0 &= (S_0(\mathbb{1}-P)v|(\mathbb{1}-P)v)_0 \\ &= (S_0(\mathbb{1}-P)v|QS_0(\mathbb{1}-P)v) \leq 0.\end{aligned} \tag{A.41}$$

Lemma A.10 then implies the uniform negativity of $\mathcal{Z}_\bullet^{(-)}$:

$$\begin{aligned}(v|(\mathbb{1}-P)S_0(\mathbb{1}-P)v)_0 &\leq -\alpha(v|(\mathbb{1}-P)v)_0 \\ &= -\alpha(S_0(\mathbb{1}-P)v|S_0(\mathbb{1}-P)v)_0.\end{aligned} \tag{A.42}$$

Clearly, $0 \neq w \in \mathcal{Z}_\bullet^{(+)} \cap \mathcal{Z}_\bullet^{(-)}$ has to be simultaneously positive and negative. Hence $\mathcal{Z}_\bullet^{(+)} \cap \mathcal{Z}_\bullet^{(-)} = \{0\}$.

Suppose that $\mathcal{Z}_\bullet^{(+)} + \mathcal{Z}_\bullet^{(-)} \neq \mathcal{W}$. Then there exists $0 \neq w \in (\mathcal{Z}_\bullet^{(+)} + \mathcal{Z}_\bullet^{(-)})^{\perp 0}$. Hence for any $v \in \mathcal{W}$,

$$0 = (w|Pv)_0 = (Pw|v)_0, \tag{A.43}$$

$$0 = (w|S_0(\mathbb{1}-P)v)_0 = ((\mathbb{1}-P)S_0W|v)_0. \tag{A.44}$$

Therefore $Pw = (\mathbb{1}-P)S_0w = 0$, and

$$\begin{aligned}\|w\|_0^2 = (S_0(\mathbb{1}-P)w|S_0(\mathbb{1}-P)w)_0 &\leq -\frac{1}{\alpha}(S_0(\mathbb{1}-P)w|QS_0(\mathbb{1}-P)w) \\ = -\frac{1}{\alpha}(S_0(\mathbb{1}-P)w|(\mathbb{1}-P)w)_0 &= -\frac{1}{\alpha}(w|(\mathbb{1}-P)S_0w)_0 = 0.\end{aligned} \tag{A.45}$$

This implies $w = 0$.

We have proved that $\mathcal{Z}_\bullet^{(+)}$ and $\mathcal{Z}_\bullet^{(-)}$ are complementary. Let $S_\bullet$ be the corresponding involution. It is obviously bounded. Besides,

$$(v|v)_\bullet \geq \alpha(v|v)_0. \tag{A.46}$$

Hence $(\cdot|\cdot)_\bullet$ is compatible. This ends the proof of 1)⇒3). □

Here is another proposition about fundamental decompositions. Note that it does not involve a reference to the topology of $\mathcal{W}$, but only to the form Q.

Proposition A.25. *Let $\mathcal{Z}_\bullet^{(+)}$ and $\mathcal{Z}_\bullet^{(-)}$ be complementary subspaces of a Krein space $(\mathcal{W}, Q)$, Q-orthogonal to one another. Assume that $\mathcal{Z}_\bullet^{(\pm)}$ are positive resp. negative, contain no neutral elements apart from 0 and are complete in the norm $\|v\|_{(\pm)} := \sqrt{\pm(v|Qv)}$. Then $\mathcal{Z}_\bullet^{(\pm)}$ is maximal uniformly positive/negative and $\mathcal{Z}_\bullet^{(-)} := \mathcal{Z}_\bullet^{(+)\perp Q}$, so that we are precisely in the setting described by Prop. A.24.*

Proof. Let $S_\bullet$ be the involution defined by $\mathcal{W} = \mathcal{Z}_\bullet^{(+)} \oplus \mathcal{Z}_\bullet^{(-)}$. As usual, we introduce the corresponding scalar product $(v|w)_\bullet := (v|QS_\bullet w)$ and the norm $\|\cdot\|_\bullet$. Note that $\|v\|_\bullet = \|v\|_{(\pm)}$ if $v \in \mathcal{Z}_\bullet^{(\pm)}$.

Let $\|\cdot\|_1$ be any compatible norm. Clearly, by the boundedness of Q, we have

$$\|v\|_\bullet \leq C\|v\|_1. \tag{A.47}$$

Consider the identity operator from $\mathcal{W}$ with $\|\cdot\|_\bullet$ to $\mathcal{W}$ with $\|\cdot\|_1$. In both norms $\mathcal{W}$ is complete. Then the identity is bounded. Hence it is closed. The operator is bijective. Hence by Banach's theorem its inverse is bounded. Therefore we have

$$\|v\|_1 \leq c\|v\|_\bullet. \tag{A.48}$$

Thus, $\mathcal{Z}_\bullet^{(\pm)}$ are uniformly positive resp. negative. □

Proposition A.26. *Let S_1, S_2 be a pair of admissible involutions. Define K, c, d as in (A.4) and (A.6). Then K is pseudo-unitary on $(\mathcal{W}, Q)$ and K is positive with respect to both $(\cdot|\cdot)_1$ and $(\cdot|\cdot)_2$. Besides, $\|c\| < 1$ and $c^* = d$ with respect to $(\cdot|\cdot)_1$.*

Proof. K is pseudo-unitary as the product of two pseudo-unitary transformations. The inequality

$$(v|Kv)_1 = (S_1v|QS_2S_1v) = (S_1v|S_1v)_2 \geq a(S_1v|S_1v)_1 = a(v|v)_1$$

with $a > 0$ shows the positivity of K with respect to $(\cdot|\cdot)_1$. Therefore, $\mathbb{1} + K$ is invertible and $\|\frac{\mathbb{1}-K}{\mathbb{1}+K}\| < 1$. Hence $\|c\| < 1$. □

We finally show that any pair consisting of a maximal uniformly positive and a maximal uniformly negative subspace is complementary. (See also Lem. V.7.6. in [13]).

Proposition A.27. *Suppose that $\mathcal{Z}_1^{(+)}$ is a maximal uniformly positive space and $\mathcal{Z}_2^{(-)}$ is a maximal uniformly negative space. Then they are complementary.*

Proof. Set $\mathcal{Z}_1^{(-)} := \mathcal{Z}_1^{(+)\perp Q}$ and $\mathcal{Z}_2^{(+)} := \mathcal{Z}_2^{(-)\perp Q}$. Let S_1 resp. S_2 be the involutions corresponding to the pairs of complementary subspaces $(\mathcal{Z}_1^{(+)}, \mathcal{Z}_1^{(-)})$, resp. $(\mathcal{Z}_2^{(+)}, \mathcal{Z}_2^{(-)})$. They are admissible. By Prop. A.26, $K = S_2 S_1$ is positive. Hence $\mathbb{1} + K$ is invertible. Thus the result follows from Prop. A.8. □

B Gegenbauer equation

For the convenience of the reader, we present in this appendix basic statements about Gegenbauer functions. The following definitions and formulas are needed in many places in Sections 6 and 7 to describe propagators on deSitter space, anti-deSitter space and the latter's universal cover. More details on Gegenbauer functions can be found in [31], on which this section is based.

Here is the *Gegenbauer equation*:

$$\left((1-w^2)\partial_w^2 - 2(1+\alpha)w\partial_w + \lambda^2 - \left(\alpha+\frac{1}{2}\right)^2\right)f(w) = 0. \tag{B.1}$$

Its solutions can be expressed in terms of the Gauss hypergeometric function $F(a,b;c;z)$. We will use this function with the so-called *Olver's normalization*

$$\mathbf{F}(a,b;c;z) := \frac{F(a,b;c;z)}{\Gamma(c)} = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n z^n}{\Gamma(c+n) n!}. \tag{B.2}$$

The defining series converges only in the unit disc, but $\mathbf{F}(a,b;c;z)$ extends to a holomorphic function on $\mathbb{C}\backslash[1,\infty[$ as well as on a universal cover of $\mathbb{C}\backslash\{0,1\}$.

In what follows complex power functions should be interpreted as their principal branches (holomorphic on $\mathbb{C}\backslash]-\infty,0]$). For example $w \mapsto (1-w)^\alpha$ is holomorphic away from $[1,\infty[$. In addition, we will frequently use the notation

$$(w^2-1)_\bullet^\alpha := (w-1)^\alpha (w+1)^\alpha. \tag{B.3}$$

The function $(w^2-1)_\bullet^\alpha$ is holomorphic on $\mathbb{C}\backslash]-\infty,1]$, whereas $(w^2-1)^\alpha$ is holomorphic on $\mathbb{C}\backslash\big([-1,1]\cup \mathrm{i}\mathbb{R}\big)$. One has $(w^2-1)_\bullet^\alpha = (w^2-1)^\alpha$ only for $\mathrm{Re}(w) > 0$. However, $(1-w^2)^\alpha = (1-w)^\alpha(1+w)^\alpha$ for all $w \notin]-\infty,-1]\cup[1,\infty[$.

Standard solutions of the Gegenbauer equations are characterized by simple behavior at one of the three singular points $1, -1, \infty$. Due to the $w \mapsto -w$ symmetry of the equation (B.1), solutions of the second type are obtained from solutions of the first type by negating the argument. Therefore we consider 4 functions, corresponding to 2 behaviors at 1 and 2 behaviors at ∞. All of them are holomorphic on $\mathbb{C}\backslash]-\infty,1]$.

- The solution characterized by asymptotics ~ 1 at 1:

$$S_{\alpha,\pm\lambda}(w) := F\Big(\frac{1}{2}+\alpha+\lambda, \frac{1}{2}+\alpha-\lambda; \alpha+1; \frac{1-w}{2}\Big) \tag{B.4}$$

$$= \Big(\frac{2}{w+1}\Big)^{\alpha} F\Big(\frac{1}{2}+\lambda, \frac{1}{2}-\lambda; \alpha+1; \frac{1-w}{2}\Big). \tag{B.5}$$

  $S_{\alpha,\lambda}$ is distinguished among the four solutions introduced here by the fact that it is holomorphic on $\mathbb{C}\backslash]-\infty,-1]$ rather than only on $\mathbb{C}\backslash]-\infty,1]$. On the right half-plane we have an alternative expression:

$$S_{\alpha,\lambda}(w) = F\Big(\frac{1}{4}+\frac{\alpha}{2}+\frac{\lambda}{2}, \frac{1}{4}+\frac{\alpha}{2}-\frac{\lambda}{2}; \alpha+1; 1-w^2\Big), \qquad \mathrm{Re}(w) > 0. \tag{B.6}$$

- The solution $\frac{2^{2\alpha}}{(w^2-1)^{\alpha}_{\bullet}} S_{-\alpha,\lambda}(w)$ is characterized by asymptotics $\sim \frac{2^{\alpha}}{(w-1)^{\alpha}}$ at 1.

- The solution characterized by asymptotics $\sim w^{-\frac{1}{2}-\alpha-\lambda}$ at ∞:

$$Z_{\alpha,\lambda}(w) = (w\pm 1)^{-\frac{1}{2}-\alpha-\lambda} F\Big(\frac{1}{2}+\lambda, \frac{1}{2}+\lambda+\alpha; 1+2\lambda; \frac{2}{1\pm w}\Big) \tag{B.7}$$

$$= w^{-\frac{1}{2}-\alpha-\lambda} F\Big(\frac{1}{4}+\frac{\alpha}{2}+\frac{\lambda}{2}, \frac{3}{4}+\frac{\alpha}{2}+\frac{\lambda}{2}; 1+\lambda; \frac{1}{w^2}\Big).$$

- The solution $Z_{\alpha,-\lambda}(w)$ is characterized by asymptotics $\sim w^{-\frac{1}{2}-\alpha+\lambda}$ at ∞.

All these 4 functions can be expressed in terms of $S_{\alpha,\lambda}$, but for typographical reasons it is convenient to introduce also $Z_{\alpha,\lambda}$. We will use Olver's normalization:

$$\mathbf{S}_{\alpha,\lambda}(w) := \frac{1}{\Gamma(\alpha+1)} S_{\alpha,\lambda}(w), \qquad \mathbf{Z}_{\alpha,\lambda}(w) := \frac{1}{\Gamma(\lambda+1)} Z_{\alpha,\lambda}(w). \tag{B.8}$$

We note the identities

$$\mathbf{S}_{\alpha,\lambda}(w) = \mathbf{S}_{\alpha,-\lambda}(w), \quad \mathbf{Z}_{\alpha,\lambda}(w) = \frac{\mathbf{Z}_{-\alpha,\lambda}(w)}{(w^2-1)^{\alpha}_{\bullet}}. \tag{B.9}$$

$\mathbf{S}_{\alpha,\lambda}$ and $\mathbf{Z}_{\lambda,\alpha}$ are related by the *Whipple transformations*:

$$\mathbf{Z}_{\alpha,\lambda}(w) := (w^2-1)_{\bullet}^{-\frac{1}{4}-\frac{\alpha}{2}-\frac{\lambda}{2}} \mathbf{S}_{\lambda,\alpha}\left(\frac{w}{(w^2-1)^{\frac{1}{2}}_{\bullet}}\right), \tag{B.10}$$

$$\mathbf{S}_{\alpha,\lambda}(w) := (w^2-1)_{\bullet}^{-\frac{1}{4}-\frac{\alpha}{2}-\frac{\lambda}{2}} \mathbf{Z}_{\lambda,\alpha}\left(\frac{w}{(w^2-1)^{\frac{1}{2}}_{\bullet}}\right), \qquad \mathrm{Re}(w) > 0. \tag{B.11}$$

Here are the connection formulas:

$$\mathbf{S}_{\alpha,\lambda}(-w) = -\frac{\cos(\pi\lambda)}{\sin(\pi\alpha)}\mathbf{S}_{\alpha,\lambda}(w) + \frac{2^{2\alpha}\pi\mathbf{S}_{-\alpha,-\lambda}(w)}{\sin(\pi\alpha)\Gamma(\frac{1}{2}+\alpha+\lambda)\Gamma(\frac{1}{2}+\alpha-\lambda)(1-w^2)^{\alpha}}, \quad \text{(B.12)}$$

$$\mathbf{Z}_{\alpha,\lambda}(w) = -\frac{2^{\lambda-\alpha-\frac{1}{2}}\sqrt{\pi}\mathbf{S}_{\alpha,\lambda}(w)}{\sin(\pi\alpha)\Gamma(\frac{1}{2}-\alpha+\lambda)} + \frac{2^{\lambda+\alpha-\frac{1}{2}}\sqrt{\pi}}{\sin(\pi\alpha)\Gamma(\frac{1}{2}+\alpha+\lambda)}\frac{\mathbf{S}_{-\alpha,-\lambda}(w)}{(w^2-1)_{\bullet}^{\alpha}}, \quad \text{(B.13)}$$

$$\mathbf{S}_{\alpha,\lambda}(w) = \frac{2^{-\lambda+\alpha-\frac{1}{2}}\sqrt{\pi}}{\sin\pi\lambda}\left(-\frac{\mathbf{Z}_{\alpha,\lambda}(w)}{\Gamma\left(\frac{1}{2}+\alpha-\lambda\right)} + \frac{2^{2\lambda}\mathbf{Z}_{\alpha,-\lambda}(w)}{\Gamma\left(\frac{1}{2}+\alpha+\lambda\right)}\right). \quad \text{(B.14)}$$

From its definition, it is clear that $\mathbf{Z}_{\alpha,\lambda}$ satisfies

$$\mathbf{Z}_{\alpha,\lambda}(-w \mp \mathrm{i}0) = \mathrm{e}^{\pm\mathrm{i}\pi\left(\frac{1}{2}+\alpha+\lambda\right)}\mathbf{Z}_{\alpha,\lambda}(w \pm \mathrm{i}0), \quad w \in \mathbb{R}. \quad \text{(B.15)}$$

For further information on Gegenbauer functions (in various conventions), consult for example [31, 41, 48, 64, 75].

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