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Original Paper

Point Potentials on Euclidean Space, Hyperbolic Space and Sphere in Any Dimension

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Abstract. In dimensions $d = 1, 2, 3$, the Laplacian can be perturbed by a point potential. In higher dimensions, the Laplacian with a point potential cannot be defined as a self-adjoint operator. However, for any dimension there exists a natural family of functions that can be interpreted as Green's functions of the Laplacian with a spherically symmetric point potential. In dimensions $1, 2, 3$, they are the integral kernels of the resolvent of well-defined self-adjoint operators. In higher dimensions, they are not even integral kernels of bounded operators. Their construction uses the so-called generalized integral, a concept going back to Riesz and Hadamard. We consider the Laplace(–Beltrami) operator on the Euclidean space, the hyperbolic space and the sphere in any dimension. We describe the corresponding Green's functions, also perturbed by a point potential. We describe their limit as the scaled hyperbolic space and the scaled sphere approach the Euclidean space. Especially interesting is the behavior of positive eigenvalues of the spherical Laplacian, which undergo a shift proportional to a negative power of the radius of the sphere. We expect that in any dimension our constructions yield possible behaviors of the integral kernel of the resolvent of a perturbed Laplacian far from the support of the perturbation. Besides, they can be viewed as toy models illustrating various aspects of renormalization in quantum field theory, especially the point-splitting method and dimensional regularization.

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1. Introduction

1.1. Euclidean, Hyperbolic and Spherical Laplacian

Let $\mathbb{H}^d$ and $\mathbb{S}^d$ denote the *hyperbolic space* and the *unit sphere*, respectively, both d-dimensional. Let Δ_d, Δ_d^{h} and Δ_d^{s} denote the *Laplace(–Beltrami) operators* on $\mathbb{R}^d$, $\mathbb{H}^d$ and $\mathbb{S}^d$, respectively. It is convenient to shift the *hyperbolic Laplacian* by $-\frac{(d-1)^2}{4}$ and the *spherical Laplacian* by $\frac{(d-1)^2}{4}$. Our paper is devoted to the operators

$$\begin{aligned} H_d &:= -\Delta_d, \\ H_d^{\mathrm{h}} &:= -\Delta_d^{\mathrm{h}} - \frac{(d-1)^2}{4}, \\ H_d^{\mathrm{s}} &:= -\Delta_d^{\mathrm{s}} + \frac{(d-1)^2}{4}, \end{aligned} \tag{1.1}$$

possibly perturbed by a *point potential.*

The operators H_d, H_d^{h} and H_d^{s} can be viewed as self-adjoint operators on $L^2(\mathbb{R}^d)$, $L^2(\mathbb{H}^d)$ and $L^2(\mathbb{S}^d)$, respectively. For $z \in \mathbb{C}$ outside of the spectrum of H_d, H_d^{h} and H_d^{s}, one can define their *resolvent* (*Green's operator*)

$$\begin{aligned} G_d(z) &:= (-z + H_d)^{-1}, \\ G_d^{\mathrm{h}}(z) &:= (-z + H_d^{\mathrm{h}})^{-1}, \\ G_d^{\mathrm{s}}(z) &:= (-z + H_d^{\mathrm{s}})^{-1}. \end{aligned} \tag{1.2}$$

The spectrum of H_d and H_d^{h} is continuous and coincides with $[0,\infty[$. The spectrum of H_d^{s} is discrete and equals $\left\{\left(l + \frac{d-1}{2}\right)^2 \mid l = 0, 1, \dots\right\} \subset [0,\infty[$. Therefore, it is often convenient to represent the spectral parameter $z \in \mathbb{C}\backslash[0,\infty[$ as $z = -\beta^2$ with $\Re\beta > 0$, so that

$$\begin{aligned} G_d(-\beta^2) &= (\beta^2 + H_d)^{-1}, \\ G_d^{\mathrm{h}}(-\beta^2) &= (\beta^2 + H_d^{\mathrm{h}})^{-1}, \\ G_d^{\mathrm{s}}(-\beta^2) &= (\beta^2 + H_d^{\mathrm{s}})^{-1}. \end{aligned} \tag{1.3}$$

Sometimes we will also write ζ^2 for z.

For $0 \le a < b$, one can define the *spectral projections onto* $[a, b]$:

$$\mathbb{P}_d(a,b) := \mathbb{1}_{[a,b]}(H_d), \quad \mathbb{P}_d^{\mathrm{h}}(a,b) := \mathbb{1}_{[a,b]}(H_d^{\mathrm{h}}), \quad \mathbb{P}_d^{\mathrm{s}}(a,b) := \mathbb{1}_{[a,b]}(H_d^{\mathrm{s}}). \tag{1.4}$$

We can also introduce the *spectral projections onto eigenvalues* of H_d^{s}:

$$\mathbb{P}_{d,l}^{\mathrm{s}} := \mathbb{1}_{\left(l+\frac{d-1}{2}\right)^2}(H_d^{\mathrm{s}}). \tag{1.5}$$

The integral kernels of the resolvents (1.3), denoted by $G_d(-\beta^2; x, x')$, $G_d^{\mathrm{h}}(-\beta^2; x, x')$ and $G_d^{\mathrm{s}}(-\beta^2; x, x')$, are often called *Green's functions*. The integral kernels of the spectral projections (1.4) are denoted $\mathbb{P}_d(a, b; x, x')$, $\mathbb{P}_d^{\mathrm{h}}(a, b; x, x')$, $\mathbb{P}_d^{\mathrm{s}}(a, b; x, x')$. The integral kernel of (1.5) is denoted $\mathbb{P}_{d,l}^{\mathrm{s}}(x, x')$. Explicit formulas for these in terms of special functions are known, and for convenience of the reader, we provide them in our paper.

The integral kernels related to H_d are expressed in terms of functions from the Bessel family. The integral kernels related to H_d^{h} and H_d^{s} are expressed in terms of Gegenbauer functions. Here are, for instance, the formulas for Green's functions:

$$G_d\big(-\beta^2;x,x'\big)=\frac{1}{(2\pi)^{\frac{d}{2}}}\Big(\frac{\beta}{r}\Big)^{\frac{d}{2}-1}K_{\frac{d}{2}-1}\big(\beta r\big), \tag{1.6}$$

$$G_d^{\mathrm{h}}\big(-\beta^2;x,x'\big)=\frac{\sqrt{\pi}\Gamma(\frac{d-1}{2}+\beta)}{\sqrt{2}(2\pi)^{\frac{d}{2}}2^{\beta}}\mathbf{Z}_{\frac{d}{2}-1,\beta}\big(\cosh(r)\big), \tag{1.7}$$

$$G_d^{\mathrm{s}}\big(-\beta^2;x,x'\big)=\frac{\Gamma(\frac{d-1}{2}+\mathrm{i}\beta)\Gamma(\frac{d-1}{2}-\mathrm{i}\beta)}{(4\pi)^{\frac{d}{2}}}\mathbf{S}_{\frac{d}{2}-1,\mathrm{i}\beta}\big(-\cos(r)\big). \tag{1.8}$$

Above, r denotes the Euclidean, hyperbolic and spherical distance between x and x', respectively. K_α is the Macdonald function (one of the functions from the Bessel family). $\mathbf{S}_{\alpha,\lambda}$ and $\mathbf{Z}_{\alpha,\lambda}$ are two kinds of Gegenbauer functions, see Appendix C.

One should note that Bessel and Gegenbauer functions have special properties when their parameter α is half-integer or integer. For half-integer α, Bessel and Gegenbauer functions can be expressed as elementary functions. For integer α, Bessel and Gegenbauer functions have a logarithmic singularity. From the point of view of Green's operators, these values are important: half-integer α is used in odd dimensions and integer α in even dimensions.

All Green's functions (1.6), (1.7) and (1.8) behave similarly for x, x' close to one another, which follows from well-known expansions of K_α, $\mathbf{S}_{\alpha,\lambda}$ and $\mathbf{Z}_{\alpha,\lambda}$. However, for large distances they are rather different. This can be seen by comparing the expansions of (1.6), (1.7) and (1.8) for large distances, which we describe in (2.19), (3.14) and (4.13).

1.2. Point Potentials

The main goal of this paper is to extend the above theory to the operators H_d, H_d^{h} and H_d^{s} perturbed by a *point potential* (also called a *contact* or *delta potential*). It is a well-known fact that the one-dimensional Laplacian can be perturbed by a delta potential in the form sense [28]. In dimensions 2 and 3, the Laplacian can also be perturbed by a point-like perturbation; however, one cannot use the naive form formalism anymore [2–4]. Thus, in dimensions $d=1,2,3$ we obtain one-parameter families of self-adjoint operators H_d^γ, $H_d^{\mathrm{h},\gamma}$ and $H_d^{\mathrm{s},\gamma}$. We denote their resolvents by $G_d^\gamma(z)$, $G_d^{\mathrm{h},\gamma}(z)$ and $G_d^{\mathrm{s},\gamma}(z)$. Their integral kernels have the form

$$G_d^\gamma(z;x,x')=G_d(z;x,x')+\frac{G_d(z;x,x_0)G_d(z;x_0,x')}{\gamma+\Sigma_d(z)}, \tag{1.9}$$

$$G_d^{\mathrm{h},\gamma}(z;x,x')=G_d^{\mathrm{h}}(z;x,x')+\frac{G_d^{\mathrm{h}}(z;x,x_0)G_d^{\mathrm{h}}(z;x_0,x')}{\gamma+\Sigma_d^{\mathrm{h}}(z)}, \tag{1.10}$$

$$G_d^{\mathrm{s},\gamma}(z;x,x')=G_d^{\mathrm{s}}(z;x,x')+\frac{G_d^{\mathrm{s}}(z;x,x_0)G_d^{\mathrm{s}}(z;x_0,x')}{\gamma+\Sigma_d^{\mathrm{s}}(z)}, \tag{1.11}$$

where x_0 is the position of the point potential (e.g., the origin of coordinates of $\mathbb{R}^d$ or the north pole of $\mathbb{S}^d$). Here, the functions Σ_d, Σ_d^{h} and Σ_d^{s} satisfy

$$\partial_z \Sigma_d(z) = -\int_{\mathbb{R}^d} G_d(z; x_0, x)^2 \mathrm{d}x, \tag{1.12}$$

$$\partial_z \Sigma_d^{\mathrm{h}}(z) = -\int_{\mathbb{H}^d} G_d^{\mathrm{h}}(z; x_0, x)^2 \mathrm{d}x, \tag{1.13}$$

$$\partial_z \Sigma_d^{\mathrm{s}}(z) = -\int_{\mathbb{S}^d} G_d^{\mathrm{s}}(z; x_0, x)^2 \mathrm{d}x. \tag{1.14}$$

The parameter $\gamma \in \mathbb{R} \cup \{\infty\}$ is a real integration constant and describes the strength of the perturbation. The function $\gamma + \Sigma_d^\bullet(z)$, where $\bullet$ is empty, h or s, will be called the *full self-energy.* $\Sigma_d^\bullet(z)$ is the *reference self-energy*, fixed by imposing some additional conditions.

In dimensions 1 and 3, there exists a natural condition that allows us to fix the reference self-energy: $\lim\limits_{z\to-\infty} \Sigma_1^\bullet(z) = 0$ and $\lim\limits_{z\to-\infty} (\Sigma_3^\bullet(z) - \frac{\sqrt{-z}}{4\pi}) = 0$.

For $d = 2$, one possible choice for the reference self-energy is to demand $\Sigma_2^\bullet(-\beta^2) \sim \frac{\ln\beta}{2\pi}$ for $\beta \to \infty$, or equivalently, $\Sigma_2^\bullet(-1) = 0$. This, however, distinguishes a certain length scale corresponding to $\beta = 1$. In order to avoid such an *a priori* unphysical distinction, we treat all possible full self-energies on an equal footing as members of a family of reference self-energies parametrized by a real parameter $\varepsilon = -2\pi\gamma$:

$$\gamma + \Sigma_2^\bullet(z) =: \Sigma_2^{\bullet,\varepsilon}(z). \tag{1.15}$$

γ (and ε in $d = 2$) are closely related to the so-called *scattering length* a used in the physical literature. Here are the relations between these two parameters:

$$d = 1, \qquad a = -2\gamma; \tag{1.16}$$

$$d = 2, \qquad a = \mathrm{e}^{2\pi\gamma} = \mathrm{e}^{-\varepsilon}; \tag{1.17}$$

$$d = 3, \qquad a = -\frac{1}{4\pi\gamma}. \tag{1.18}$$

It is well known that the Laplacian is essentially self-adjoint on $C_{\mathrm{c}}^\infty(\mathbb{R}^d\backslash\{0\})$ in dimensions $d \geq 4$ [28]. In other words, there are no point-like perturbations of the Laplacian in dimensions $d \geq 4$, if we stick to the usual Hilbert space setting. This corresponds to the divergence of the integrals in (1.12), (1.13) and (1.14) defining the self-energies.

The description of Green's functions for the Laplacians with a point potential in dimensions $d \geq 4$ is probably the main novelty of our paper. Our starting points are Eqs. (1.9), (1.10) and (1.11). Hence, we need to give meaning to divergent self-energies. We will consider two different but consistent methods to do this. The first will be called the *point-splitting method* and the second the *minimal subtraction method*.

In the first method, we start with replacing the integrals (1.12), (1.13), (1.13) by their "point-split versions," which are then repeatedly differentiated in z (the "energy") until convergent integrals are obtained. Then we repeatedly integrate them to get the self-energy. Integration constants from multiple integrations can be gathered in a polynomial $\gamma(z)$, which replaces the integration constant γ used in lower dimensions. $\gamma(z)$ is a polynomial of degree $\leq n = \lfloor \frac{d-2}{2} \rfloor$, i.e., $n = \frac{d-3}{2}$ if d is odd and degree $n = \frac{d-2}{2}$ if d is even.

The second approach to define self-energies is to replace (1.12), (1.13), (1.14) with the corresponding *generalized integrals*. Then the self-energies $\Sigma_d^{\bullet}(z)$ are well defined in all dimensions up to only one integration constant.

As we explain in Appendix A, the generalized integral is a natural extension of the classical integration to a certain class of not necessarily integrable functions. It resembles the minimal subtraction scheme in QFT. Clearly, it is only one of many linear extensions of the integration functional. Other extensions differ by an additional polynomial of degree $\leq \lfloor \frac{d-2}{2} \rfloor$, whose parameters can be viewed as arbitrary "renormalization constants." Thus, both approaches to defining self-energies agree.

A generalized integral is said to have a *scaling anomaly* if it transforms inhomogeneously upon a rescaling of the integration variable. There is a big difference between non-anomalous and anomalous generalized integrals. In the non-anomalous case, the computation of a generalized integral essentially reduces to the analytic continuation of the usual integral in a certain parameter, which often (in particular, in our case) can be interpreted as the dimension. In the anomalous case, in addition to analytic continuation one has to perform an appropriate subtraction.

One could ask whether it is natural to fix a certain full self-energy, and to call it the *reference self-energy*. We would like our reference self-energies to be algebraically as simple as possible; in particular, they should be factorized in simple factors.

The generalized integral suggests a certain expression, which we denote $\Sigma_d^{\bullet,\mathrm{ms}}$, (where we fix a single integration constant in some natural way and ms stands for "minimal subtraction"). In odd dimensions $d \geq 5$, there is an obvious choice of reference self-energy which is given by an algebraically simple expression. This reference self-energy is equal to $\Sigma_d^{\bullet,\mathrm{ms}}$ and can also be obtained by formally extending $\Sigma_d^{\bullet}(z)$ to complex d in the region $|\Re d - 2| < 2$ $(d \neq 2)$, and then by using analytic continuation.

In even dimensions $d \geq 4$, we are in the anomalous case, which is much more complicated. The generalized integrals on the right-hand side of (1.12), (1.13) and (1.14) involve non-elementary functions: the logarithm or the digamma function $\psi(z) := \frac{\Gamma'(z)}{\Gamma(z)}$.

The anomalous generalized integral is not invariant under a change of variables. In the Euclidean case, the natural variable is r, the distance from the origin in some fixed units. Since the generalized integral is invariant under a change of variable $r \to r^{\alpha}$ for any $\alpha > 0$, one can equivalently use the coordinate r^2.

In the hyperbolic and spherical cases, the variables r (or r^2), now denoting the hyperbolic and spherical distance, respectively, seem not convenient to compute self-energies. Instead, in [10], to this end we used the variables $w = 2(\cosh(r) - 1)$ in the hyperbolic case and $w = 2(1 - \cos(r))$ in the spherical case. These variables are convenient in calculations involving resolvents of the Laplacian, and they seem to be a natural choice. Note that $w = r^2 + \mathcal{O}(r^4)$ is a function of r^2 in both cases. Anyway, if we change the variable in the generalized integral according to (A.8), the resulting change in the self-energy is a polynomial of degree $\le \left\lfloor \frac{d-2}{2} \right\rfloor$, which is consistent with the ambiguity in the point-splitting approach.[1]

Thus, for even $d \ge 4$, selecting in some way the integration constant, we can introduce the self-energy given by the generalized integral $\Sigma_d^{\bullet,\mathrm{ms}}$. All self-energies are given by $\gamma(z) + \Sigma_d^{\bullet,\mathrm{ms}}(z)$, where γ is of degree $\le \frac{d-2}{2}$. In the hyperbolic and spherical cases, $\Sigma_d^{\bullet,\mathrm{ms}}$ is rather complicated and has no obvious factorization. There exists, however, a one-parameter family of factorized expressions $\Sigma_d^{\bullet,\varepsilon}(z)$, $\varepsilon \in \mathbb{R}$, which one can use as reference self-energies. We absorb the highest term of the polynomial γ in ε, so that now the remaining freedom consists of a polynomial $\eta(z)$ of degree only $\le \frac{d-4}{2}$. Thus, the general form of a full self-energy in even dimensions is now given by $\eta(z) + \Sigma_d^{\bullet,\varepsilon}(z)$.

Summarizing, for odd d we obtained the families of functions

$$G_d^{\gamma}(z;x,x') = G_d(z;x,x') + \frac{G_d(z;x,x_0)G_d(z;x_0,x')}{\gamma(z) + \Sigma_d(z)}, \tag{1.19}$$

$$G_d^{\mathrm{h},\gamma}(z;x,x') = G_d^{\mathrm{h}}(z;x,x') + \frac{G_d^{\mathrm{h}}(z;x,x_0)G_d^{\mathrm{h}}(z;x_0,x')}{\gamma(z) + \Sigma_d^{\mathrm{h}}(z)}, \tag{1.20}$$

$$G_d^{\mathrm{s},\gamma}(z;x,x') = G_d^{\mathrm{s}}(z;x,x') + \frac{G_d^{\mathrm{s}}(z;x,x_0)G_d^{\mathrm{s}}(z;x_0,x')}{\gamma(z) + \Sigma_d^{\mathrm{s}}(z)}, \tag{1.21}$$

parametrized by an arbitrary polynomial γ of degree $\le \frac{d-3}{2}$.

For even d, we need to slightly modify (1.19), (1.20) and (1.21): We replace the superscript γ with ε, η and $\gamma(z) + \Sigma_d^{\bullet}(z)$ with $\eta(z) + \Sigma_d^{\bullet,\varepsilon}(z)$. Here ε is a real number and η is an arbitrary polynomial of degree $\le \frac{d-4}{2}$.

In what follows, abusing the notation, we will sometimes write γ for a pair ε, η. $G_d^{\bullet,\gamma}(z)$ will be called *Green's functions*. For $d \ge 4$, they are not integral kernels of bounded operators. Hence, for such d, they are not resolvents of well-defined self-adjoint operators.

Here is the list of the reference self-energies in various dimensions:

$$\begin{aligned}
d=1: \qquad & \Sigma_1(-\beta^2) = -\frac{1}{2\beta}, \\
& \Sigma_1^{\mathrm{h}}(-\beta^2) = -\frac{1}{2\beta}, \\
& \Sigma_1^{\mathrm{s}}(-\beta^2) = -\frac{\coth \pi\beta}{2\beta}; \qquad (1.22\mathrm{a}) \\
d=2: \qquad & \Sigma_2^{\varepsilon}(-\beta^2) = \frac{1}{2\pi}\big(\ln \beta - \varepsilon\big),
\end{aligned}$$

[1] We remark that it makes a difference whether we allow for coordinate changes that are functions of r or only of r^2. In the latter case, the generalized integral transforms anomalously only in even dimensions. In the former case, it transforms anomalously also in odd dimensions, giving a polynomial freedom in any dimension. Since w is a function of r^2, the change of variables $r^2 \to w$ does not affect the generalized integral in odd dimensions.

$$\Sigma_2^{\mathrm{h},\varepsilon}(-\beta^2) = \frac{1}{2\pi}\Big(\psi(\tfrac12+\beta)-\varepsilon\Big),$$

$$\Sigma_2^{\mathrm{s},\varepsilon}(-\beta^2) = \frac{1}{4\pi}\Big(\psi(\tfrac12+\mathrm{i}\beta)+\psi(\tfrac12-\mathrm{i}\beta)-2\varepsilon\Big); \tag{1.22b}$$

$d=3:$

$$\Sigma_3(-\beta^2) = \frac{\beta}{4\pi},$$

$$\Sigma_3^{\mathrm{h}}(-\beta^2) = \frac{\beta}{4\pi},$$

$$\Sigma_3^{\mathrm{s}}(-\beta^2) = \frac{\beta\coth\pi\beta}{4\pi}; \tag{1.22c}$$

even $d\geq 4:$

$$\Sigma_d^{\varepsilon}(-\beta^2) = \frac{1}{(4\pi)^{\frac d2}\Gamma(\frac d2)}\big(\ln(\beta^2)-2\varepsilon\big)(-\beta^2)^{\frac{d-2}{2}},$$

$$\Sigma_d^{\mathrm{h},\varepsilon}(-\beta^2) = \frac{\psi\big(\frac{3-d}{2}+\beta\big)+\psi\big(\frac{d-1}{2}+\beta\big)-2\varepsilon}{(4\pi)^{\frac d2}\Gamma\big(\frac d2\big)}\prod_{j=0}^{\frac{d-4}{2}}\Big(-\beta^2+\big(\tfrac12+j\big)^2\Big),$$

$$\Sigma_d^{\mathrm{s},\varepsilon}(-\beta^2) = \frac{\psi\big(\frac{d-1}{2}+\mathrm{i}\beta\big)+\psi\big(\frac{d-1}{2}-\mathrm{i}\beta\big)-2\varepsilon}{(4\pi)^{\frac d2}\Gamma\big(\frac d2\big)}\prod_{j=0}^{\frac{d-4}{2}}\big(-\beta^2-(\tfrac12+j)^2\big); \tag{1.22d}$$

odd $d\geq 5:$

$$\Sigma_d(-\beta^2) = \frac{\pi}{(4\pi)^{\frac d2}\Gamma(\frac d2)}\beta(-\beta^2)^{\frac{d-3}{2}},$$

$$\Sigma_d^{\mathrm{h}}(-\beta^2) = \frac{\pi}{(4\pi)^{\frac d2}\,\Gamma\big(\frac d2\big)}\beta\prod_{k=1}^{\frac{d-3}{2}}\big(-\beta^2+k^2\big),$$

$$\Sigma_d^{\mathrm{s}}(-\beta^2) = \frac{\pi\coth(\pi\beta)}{(4\pi)^{\frac d2}\Gamma\big(\frac d2\big)}\beta\prod_{k=1}^{\frac{d-3}{2}}\big(-\beta^2-k^2\big). \tag{1.22e}$$

Of course, some items of the above list are well known. The self-energy in the Euclidean case for $d=1,2,3$ belongs to standard knowledge of contemporary quantum physics. The Euclidean self-energy for $d\geq 4$ obtained with help of the generalized integral is partially covered in the literature, see, e.g., [21] for odd dimensions. The self-energies for the hyperbolic and spherical Laplacian appear to be new.

As we stressed above, for $d\geq 4$, the functions (1.9), (1.10) and (1.11) do not define bounded operators and their inverses do not define self-adjoint operators. It is natural to ask what is their meaning.

One approach that can be found in the literature is to extend the Hilbert space, typically, to a Pontryagin space (with an indefinite metric product). This approach is described, e.g., in [20].

One can also consider a different interpretation. Fix a point x_0 in $\mathbb{R}^d$, $\mathbb{H}^d$ or $\mathbb{S}^d$. Suppose that $H_d^\bullet+V$ is a self-adjoint operator obtained by perturbing H_d in a ball around x_0 of small radius r. We expect that far away from that ball, the integral kernel of $(H_d^\bullet+V-z)^{-1}$ is well approximated by $G_d^{\bullet,\gamma}$ for some γ determined by V. Thus, coefficients of γ summarize universal long distance properties of V. We will discuss this idea further in a separate paper, which is in preparation.

The reference self-energies (corresponding to $\gamma = 0$ in odd dimensions and $\eta = 0$ in even dimensions) are in some sense distinguished—the poles of the corresponding Green's functions can be easily computed. In the Euclidean case, they are also distinguished by their scaling property. (They are "fixed points of the renormalization group.")

Our analysis of point interactions in dimensions $d \geq 4$ resembles renormalization in quantum field theory. In QFT, especially in the Wilsonian approach, one does not worry too much whether the quantities computed by renormalization techniques correspond to a well-defined Hamiltonian. They should reproduce the "infrared behavior of correlation functions." We apply a similar philosophy to Green's functions. Note in particular that the borderline case when the perturbed Green's functions do not correspond to self-adjoint operators is $d = 4$—the physical dimension of our space-time. (Our space-time has a Lorentzian signature; however, using the Wick rotation it can often be replaced by the Euclidean $\mathbb{R}^4$.)

Our analysis can be viewed as a toy model illustrating various aspects of renormalization in QFT. As explained above, we use two methods to define self-energies. The first applies the so-called *point splitting* and then regularization by *differentiation in the energy*. The second method, using generalized integrals, resembles the *minimal subtraction* method. To compute them, we use *dimensional regularization*. Both methods have their widely used counterparts in QFT.

1.3. Flat Limit

Let $R > 0$ and let $\mathbb{H}_R^d$, $\mathbb{S}_R^d$ denote the rescaled hyperbolic space of curvature $-\frac{1}{R^2}$ and the rescaled sphere of curvature $\frac{1}{R^2}$ (that means, of radius R). Intuitively it is clear that in some sense $\mathbb{H}_R^d$, $\mathbb{S}_R^d$ converge to $\mathbb{R}^d$ as $R \to \infty$.

Green's functions on the rescaled spaces are

$$G_{d,R}^{\mathrm{h}}(-\beta^2; x, x') = R^{-d+2} G_d^{\mathrm{h}}\Big(- (\beta R)^2, \frac{x}{R}, \frac{x'}{R}\Big), \tag{1.23}$$

$$G_{d,R}^{\mathrm{s}}(-\beta^2; x, x') = R^{-d+2} G_d^{\mathrm{s}}\Big(- (\beta R)^2, \frac{x}{R}, \frac{x'}{R}\Big). \tag{1.24}$$

We describe the convergence of these Green's functions to the Euclidean ones $G_d(-\beta^2; x, x')$. This is of course well known, see, e.g., [5].

On the rescaled spaces, the reference self-energies are defined as follows:

$$\begin{aligned}
&\Sigma_{d,R}^{\mathrm{h}}(-\beta^2) := R^{2-d}\Sigma_d^{\mathrm{h}}\big(- (\beta R)^2\big), && \text{odd } d,\\
&\Sigma_{d,R}^{\mathrm{h},\varepsilon}(-\beta^2) := R^{2-d}\Sigma_d^{\mathrm{h},\varepsilon+\ln R}\big(- (\beta R)^2\big), && \text{even } d,\\
&\Sigma_{d,R}^{\mathrm{s}}(-\beta^2) := R^{2-d}\Sigma_d^{\mathrm{s}}\big(- (\beta R)^2\big), && \text{odd } d,\\
&\Sigma_{d,R}^{\mathrm{s},\varepsilon}(-\beta^2) := R^{2-d}\Sigma_d^{\mathrm{s},\varepsilon+\ln R}\big(- (\beta R)^2\big), && \text{even } d.
\end{aligned} \tag{1.25}$$

Note that in even dimensions we need an additional additive renormalization, which can be traced back to rescaling of the variable in a generalized integral.

Using the above self-energies, we define the corresponding Green's functions. We prove that they converge to the Euclidean Green's function with a point potential and the same parameters. That is, in odd dimensions $G^{\mathrm{h}\gamma}_{d,R}(-\beta^2;x,x')$ and $G^{\mathrm{s},\gamma}_{d,R}(-\beta^2;x,x')$ converge to $G^{\gamma}_{d}(-\beta^2;x,x')$, and in even dimensions, $G^{\mathrm{h},\varepsilon,\eta}_{d,R}(-\beta^2;x,x')$ and $G^{\mathrm{s},\varepsilon,\eta}_{d,R}(-\beta^2;x,x')$ converge to $G^{\varepsilon,\eta}_{d}(-\beta^2;x,x')$. This convergence is, perhaps, not very surprising. However, it requires a rather careful treatment of the self-energy (including the choice of renormalization), especially for even $d \geq 4$, when there is the scaling anomaly.

1.4. Poles of Green's Functions

In dimensions 1,2,3, the singularities of Green's functions $G^{\bullet,\gamma}_{d}(z)$ and $G^{\bullet,\varepsilon,\eta}_{d}(z)$ are located at the spectrum $H^{\bullet,\gamma}_{d}(z)$ and $H^{\bullet,\varepsilon,\eta}_{d}(z)$. In the Euclidean and hyperbolic case, the continuous spectrum remains $[0,\infty[$, but the point potential may introduce an additional eigenvalue. In the spherical case, the point potential shifts the old eigenvalues and may introduce a new one. For example, in the Euclidean case we have the following new eigenvalues:

$$H^{\gamma}_{1}: \quad -\frac{1}{a^2}, \quad \text{if } a<0, \tag{1.26}$$

$$H^{\varepsilon}_{2}: \quad -\frac{1}{a^2}, \tag{1.27}$$

$$H^{\gamma}_{3}: \quad -\frac{1}{a^2}, \quad \text{if } a>0, \tag{1.28}$$

where we use the scattering length a (see Subsect. 2.2 for its relation to γ and ε).

For dimensions $d \geq 4$, the point potential may introduce additional poles of Green's functions located at z satisfying

$$\gamma(z) + \Sigma^{\bullet}_{d}(z) = 0, \qquad \text{odd } d; \tag{1.29}$$

$$\eta(z) + \Sigma^{\bullet,\varepsilon}_{d}(z) = 0, \qquad \text{even } d. \tag{1.30}$$

The interpretation of these singularities is less clear. We may call them *eigenvalues* of $H^{\bullet,\gamma}_{d}$, $H^{\bullet,\varepsilon,\eta}_{d}$, even though strictly speaking these Hamiltonians do not exist in the Hilbert space sense. For $d \geq 4$, these poles may appear outside of the real line. (After all, they are not eigenvalues of a true self-adjoint operator.)

For pure reference self-energies, the additional singularities are easy to determine. If $d \geq 3$ is odd, the singularities originating from (1.29) with $\gamma = 0$ are as follows:

$$\begin{aligned} H^{0}_{d}: &\qquad z = 0;\\ H^{\mathrm{h},0}_{d}: &\qquad z = -k^2, \quad k = 0,1,\dots,\frac{d-3}{2};\\ H^{\mathrm{s},0}_{d}: &\qquad z = \big(k+\tfrac{1}{2}\big)^2, \quad k \in \mathbb{N}_0. \end{aligned} \tag{1.31}$$

If $d \geq 2$ is even, the singularities originating from (1.30) with $\eta = 0$ are

$$\begin{aligned} H^{0}_{d}: &\qquad z = -\mathrm{e}^{2\varepsilon}, \quad \text{and if } d \geq 4 \text{ also } z = 0;\\ H^{\mathrm{h},0}_{d}: &\qquad z \text{ solving } \psi\big(\tfrac{3-d}{2} + \sqrt{-z}\big) + \psi\big(\tfrac{d-1}{2} + \sqrt{-z}\big) = 2\varepsilon, \end{aligned}$$

$$\begin{aligned} &\text{and} \quad z = -(k+\tfrac{1}{2})^2, \quad k = 0, 1, \ldots, \frac{d-4}{2}; \\ H_d^{\mathrm{s},0}: \quad &z \text{ solving } \psi\big(\tfrac{d-1}{2} + \mathrm{i}\sqrt{-z}\big) + \psi\big(\tfrac{d-1}{2} - \mathrm{i}\sqrt{-z}\big) = 2\varepsilon \\ &\text{and} \quad z = (k+\tfrac{1}{2})^2, \quad k = 0, 1, \ldots, \frac{d-4}{2}. \end{aligned} \tag{1.32}$$

What is especially interesting are eigenvalues (or poles of Green's functions) in the spherical case inside $[0,\infty[$ for a general point potential, which we discuss in Subsect. 4.4. In the unperturbed case for the sphere of radius R, they are located at

$$\frac{(l+\frac{d-1}{2})^2}{R^2}, \quad \text{with multiplicity } \frac{(2l+d-1)(d+l-2)!}{(d-1)!l!}, \quad l = 0, 1, \ldots. \tag{1.33}$$

One effect of the perturbation is that the multiplicity of each of these eigenvalues is decreased by one (in particular $\big(\frac{d-1}{2R}\big)^2$ is not an eigenvalue) and a shifted eigenvalue appears.

Let $E^{\gamma}_{d,l,R}$ be the lth shifted eigenvalue in the odd case. Below we give formulas for $E^{\gamma}_{d,l,R}$ in the generic case $\gamma(z) \neq 0$. If ν denotes the order of vanishing of $\gamma(z)$ at $z = 0$, we find:

$$E^{\gamma}_{1,l,R} = \frac{(l+\frac{1}{2})^2}{R^2} + \frac{4\gamma(l+\frac{1}{2})^2}{\pi R^3} + \mathcal{O}\left(\frac{\gamma^2}{R^4}\right), \tag{1.34}$$

$$\begin{aligned} E^{\gamma}_{d,l,R} = &\frac{(l+\frac{d-1}{2})^2}{R^2} - \frac{2(l+\frac{d-1}{2})^2 \prod_{k=0}^{\frac{d-3}{2}} \big((l+\frac{d-1}{2})^2 - k^2\big)}{(4\pi)^{\frac{d}{2}}\Gamma(\frac{d}{2})R^d \gamma\left(\frac{(l+\frac{d-1}{2})^2}{R^2}\right)} \\ &+ \mathcal{O}(R^{2-2d+4\nu}), \quad \text{odd } d \geq 3. \end{aligned} \tag{1.35}$$

Note that $d = 1$ is special.

Now consider the lth shifted eigenvalue $E^{\varepsilon,\eta}_{d,l,R}$ in the even-dimensional case. We have

$$E^{\varepsilon}_{2,l,R} = \frac{(l+\frac{1}{2})^2}{R^2} + \frac{l+\frac{1}{2}}{R^2 \ln(R\mathrm{e}^{\varepsilon})} + \mathcal{O}\Big(\frac{1}{R^2 \ln^2(R\mathrm{e}^{\varepsilon})}\Big), \tag{1.36}$$

$$\begin{aligned} E^{\varepsilon,\eta}_{d,l,R} = &\frac{(l+\frac{d-1}{2})^2}{R^2} - \frac{2(l+\frac{d-1}{2}) \prod_{j=0}^{\frac{d-4}{2}} \big((l+\frac{d-1}{2})^2 - (j+\frac{1}{2})^2\big)}{(4\pi)^{\frac{d}{2}}\Gamma(\frac{d}{2})R^d \eta\left(\frac{(l+\frac{d-1}{2})^2}{R^2}\right)} \\ &+ \mathcal{O}\Big(\ln(\mathrm{e}^{\varepsilon}R)R^{-2d+2+4\nu}\Big), \quad \text{even } d\text{gen}4. \end{aligned} \tag{1.37}$$

Note that for $d \geq 3$ we have a systematic shift of the lth eigenvalue asymptotically proportional to $\gamma(0)^{-1}$ or $\eta(0)^{-1}$, respectively, and inversely proportional to the volume of $\mathbb{S}^d$. The scaling of the shift with R is changed if $\gamma(0) = 0$ and $\eta(0) = 0$, respectively.

In particular, for $d = 3$ the $l = 0$ eigenvalue moves up by

$$\approx -\frac{1}{|\mathbb{S}^3|\gamma} = \frac{4\pi a}{|\mathbb{S}^3|}, \tag{1.38}$$

where $|\mathbb{S}^3|$ is the volume of $\mathbb{S}^3$. It is interesting to ask whether a similar formula is true for other compact manifolds.

1.5. Comparison with the Literature

Explicit formulas for the Euclidean, hyperbolic and spherical Green's functions in any dimension are known in the literature, see, e.g., [5,13]. In our presentation, we made an effort to describe various facets of these Green's functions in a (hopefully) complete and transparent way. In particular, we use the Gegenbauer functions with the conventions of Appendix C, because they yield much simpler expressions than the so-called associated Legendre functions, which are commonly found in the literature [5,26] in this context.

Point potentials in dimension $d = 3$ go back to Fermi [14], and since then, they have been often used in the physics literature. Berezin and Faddeev [4] seem to have been the first who interpreted them in a rigorous way. They are the subject of an extensive mathematical literature, confer, for example, [2,3]. Point potentials in dimension $d = 1, 2, 3$ are special cases of *singular perturbations*, that is, perturbations which cannot be interpreted as operators. As we mentioned above, the case $d = 1$ can be interpreted as a form perturbation, so that one can use the so-called KLMN theorem [28]. For $d = 2, 3$, the form technique is not applicable; therefore, in these dimensions point potentials belong to the class of *form singular perturbations*.

The formula for the resolvent of the form (1.9) is often called the *Krein formula* [19]. One can also find the name *Aronszajn–Donoghue theory* for this kind of treatment of singular rank one perturbations, see, e.g., [9]. (1.9) is essentially the singular version of the formula for the resolvent of an operator with a rank one perturbation, sometimes called the *Sherman–Morrison formula*.

There exist a large literature about point potentials on $\mathbb{R}^d$ for $d \geq 4$. These potentials are examples of *supersingular perturbations*. In order to interpret them as true linear operators, one needs to extend the Hilbert space by adding additional dimensions, see, e.g., [21]. This is reviewed in [20]. Note that our approach is different: We do not look for the perturbed operator, and we try to compute Green's function associated with point potentials in all dimensions. It seems that the formulas (1.22e), (1.22d) are new, at least in the hyperbolic and spherical case.

It is clear that point potentials can be defined on a manifold of dimension 1,2,3, and have then similar properties as on the Euclidean space. Nevertheless, we have never seen their analysis on hyperbolic and spherical spaces including an explicit formula for their resolvent. So we think that also the identities (1.22b) and (1.22c) in the hyperbolic and spherical case are new.

The concept of a generalized integral goes back to independent considerations of Hadamard [16,17] and Riesz [29]. The generalized integral is a linear extension of the integration functional to not necessarily integrable functions. It is closely related to the extension of homogeneous distributions [18]. More recent accounts are given in [22,27]. In a parallel work [10], we revisited this concept in a manner that is well suited for our applications.

The flat limit of hyperbolic and spherical Green's functions (without point potentials) is discussed in [5]. Note that the latter reference uses the *associated Legendre equation* instead of the Gegenbauer equation. The two equations are equivalent. The relation between Gegenbauer functions in our convention and associated Legendre functions can be found in [10].

Our results about the energy shift of eigenvalues of the spherical Laplacian in dimensions $d \geq 3$ seem to be new. They are consistent with the following known fact, implicit, e.g., in [24]: In dimension 3, in a large box the ground state energy of the Schrödinger Hamiltonian with a short-range potential characterized by scattering length a has the ground state energy $\approx \frac{4\pi a}{\mathrm{Vol}}$, where Vol is the volume of the box (compare with (1.38)). This fact plays an important role in the well-known asymptotics of the bound state energy of the three-dimensional N-body Bose gas.

The ground state energy of a dilute Bose gas as in dimensions $d \geq 4$ was studied in [1], where a similar asymptotics as for $d = 3$ was obtained and analogs of the concept of the scattering length for higher dimensions proved useful. Our interest in point potentials in higher dimension was partially sparked by this paper.

Our paper extensively uses results of the companion paper [10] by the same authors. In particular, we use the conventions for special functions described in [10]. These conventions are also explained in [7,8].

An important source of inspiration for our paper is renormalization in quantum field theory. In fact, our paper applies the ideas of two distinct methods: the point splitting followed by differentiation in the energy, as well as dimensional regularization followed by the minimal subtraction. This is described, e.g., in [6].

1.6. Strategy of the Paper

The study of the operators H_d, H_d^{h} and H_d^{s} and their point-like perturbations is carried out in Sects. 2, 3 and 4, respectively. In all three cases ($\mathbb{R}^d, \mathbb{H}^d$ and $\mathbb{S}^d$), the general strategy to determine the family of renormalized Green's function is the same:

1. We first describe (the well-known) Green's functions and spectral projections of the *unperturbed* operators.
2. The subsequent analysis of point potentials greatly varies with the dimension. The computations in dimensions $d = 1, 2, 3$ are straightforward and only employ the standard integral. These three dimensions are spelled out separately. Odd and even dimensions $d \geq 4$ need a more careful treatment. We present two methods: the point-splitting method and the minimal subtraction method. In particular, following the latter approach we compute the self-energy derived from the generalized integral. In the case of odd dimensions $d \geq 5$, this integral is *non-anomalous* and yields easily the reference self-energy. In the case of even dimensions $d \geq 4$, the generalized integral is *anomalous* and was computed in [10] via the method of dimensional regularization. We describe how to pass from the

minimal subtraction self-energy to the family of reference self-energies that seem to be preferable.

3. In the hyperbolic and spherical cases, we describe the flat limit of the free and perturbed Green's functions.
4. In the spherical case, we discuss the poles of the perturbed Green's functions $G_d^{\mathrm{s},\gamma}$ and $G_d^{\mathrm{s},\varepsilon,\eta}$.

This paper has four appendices. In Appendix A, we briefly introduce the generalized integral and list its relevant properties. Appendices B and C contain basic information on Bessel and Gegenbauer functions as well as a collection of their bilinear generalized integrals, which are needed in our analysis of point-like perturbations. Moreover, Appendix C contains a list of the asymptotic behaviors of Gegenbauer functions, relevant in our description of the flat limit. Both appendices are based on our parallel work [10], where the focus lies on properties of the generalized integral and of Bessel and Gegenbauer functions. Finally, Appendix D contains some useful formulas related to Pochhammer symbols and harmonic numbers.

1.7. Notation for Operators on Hilbert Spaces

The *integral kernel* $A(x,y)$ of an operator A on $L^2(\mathbb{R}^d)$ is the function, or sometimes distribution, on $\mathbb{R}^d \times \mathbb{R}^d$, such that

$$(f|Ag) = \int \mathrm{d}x \int \mathrm{d}y \overline{f(x)} A(x,y) g(y). \tag{1.39}$$

More generally, one can also define the integral kernel of an operator on $L^2(M, \mathrm{d}\mu)$, where M is a manifold with a measure $\mathrm{d}\mu$, and the scalar product is given by $\int \overline{f(x)} g(x) \mathrm{d}\mu(x)$. Then we define the integral kernel of an operator A, also denoted $A(x,y)$, by

$$(f|Ag) = \int \mathrm{d}\mu(x) \int \mathrm{d}\mu(y) \overline{f(x)} A(x,y) g(y).$$

Let H be a self-adjoint operator. $\sigma(H)$ will denote its *spectrum*. For $z \in \mathbb{C}\backslash\sigma(H)$, we can define its *resolvent* $(-z+H)^{-1}$. We will denote the *spectral projection* of H corresponding to $a \in \sigma(H)$ by $\mathbb{1}_a(H)$. We will denote by $\mathbb{1}_{[a,b]}(H)$ the spectral projection of H corresponding to the closed interval $[a,b]$.

The spectral projections can be computed with help of the resolvent:

$$\mathbb{1}_a(H) = -\mathrm{s-}\lim_{\epsilon \searrow 0} \mathrm{i}\epsilon(-a - \mathrm{i}\epsilon + H)^{-1}, \tag{1.40}$$

$$\begin{aligned} &\mathbb{1}_{[a,b]}(H) - \frac{1}{2}\big(\mathbb{1}_a(H) + \mathbb{1}_b(H)\big) \\ &\quad = \mathrm{s-}\lim_{\epsilon \searrow 0} \int_a^b \frac{\mathrm{d}s}{2\pi\mathrm{i}} \big((-s - \mathrm{i}\epsilon + H)^{-1} - (-s + \mathrm{i}\epsilon + H)^{-1}\big). \end{aligned} \tag{1.41}$$

$\mathrm{s-}\lim$ denotes the limit in the strong operator topology. (1.41) is called the *Stone formula*.

If $z \in \sigma(H)$, then $(-z+H)^{-1}$ is not well defined. However, in an appropriate topology $(-z+H)^{-1}$ may have well-defined limits on the spectrum.

Usually, the limit is different when we approach the spectrum from above and from below. These limits are denoted by $(-z \mp \mathrm{i}0 + H)^{-1}$.

Let $G(z) := (-z+H)^{-1}$ be the resolvent of a self-adjoint operator H. Then

$$(H-z)G(z) = \mathbb{1}, \qquad G(z)^* = G(\bar{z}), \qquad \frac{\mathrm{d}}{\mathrm{d}z}G(z) = G(z)^2. \tag{1.42}$$

2. Green's Operators on Euclidean Space

2.1. The Euclidean Laplacian

As explained in the introduction, in this subsection we consider $H_d := -\Delta_d$, where Δ_d is the Laplacian on $L^2(\mathbb{R}^d)$. We first recall the well-known formulas for the integral kernel of its resolvent

$$G_d(z) := (-z + H_d)^{-1} = (\beta^2 + H_d)^{-1}, \tag{2.1}$$

where we indicate two notations for the spectral parameter that we will use, $z = -\beta^2$. We also describe the integral kernel of $\mathbb{P}_d(a, b)$, the spectral projection of H_d onto $[a, b]$. We express them in terms of various functions from the Bessel family: K_α, J_α and $H_\alpha^\pm$. Relevant properties of these functions are listed in Appendix B. For completeness, we sketch a proof of this theorem.

Theorem 2.1. *1. For $\Re\beta > 0$, we have*

$$G_d(-\beta^2; x, x') = \frac{1}{(2\pi)^{\frac{d}{2}}} \Big(\frac{\beta}{|x-x'|}\Big)^{\frac{d}{2}-1} K_{\frac{d}{2}-1}\big(\beta|x-x'|\big). \tag{2.2}$$

2. Green's function possesses limits on $]0, \infty[$ from above and below. For $\zeta \in \mathbb{R}$, $\zeta > 0$, these limits are

$$G_d(\zeta^2 \pm \mathrm{i}0; x, x') = \pm\frac{\mathrm{i}}{4}\Big(\frac{\zeta}{2\pi|x-x'|}\Big)^{\frac{d}{2}-1} H^{\pm}_{\frac{d}{2}-1}\big(\zeta|x-x'|\big). \tag{2.3}$$

3. For $d > 2$, there exists also a limit at $z = 0$:

$$G_d(0; x, x') = \frac{\Gamma(\frac{d}{2}-1)}{4\pi^{\frac{d}{2}}|x-x'|^{d-2}}. \tag{2.4}$$

4. Finally, the integral kernels of the spectral projections are

$$\mathbb{P}_d(a, b; x, x') = \int_{\sqrt{a}}^{\sqrt{b}} \Big(\frac{\zeta}{2\pi}\Big)^{\frac{d}{2}} \frac{J_{\frac{d}{2}-1}\big(\zeta|x-x'|\big)}{|x-x'|^{\frac{d}{2}-1}} \mathrm{d}\zeta. \tag{2.5}$$

Proof. Let $r := |x-x'|$. By Euclidean invariance, there exists a function $G_d(z, r)$ such that $G_d(z; x, x') = G_d(z, r)$. Away from $r = 0$, we can write

$$\begin{aligned} 0 &= \Big(-\partial_r^2 - \frac{d-1}{r}\partial_r + \beta^2\Big)G_d(-\beta^2, r) \\ &= r^{1-\frac{d}{2}}\Big(-\partial_r^2 - \frac{1}{r}\partial_r + \frac{(\frac{d}{2}-1)^2}{r^2} + \beta^2\Big)r^{-1+\frac{d}{2}}G_d(-\beta^2, r). \end{aligned} \tag{2.6}$$

Then we find the solution vanishing at infinity and behaving for $r \to 0$ as

$$G_d(-\beta^2, r) \sim \begin{cases} \frac{\Gamma(\frac{d}{2}-1)}{4\pi^{\frac{d}{2}}} r^{2-d}, & d \neq 2, \\ -\frac{1}{2\pi} \ln r, & d = 2, \end{cases} \tag{2.7}$$

which implies the distributional differential equation

$$(-\Delta_d - z) G_d(z; x, x') = \delta(x, x'). \tag{2.8}$$

Since $G_d(-z; x, x')$ is an integrable function of $x - x'$, i.e.,

$$\int \Big| G_d(-z; x, x') \Big| \mathrm{d}x = \int \Big| G_d(-z; x, 0) \Big| \mathrm{d}x =: C, \tag{2.9}$$

Young's inequality for convolutions [23, Theorem 4.2] implies

$$\left\| \int G_d(-z; \cdot, x) f(x) \mathrm{d}x \right\|_2 \leq C \|f\|_2, \quad f \in L^2(\mathbb{R}^d). \tag{2.10}$$

Thus, $G_d(-z; x, x')$ is the integral kernel of a bounded operator, and hence, by (2.8) the integral kernel of the resolvent of H_d. This shows (2.2).

To derive the limits (2.3), write $\beta = \beta_{\mathrm{R}} + \mathrm{i}\beta_{\mathrm{I}}$ with $\beta_{\mathrm{R}} > 0$. Then

$$z = -(\beta_{\mathrm{R}} + \mathrm{i}\beta_{\mathrm{I}})^2 = \big(|\beta_{\mathrm{I}}| - \mathrm{i}\beta_{\mathrm{R}} \mathrm{sgn}(\beta_{\mathrm{I}}) \big)^2 \underset{\beta_{\mathrm{R}} \searrow 0}{\longrightarrow} |\beta_{\mathrm{I}}|^2 - \mathrm{i}0 \, \mathrm{sgn}(\beta_{\mathrm{I}}). \tag{2.11}$$

Hence, to get $z = \zeta^2 \pm \mathrm{i}0$ we need to insert $\beta = \mp\mathrm{i}(\zeta \pm \mathrm{i}0)$ with $\zeta > 0$.

(2.5) follows from (2.3) by (1.41):

$$\text{lhs of (2.5)} = \frac{1}{2\pi\mathrm{i}} \int_{\sqrt{a}}^{\sqrt{b}} 2\zeta \Big(G_d(\zeta^2 + \mathrm{i}0; x, x') - G_d(\zeta^2 - \mathrm{i}0; x, x') \Big) \mathrm{d}\zeta. \tag{2.12}$$

□

We remark that the function

$$G_d(-\beta^2, r) := \frac{1}{(2\pi)^{\frac{d}{2}}} \left(\frac{\beta}{r} \right)^{\frac{d}{2}-1} K_{\frac{d}{2}-1}(\beta r) \tag{2.13}$$

is well defined for all $d \in \mathbb{C}$, and not only for positive integers. Note its symmetry:

$$G_{4-d}(-\beta^2, r) = \left(\frac{\beta}{2\pi r} \right)^{2-d} G_d(-\beta^2, r), \tag{2.14}$$

coming from the symmetry of the Macdonald function $K_\alpha(z) = K_{-\alpha}(z)$ [25]. We have also the homogeneity relation

$$G_d(-(\lambda\beta)^2, \lambda^{-1} r) = \lambda^{d-2} G_d(-\beta^2, r), \tag{2.15}$$

which is equivalent to the fact that H_d is an operator homogeneous of degree -2.

Let us sum up the properties of Green's function for various d (not only positive integers). The behavior near zero is described by the power series (cf. (B.2), (B.5)):

$$G_d(-\beta^2;r) = \begin{cases} \frac{1}{4\pi^{\frac{d}{2}} r^{d-2}} \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(\frac{d-2}{2}-k)}{k!} \left(\frac{\beta r}{2}\right)^{2k} \\ + \frac{\beta^{d-2}}{(4\pi)^{\frac{d}{2}}} \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(\frac{2-d}{2}-j)}{j!} \left(\frac{\beta r}{2}\right)^{2j}, & d \notin 2\mathbb{Z}; \quad (2.16a) \\ \frac{1}{4\pi^{\frac{d}{2}} r^{d-2}} \sum_{k=0}^{\frac{d-4}{2}} \frac{(-1)^k (\frac{d-4}{2}-k)!}{k!} \left(\frac{\beta r}{2}\right)^{2k} + \frac{\beta^{d-2}(-1)^{\frac{d}{2}}}{(4\pi)^{\frac{d}{2}}} \\ \sum_{j=0}^{\infty} \frac{2\ln(\frac{\beta r}{2}) + 2\gamma_{\mathrm{E}} - H_j - H_{\frac{d-2}{2}+j}}{j!(\frac{d-2}{2}+j)!} \left(\frac{\beta r}{2}\right)^{2j}, & d = 2,4,6,\dots . \quad (2.16b) \end{cases}$$

Note that the singular part of the first line of (2.16a) has the same form as the first line of (2.16a), and that for $\Re d < 2 + 2n$ there exists the limit

$$\lim_{x\to 0} \frac{\partial^n}{\partial z^n} G_d(z;x,0) = \frac{\Gamma(\frac{2-d}{2}+n)}{(4\pi)^{\frac{d}{2}}} \beta^{d-2-2n}. \tag{2.17}$$

The latter equation can be derived from the power series (2.16a) and (2.16a). Note that there is no logarithmic singularity as $r \to 0$ for $d = 2,4,6,\dots$ because $2n > d-2$, and hence, at least one derivative must act on $\ln\left(\frac{\beta r}{2}\right)$.

There is an elementary formula for odd dimensions (cf. (B.7)):

$$G_d(-\beta^2;x,x') = \frac{1}{2(2\pi)^{\frac{d-1}{2}}} \left(-\frac{1}{r}\partial_r\right)^{\frac{d-3}{2}} \frac{\mathrm{e}^{-\beta r}}{r}, \quad d = 3,5,\dots . \tag{2.18}$$

For $\beta > 0$, the behavior for large r can be described by the following asymptotic (divergent) series (cf. (B.6)):

$$G_d(-\beta^2;x,x') \simeq \frac{1}{2\beta}\left(\frac{\beta}{2\pi r}\right)^{\frac{d-1}{2}} \mathrm{e}^{-\beta r} \sum_{j=0}^{\infty} \frac{(\frac{d-1}{2}-j)_{2j}}{j!(2\beta r)^j}. \tag{2.19}$$

2.2. Point Potentials on Euclidean Space

Suppose that H_d^γ is a self-adjoint extension of the restriction of H_d to $C_{\mathrm{c}}^\infty(\mathbb{R}^d\backslash\{0\})$. Consider its resolvent

$$G_d^\gamma(z) = (H_d^\gamma + \beta^2)^{-1} = (H_d^\gamma - z)^{-1}. \tag{2.20}$$

By (1.42), the integral kernel of $G_d^\gamma(z)$, denoted $G_d^\gamma(z;x,x')$, satisfies

$$(-\Delta_x + \beta^2) G_d^\gamma(z;x,x') = \delta(x-x'), \qquad x,x' \neq 0, \tag{2.21}$$

$$G_d^\gamma(z;x,x') = G_d^\gamma(z;x',x), \tag{2.22}$$

$$\partial_z G_d^\gamma(z;x,x') = \int G_d^\gamma(z;x,y) G_d^\gamma(z;y,x') \mathrm{d}y. \tag{2.23}$$

To solve these equations, we make an ansatz

$$G_d^\gamma(z;x,x') = G_d(z;x,x') + \frac{1}{\gamma(z)+\Sigma_d(z)} G_d(z;x,0)G_d(z;0,x'), \tag{2.24}$$

which already incorporates the conditions (2.21) and (2.22). The denominator of the second term is split as $\gamma(z)+\Sigma_d(z)$ because this expression will depend on some number of free parameters. We will fix $\Sigma_d(z)$ for every dimension and collect all free parameters in $\gamma(z)$.

Remark 2.2. Note that (2.24) describes a spherically symmetric perturbation. In particular, it excludes the δ' potential for $d=1$.

Let us insert (2.24) into (2.23) to determine $\Sigma_d(z)$. $G_d^\gamma(z;x,x')$ satisfies (2.23) if

$$\frac{\mathrm{d}}{\mathrm{d}z}(\gamma(z)+\Sigma_d(z)) = -\sigma_d(z), \tag{2.25}$$

where

$$\sigma_d(z) := \int_{\mathbb{R}^d} G_d(z;0,y)G_d(z;y,0)\mathrm{d}y = \frac{(\beta^2)^{\frac{d}{2}-1}2\pi^{\frac{d}{2}}}{(2\pi)^d\Gamma(\frac{d}{2})}\int_0^\infty K_{\frac{d}{2}-1}(\beta r)^2 r\mathrm{d}r. \tag{2.26}$$

Note that the rightmost integral in (2.26) makes sense for complex d. It converges only for $|\Re(d-2)|<2$, which includes the dimensions $d=1,2,3$ [15]:

$$\sigma_d(-\beta^2) = \frac{\Gamma\left(\frac{4-d}{2}\right)}{(4\pi)^{\frac{d}{2}}}\beta^{d-4}, \quad |\Re(d-2)|<2. \tag{2.27}$$

For these dimensions, we take $\Sigma_d(z)$ to be a fixed antiderivative of $\sigma_d(z)$. Then (2.25) says that γ is a constant. G_d^γ is the integral kernel of the resolvent of a closed operator H_d^γ, which is self-adjoint if γ is real. One has $H_d = H_d^\infty$.

Below we propose how to define $\Sigma_d(z)$ in all dimensions. It will be seen that in contrast to $d=1,2,3$, it is natural to take $\gamma(z)$ to be a polynomial in z of degree depending on d. Hence, G_d^γ depends on several parameters. For every choice of $\gamma(z)$, $G_d^\gamma(z;x,x')$ is a well-defined locally integrable function, but for $d\geq 4$ it is not the integral kernel of a bounded operator. It describes the asymptotic behavior of Green's function of a Laplacian with a perturbation of a very small support, as explained in a separate paper. Let us discuss various dimensions separately.

Dimension 1. We have $\sigma_1(-\beta^2)=\frac{1}{4\beta^3}$, so we define $\Sigma_1(-\beta^2)=-\frac{1}{2\beta}$, homogeneous and vanishing at infinity. We have

$$G_1^\gamma(-\beta^2;x,x') = \frac{\mathrm{e}^{-\beta|x-x'|}}{2\beta} + \frac{\mathrm{e}^{-\beta|x|}\mathrm{e}^{-\beta|x'|}}{(2\beta)^2\left(\gamma-\frac{1}{2\beta}\right)}. \tag{2.28}$$

Sometimes $a{:=}-2\gamma$ is called the *scattering length*. The operator H_1^γ is the perturbation of H_1 by the quadratic form $\frac{2}{a}\delta(x)$. If $\gamma=0$, it is $-\Delta_d$ with Dirichlet boundary condition at 0, and it is homogeneous of degree -2. Functions in the domain of H_1^γ with $\gamma \neq 0$ have the leading singularity near zero proportional to $\frac{|x|}{a} - 1$. For $a < 0$, there exists a bound state $\mathrm{e}^{\frac{|x|}{a}}$ with eigenvalue $-\frac{1}{a^2}$.

Dimension 2. The full self-energy is now

$$\gamma + \Sigma_2(-\beta^2) = \gamma + \frac{\ln \beta}{2\pi}. \tag{2.29}$$

$\Sigma_2(-\beta^2)$ diverges if both $\beta \to 0$ and $\beta \to \infty$. γ is an arbitrary constant of integration. It is convenient to replace (2.29) by a family of reference self-energies introducing $\varepsilon := -2\pi\gamma$ and

$$\Sigma_2^\varepsilon(-\beta^2) := \frac{1}{2\pi}(\ln \beta - \varepsilon). \tag{2.30}$$

Then $a := \exp(2\pi\gamma) = \mathrm{e}^{-\varepsilon}$ specifies a length scale, in the physics literature again called the *scattering length*. We find

$$G_2^\varepsilon(-\beta^2; x, x') = \frac{K_0(\beta|x - x'|)}{2\pi} + \frac{K_0(\beta|x|)K_0(\beta|x'|)}{2\pi \ln(\beta \mathrm{e}^{-\varepsilon})}. \tag{2.31}$$

In contrast to $d = 1$, the scattering length cannot be negative. The denominator of the second term of (2.31) can be rewritten as $2\pi \ln(\beta a)$. Functions in the domain of H_2^ε behave near zero as $\ln(\frac{|x|}{2a}) + \gamma_{\mathrm{E}}$, where γ_{E} is the Euler–Mascheroni constant. For all a, there is a bound state $K_0\big(\frac{|x|}{a}\big)$ with eigenvalue $-\frac{1}{a^2}$.

Dimension 3. In this case, we take $\Sigma_3(-\beta^2) = \frac{\beta}{4\pi}$, homogeneous and vanishing at 0. We have

$$G_3^\gamma(-\beta^2; x, x') = \frac{\mathrm{e}^{-\beta|x-x'|}}{4\pi|x - x'|} + \frac{\mathrm{e}^{-\beta|x|}\mathrm{e}^{-\beta|x'|}}{(4\pi)^2|x||x'|(\gamma + \frac{\beta}{4\pi})}. \tag{2.32}$$

As in lower dimensions, one usually introduces the *scattering length*, now given by $a = -\frac{1}{4\pi\gamma}$. The denominator of the second term of (2.32) can be rewritten as $4\pi(\beta - \frac{1}{a})|x||x'|$. Functions in the domain of H_3^γ behave as $1 - \frac{a}{|x|}$ near 0. Operator H_3^0 is homogeneous of degree -2. For $a > 0$, there is a bound state $\frac{1}{|x|}\exp(-\frac{|x|}{a})$ with eigenvalue $-\frac{1}{a^2}$.

Higher dimensions. We will describe two methods of introducing self-energy in higher dimensions. The first, in the physical terminology, is based on the differentiation with respect to the energy and point splitting. First we rewrite the definition of σ_d from (2.26) as

$$\sigma_d(z) = \lim_{x \to 0} \int G_d(z; x, y)G_d(z; y, 0)\mathrm{d}y = \lim_{x \to 0} \frac{\partial}{\partial z} G_d(z; x, 0). \tag{2.33}$$

The limit on the right in general does not exist. However, if $d < 2 + 2n$ and we differentiate both sides n times in z, then the limit becomes finite:

$$\sigma_d^{(n-1)}(z) = \lim_{x \to 0} \frac{\partial^n}{\partial z^n} G_d(z; x, 0) = \frac{\Gamma(\frac{2-d}{2} + n)}{(4\pi)^{\frac{d}{2}}} \beta^{d-2-2n}, \tag{2.34}$$

where we inserted (2.17) in the last step.

We take n to be the smallest integer greater than $\frac{d-2}{2}$ and choose some $\Sigma_d(z)$ satisfying

$$\Sigma_d^{(n)}(z) = -\frac{\Gamma(\frac{2-d}{2}+n)}{(4\pi)^{\frac{d}{2}}}\beta^{d-2-2n}. \tag{2.35}$$

Then differentiating (2.25) $n-1$ times, we find that $\gamma(z)$ is a polynomial of degree $n-1$.

The second method of defining the self-energy yields a concrete Σ_d satisfying (2.35) for every dimension. To this end, we define $\sigma_d(z)$ by replacing the Lebesgue integral in (2.26) with the generalized integral $\operatorname{gen}\int_0^\infty$, which is defined in (A.2). Then we can choose Σ_d to be an antiderivative of $-\sigma_d(z)$.

We will check that thus defined Σ_d satisfies (2.35) by explicit computation, separately for odd and even dimensions. One can also see this by the following general argument.

Consider

$$\sigma_d(z) := \frac{2\pi^{\frac{d}{2}}}{(2\pi)^d\Gamma(\frac{d}{2})}\operatorname{gen}\int_0^\infty \beta^{d-2}K_{\frac{d}{2}-1}(\beta r)^2 r\mathrm{d}r. \tag{2.36}$$

Since the exponents of terms non-integrable near 0 do not depend on z, one can check that

$$\frac{\partial^{n-1}}{\partial z^{n-1}}\sigma_d(z) = \frac{2\pi^{\frac{d}{2}}}{(2\pi)^d\Gamma(\frac{d}{2})}\operatorname{gen}\int_0^\infty \frac{\partial^{n-1}}{\partial z^{n-1}}\Big(\beta^{d-2}K_{\frac{d}{2}-1}(\beta r)^2\Big) r\mathrm{d}r. \tag{2.37}$$

If $d < 2+2n$, the generalized integral on the right-hand side converges in the classical sense, and we can write

$$\begin{aligned}\frac{\partial^{n-1}}{\partial z^{n-1}}\sigma_d(z) &= \frac{2\pi^{\frac{d}{2}}}{(2\pi)^d\Gamma(\frac{d}{2})}\int_0^\infty \frac{\partial^{n-1}}{\partial z^{n-1}}\Big(\beta^{d-2}K_{\frac{d}{2}-1}(\beta r)^2\Big) r\mathrm{d}r \\ &= \int \frac{\partial^{n-1}}{\partial z^{n-1}}\lim_{x\to 0}\Big(G_d(z;x,y)G_d(z;y,0)\Big)\mathrm{d}y \\ &= \int \lim_{x\to 0}\frac{\partial^{n-1}}{\partial z^{n-1}}\Big(G_d(z;x,y)G_d(z;y,0)\Big)\mathrm{d}y. \end{aligned} \tag{2.38}$$

Next, we rename $y = y_n$ and express the derivative in the last line as an $(n-1)$-fold integral using the resolvent identity repeatedly:

$$\frac{\partial^{n-1}}{\partial z^{n-1}}\sigma_d(z) = \int \lim_{x\to 0}\Big(\int G_d(z;x,y_1)\cdots G_d(z;y_n,0)\mathrm{d}y_1\cdots\mathrm{d}y_{n-1}\Big)\mathrm{d}y_n. \tag{2.39}$$

Since $G_d(z;x,y_1)\cdots G_d(z;y_n,0)$ is an integrable function of $y_1,\ldots,y_n$, we may take the limit out of the integral.[2] Therefore,

$$\frac{\partial^{n-1}}{\partial z^{n-1}}\sigma_d(z) = \lim_{x\to 0}\int G_d(z;x,y_1)\cdots G_d(z;y_n,0)\mathrm{d}y_1\cdots\mathrm{d}y_n. \tag{2.40}$$

The integral can now be computed using the resolvent identity again. This shows that (2.34), and hence (2.35), is satisfied.

[2] If $f\in L^1(\mathbb{R}^N)$ and $f_a(x) := f(x-a)$ for $a\in\mathbb{R}^N$, then $f_a\to f$ strongly in $L^1(\mathbb{R}^N)$ as $a\to 0$.

One could use other definitions of generalized integration, for example, in other coordinates (cf. (A.8)). An inspection of formulas (2.16a), (2.16a) and (A.4) shows that the resulting $\sigma_d(z)$ differs only by a polynomial of degree $n-2$, where n is the smallest integer greater than $\frac{d-2}{2}$. Integrating to find $\Sigma_d(z)$ leads to another integration constant, and this accounts for the same freedom in the choice of $\Sigma_d(z)$ as suggested by (2.35).

We note that the leading term of $\Sigma_d(z)$ for large z is uniquely determined either by (2.35) or by calculation of generalized integrals. The term $\gamma(z)$ containing free parameters is of lower order for large z.

Odd dimensions $d \geq 5$. If (2.26) is understood as a generalized integral and d is not an even integer, then expression (2.27) remains valid. It is convenient to rewrite (2.27) as

$$\sigma_d(-\beta^2) = \beta^{d-4}\frac{\pi}{(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{d-2}{2}\right)\cos\left(\pi\frac{d-3}{2}\right)}, \quad d \in \mathbb{C}\setminus 2\mathbb{Z}. \tag{2.41}$$

Therefore, we take

$$\Sigma_d(-\beta^2) = \beta^{d-2}\frac{\pi}{(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)\cos\left(\pi\frac{d-3}{2}\right)}, \quad d \in \mathbb{C}\setminus 2\mathbb{Z}, \tag{2.42}$$

homogeneous and vanishing at 0. Since the generalized integral is non-anomalous for $d \in \mathbb{C}\setminus 2\mathbb{Z}$, the same result is obtained using the generalized integral with the integration variable λr for any constant $\lambda > 0$. Specifying d to be an odd integer, we obtain

$$\Sigma_d(-\beta^2) = (-\beta^2)^{\frac{d-3}{2}}\beta\frac{\pi}{(4\pi)^{\frac{d}{2}}\Gamma(\frac{d}{2})}. \tag{2.43}$$

The polynomial $\gamma(z)$ is of degree $\frac{d-3}{2}$, so it depends on 2 parameters for $d=5$, on 3 parameters for $d=7$, etc. For large z, it is subleading with respect to $\Sigma_d(z)$ by at least one power of β, and in contrast to Σ_d, it may contain even powers of β. The function G_d^0 (i.e., with $\gamma(z)=0$) satisfies the same homogeneity relation (2.15) as G_d.

Even dimensions $d \geq 4$. The expression (2.27) for $\sigma_d(-\beta^2)$ is not valid, even in the generalized sense because of the scaling anomaly. For $d = 4, 6, \ldots$, we find (cf. App. B)

$$\sigma_d(-\beta^2) = -\frac{(-\beta^2)^{\frac{d-4}{2}}}{(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)}\Big(1+(d-2)\left(\ln\tfrac{\beta}{2}+1-\psi\left(\tfrac{d}{2}\right)\right)\Big), \tag{2.44}$$

and therefore, the self-energy given by the minimal subtraction method is

$$\Sigma_d^{\mathrm{ms}}(-\beta^2) = \frac{(-\beta^2)^{\frac{d-2}{2}}}{(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)}\left(2-2\psi\left(\tfrac{d}{2}\right)+\ln\tfrac{\beta^2}{4}\right), \tag{2.45}$$

to which we can add a polynomial γ of degree $\leq \frac{d-2}{2}$. If a rescaled radial coordinate is used in the generalized integral, the result is shifted by a multiple of $(-\beta^2)^{\frac{d-2}{2}}$. Note that this power of $-\beta^2$ is also the leading term of $\gamma(-\beta^2)$, but it is still subleading in the self-energy due to the presence of the logarithm in (2.45). Therefore, there are 2 parameters for $d = 4$, 3 parameters for $d = 6$, etc.

It is convenient to introduce a scale-dependent reference self-energy

$$\Sigma_d^{\varepsilon}(-\beta^2){:=}\frac{1}{(4\pi)^{\frac{d}{2}}\Gamma(\frac{d}{2})}(-\beta^2)^{\frac{d-2}{2}}(\ln\beta^2 - 2\varepsilon), \tag{2.46}$$

where ε is used to absorb the highest term $\gamma_{\frac{d-2}{2}}\,(-\beta^2)^{\frac{d-2}{2}}$ in γ:

$$-2\varepsilon = (4\pi)^{\frac{d}{2}}\Gamma\left(\tfrac{d}{2}\right)\gamma_{\frac{d-2}{2}} + 2 - 2\psi\left(\tfrac{d}{2}\right) - \ln 4. \tag{2.47}$$

Thus, we obtain a family of Green's functions

$$G_d^{\varepsilon,\eta}(z;x,x') = G_d(z;x,x') + \frac{1}{\eta(z)+\Sigma_d^{\varepsilon}(z)}G_d(z;x,x_0)G_d(z;x_0,x'), \tag{2.48}$$

where η is an arbitrary polynomial of degree $\leq \frac{d-4}{2}$.

Remark 2.3 (Scattering length in higher dimensions). Let $d \geq 3$ be odd. If x is small, y large and $z = 0$, then

$$G_d^{\gamma}(0;x,y) \approx G_d(0;0,y)\left(1 + \frac{G_d(0;x,0)}{\gamma(0)}\right) = G_d(0;0,y)\left(1 - \frac{a}{|x|^{d-2}}\right), \tag{2.49}$$

where, following [24, Appendix C], we introduced the scattering length

$$a{:=} - \frac{\Gamma(\frac{d}{2}-1)}{4\pi^{\frac{d}{2}}\gamma(0)}. \tag{2.50}$$

If $d \geq 4$ is even, we can do the same, replacing G_d^{γ} with $G_d^{\varepsilon,\eta}$ and $\gamma(0)$ with $\eta(0)$.

Calling a a *length* is actually a misnomer, since now (2.50) does not have the dimension of length, unlike for $d = 1, 2, 3$. Moreover, (2.50) is not consistent with the definition of a for $d = 1, 2$.

3. Green's Operators on Hyperbolic Space

3.1. Hyperbolic Laplacian

The space $\mathbb{R}^{1+d}$ equipped with the bilinear form

$$[x|y] = x^0y^0 - x^1y^1 - \cdots - x^dy^d$$

will be denoted $\mathbb{R}^{1,d}$. The set

$$\mathbb{H}^d{:=}\{x \in \mathbb{R}^{1,d} \mid [x|x] = 1,\ x^0 > 0\}$$

equipped with the Riemannian metric inherited from $\mathbb{R}^{1,d}$ is called the *hyperbolic space*. The geodesic distance between $x, x' \in \mathbb{H}^d$ is given by

$$d^{\mathrm{h}}(x,x') = \cosh^{-1}\left([x|x']\right), \qquad \cosh d^{\mathrm{h}}(x,x') = [x|x']. \tag{3.1}$$

$\mathbb{H}^d$ has also a measure induced by the metric.

In this section, we study

$$H_d^{\mathrm{h}} := -\Delta_d^{\mathrm{h}} - \frac{(d-1)^2}{4}, \tag{3.2}$$

where Δ_d^{h} is the Laplacian on $L^2(\mathbb{H}^d)$ induced by the metric. H_d^{h} is a self-adjoint operator. For $z \in \mathbb{C}\backslash\sigma(H_d^{\mathrm{h}}) = \mathbb{C}\backslash[0,\infty[$, we define the *hyperbolic Green's operator* $G_d^{\mathrm{h}}(z):=(-z+H_d^{\mathrm{h}})^{-1}$. The spectral projection of H_d^{h} onto $[a,b[\subset[0,\infty[$ is denoted $\mathbb{P}_d^{\mathrm{h}}(a,b)$. In the following theorem, we express the integral kernels of $G_d^{\mathrm{h}}(z)$ and $\mathbb{P}_d^{\mathrm{h}}(a,b)$ in terms of two kinds of Gegenbauer functions, $\mathbf{S}_{\alpha,\beta}$ and $\mathbf{Z}_{\alpha,\beta}$, which are defined in Appendix C.

Theorem 3.1 *1. For $\Re\beta > 0$, the integral kernel of $G_d^{\mathrm{h}}(-\beta^2)$ is*

$$G_d^{\mathrm{h}}\Big(-\beta^2; x, x'\Big) = \frac{\sqrt{\pi}\Gamma(\frac{d-1}{2}+\beta)}{\sqrt{2}(2\pi)^{\frac{d}{2}}2^{\beta}}\mathbf{Z}_{\frac{d}{2}-1,\beta}\big([x|x']\big). \tag{3.3}$$

2. For $\zeta > 0$, it has the following limits

$$G_d^{\mathrm{h}}(\zeta^2 \pm \mathrm{i}0; x, x') = \frac{\sqrt{\pi}\Gamma(\frac{d-1}{2}\mp\mathrm{i}\zeta)}{\sqrt{2}(2\pi)^{\frac{d}{2}}2^{\mp\mathrm{i}\zeta}}\mathbf{Z}_{\frac{d}{2}-1,\mp\mathrm{i}\zeta}\big([x|x']\big). \tag{3.4}$$

3. The integral kernel of $\mathbb{P}_d^{\mathrm{h}}(a,b)$ is

$$\mathbb{P}_d^{\mathrm{h}}(a,b;x,x') = \int_{\sqrt{a}}^{\sqrt{b}} \frac{2\zeta\sinh(\pi\zeta)\,\Gamma(\frac{d-1}{2}+\mathrm{i}\zeta)\Gamma(\frac{d-1}{2}-\mathrm{i}\zeta)}{\pi(4\pi)^{\frac{d}{2}}}\mathbf{S}_{\frac{d}{2}-1,\mathrm{i}\zeta}\big([x|x']\big)\mathrm{d}\zeta. \tag{3.5}$$

Proof The isometry group of $\mathbb{H}^d$ acts transitively on pairs (x,x') with fixed $[x|x']$, so there exists a function $G_d^{\mathrm{h}}(z;w)$ such that $G_d^{\mathrm{h}}(z;x,x') = G_d^{\mathrm{h}}(z;w)$, $w:=[x|x']$. We have

$$0 = \Big((1-w^2)\partial_w^2 - dw\partial_w + \beta^2 - \big(\tfrac{d-1}{2}\big)^2\Big)\,G_d^{\mathrm{h}}(-\beta^2, w). \tag{3.6}$$

Near the diagonal, the hyperbolic Green's function should have the same asymptotics as Green's function of the Laplacian:

$$G_d^{\mathrm{h}}\big(-\beta^2, \cosh(r)\big) \sim \begin{cases} \frac{\Gamma(\frac{d}{2}-1)}{4\pi^{\frac{d}{2}}}r^{2-d}, & d \neq 2, \\ -\frac{1}{2\pi}\ln r, & d = 2. \end{cases} \tag{3.7}$$

Besides, it should vanish for $w \to \infty$. This fixes uniquely $G_d^{\mathrm{h}}(-\beta^2, w)$ to be (3.3). (3.4) follows immediately.

To derive (3.5), we note that by (1.41)

$$\text{lhs of (3.5)} = \frac{1}{2\pi\mathrm{i}}\int_{\sqrt{a}}^{\sqrt{b}} 2\zeta\Big(G_d^{\mathrm{h}}\Big(\zeta^2+\mathrm{i}0;x,x'\Big) - G_d^{\mathrm{h}}\Big(\zeta^2-\mathrm{i}0;x,x'\Big)\Big)\mathrm{d}\zeta. \tag{3.8}$$

Then we use the following identity, which is a consequence of (C.13):

$$\begin{aligned}&\frac{\sqrt{\pi}}{\sqrt{2}}\Big(2^{\mathrm{i}\zeta}\mathbf{Z}_{\alpha,-\mathrm{i}\zeta}(z)\Gamma\Big(\frac{1}{2}+\alpha-\mathrm{i}\zeta\Big) - 2^{-\mathrm{i}\zeta}\mathbf{Z}_{\alpha,\mathrm{i}\zeta}(z)\Gamma\Big(\frac{1}{2}+\alpha+\mathrm{i}\zeta\Big)\Big)\\ &\quad= \mathrm{i}2^{-\alpha}\sinh(\pi\zeta)\Gamma\Big(\frac{1}{2}+\alpha+\mathrm{i}\zeta\Big)\Gamma\Big(\frac{1}{2}+\alpha-\mathrm{i}\zeta\Big)\mathbf{S}_{\alpha,\mathrm{i}\zeta}(z).\end{aligned} \tag{3.9}$$

□

We can view $G_d^{\mathrm{h}}(-\beta^2, w)$ as defined for all $d \in \mathbb{C}$. It satisfies the symmetry

$$G_{4-d}^{\mathrm{h}}(-\beta^2, \cosh r) = \frac{1}{(\frac{3-d}{2} + \beta)_{d-2}(2\pi \sinh r)^{2-d}} G_d^{\mathrm{h}}(-\beta^2, \cosh r). \tag{3.10}$$

Here are explicit formulas for the hyperbolic Green's function useful for small r. They follow from (3.3), the connection formula (C.13), (C.2), (C.3) and $\frac{\cosh(r)-1}{2} = \sinh^2 \frac{r}{2}$:

$$G_d^{\mathrm{h}}(-\beta^2; r) = \begin{cases} \dfrac{1}{(4\pi)^{\frac{d}{2}} \sinh(\frac{r}{2})^{d-2}} \sum_{k=0}^{\infty} \dfrac{(-1)^k \Gamma(\frac{d-2}{2} - k)(\frac{1}{2} + \beta - k)_{2k}}{k!} \sinh^{2k}(\tfrac{r}{2}) \\ + \dfrac{1}{(4\pi)^{\frac{d}{2}}} \sum_{j=0}^{\infty} \dfrac{(-1)^j (\frac{3-d}{2} + \beta - j)_{d-2+2j} \Gamma(\frac{2-d}{2} - j)}{j!} \sinh^{2j}(\tfrac{r}{2}), \qquad d \notin 2\mathbb{Z}; & (3.11a) \\ \dfrac{1}{(4\pi)^{\frac{d}{2}} \sinh(\frac{r}{2})^{d-2}} \sum_{k=0}^{\frac{d-4}{2}} \dfrac{(\frac{1}{2} + \beta - k)_{2k} (\frac{d-4}{2} - k)! (-1)^k}{k!} \sinh^{2k}(\tfrac{r}{2}) \\ + \dfrac{(-1)^{\frac{d-2}{2}}}{(4\pi)^{\frac{d}{2}}} \sum_{j=0}^{\infty} \dfrac{(\frac{3-d}{2} + \beta - j)_{d-2+2j}}{j!(j + \frac{d-2}{2})!} \sinh^{2j}(\tfrac{r}{2}) \Big(H_{\frac{d-2}{2}+j} + H_j - 2\gamma_{\mathrm{E}} \\ -\psi(\frac{d-1}{2} + \beta + j) - \psi(\frac{3-d}{2} + \beta - j) - \ln\big(\sinh^2(\tfrac{r}{2})\big)\Big), \qquad d = 2, 4, \dots. & (3.11b) \end{cases}$$

From this, one easily gets for $2 + 2n > \Re d$

$$\begin{aligned} &\lim_{x \to x'} \frac{\partial^n}{\partial z^n} G_d^{\mathrm{h}}(z; x, x') \\ &= \frac{\partial^n}{\partial z^n} \frac{\Gamma(\frac{2-d}{2})(\frac{3-d}{2} + \sqrt{-z})_{d-2}}{(4\pi)^{\frac{d}{2}}}, \quad d \notin 2\mathbb{Z}, \\ &= \frac{\partial^n}{\partial z^n} \frac{(-1)^{\frac{d}{2}} (\frac{3-d}{2} + \sqrt{-z})_{d-2}}{(4\pi)^{\frac{d}{2}} \Gamma(\frac{d}{2})} \left(\psi\big(\tfrac{d-1}{2} + \sqrt{-z}\big) + \psi\big(\tfrac{3-d}{2} + \sqrt{-z}\big)\right), \ d = 2, 4, \dots \end{aligned} \tag{3.12}$$

For odd dimensions, we have an expression in terms of elementary functions:

$$G_d^{\mathrm{h}}(-\beta^2; x, x') = \frac{1}{2(2\pi)^{\frac{d-1}{2}}} \left(-\frac{1}{\sinh r} \partial_r \right)^{\frac{d-3}{2}} \frac{\mathrm{e}^{-\beta r}}{\sinh r}, \qquad d = 3, 5, \dots \tag{3.13}$$

To describe the behavior near infinity, we use the expansions (C.5), respectively (C.6), as well as $\frac{\cosh(r)+1}{2} = \cosh^2(\frac{r}{2})$ and $\frac{\cosh(r)-1}{2} = \sinh^2(\frac{r}{2})$:

$$\begin{aligned} G_d^{\mathrm{h}}(-\beta^2; x, x') &= \frac{1}{(4\pi)^{\frac{d}{2}}} \sum_{j=0}^{\infty} \frac{\Gamma(\frac{1}{2} + \beta + j)\Gamma(\frac{d-1}{2} + \beta + j)}{j!\Gamma(1 + 2\beta + j)\big(\cosh(\frac{r}{2})\big)^{2j+d-1+2\beta}} \\ &= \frac{\Gamma\big(\frac{d-1}{2} + \beta\big)}{(4\pi)^{\frac{d}{2}}\big(\sinh(\frac{r}{2})\big)^{d-2}} \sum_{j=0}^{\infty} \frac{\Gamma\big(\frac{1}{2} + \beta + j\big)\big(-\frac{d+1}{2} + \beta\big)_j}{j!\Gamma(1 + 2\beta + j)\big(\cosh(\frac{r}{2})\big)^{2j+1+2\beta}}. \end{aligned} \tag{3.14}$$

Note that (3.11a), (3.11b), (3.13) and (3.14) are the analogs of (2.16a), (2.16a), (2.18) and (2.19).

3.2. Point Potentials on Hyperbolic Space

We fix the point $x_0:=(1,0,\dots,0)$ in $\mathbb{H}^d$. Green's function of the hyperbolic Laplacian with a point-like potential located at x_0 has the form

$$G_d^{\mathrm{h},\gamma}(z;x,x') = G_d^{\mathrm{h}}(z;x,x') + \frac{1}{\gamma(z)+\Sigma_d^{\mathrm{h}}(z)} G_d^{\mathrm{h}}(z;x,x_0)G_d^{\mathrm{h}}(z;x_0,x'), \quad (3.15)$$

where[3]

$$\begin{aligned} -\frac{\mathrm{d}}{\mathrm{d}z}\big(\gamma(z)+\Sigma_d^{\mathrm{h}}(z)\big) &= \sigma_d^{\mathrm{h}}(z) := \int_{\mathbb{H}^d} G_d^{\mathrm{h}}(z;x_0,x)^2 \mathrm{d}x \\ &= \int_1^\infty G_d^{\mathrm{h}}(z,w)^2 |\mathbb{S}^{d-1}|(w^2-1)^{\frac{d}{2}-1}\mathrm{d}w \quad (3.16) \\ &= \frac{\pi\Gamma\big(\frac{d-1}{2}+\beta\big)^2}{2^{2\beta+1}(4\pi)^{\frac{d}{2}}\Gamma\big(\frac{d}{2}\big)} \int_2^\infty \mathbf{Z}_{\frac{d}{2}-1,\beta}(w)^2(w^2-1)^{\frac{d}{2}-1}\mathrm{d}2w. \quad (3.17) \end{aligned}$$

The integrand of (3.17) is well defined for any complex d. As in the flat case, the integral is convergent only for $|\Re(d-2)|<2$, which includes the dimensions $d=1,2,3$:

$$\sigma_d^{\mathrm{h}}(-\beta^2) = \begin{cases} \dfrac{\pi\big(\frac{3-d}{2}+\beta\big)_{d-2} H_{d-2}\big(\frac{3-d}{2}+\beta\big)}{(4\pi)^{\frac{d}{2}}\, 2\beta\Gamma\big(\frac{d}{2}\big)\,\sin\big(\pi\frac{d}{2}\big)}, & |\Re d-2|<2,\ d\neq 2; \\ \dfrac{\psi'\big(\frac{1}{2}+\beta\big)}{4\pi\beta}, & d=2. \end{cases} \quad (3.18)$$

In terms of $z=-\beta^2$, these formulas can be conveniently rewritten as

$$\sigma_d^{\mathrm{h}}(z) = \begin{cases} \dfrac{\pi\,\partial_z\big(\frac{3-d}{2}+\sqrt{-z}\big)_{d-2}}{(4\pi)^{\frac{d}{2}}\Gamma\big(\frac{d}{2}\big)\,\sin\big(\pi\frac{d}{2}\big)}, & |\Re d-2|<2,\ d\neq 2; \\ -\dfrac{1}{2\pi}\partial_z\psi\big(\frac{1}{2}+\sqrt{-z}\big), & d=2. \end{cases} \quad (3.19)$$

Based on the same arguments as in the Euclidean case, we define $\sigma_d^{\mathrm{h}}(z)$ for higher dimensions as a generalized integral, choose an antiderivative Σ_d^{h} and let $\gamma(z)$ be a polynomial of degree the smallest integer greater than $\frac{d-4}{2}$. As in the Euclidean case, this Σ_d^{h} satisfies for n large enough

$$\frac{\partial^n}{\partial z^n}\Sigma_d^{\mathrm{h}}(z) = -\lim_{x'\to x}\frac{\partial^n}{\partial z^n}G_d^{\mathrm{h}}(z;x,x'), \quad (3.20)$$

[3] Note that we choose the integration variable $2w$ in (3.17). This choice is only important when replacing the standard integral by the anomalous generalized integral, which is needed in even $d\geq 4$ and has the scaling anomaly (A.9). Using $2w$ instead of w ensures that the reference self-energy given by the generalized integral is asymptotic to the reference self-energy given by the flat generalized integral in any dimension—c.f. Subsection 3.3.

and the right-hand side was computed in (3.12). For the following discussion of special cases, we introduce the notation $[x|x'] = \cosh r$, $[x|x_0] = \cosh\theta$ and $[x'|x_0] = \cosh\theta'$.

Dimension 1. We have

$$\mathbf{Z}_{-\frac{1}{2},\beta}(\cosh r) = \frac{2^\beta}{\Gamma(1+\beta)}\mathrm{e}^{-\beta r} \quad \text{and} \quad G_1^{\mathrm{h}}(-\beta^2; x, x') = \frac{\mathrm{e}^{-\beta r}}{2\beta}. \tag{3.21}$$

Moreover, (3.19) gives

$$\sigma_1^{\mathrm{h}}(z) = \frac{1}{2}\partial_z \frac{1}{\sqrt{-z}}. \tag{3.22}$$

Imposing $\Sigma_1^{\mathrm{h}}(-\infty) = 0$, we find

$$\Sigma_1^{\mathrm{h}}(-\beta^2) = -\frac{1}{2\beta}, \tag{3.23}$$

which coincides with the Euclidean $\Sigma_1(-\beta^2)$. Thus,

$$G_1^{\mathrm{h},\gamma}(-\beta^2; x, x') = \frac{\mathrm{e}^{-\beta r}}{2\beta} + \frac{\mathrm{e}^{-\beta\theta}\mathrm{e}^{-\beta\theta'}}{(2\beta)^2\left(\gamma - \frac{1}{2\beta}\right)}, \tag{3.24}$$

and we obtain the same expression as in the Euclidean case, in accord with the isometry $\mathbb{H}^1 \cong \mathbb{R}^1$.

Dimension 2. We have

$$G_2^{\mathrm{h}}(-\beta^2; x, x') = \frac{\Gamma\left(\frac{1}{2}+\beta\right)}{\sqrt{2\pi}2^{\beta+1}}\mathbf{Z}_{0,\beta}(\cosh r). \tag{3.25}$$

From (3.19), we obtain a family of self-energies depending on a parameter $\varepsilon := -2\pi\gamma$:

$$\Sigma_2^{\mathrm{h},\varepsilon}(-\beta^2) = \frac{1}{2\pi}\left(\psi\left(\tfrac{1}{2}+\beta\right) - \varepsilon\right). \tag{3.26}$$

Thus,

$$\begin{aligned} G_2^{\mathrm{h},\varepsilon}(-\beta^2; x, x') &= \frac{\Gamma\left(\frac{1}{2}+\beta\right)}{\sqrt{2\pi}2^{\beta+1}}\mathbf{Z}_{0,\beta}(\cosh r) \\ &\quad + \frac{\Gamma\left(\frac{1}{2}+\beta\right)^2}{2^{2\beta+2}}\,\frac{\mathbf{Z}_{0,\beta}(\cosh\theta)\mathbf{Z}_{0,\beta}(\cosh\theta')}{\psi\left(\frac{1}{2}+\beta\right) - \varepsilon}. \end{aligned} \tag{3.27}$$

Dimension 3. We have

$$\mathbf{Z}_{\frac{1}{2},\beta}(\cosh r) = \frac{2^\beta}{\Gamma(1+\beta)}\frac{\mathrm{e}^{-\beta r}}{\sinh r} \quad \text{and} \quad G_3^{\mathrm{h}}(-\beta^2; x, x') = \frac{\mathrm{e}^{-\beta r}}{4\pi\,\sinh r}. \tag{3.28}$$

Moreover, (3.19) gives

$$\sigma_3^{\mathrm{h}}(z) = -\tfrac{1}{4\pi}\partial_z\sqrt{-z}, \tag{3.29}$$

so that imposing $\Sigma_3^{\mathrm{h}}(0) = 0$ yields

$$\Sigma_3^{\mathrm{h}}(-\beta^2) = \frac{\beta}{4\pi}. \tag{3.30}$$

In dimension 3, the hyperbolic self-energy equals the Euclidean self-energy: $\Sigma_3^{\rm h}(-\beta^2)=\Sigma_3(-\beta^2)$. However, Green's function is different:

$$G_3^{{\rm h},\gamma}(-\beta^2;x,x')=\frac{{\rm e}^{-\beta r}}{4\pi\,\sinh r}+\frac{{\rm e}^{-\beta\theta}{\rm e}^{-\beta\theta'}}{(4\pi)^2\,\sinh\theta\,\sinh\theta'\big(\gamma+\frac{\beta}{4\pi}\big)}. \tag{3.31}$$

Odd dimensions $d\geq 5$. The identities (3.18) and (3.19) remain valid for $d\in\mathbb{C}\backslash 2\mathbb{Z}$ if the integrals are interpreted in the generalized sense. Therefore, we can set

$$\Sigma_d^{\rm h}(-\beta^2)=-\frac{\pi\,(\frac{3-d}{2}+\beta)_{d-2}}{(4\pi)^{\frac{d}{2}}\Gamma\big(\frac{d}{2}\big)\,\sin\big(\pi\frac{d}{2}\big)}. \tag{3.32}$$

If d is an odd integer, we can rewrite (3.32) as

$$\Sigma_d^{\rm h}(-\beta^2)=\frac{\pi}{(4\pi)^{\frac{d}{2}}\,\Gamma\big(\frac{d}{2}\big)}\beta\prod_{j=1}^{\frac{d-3}{2}}\big(-\beta^2+j^2\big). \tag{3.33}$$

Inserting (3.13) and (3.33) into (3.15), we obtain an expression for $G_d^{{\rm h},\gamma}(z;x,x')$.

Even dimensions $d\geq 4$.

In this case, the formula (3.32) is not applicable. Instead we introduce a family of reference self-energies parametrized by $\varepsilon\in\mathbb{R}$:

$$\Sigma_d^{{\rm h},\varepsilon}(-\beta^2):=\frac{\psi\big(\frac{3-d}{2}+\beta\big)+\psi\big(\frac{d-1}{2}+\beta\big)-2\varepsilon}{(4\pi)^{\frac{d}{2}}\Gamma\big(\frac{d}{2}\big)}\prod_{j=0}^{\frac{d-4}{2}}\Big(-\beta^2+\big(\tfrac{1}{2}+j\big)^2\Big). \tag{3.34}$$

We obtain a family of Green's function

$$G_d^{{\rm h},\varepsilon,\eta}(z;x,x')=G_d^{\rm h}(z;x,x')+\frac{G_d^{\rm h}(z;x,x_0)G_d^{\rm h}(z;x_0,x')}{\eta(z)+\Sigma_d^{{\rm h},\varepsilon}(z)}, \tag{3.35}$$

where $\deg\eta\leq\frac{d-4}{2}$.

Let us derive (3.34) from the integral (3.17). For $d\in 2\mathbb{Z}$, $d>2$, it has to be understood in the generalized anomalous sense and is not equal to (3.18). Instead, the anomalous integral given by (C.21) yields

$$\begin{aligned}\sigma_d^{\rm h}(-\beta^2)=\frac{(-1)^{\frac{d}{2}-1}(\frac{3-d}{2}+\beta)_{d-2}}{(4\pi)^{\frac{d}{2}}\Gamma(\frac{d}{2})}\Bigg(&\frac{\psi'(\frac{3-d}{2}+\beta)+\psi'(\frac{d-1}{2}+\beta)}{2\beta}\\ &+\frac{H_{\frac{d-2}{2}}(\frac{1}{2}-\beta)-H_{\frac{d-2}{2}}(\frac{1}{2}+\beta)}{2\beta}\ln 4\\ &+\sum_{k=0}^{\frac{d-4}{2}}\frac{\psi(\frac{3}{2}+k+\beta)+\psi(-\frac{1}{2}-k+\beta)-\psi(\frac{d}{2}-1-k)-\psi(1+k)}{\beta^2-(\frac{1}{2}+k)^2}\Bigg).\end{aligned} \tag{3.36}$$

We notice that $\partial_z=-\frac{1}{2\beta}\partial_\beta$ and $\partial_z(z)_k=H_k(z)(z)_k$. Using this, the Leibniz rule for ∂_z and identities satisfied by the Pochhammer symbol, harmonic numbers and the digamma function (see Appendix D) yields

$$\sigma_d^{\mathrm{h}}(z) = \frac{(-1)^{\frac{d}{2}}}{(4\pi)^{\frac{d}{2}}\Gamma\big(\frac{d}{2}\big)}\partial_z\Bigg(\Big(\psi\big(\tfrac{3-d}{2}+\sqrt{-z}\big)+\psi\big(\tfrac{d-1}{2}+\sqrt{-z}\big)-\ln 4\Big)$$
$$\prod_{j=0}^{\frac{d-4}{2}}\Big(-z-\big(\tfrac12+j\big)^2\Big)\Bigg)+\frac{1}{(4\pi)^{\frac{d}{2}}\Gamma\big(\frac{d}{2}\big)}\pi_d^{\mathrm{h}}(z), \tag{3.37}$$

where $\pi_d^{\mathrm{h}}(z)$ is the polynomial of degree $\frac{d-4}{2}$ defined by

$$\pi_d^{\mathrm{h}}(z) = \Bigg(\sum_{k=0}^{\frac{d-4}{2}}\frac{\psi\big(\frac{d-2}{2}-k\big)+\psi(1+k)}{z+\big(\frac12+k\big)^2}$$
$$+\sum_{k=0}^{\frac{d-4}{2}}\sum_{l=k+1}^{\frac{d-4}{2}}\frac{2l+1}{\Big(z+\big(\frac12+k\big)^2\Big)\Big(z+\big(\frac12+l\big)^2\Big)}\Bigg)\prod_{j=0}^{\frac{d-4}{2}}\Big(z+\big(\tfrac12+j\big)^2\Big).$$

Therefore, the following function is an antiderivative of minus (3.37) and is a possible self-energy:

$$\Sigma_d^{\mathrm{h,ms}}(-\beta^2) = \frac{\psi\big(\frac{3-d}{2}+\beta\big)+\psi\big(\frac{d-1}{2}+\beta\big)-\ln 4}{(4\pi)^{\frac{d}{2}}\Gamma\big(\frac{d}{2}\big)}\prod_{j=0}^{\frac{d-4}{2}}\Big(-\beta^2+\big(\tfrac12+j\big)^2\Big)$$
$$+\frac{1}{(4\pi)^{\frac{d}{2}}\Gamma\big(\frac{d}{2}\big)}\Pi_d^{\mathrm{h}}(-\beta^2), \tag{3.38}$$

where

$$\Pi_d^{\mathrm{h}}(z) := -\int_0^z \pi_d^{\mathrm{h}}(\tau)\mathrm{d}\tau, \tag{3.39}$$

which is a polynomial of degree $\frac{d-2}{2}$ with $\Pi_d^{\mathrm{h}}(0)=0$. (3.38) will be called the *reference self-energy based on minimal subtraction*. (The superscript ms stands for the "minimal subtraction.")

As in the Euclidean case, we can add to (3.38) an arbitrary polynomial $\gamma(-\beta^2)$ of degree $\le \frac{d-2}{2}$. Let $\frac{\ln 4-2\varepsilon}{(4\pi)^{\frac{d}{2}}\Gamma(\frac{d}{2})}$ be the coefficient at the term $z^{\frac{d-2}{2}}$ of $\frac{\Pi_d^{\mathrm{h}}(z)}{(4\pi)^{\frac{d}{2}}\Gamma(\frac{d}{2})}+\gamma(z)$. Then we can write

$$\Sigma_d^{\mathrm{h,ms}}(-\beta^2)+\gamma(-\beta^2) = \Sigma_d^{\mathrm{h},\varepsilon}(-\beta^2)+\eta(-\beta^2), \tag{3.40}$$

where η is a polynomial of degree $\le \frac{d-4}{2}$ and $\Sigma_d^{\mathrm{h},\varepsilon}$ was introduced in (3.34). We prefer the latter as the family of reference self-energies because of the factorized form.

Remark 3.2 We have two proposals for the reference self-energy for even $d \ge 4$: $\Sigma_d^{\mathrm{h},\varepsilon}$, $\varepsilon\in\mathbb{R}$, in (3.34) and $\Sigma_d^{\mathrm{h,ms}}$ in (3.38). We choose the former as the standard one, because of its simplicity. In particular, it is factorized, which allows to determine easily its zeros responsible for singularities of the corresponding Green's functions.

However, $\Sigma_d^{\mathrm{h,ms}}$ is also in some sense special. It is obtained with the help of the generalized integral, a concept closely related to the minimal subtraction method in QFT. One can criticize it saying that because of anomaly it depends on the choice of the integration variable, which in the hyperbolic case is chosen to be $2(w-1) = 2(\cosh r - 1)$. However, this is actually a natural variable. It is closely related to the family of conformal transformations, which are best expressed in the variable w:

$$\phi_\lambda(w) = \frac{\lambda^{-1}(w+1) + \lambda(w-1)}{\lambda^{-1}(w+1) - \lambda(w-1)}. \tag{3.41}$$

ϕ_λ form a one-parameter group $\phi_\lambda \circ \phi_\mu = \phi_{\lambda\mu}$ and $\phi_1 = \mathrm{id}$.

As outlined in the introduction, Green's functions that we introduce most likely describe the asymptotics of the resolvent of the Laplacian with a perturbation supported in a shrinking region. We expect that fine-tuning the perturbation we should be able to see Green's functions corresponding to various ε, η. One can ask the question whether the self-energy $\Sigma_d^{\mathrm{h,ms}}$ (based on the generalized integral) is distinguished and obtained by a special way of shrinking the perturbation. As of now, we do not know.

Spectral properties. Green's functions $G_d^{\mathrm{h},\gamma}(z)$ and $G_d^{\gamma}(z)$ have a cut at $z \in [0, \infty[$, which in dimensions $1, 2, 3$ corresponds to the continuous spectrum of $H_d^{\mathrm{h},\gamma}$ and H_d^{γ}. There may be also some singularities outside of $[0, \infty[$, which in the hyperbolic case, apart from dimension $d = 1, 3$, have a more complicated structure than in the flat case, because the logarithm is replaced by the digamma function. Note that in dimensions $d = 1, 3$ the poles are exactly the same as in the flat case, because

$$\Sigma_1^{\mathrm{h}}(-\beta^2) = \Sigma_1(-\beta^2) \quad \text{and} \quad \Sigma_3^{\mathrm{h}}(-\beta^2) = \Sigma_3(-\beta^2). \tag{3.42}$$

3.3. Flat Limit of the Hyperbolic Laplacian

Let $R > 0$. Instead of the hyperbolic space of curvature -1, we can use its scaled version of curvature $-\frac{1}{R^2}$:

$$\mathbb{H}_R^d := \{x \in \mathbb{R}^{1,d} \mid [x|x] = R^2\}.$$

We can introduce various objects from the previous subsection corresponding to $\mathbb{H}_R^d$, which will be distinguished by the subscript R. Clearly, $\mathbb{H}_1^d = \mathbb{H}^d \ni x \mapsto Rx \in \mathbb{H}_R^d$ is a bijection and

$$d_R^{\mathrm{h}}(Rx, Rx') = R\, d^{\mathrm{h}}(x, x').$$

The map $U_R : L^2(\mathbb{H}^d) \to L^2(\mathbb{H}_R^d)$ given by

$$U_R f(x) := R^{-\frac{d}{2}} f\left(\frac{x}{R}\right)$$

is unitary. If $K(x,x')$ is the integral kernel of K on $L^2(\mathbb{H}^d)$ and $K_R(x,x')$ is the integral kernel of $U_R K U_R^{-1}$ on $L^2(\mathbb{H}_R^d)$, then

$$K_R(x,x') = R^{-d} K\Big(\frac{x}{R}, \frac{x'}{R}\Big). \tag{3.43}$$

The hyperbolic Laplacian on $L^2(\mathbb{H}_R^d)$ is

$$\Delta_{d,R}^{\mathrm{h}} = \frac{1}{R^2} U_R \Delta_d^{\mathrm{h}} U_R^{-1}. \tag{3.44}$$

We set

$$H_{d,R}^{\mathrm{h}} := -\Delta_{d,R}^{\mathrm{h}} - \frac{(d-1)^2}{4R^2}, \tag{3.45}$$

so that $\sigma(H_{d,R}^{\mathrm{h}}) = [0,\infty[$. For $z \in \mathbb{C}\backslash[0,\infty[$, we set

$$G_{d,R}^{\mathrm{h}}(z) := (-z + H_{d,R}^{\mathrm{h}})^{-1}, \tag{3.46}$$

$$\mathbb{P}_{d,R}^{\mathrm{h}}(a,b) := 1\!\!1_{[a,b]}(H_{d,R}^{\mathrm{h}}). \tag{3.47}$$

Note that

$$G_{d,R}^{\mathrm{h}}(-\beta^2; x, x') = R^{-d+2} G_d^{\mathrm{h}}\Big(-(\beta R)^2; \frac{x}{R}, \frac{x'}{R}\Big), \tag{3.48}$$

$$\mathbb{P}_{d,R}^{\mathrm{h}}(a,b;x,x') = R^{-d} \mathbb{P}_d^{\mathrm{h}}\Big(aR^2, bR^2; \frac{x}{R}, \frac{x'}{R}\Big). \tag{3.49}$$

Proceeding as for $R=1$, we introduce

$$G_{d,R}^{\mathrm{h},\gamma}(z;x,x') = G_{d,R}^{\mathrm{h}}(z;x,x') + \frac{G_{d,R}^{\mathrm{h}}\big(z;x,Rx_0\big) G_{d,R}^{\mathrm{h}}\big(z;Rx_0,x'\big)}{\gamma(z) + \Sigma_{d,R}^{\mathrm{h}}(z)}, \tag{3.50}$$

where $\gamma(z)$ is a polynomial (of degree as for $R=1$) and $\Sigma_{d,R}^{\mathrm{h}}$ is a particular solution of

$$-\frac{\mathrm{d}}{\mathrm{d}z} \Sigma_{d,R}^{\mathrm{h}}(z) = \sigma_{d,R}^{\mathrm{h}}(z) := \int_{\mathbb{H}_R^d} G_{d,R}^{\mathrm{h}}(z;x,Rx_0)^2 \mathrm{d}x. \tag{3.51}$$

For dimensions d for which the integral does not converge, we integrate over angles and then compute the generalized integral with respect to the radial coordinate $2Rw_R = R^2[\frac{x}{R}|x_0]$:

$$\begin{aligned}
\sigma_{d,R}^{\mathrm{h}}(z) &= \operatorname{gen} \int_{2R^2}^{\infty} |\mathbb{S}^{d-1}| G_{d,R}^{\mathrm{h}}(z;x,Rx_0)^2 (w_R^2 - R^2)^{\frac{d-2}{2}} R \frac{\mathrm{d}(2Rw_R)}{2R} \\
&= R^d \operatorname{gen} \int_{2R^2}^{\infty} |\mathbb{S}^{d-1}| (R^{-d+2} G_d^{\mathrm{h}}(R^2 z; w))^2 (w^2-1)^{\frac{d-2}{2}} \frac{d(2R^2 w)}{2R^2} \\
&= R^{-d+4} \begin{cases} \sigma_d^{\mathrm{h}}(R^2 z), & d \notin \{4,6,\dots\}, \\ \sigma_d^{\mathrm{h}}(R^2 z) - \frac{2(-1)^{\frac{d}{2}}}{(4\pi)^{\frac{d}{2}} \Gamma(\frac{d}{2})} \frac{\partial}{\partial(R^2 z)} \frac{\Gamma\big(\frac{d-1}{2} + \sqrt{-R^2 z}\big)}{\Gamma\big(\frac{3-d}{2} + \sqrt{-R^2 z}\big)} \ln R, & d \in \{2,4,\dots\}. \end{cases}
\end{aligned} \tag{3.52}$$

In the last equality, we used (A.11); the scaling anomaly coefficient f_{-1} was computed in [10]. In even dimensions, we prefer to use

$$G_{d,R}^{\mathrm{h},\varepsilon,\eta}(z;x,x') = G_{d,R}^{\mathrm{h}}(z;x,x') + \frac{G_{d,R}^{\mathrm{h}}\big(z;x,Rx_0\big) G_{d,R}^{\mathrm{h}}\big(z;Rx_0,x'\big)}{\eta(z) + \Sigma_{d,R}^{\mathrm{h},\varepsilon}(z)}. \tag{3.53}$$

We choose the reference self-energies to be

$$\Sigma^{\mathrm{h}}_{d,R}(-\beta^2){:=}R^{2-d}\Sigma^{\mathrm{h}}_{d}\big(-(\beta R)^2\big), \qquad d \text{ odd};$$
$$\Sigma^{\mathrm{h},\varepsilon}_{d,R}(-\beta^2){:=}R^{2-d}\Sigma^{\mathrm{h},\varepsilon+\ln R}_{d}\big(-(\beta R)^2\big), \qquad d \text{ even}.$$

Theorem 3.3 *Let* $-\beta^2 \in \mathbb{C}\backslash[0,\infty[$. *We have*

$$G^{\mathrm{h}}_{d,R}\big(-\beta^2;r\big) = G_d\big(-\beta^2;r\big)\Big(1+\mathcal{O}\big(\tfrac{1}{\beta R}\big)+\mathcal{O}\big(\tfrac{r}{R}\big)\Big) \tag{3.54}$$

and

$$\Sigma^{\mathrm{h}}_{d,R}\big(-\beta^2\big) = \Sigma_d\big(-\beta^2\big)\Big(1+\mathcal{O}\big(\tfrac{1}{\beta R}\big)\Big), \quad d \text{ odd}; \tag{3.55}$$
$$\Sigma^{\mathrm{h},\varepsilon}_{d,R}\big(-\beta^2\big) = \Sigma^{\varepsilon}_d\big(-\beta^2\big)\Big(1+\mathcal{O}\big(\tfrac{1}{\beta R}\big)\Big), \quad d \text{ even}. \tag{3.56}$$

Thus, if we have a family $x_R, x'_R \in \mathbb{H}^d_R$ *and* $x, x' \in \mathbb{R}^d$ *such that*

$$\lim_{R\to\infty} d^{\mathrm{h}}_R(x_R, x'_R) = |x-x'|,$$
$$\lim_{R\to\infty} d^{\mathrm{h}}_R(x_R, Rx_0) = |x|,$$
$$\lim_{R\to\infty} d^{\mathrm{h}}_R(x'_R, Rx_0) = |x'|, \tag{3.57}$$

then

$$\lim_{R\to\infty} G^{\mathrm{h},\gamma}_{d,R}\big(-\beta^2;x_R,x'_R\big) = G^{\gamma}_d(-\beta^2;x,x'), \qquad d \text{ odd}; \tag{3.58}$$
$$\lim_{R\to\infty} G^{\mathrm{h},\varepsilon,\eta}_{d,R}\big(-\beta^2;x_R,x'_R\big) = G^{\varepsilon,\eta}_d(-\beta^2;x,x'), \qquad d \text{ even}. \tag{3.59}$$

Proof Using the asymptotics of the Gegenbauer functions from Thm. C.1, we find

$$\begin{aligned} G^{\mathrm{h}}_{d,R}(-\beta^2,r_R) &= R^{-d+2}G^{\mathrm{h}}_d\Big(-(\beta R)^2, \cosh\frac{r_R}{R}\Big) \\ &= \frac{R^{-d+2}\Gamma(\frac{d-1}{2}+\beta R)}{\Gamma(\frac{3-d}{2}+\beta R)}\frac{\sqrt{\pi}\Gamma(\frac{3-d}{2}+\beta R)}{\sqrt{2}(2\pi)^{\frac{d}{2}}2^{\beta R}}\mathbf{Z}_{\frac{d-2}{2},\beta R}\Big(\cosh\frac{r_R}{R}\Big) \\ &= \frac{(\frac{r_R}{R})^{\frac{d-1}{2}}}{(\sinh\frac{r_R}{R})^{\frac{d-1}{2}}(2\pi)^{\frac{d}{2}}}\Big(\frac{\beta}{r_R}\Big)^{\frac{d-2}{2}}K_{\frac{d-2}{2}}(\beta r_R)\Big(1+\mathcal{O}\big(\tfrac{1}{\beta R}\big)\Big) \\ &= \frac{1}{(2\pi)^{\frac{d}{2}}}\Big(\frac{\beta}{r_R}\Big)^{\frac{d-2}{2}}K_{\frac{d-2}{2}}(\beta r_R)\Big(1+\mathcal{O}\big(\tfrac{1}{\beta R}\big)+\mathcal{O}\big(\tfrac{r_R}{R}\big)\Big). \end{aligned} \tag{3.60}$$

This proves (3.54). (3.55) follows from C.2 and

$$\psi\big(\tfrac12 \pm \alpha + \beta R\big) - \ln(\beta R) = \mathcal{O}\big(\tfrac{1}{\beta R}\big). \tag{3.61}$$

Now let

$$r_R := d^{\mathrm{h}}_R(x_R,x'_R), \quad \theta_R := d^{\mathrm{h}}_R(x_R,Rx_0), \quad \theta'_R := d^{\mathrm{h}}_R(x'_R,Rx_0), \tag{3.62}$$
$$r := |x-x'|, \quad \theta := |x-x_0|, \quad \theta' := |x'-x_0|. \tag{3.63}$$

By (3.54) and (3.55), we obtain

$$G^{\mathrm{h},\gamma}_{d,R}(-\beta^2;x_R,x'_R) = G^{\mathrm{h}}_{d,R}(-\beta^2,r_R) + \frac{G^{\mathrm{h}}_{d,R}\big(-\beta^2,\theta_R\big)G^{\mathrm{h}}_{d,R}\big(-\beta^2,\theta'_R\big)}{\gamma(-\beta^2)+\Sigma^{\mathrm{h}}_{d,R}(-\beta^2)}$$

$$= G_d(-\beta^2, r_R)\Big(1 + \mathcal{O}\big(\tfrac{1}{\beta R}\big) + \mathcal{O}\big(\tfrac{r_R}{R}\big)\Big) + \frac{G_d\big(-\beta^2, \theta_R\big)G_d\big(-\beta^2, \theta'_R\big)}{\gamma(-\beta^2) + \Sigma_d\big(-\beta^2\big)\big(1 + \mathcal{O}\big(\tfrac{1}{\beta R}\big)\big)} \Big(1 + \mathcal{O}\big(\tfrac{1}{\beta R}\big) + \mathcal{O}\big(\tfrac{\theta_R}{R}\big) + \mathcal{O}\big(\tfrac{\theta'_R}{R}\big)\Big). \tag{3.64}$$

Now (3.57) implies $\frac{r_R}{R}$, $\frac{\theta_R}{R}$ and $\frac{\theta'}{R}$ to be $\mathcal{O}(\frac{1}{R})$. Hence, the limit $R \to \infty$ of the right-hand side is

$$G_d(-\beta^2, r) + \frac{G_d\big(-\beta^2, \theta\big)G_d\big(-\beta^2, \theta'\big)}{\gamma(-\beta^2) + \Sigma_d\big(-\beta^2\big)} = G_d^{\gamma}(-\beta^2; x, x'), \tag{3.65}$$

which proves (3.58). □

4. Green's Operators on the Sphere

4.1. Spherical Laplacian

Equip the space $\mathbb{R}^{1+d}$ with the Euclidean bilinear form

$$(x|y) = x^0y^0 + x^1y^1 + \cdots + x^dy^d.$$

The set

$$\mathbb{S}^d := \{x \in \mathbb{R}^{1+d} \mid (x|x) = 1\}$$

equipped with the Riemannian metric inherited from $\mathbb{R}^{1+d}$ is called the *(unit) sphere*. The geodesic distance between $x, x' \in \mathbb{S}^d$ is given by

$$d_{\mathrm{s}}(x, x') = \cos^{-1}(x|x'), \qquad \cos d_{\mathrm{s}}(x, x') = (x|x'). \tag{4.1}$$

$\mathbb{S}^d$ has also a natural measure, so one can define $L^2(\mathbb{S}^d)$.

In this section, we study the operator

$$H_d^{\mathrm{s}} := -\Delta_d^{\mathrm{s}} + \frac{(d-1)^2}{4}, \tag{4.2}$$

where Δ_d^{s} is the *spherical Laplacian*. For $z \in \mathbb{C}\backslash\sigma(H_d^{\mathrm{s}}) \supset \mathbb{C}\backslash[0, \infty[$, we define the *spherical Green's operator* $G_d^{\mathrm{s}}(z) := (-z + H_d^{\mathrm{h}})^{-1}$. Eigenfunctions of H_d^{s} with eigenvalue $(l + \frac{d-1}{2})^2$ are called *spherical harmonics of degree l*. The corresponding spectral projection will be denoted

$$\mathbb{P}_{d,l}^{\mathrm{s}} := \mathbb{1}_{l(l+d-1)}\big(-\Delta_d^{\mathrm{s}}\big) = \mathbb{1}_{(l+\frac{d-1}{2})^2}\big(H_d^{\mathrm{s}}\big). \tag{4.3}$$

In the following theorem, we express the integral kernels of $G_d^{\mathrm{s}}(z)$ and $\mathbb{P}_{d,l}^{\mathrm{s}}$ in terms of Gegenbauer function $\mathbf{S}_{\alpha,\beta}$ and Gegenbauer polynomials C_l^{α}, see Appendix C.

Theorem 4.1 *1. For $\Re\beta > 0$, the integral kernel of $G_d^{\mathrm{s}}(-\beta^2)$ is*

$$G_d^{\mathrm{s}}(-\beta^2; x, x') = \frac{\Gamma(\frac{d-1}{2} + \mathrm{i}\beta)\Gamma(\frac{d-1}{2} - \mathrm{i}\beta)}{(4\pi)^{\frac{d}{2}}}\mathbf{S}_{\frac{d}{2}-1,\mathrm{i}\beta}\big(-(x|x')\big). \tag{4.4}$$

2. *(4.4) is true also on the real line away of the spectrum of* H_d^{s}*. It is convenient to rewrite it as follows: For* $\zeta\in\mathbb{R}$, $\zeta-\frac{d-1}{2}\notin\{0,1,2,\dots\}$,

$$G_d^{\mathrm{s}}(\zeta^2;x,x')=\frac{\Gamma(\frac{d-1}{2}+\zeta)\Gamma(\frac{d-1}{2}-\zeta)}{(4\pi)^{\frac{d}{2}}}\mathbf{S}_{\frac{d}{2}-1,\zeta}\big(-(x|x')\big). \tag{4.5}$$

3. *The integral kernel of* $\mathbb{P}^{\mathrm{s}}_{d,l}$ *is*

$$\mathbb{P}^{\mathrm{s}}_{d,l}(x,x')=\frac{(2l+d-1)\Gamma(\frac{d-1}{2})}{4\pi^{\frac{d+1}{2}}}C_l^{\frac{d-1}{2}}\big((x|x')\big) \tag{4.6}$$

Proof Let $w:=-(x|x')$. By the spherical symmetry, there exists a function $G_d^{\mathrm{s}}(z,w)$ such that, $G_d^{\mathrm{s}}(z,x,x')=:G_d^{\mathrm{s}}(z,w)$. We obtain the differential equation

$$0=\left((1-w^2)\partial_w^2-dw\partial_w-\beta^2-\Big(\frac{d-1}{2}\Big)^2\right)G_d^{\mathrm{s}}(-\beta^2,w). \tag{4.7}$$

Then we need to find the solution regular near $w=-1$ and such that

$$G_d^{\mathrm{s}}\big(-\beta^2,-\cos(r)\big)\sim\begin{cases}\frac{\Gamma(\frac{d}{2}-1)}{4\pi^{\frac{d}{2}}}r^{2-d}, & d\neq 2,\\ -\frac{1}{2\pi}\ln r, & d=2,\end{cases} \tag{4.8}$$

This yields (4.4), (4.5). To see (4.6), we use (1.40):

$$\begin{aligned}&\text{lhs of (4.6)}\\ &=\lim_{\epsilon\searrow 0}\left(\Big(l+\mathrm{i}\epsilon+\frac{d-1}{2}\Big)^2-\Big(l+\frac{d-1}{2}\Big)^2\right)G_d^{\mathrm{s}}\left(\Big(l+\mathrm{i}\epsilon+\frac{d-1}{2}\Big)^2;x,x'\right)\\ &=\lim_{\epsilon\searrow 0}\mathrm{i}\epsilon(2l+d-1)\frac{\Gamma(-l-\mathrm{i}\epsilon)\Gamma(d-1+l+\mathrm{i}\epsilon)}{2^d\pi^{\frac{d}{2}}}\mathbf{S}_{\frac{d}{2}-1,\frac{d-1}{2}+l+\mathrm{i}\epsilon}\big(-(x|x')\big).\\ &=\frac{(-1)^l(2l+d-1)\Gamma(d-1+l)}{l!2^d\pi^{\frac{d}{2}}}\mathbf{S}_{\frac{d}{2}-1,\frac{d-1}{2}+l}\big(-(x|x')\big)\\ &=\frac{(2l+d-1)\Gamma(\frac{d-1}{2})}{4\pi^{\frac{d+1}{2}}}C_l^{\frac{d-1}{2}}\big((x|x')\big).\end{aligned} \tag{4.9}$$

At the end, we used (C.11) and $\sqrt{\pi}\Gamma(d-1)=2^{d-2}\Gamma(\frac{d-1}{2})\Gamma(\frac{d}{2})$.

□

The function $G_d^{\mathrm{s}}(-\beta^2,-\cos r)$ can be defined for any $d\in\mathbb{C}$. However, we do not have a symmetry similar to (3.10), except for even integers. To obtain formulas describing the behavior close to $r=0$, we use (C.12), (C.2), (C.3) and $\frac{1-\cos r}{2} = \sin^2(\frac{r}{2})$:

$$G_d^{\rm s}(-\beta^2, r) = \begin{cases} \frac{1}{\sin^{d-2}(\frac{r}{2})} \sum_{k=0}^{\infty} \frac{(\frac12+{\rm i}\beta-k)_{2k}\Gamma(\frac{d-2}{2}-k)}{(4\pi)^{\frac{d}{2}}k!} \sin^{2k}\left(\frac{r}{2}\right) \\ + \sum_{j=0}^{\infty} \frac{(-1)^j(\frac12+{\rm i}\beta)_{\frac{d-2}{2}+j}(\frac12-{\rm i}\beta)_{\frac{d-2}{2}+j}\Gamma(\frac{2-d}{2}-j)}{(4\pi)^{\frac{d}{2}}j!} \sin^{2j}\left(\frac{r}{2}\right), \quad d \notin 2\mathbb{Z}; & (4.10a) \\ \frac{1}{\sin^{d-2}(\frac{r}{2})} \sum_{k=0}^{\frac{d-4}{2}} \frac{(\frac12+{\rm i}\beta-k)_{2k}(\frac{d-4}{2}-k)!}{(4\pi)^{\frac{d}{2}}k!} \sin^{2k}\left(\frac{r}{2}\right) \\ + \sum_{j=0}^{\infty} \frac{(-1)^j(\frac{3-d}{2}+{\rm i}\beta-j)_{d-2+2j}}{(4\pi)^{\frac{d}{2}}j!(j+\frac{d-2}{2})!} \sin^{2j}\left(\frac{r}{2}\right)\Big(H_{\frac{d-2}{2}+j}+H_j-2\gamma_{\rm E} \\ -\psi(\frac{d-1}{2}+{\rm i}\beta+j)-\psi(\frac{d-1}{2}-{\rm i}\beta+j)-\ln\left(\sin^2(\frac{r}{2})\right)\Big), \quad d=2,4,\dots. & (4.10b) \end{cases}$$

If $2+2n > \Re d$, then

$$\lim_{x\to x'} \partial_z^n G_d^{\rm s}(z;x,x') = \begin{cases} \frac{\Gamma\left(\frac{2-d}{2}\right)}{(4\pi)^{\frac{d}{2}}} \partial_z^n \Big((\frac12+{\rm i}\sqrt{-z})_{\frac{d-2}{2}}(\frac12-{\rm i}\sqrt{-z})_{\frac{d-2}{2}}\Big), & d\notin 2\mathbb{Z}; \\ \frac{(-1)^{\frac{d}{2}}}{(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)} \partial_z^n \Big((\psi(\frac{d-1}{2}+{\rm i}\sqrt{-z})+\psi(\frac{d-1}{2}-{\rm i}\sqrt{-z}))\prod_{j=0}^{\frac{d-4}{2}}((\frac12+j)^2-z)\Big), & d=2,4,\dots. \end{cases} \tag{4.11}$$

For odd integers, we have an expression in terms of elementary functions:

$$G_d^{\rm s}(-\beta^2;x,x') = \frac{1}{2(2\pi)^{\frac{d-1}{2}}\sinh\pi\beta}\Big(-\frac{1}{\sin r}\partial_r\Big)^{\frac{d-3}{2}} \frac{\sinh\big(\beta(\pi-r)\big)}{\sin r}, \quad d=3,5,\dots \tag{4.12}$$

To describe the behavior close to pairs of antipodal points, that is close to $r=\pi$, we use (C.2), (C.3) as well as $\frac{1+\cos r}{2} = \cos^2(\frac{r}{2})$ and $\frac{1-\cos r}{2} = \sin^2(\frac{r}{2})$:

$$\begin{aligned} G_d^{\rm s}(-\beta^2;x,x') &= \frac{1}{(4\pi)^{\frac{d}{2}}} \sum_{j=0}^{\infty} \frac{\Gamma(\frac{d-1}{2}+{\rm i}\beta+j)\Gamma(\frac{d-1}{2}-{\rm i}\beta+j)}{\Gamma(\frac{d}{2}+j)j!} \cos^{2j}\left(\frac{r}{2}\right) \\ &= \frac{\Gamma(\frac{d-1}{2}+{\rm i}\beta)\Gamma(\frac{d-1}{2}-{\rm i}\beta)}{(4\pi)^{\frac{d}{2}}\left(\sin^2\left(\frac{r}{2}\right)\right)^{\frac{d-2}{2}}} \sum_{j=0}^{\infty} \frac{\left(\frac12+{\rm i}\beta\right)_j\left(\frac12-{\rm i}\beta\right)_j}{\Gamma\left(\frac{d}{2}+j\right)j!} \cos^{2j}\left(\frac{r}{2}\right). \end{aligned} \tag{4.13}$$

For integer d, (4.13) can be simplified as follows:

$$G_d^{\rm s}(-\beta^2, r) = \begin{cases} \frac{(-1)^{\frac{d-2}{2}}\pi}{(4\pi)^{\frac{d}{2}}\cosh\pi\beta} \sum_{j=0}^{\infty} \frac{(-1)^j(\frac{3-d}{2}-j+{\rm i}\beta)_{d-2+2j}}{(\frac{d-2}{2}+j)!j!} \cos^{2j}\left(\frac{r}{2}\right), \quad d=2,4,\dots; & (4.14a) \\ \frac{(-1)^{\frac{d-1}{2}}\pi}{(4\pi)^{\frac{d}{2}}{\rm i}\sinh\pi\beta} \sum_{j=0}^{\infty} \frac{(-1)^j(\frac{3-d}{2}-j+{\rm i}\beta)_{d-2+2j}}{\Gamma(\frac{d}{2}+j)j!} \cos^{2j}\left(\frac{r}{2}\right), \quad d=3,5,\dots. & (4.14b) \end{cases}$$

Note that (4.10a), (4.10a), (4.12) and (4.13) are the analogs of (2.16a), (2.16a), (2.18) and (2.19).

4.2. Point Potentials on the Sphere

Denote the *north pole* of the sphere by $x_0 = (1, 0, \dots, 0)$. Green's function of the spherical Laplacian with a point-like potential located at $x_0 \in \mathbb{S}^d$ has the form

$$G_d^{\mathrm{s},\gamma}(z; x, x') = G_d^{\mathrm{s}}(z; x, x') + \frac{G_d^{\mathrm{s}}(z; x, x_0) G_d^{\mathrm{s}}(z; x_0, x')}{\gamma(z) + \Sigma_d^{\mathrm{s}}(z)}, \tag{4.15}$$

where (with $z = -\beta^2$)

$$\begin{aligned} -\partial_z \Sigma_d^{\mathrm{s}}(z) = \sigma_d^{\mathrm{s}}(z) &:= \int_{\mathbb{S}^d} G_d^{\mathrm{s}}(z; x_0, x)^2 \mathrm{d}x \\ &= \int_{-1}^{1} G_d^{\mathrm{s}}(z; w)^2 |\mathbb{S}^{d-1}| (1 - w^2)^{\frac{d}{2}-1} \mathrm{d}w \qquad (4.16) \\ &= \frac{\Gamma\big(\frac{d-1}{2} + \mathrm{i}\beta\big)^2 \Gamma\big(\frac{d-1}{2} - \mathrm{i}\beta\big)^2}{2^{2d} \pi^{\frac{d}{2}} \Gamma\big(\frac{d}{2}\big)} \int_{-2}^{2} \mathbf{S}_{\frac{d}{2}-1, \mathrm{i}\beta}(w)^2 (1 - w^2)^{\frac{d}{2}-1} \mathrm{d}2w. \qquad (4.17) \end{aligned}$$

The integrand of (4.17) is well defined for any complex d. Again, the integral is convergent only for $|\Re(d-2)| < 2$, which includes the dimensions $d = 1, 2, 3$:

$$\sigma_d^{\mathrm{s}}(z) = \begin{cases} \dfrac{\Gamma(\frac{2-d}{2})}{(4\pi)^{\frac{d}{2}}} \partial_z \Big(\big(\tfrac{1}{2} + \mathrm{i}\sqrt{-z}\big)_{\frac{d-2}{2}} \big(\tfrac{1}{2} - \mathrm{i}\sqrt{-z}\big)_{\frac{d-2}{2}} \Big), & |\Re d - 2| < 2,\ d \neq 2; \\ -\dfrac{1}{4\pi} \partial_z \big(\psi\big(\tfrac{1}{2} + \mathrm{i}\sqrt{-z}\big) + \psi\big(\tfrac{1}{2} - \mathrm{i}\sqrt{-z}\big) \big), & d = 2. \end{cases} \tag{4.18}$$

Let us discuss specific dimensions. Except for dimension 1, we will use the notation $\cos r = (x|x')$, $\cos\theta = (x|x_0)$, $\cos\theta' = (x'|x_0)$.

Dimension 1. This case can be solved in an elementary way. Solving $(-\partial_\theta^2 + \beta^2) g(\theta) = \delta(\theta)$ on $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$, we obtain $g(\theta) = \frac{\cosh(\beta(\theta-\pi))}{2\beta \sinh(\pi\beta)}$, $|\theta - \pi| < \pi$. This yields the following description of the one-dimensional Green's function:

$$G_1^{\mathrm{s}}(-\beta^2; \theta, \theta') = \frac{\cosh(\beta(\theta - \theta' - \pi))}{2\beta \sinh(\pi\beta)}, \tag{4.19}$$

where $\theta - \theta' \in]0, 2\pi[$. We put the contact potential at $\theta = \pi$ and compute

$$\sigma_1^{\mathrm{s}}(-\beta^2) = \int_{-\pi}^{\pi} G_1^{\mathrm{s}}(-\beta^2, \pi, \theta)^2 \mathrm{d}\theta = \frac{\pi}{4\beta^2 \sinh^2(\pi\beta)} + \frac{\cosh(\pi\beta)}{4\beta^3 \sinh(\pi\beta)}, \tag{4.20}$$

$$\Sigma_1^{\mathrm{s}}(-\beta^2) = -\int_{\beta^2}^{\infty} \sigma_1^{\mathrm{s}}(-\rho) \mathrm{d}\rho = -\frac{\coth(\pi\beta)}{2\beta}. \tag{4.21}$$

Thus,

$$G_1^{\mathrm{s},\gamma}(-\beta^2; \theta, \theta') = \frac{\cosh(\beta(\theta - \theta' - \pi))}{2\beta \sinh(\pi\beta)} + \frac{\cosh(\theta\beta) \cosh(\theta'\beta)}{(2\beta)^2 \sinh^2(\pi\beta) \big(\gamma - \frac{1}{2\beta} \coth(\pi\beta)\big)}. \tag{4.22}$$

Let us show that it also follows from the general theory. We check that

$$\mathbf{S}_{-\frac{1}{2},\mathrm{i}\beta}\big(-\cos(\theta-\theta')\big) = \frac{1}{\sqrt{\pi}}\cosh(\beta(\theta-\theta'-\pi)), \tag{4.23}$$

$$\Gamma(\mathrm{i}\beta)\Gamma(-\mathrm{i}\beta) = \frac{\pi}{\beta\sinh(\pi\beta)}. \tag{4.24}$$

Thus,

$$G_1^{\mathrm{s}}(-\beta^2;\theta,\theta') = \frac{\Gamma(\mathrm{i}\beta)\Gamma(-\mathrm{i}\beta)}{2\sqrt{\pi}}\mathbf{S}_{-\frac{1}{2},\mathrm{i}\beta}\big(-\cos(\theta-\theta')\big) \tag{4.25}$$

yields (4.19). The formula (4.18) specified to $d=1$ gives

$$\sigma_1^{\mathrm{s}}(z) = \partial_z\frac{\coth(\pi\sqrt{-z})}{2\sqrt{-z}}. \tag{4.26}$$

Imposing the condition $\Sigma_1^{\mathrm{s}}(-\infty)=0$ yields

$$\Sigma_1^{\mathrm{s}}(-\beta^2) = -\frac{\coth(\pi\beta)}{2\beta}. \tag{4.27}$$

Dimension 2. We have

$$G_2^{\mathrm{s}}(-\beta^2;x,x') = \frac{\mathbf{S}_{0,\mathrm{i}\beta}(-\cos r)}{4\cosh(\pi\beta)}. \tag{4.28}$$

From (4.18), we obtain a family of self-energies depending on the parameter $\varepsilon := -2\pi\gamma$:

$$\Sigma_2^{\mathrm{s},\varepsilon}(-\beta^2) = \frac{1}{4\pi}\Big(\psi\big(\tfrac{1}{2}+\mathrm{i}\beta\big)+\psi\big(\tfrac{1}{2}-\mathrm{i}\beta\big)-2\varepsilon\Big). \tag{4.29}$$

Thus,

$$\begin{aligned} G_2^{\mathrm{s},\varepsilon}(-\beta^2;x,x') &= \frac{\mathbf{S}_{0,\mathrm{i}\beta}(-\cos r)}{4\cosh(\pi\beta)} \\ &\quad + \frac{\pi}{4\cosh^2(\pi\beta)}\,\frac{\mathbf{S}_{0,\mathrm{i}\beta}(-\cos\theta)\mathbf{S}_{0,\mathrm{i}\beta}(-\cos\theta')}{\psi(\frac{1}{2}+\mathrm{i}\beta)+\psi(\frac{1}{2}-\mathrm{i}\beta)-2\varepsilon}. \end{aligned} \tag{4.30}$$

In contrast to the Euclidean case, $\Sigma_d^{\mathrm{s},\varepsilon}(-\beta^2)$ has a singularity at $-\infty$ but not at 0.

Dimension 3. We have

$$\mathbf{S}_{\frac{1}{2},\mathrm{i}\beta}(-\cos r) = \frac{2\sinh\big((\pi-r)\beta\big)}{\beta\sqrt{\pi}\sin r}, \tag{4.31}$$

$$G_3^{\mathrm{s}}(-\beta^2,x,x') = \frac{1}{4\pi}\frac{\sinh\big((\pi-r)\beta\big)}{\sinh(\pi\beta)\,\sin r}. \tag{4.32}$$

Using (4.18) and choosing the integration constant so that $\Sigma_3^{\mathrm{s}}(-\beta^2)=\Sigma_3(-\beta^2)$ $+\,o(1)_{\beta\to\infty}$,

$$\Sigma_3^{\mathrm{s}}(-\beta^2) = \frac{\beta\coth(\pi\beta)}{4\pi}. \tag{4.33}$$

Therefore,

$$G_3^{\mathrm{s},\gamma}(-\beta^2,x,x') = \frac{\sinh\big(\beta(\pi-r)\big)}{4\pi\sinh(\beta\pi)\sin r} + \frac{\sinh\big(\beta(\pi-\theta)\big)\sinh\big(\beta(\pi-\theta')\big)}{(4\pi)^2\sin\theta\sin\theta'\sinh^2(\beta\pi)\big(\gamma+\frac{\beta\coth(\pi\beta)}{4\pi}\big)}. \tag{4.34}$$

Odd dimensions $d \geq 5$. Equation (4.18) is still valid if understood in the generalized sense for all $d \in \mathbb{C}\setminus 2\mathbb{Z}$. Therefore, we can set for such d

$$\Sigma_d^{\mathrm{s}}(-\beta^2) = -\frac{\Gamma(\frac{2-d}{2})}{(4\pi)^{\frac{d}{2}}}\,(\tfrac12+\mathrm{i}\beta)_{\frac{d-2}{2}}(\tfrac12-\mathrm{i}\beta)_{\frac{d-2}{2}}. \tag{4.35}$$

Due to (D.6), (4.35) specified to odd integer values is

$$\Sigma_d^{\mathrm{s}}(-\beta^2) = \frac{\pi\beta\coth(\pi\beta)}{(4\pi)^{\frac{d}{2}}\Gamma(\frac{d}{2})}\prod_{k=1}^{\frac{d-3}{2}}\Big(-k^2-\beta^2\Big). \tag{4.36}$$

Combining (4.12), (4.36) and (4.15), we obtain $G_d^{\mathrm{s},\gamma}(-\beta^2;x,x')$.

Even dimensions $d \geq 4$. Similarly as in the flat and hyperbolic case, the formula (4.35) is not applicable. As we will argue below, for $d \in 2\mathbb{Z}$, $d \geq 4$, we will introduce a family of reference self-energies parametrized by $\varepsilon \in \mathbb{R}$:

$$\Sigma_d^{\mathrm{s},\varepsilon}(-\beta^2) := \frac{\psi\big(\frac{d-1}{2}+\mathrm{i}\beta\big)+\psi\big(\frac{d-1}{2}-\mathrm{i}\beta\big)-2\varepsilon}{(4\pi)^{\frac{d}{2}}\Gamma\big(\frac{d}{2}\big)}\prod_{j=0}^{\frac{d-4}{2}}\Big(-\beta^2-\big(\tfrac12+j\big)^2\Big). \tag{4.37}$$

Thus, we obtain a family of Green's functions

$$G_d^{\mathrm{s},\varepsilon,\eta}(z;x,x') = G_d^{\mathrm{s}}(z;x,x') + \frac{G_d^{\mathrm{s}}(z;x,x_0)G_d^{\mathrm{s}}(z;x_0,x')}{\eta(z)+\Sigma_d^{\mathrm{s},\varepsilon}(z)}, \tag{4.38}$$

parametrized by $\varepsilon \in \mathbb{R}$ and a polynomial η with $\deg\eta \leq \frac{d-4}{2}$.

Let us derive (4.37) using the integral in (4.17) as the starting point. For $d \in 2\mathbb{Z}$, $d \geq 4$, unfortunately, (4.17) has to be understood in the anomalous generalized sense and is not equal to (4.18). Instead, it is given by (C.16) and we have

$$\begin{aligned}\sigma_d^{\mathrm{s}}(-\beta^2) =& \frac{(-1)^{\frac{d-2}{2}}(\frac12+\mathrm{i}\beta)_{\frac{d-2}{2}}(\frac12-\mathrm{i}\beta)_{\frac{d-2}{2}}}{\Gamma(\frac{d}{2})(4\pi)^{\frac{d}{2}}}\Bigg(\frac{\mathrm{i}}{2\beta}\Big(\psi'\big(\tfrac{d-1}{2}+\mathrm{i}\beta\big)-\psi'\big(\tfrac{d-1}{2}-\mathrm{i}\beta\big)\Big)\\ &-\frac{\mathrm{i}}{2\beta}\Big(H_{\frac{d-2}{2}}\big(\tfrac12+\mathrm{i}\beta\big)-H_{\frac{d-2}{2}}\big(\tfrac12-\mathrm{i}\beta\big)\Big)\ln 4\\ &+\sum_{k=0}^{\frac{d-4}{2}}\frac{\psi\big(-\frac12-k+\mathrm{i}\beta\big)+\psi\big(-\frac12-k-\mathrm{i}\beta\big)-\psi\big(\frac{d-2}{2}-k\big)-\psi(1+k)}{\big(\frac12+k\big)^2+\beta^2}\Bigg).\end{aligned} \tag{4.39}$$

As in the hyperbolic case, application of the Leibniz rule and identities satisfied by the Pochhammer symbol, harmonic numbers and the digamma function (see Appendix D) yields

$$\begin{aligned}\sigma_d^{\mathrm{s}}(z) = &-\frac{1}{\Gamma(\frac{d}{2})(4\pi)^{\frac{d}{2}}}\partial_z\Bigg(\Big(\psi\big(\tfrac{d-1}{2}+\mathrm{i}\sqrt{-z}\big)+\psi\big(\tfrac{d-1}{2}-\mathrm{i}\sqrt{-z}\big)-\ln 4\Big)\prod_{j=0}^{\frac{d-4}{2}}\big(z-(\tfrac{1}{2}+j)^2\big)\Bigg)\\ &+\frac{1}{\Gamma(\frac{d}{2})(4\pi)^{\frac{d}{2}}}\pi_d^{\mathrm{s}}(z), \end{aligned} \tag{4.40}$$

where π_d^{s} is a polynomial of degree $\frac{d-4}{2}$ given by

$$\begin{aligned}\pi_d^{\mathrm{s}}(z) = \prod_{j=0}^{\frac{d-4}{2}}\big(z-(\tfrac{1}{2}+j)^2\big)\Bigg(&\sum_{k=0}^{\frac{d-4}{2}}\frac{\psi\big(\frac{d-2}{2}-k\big)+\psi(1+k)}{z-(\frac{1}{2}+k)^2}\\ &-\sum_{k=0}^{\frac{d-4}{2}}\sum_{l=k+1}^{\frac{d-4}{2}}\frac{2l+1}{\Big(z-\big(\frac{1}{2}+k\big)^2\Big)\Big(z-\big(\frac{1}{2}+l\big)^2\Big)}\Bigg).\end{aligned} \tag{4.41}$$

We define

$$\Pi_d^{\mathrm{s}}(z) := -\int_0^z \pi_d^{\mathrm{s}}(\tau)\mathrm{d}\tau, \tag{4.42}$$

a polynomial of degree $\frac{d-2}{2}$ with $\Pi_d^{\mathrm{s}}(0)=0$ and

$$\begin{aligned}\Sigma_d^{\mathrm{s,ms}}(z) = &\frac{1}{\Gamma(\frac{d}{2})(4\pi)^{\frac{d}{2}}}\Big(\psi\big(\tfrac{d-1}{2}+\mathrm{i}\sqrt{-z}\big)+\psi\big(\tfrac{d-1}{2}-\mathrm{i}\sqrt{-z}\big)-\ln 4\Big)\prod_{j=0}^{\frac{d-4}{2}}\big(z-(\tfrac{1}{2}+j)^2\big)\\ &+\frac{1}{\Gamma(\frac{d}{2})(4\pi)^{\frac{d}{2}}}\Pi_d^{\mathrm{s}}(z).\end{aligned} \tag{4.43}$$

Then $\Sigma_d^{\mathrm{s,ms}}$ is an antiderivative of minus (4.40) and will be called the *reference self-energy based on the minimal subtraction*. From $\Sigma_d^{\mathrm{s,ms}}$, we pass to the family of reference self-energies $\Sigma_d^{\mathrm{s},\varepsilon}$ by absorbing $\Pi_d^{\mathrm{s}}(z)$ into ε and $\eta(z)$ as in the hyperbolic case.

4.3. Flat Limit of the Spherical Laplacian

Let $R>0$. Instead of the unit sphere, we can consider the sphere of radius R

$$\mathbb{S}_R^d := \{x\in\mathbb{R}^{1+d} \mid (x|x)=R^2\}.$$

Various objects defined using $\mathbb{S}_R^d$ instead of $\mathbb{S}^d$ will have the subscript R. We have a bijection $\mathbb{S}^d=\mathbb{S}_1^d \ni x\mapsto Rx\in\mathbb{S}_R^d$. The distance in $\mathbb{S}_R^d$ satisfies

$$d_R^{\mathrm{s}}(Rx,Rx') = Rd^{\mathrm{s}}(x,x'). \tag{4.44}$$

The Laplace–Beltrami operator on $\mathbb{S}_R^d$, denoted $\Delta_{d,R}^{\mathrm{s}}$, is a self-adjoint operator on $L^2(\mathbb{S}_R^d)$ and

$$\sigma(-\Delta_{d,R}^{\mathrm{s}}) = \left\{\frac{l(l+d-1)}{R^2}\;\Big|\; l=0,1,2,\dots\right\}. \tag{4.45}$$

We set $H^{\mathrm{s}}_{d,R} := -\Delta^{\mathrm{s}}_{d,R} + \frac{(d-1)^2}{4R^2}$. For $\Re\beta > 0$ and $a < b$, we set

$$G^{\mathrm{s}}_{d,R}(z) := \big(-z + H^{\mathrm{s}}_{d,R}\big)^{-1}, \tag{4.46}$$

$$\mathbb{P}^{\mathrm{s}}_{d,l,R} := 1\!\!1_{\frac{1}{R^2}(l+\frac{d-1}{2})^2}\big(H^{\mathrm{s}}_{d,R}\big), \tag{4.47}$$

$$\mathbb{P}^{\mathrm{s}}_{d,R}(a,b) := 1\!\!1_{[a,b]}\big(H^{\mathrm{s}}_{d,R}\big). \tag{4.48}$$

We have

$$G^{\mathrm{s}}_{d,R}(-\beta^2;x,x') = R^{-d+2}G^{\mathrm{s}}_d\Big(-(\beta R)^2;\frac{x}{R},\frac{x'}{R}\Big), \tag{4.49}$$

$$\mathbb{P}^{\mathrm{s}}_{d,l,R}(x,x') = R^{-d}\mathbb{P}^{\mathrm{s}}_{d,l}\Big(\frac{x}{R},\frac{x'}{R}\Big), \tag{4.50}$$

$$\mathbb{P}^{\mathrm{s}}_{d,R}(a,b;x,x') = R^{-d}\mathbb{P}^{\mathrm{s}}_d\Big(R^2a,R^2b;\frac{x}{R},\frac{x'}{R}\Big). \tag{4.51}$$

The self-energy on $\mathbb{S}^d_R$ is defined analogously to the hyperbolic case and comes out to be

$$\Sigma^{\mathrm{s}}_{d,R}(-\beta^2) := R^{2-d}\Sigma^{\mathrm{s}}_d\big(-(\beta R)^2\big), \qquad d \text{ odd};$$
$$\Sigma^{\mathrm{s},\varepsilon}_{d,R}(-\beta^2) := R^{2-d}\Sigma^{\mathrm{s},\varepsilon+\ln R}_d\big(-(\beta R)^2\big), \qquad d \text{ even}.$$

Let $x,x' \in \mathbb{S}^d_R$. The perturbed Green's functions on $\mathbb{S}^d_R$ in odd and even dimensions, respectively, are

$$G^{\mathrm{s},\gamma}_{d,R}(z;x,x') = G^{\mathrm{s}}_{d,R}(z;x,x') + \frac{G^{\mathrm{s}}_{d,R}(z;x,Rx_0)G^{\mathrm{s}}_{d,R}(z;Rx_0,x')}{\gamma(z)+\Sigma^{\mathrm{s}}_{d,R}(z)}; \tag{4.52}$$

$$G^{\mathrm{s},\varepsilon,\eta}_{d,R}(z;x,x') = G^{\mathrm{s}}_{d,R}(z;x,x') + \frac{G^{\mathrm{s}}_{d,R}(z;x,Rx_0)G^{\mathrm{s}}_{d,R}(z;Rx_0,x')}{\eta(z)+\Sigma^{\mathrm{s},\varepsilon}_{d,R}(z)}. \tag{4.53}$$

Note that $\gamma(z)$ and $\eta(z)$ on the right-hand sides of (4.52) and (4.53) do not depend on R. This choice of renormalization is analogous to the hyperbolic case. Then all Green's functions have the correct flat limit in the following sense:

Theorem 4.2 *Let $-\beta^2 \in \mathbb{C}\backslash[0,\infty[$. Then*

$$G^{\mathrm{s}}_{d,R}\big(-\beta^2,r\big) = G_d\big(-\beta^2,r\big)\Big(1+\mathcal{O}\big(\tfrac{1}{\beta R}\big)+\mathcal{O}\big(\tfrac{r}{R}\big)\Big), \tag{4.54}$$

$$\Sigma^{\mathrm{s}}_{d,R}(-\beta^2) = \Sigma_d\big(-\beta^2\big)\Big(1+\mathcal{O}\big(\tfrac{1}{\beta R}\big)\Big), \quad d \textit{ odd}; \tag{4.55}$$

$$\Sigma^{\mathrm{s},\varepsilon}_{d,R}(-\beta^2) = \Sigma^{\varepsilon}_d\big(-\beta^2\big)\Big(1+\mathcal{O}\big(\tfrac{1}{\beta R}\big)\Big), \quad d \textit{ even}. \tag{4.56}$$

Thus, if we have a family $x_R, x'_R \in \mathbb{S}^d_R$ and $x,x' \in \mathbb{R}^d$ such that

$$\lim_{R\to\infty} d^{\mathrm{s}}_R(x_R,x'_R) = |x-x'|,$$
$$\lim_{R\to\infty} d^{\mathrm{s}}_R(x_R,Rx_0) = |x|,$$
$$\lim_{R\to\infty} d^{\mathrm{s}}(x_R,Rx_0) = |x'|, \tag{4.57}$$

then

$$\lim_{R\to\infty} G^{\mathrm{s},\gamma}_{d,R}\big(-\beta^2;x_R,x_R'\big)=G^\gamma_d(-\beta^2,x,x'),\quad d\ \textit{odd}; \tag{4.58}$$

$$\lim_{R\to\infty} G^{\mathrm{s},\varepsilon,\eta}_{d,R}\big(-\beta^2;x_R,x_R'\big)=G^{\varepsilon,\eta}_d(-\beta^2,x,x'),\quad d\ \textit{even}. \tag{4.59}$$

Proof Using the asymptotics of the Gegenbauer functions from Thm. C.1, we find

$$\begin{aligned}
G^{\mathrm{s}}_{d,R}(-\beta^2;r_R)&=R^{-d+2}G^{\mathrm{s}}_d\Big(-(\beta R)^2;-\cos\frac{r_R}{R}\Big)\\
&=\frac{R^{-d+2}\Gamma(\frac{d-1}{2}+\mathrm{i}\beta R)\Gamma(\frac{d-1}{2}-\mathrm{i}\beta R)}{(4\pi)^{\frac{d}{2}}}\mathbf{S}_{\frac{d-2}{2},\mathrm{i}\beta R}\Big(-\cos\frac{r_R}{R}\Big)\\
&=\frac{\beta^{d-2}\pi\mathrm{e}^{-\pi R\beta}}{2^{\frac{d-2}{2}}(2\pi)^{\frac{d}{2}}}\mathbf{S}_{\frac{d-2}{2},\mathrm{i}\beta R}\Big(-\cos\frac{r_R}{R}\Big)\Big(1+\mathcal{O}\big(\tfrac{1}{\beta R}\big)\Big)\\
&=\frac{(\frac{r_R}{R})^{\frac{d-1}{2}}}{(\sin\frac{r_R}{R})^{\frac{d-1}{2}}(2\pi)^{\frac{d}{2}}}\Big(\frac{\beta}{r_R}\Big)^{\frac{d-2}{2}}K_{\frac{d-2}{2}}(\beta r_R)\Big(1+\mathcal{O}\big(\tfrac{1}{\beta R}\big)\Big).
\end{aligned} \tag{4.60}$$

This proves (4.54). To prove (4.55), we use Thm. C.2 and

$$\psi\big(\tfrac12+\alpha\pm\mathrm{i}\beta R\big)-\ln(\beta R)\mp\mathrm{i}\tfrac{\pi}{2}=\mathcal{O}\big(\tfrac{1}{\beta R}\big). \tag{4.61}$$

Then we argue as in the hyperbolic case. □

4.4. Poles of Green's Functions and Spectral Properties

All singularities of Green's functions $G^{\mathrm{s},\gamma}_{d,R}(z)$ are isolated. In dimensions $d=1,2,3$, they correspond to the point spectrum of $H^{\mathrm{s},\gamma}_{d,R}$. In this section, we analyze the location of these singularities. They come in two types: poles of $G^{\mathrm{s}}_{d,R}$ and zeros of $\gamma(z)+\Sigma^{\mathrm{s}}_{d,R}(z)$, and $\eta(z)+\Sigma^{\mathrm{s},\varepsilon}_{d,R}(z)$. First we discuss the former.

Let $l\in\mathbb{N}_0$ and let $\omega_{d,l}=\frac{d-1}{2}+l$ parametrize the eigenvalues $R^{-2}\omega_{d,l}^2$ of the unperturbed operator. It is well known that the multiplicity of $R^{-2}\omega_{d,l}^2$, or in other words the dimension of the range of $\mathbb{P}^{\mathrm{s}}_{d,l,R}$, is $m_{d,l}=\binom{d+l}{d}-\binom{d+l-2}{d}$. Therefore,

$$\int\mathbb{P}^{\mathrm{s}}_{d,l,R}(x,Rx_0)^2\mathrm{d}x=\mathbb{P}^{\mathrm{s}}_{d,l,R}(Rx_0,Rx_0)=\frac{m_{d,l}}{|\mathbb{S}^d|R^d}. \tag{4.62}$$

The right-hand side of (4.62) can be verified explicitly using an appropriately rescaled version of (4.6). Moreover, the rank of the residue of the unperturbed Green's operator $G^{\mathrm{s}}_{d,R}(z)$ at $z=R^{-2}\omega_{d,l}^2$ is $m_{d,l}$. Let us show that after perturbation this rank drops by 1.

Theorem 4.3 *$G^{\mathrm{s},\gamma}_{d,R}(z)$ has a pole of rank $m_{d,l}-1$ at $R^{-2}\omega_{d,l}^2$. In particular, since $m_{d,0}=1$, the perturbed Green's function does not have a pole at $R^{-2}\omega_{d,0}^2$.*

Proof We have

$$G^{\mathrm{s}}_{d,R}(z;x,x')=\frac{\mathbb{P}^{\mathrm{s}}_{d,l,R}(x,x')}{R^{-2}\omega_{d,l}^2-z}+R(z;x,x'), \tag{4.63}$$

with a remainder $R(z;x,x')$ non-singular at $z=R^{-2}\omega_{d,l}^2$ and satisfying

$$\int_{\mathbb{S}_R^d} \mathbb{P}^{\mathrm{s}}_{d,l,R}(x,x')R(z;x',x'')\mathrm{d}x'=0. \tag{4.64}$$

From this and (4.62), we can deduce that near $z=R^{-2}\omega_{d,l}^2$, function $\sigma^{\mathrm{s}}_{d,R}(z)$ is given by

$$\begin{aligned}\sigma^{\mathrm{s}}_{d,R}(z) &= \int G^{\mathrm{s}}_{d,R}(z;0,y)^2\mathrm{d}y && (4.65)\\ &= \frac{1}{(R^{-2}\omega_{d,l}^2-z)^2}\int \mathbb{P}^{\mathrm{s}}_{d,l,R}(x,Rx_0)^2\mathrm{d}x+\mathcal{O}(1) && (4.66)\\ &= \frac{1}{(R^{-2}\omega_{d,l}^2-z)^2}\frac{m_{d,l}}{|\mathbb{S}^d|R^d}+\mathcal{O}(1). && (4.67)\end{aligned}$$

The self-energy thus satisfies

$$\Sigma^{s}_{d,R}(z)=-\frac{1}{R^{-2}\omega_{d,l}^2-z}\frac{m_{d,l}}{|\mathbb{S}^d|R^d}+\mathcal{O}(1). \tag{4.68}$$

In particular, $\gamma(z)+\Sigma^{\mathrm{s}}_{d,R}(z)\neq 0$ at $z=R^{-2}\omega_{d,l}^2$ due to the singularity of $\Sigma^{\mathrm{s}}_{d,R}(z)$ at this point. Hence,

$$G^{\mathrm{s},\gamma}_{d,R}(z;x,x')=\frac{\mathbb{P}^{\mathrm{s}}_{d,l,R}(x,x')-\frac{|\mathbb{S}^d|R^d}{m_{d,l}}\mathbb{P}^{\mathrm{s}}_{d,l,R}(x,Rx_0)\mathbb{P}^{\mathrm{s}}_{d,l,R}(Rx_0,x')}{R^{-2}\omega_{d,l}^2-z}+\mathcal{O}(1). \tag{4.69}$$

Now note that $\sqrt{\frac{|\mathbb{S}^d|R^d}{m_{d,l}}}\mathbb{P}^{\mathrm{s}}_{d,l,R}(\cdot,Rx_0)=\sqrt{\frac{|\mathbb{S}^d|R^d}{m_{d,l}}}\mathbb{P}^{\mathrm{s}}_{d,l,R}(Rx_0,\cdot)$ is a real-valued and L^2-normalized vector in the range of $\mathbb{P}^{s}_{d,l,R}$. We remark that $\sqrt{\frac{|\mathbb{S}^d|R^d}{m_{d,l}}}\mathbb{P}^{\mathrm{s}}_{d,l,R}$ $(\cdot,Rx_0)$ may be characterized as the unique normalized vector in the range of $\mathbb{P}^{s}_{d,l,R}$ which is invariant under orthogonal transformations preserving Rx_0 and is nonnegative at Rx_0. Hence, the numerator in (4.69) is (the integral kernel of) the orthogonal projection onto the orthogonal complement of $\sqrt{\frac{|\mathbb{S}^d|R^d}{m_{d,l}}}\mathbb{P}^{\mathrm{s}}_{d,l,R}$ $(\cdot,Rx_0)$ in the range of $\mathbb{P}^{\mathrm{s}}_{d,l,R}$. In particular, it is the kernel of a projection of rank $m_{d,l}-1$. □

The rank of the residue at $R^{-2}\omega_{d,l}^2$ drops by one because the pole corresponding to one "eigenvector" is shifted. We find the shifted poles by solving the equations

$$\begin{aligned}\gamma(z)+\Sigma^{\mathrm{s}}_{d,R}(z)&=0, \quad d \text{ odd},\\ \eta(z)+\Sigma^{\mathrm{s},\varepsilon}_{d,R}(z)&=0, \quad d \text{ even}.\end{aligned} \tag{4.70}$$

First let us consider dimension 1. The unperturbed poles are at $z=R^{-2}l^2$, with multiplicity 1 for $l=0$ and multiplicity 2 for $l=1,2,\ldots$. The perturbation cancels the pole at 0 and decreases the multiplicity for $l\geq 1$ to 1. Putting $\beta R=it$ and $\epsilon=\frac{\gamma}{R}$, the equation for shifted poles takes the form

$$-\frac{1}{2t}\cot(\pi t)=\epsilon. \tag{4.71}$$

If $\gamma = 0$, then $\epsilon = 0$ and the solutions are half-integers. Negative solutions correspond to the same z, so we focus at positive half-integers. If $\gamma \neq 0$, then ϵ is nonzero but small, so we can use the implicit function theorem to find a solution at

$$t = l + \frac{1}{2} + \frac{2l+1}{\pi}\epsilon + \mathcal{O}(\epsilon^2). \tag{4.72}$$

We remark that if $\gamma \in \mathbb{R}$, this solution is in $]l, l+1[$ (no matter the size of ϵ).

Unpacking the notation, the above calculation proves:

Theorem 4.4 *$G^{s,\gamma}_{1,R}(z)$ has poles at $z = R^{-2}l^2$ with $l = 1, 2, \dots$ and at*

$$E^{\gamma}_{1,l,R} = \left(\frac{l+\frac{1}{2}}{R}\right)^2 \left(1 + \frac{4\gamma}{\pi R} + \mathcal{O}\left(\frac{\gamma^2}{R^2}\right)\right), \qquad l = 0, 1, \dots. \tag{4.73}$$

All residues are rank one projections.

We see that for $d = 1$ and large R, the eigenvalues are approximated by these for $\gamma = 0$. Starting from dimension 2, the eigenvalues approach the unperturbed ones (infinite γ) instead.

Theorem 4.5 *For $d = 2, 3$, the lth ($l = 0, 1, \dots$) shifted pole of $G^{s,\gamma}_{d,R}$ is at*

- *$d = 2$: $E^{\gamma}_{2,l,R} = \frac{1}{R^2}\left((l+\frac{1}{2})^2 + \frac{l+\frac{1}{2}}{\ln\frac{R}{a}} + \mathcal{O}\left(\frac{1}{\ln^2\frac{R}{a}}\right)\right)$, where $a = \mathrm{e}^{2\pi\gamma} = \mathrm{e}^{-\varepsilon}$,*
- *$d = 3$: $E^{\gamma}_{3,l,R} = \frac{(l+1)^2}{R^2}\left(1 - \frac{1}{2\pi^2 R\gamma} + \mathcal{O}\left(\frac{1}{R^2\gamma^2}\right)\right)$, except for the case $\gamma = 0$, in which the pole is at $E^{0}_{3,l,R} = \frac{(l+\frac{1}{2})^2}{R^2}$.*

The residues are rank one projections.

Proof The claim about residues is obvious. We consider the equation for the shifted eigenvalue. Throughout the proof, we set $\beta R = \mathrm{i}t$. First consider $d = 2$. We have the equation

$$\psi(\tfrac{1}{2} + t) + \psi(\tfrac{1}{2} - t) = 2\ln\frac{R}{a}. \tag{4.74}$$

The right-hand side blows up for $R \to \infty$, so we denote it $\frac{1}{\epsilon}$. We expand the left-hand side around the unperturbed pole: Writing $t = l + \frac{1}{2} + \delta$, we obtain

$$\frac{1}{\delta} + \mathcal{O}(1)_{\delta\to 0} = \frac{1}{\epsilon}. \tag{4.75}$$

By the implicit function theorem, there exists a solution $\delta = \epsilon + \mathcal{O}(\epsilon^2)$.

Next consider $d = 3$. We have the equation

$$-t\cot(\pi t) = 4\pi R\gamma. \tag{4.76}$$

We denote the right-hand side by $\frac{1}{\epsilon}$ and expand the left-hand side around $l+1$, finding a solution of the form

$$t = (l+1)\left(1 - \frac{\epsilon}{\pi} + \mathcal{O}(\epsilon^2)\right). \tag{4.77}$$

Separate analysis of $\gamma = 0$ (hence $\epsilon = \infty$) is elementary. □

Due to the presence of the polynomials $\gamma(z)$ and $\eta(z)$, the situation is more complicated in higher dimensions, especially in even dimensions $d \geq 4$. We distinguish three cases:

1. $\gamma(z) \equiv 0$ and $\eta(z) \equiv 0$, respectively, where the poles correspond to the zeros of the reference self-energies. We might call this the *unitary gas case.*
2. $\gamma(z)$ and $\eta(z)$, respectively, are non-constant polynomials. The shifted poles of the perturbed Green's functions are located near the poles of the unperturbed Green's function. As we will see, the rate of convergence of perturbed poles to unperturbed ones as $R \to \infty$ is modified if $\gamma(z)$ and $\eta(z)$ vanish at zero and depend on the degree of vanishing.
3. $\gamma(z) = \gamma_0$ and $\eta(z) = \eta_0$ respectively, are nonzero constants. This could be treated on the same footing as case 2. with $\gamma(z)$ and $\eta(z)$ not vanishing at zero, but since the conclusions are particularly simple we prefer to state them separately.

Odd dimensions $d \geq 5$. Let us first find the zeros and poles of the reference self-energy:

Lemma 4.6 *Let $d \geq 5$ be an odd integer. The zeros and poles of the reference self-energy are located at $\zeta = \mathrm{i}\beta \geq 0$ such that*

$$\begin{aligned} &\Sigma^{\mathrm{s}}_{d,R}(\zeta^2) = 0 &&\Leftrightarrow \quad \zeta = \frac{l+\frac{1}{2}}{R}, \quad l \in \mathbb{N}_0, \\ &\Sigma^{\mathrm{s}}_{d,R}(\zeta^2) \quad \textit{has a pole} &&\Leftrightarrow \quad \zeta = \frac{k}{R}, \quad k = l + \frac{d-1}{2}, \quad l \in \mathbb{N}_0. \end{aligned} \tag{4.78}$$

Proof The reference self-energy is

$$\Sigma^{\mathrm{s}}_{d,R}(-\beta^2) = \frac{\pi \coth(\pi\beta R)\beta}{(4\pi)^{\frac{d}{2}}\Gamma(\frac{d}{2})} \prod_{k=1}^{\frac{d-3}{2}} \Big(-\frac{k^2}{R^2} - \beta^2 \Big), \quad l \in \mathbb{N}_0. \tag{4.79}$$

Writing $\beta = \mathrm{i}\zeta$ with $\zeta \geq 0$, we find

$$\Sigma^{\mathrm{s}}_{d,R}(\zeta^2) = \frac{\pi}{(4\pi)^{\frac{d}{2}}\Gamma(\frac{d}{2})} \cos(\pi\zeta R) \frac{\zeta \prod_{k=1}^{\frac{d-3}{2}} \Big(\zeta^2 - \frac{k^2}{R^2}\Big)}{\sin(\pi\zeta R)}. \tag{4.80}$$

The zeros of $\Sigma^{\mathrm{s}}_{d,R}$ are located at the zeros of the cosine. The poles are located at the zeros of the sine, except for the first few, which are canceled by the zeros of the numerator. □

Note that the location of the poles of $\Sigma^{\mathrm{s}}_{d,R}$ precisely corresponds to the unperturbed eigenvalues.

Theorem 4.7 *Let $d \geq 5$ be an odd integer.*

1. *Suppose that $\gamma(z) \equiv 0$ vanishes identically, then $G^{s,\gamma}_{d,R}$ has a sequence of isolated poles located at*

$$E^0_{d,l,R} = \frac{\big(l+\frac{1}{2}\big)^2}{R^2}, \quad l \in \mathbb{N}_0. \tag{4.81}$$

2. *Suppose that $\gamma(z)$ does not vanish identically and let ν be the order of vanishing of $\gamma(z)$ at 0 ($\nu = 0$ if $\gamma(0) \neq 0$). The lth shifted pole of $G^{s,\gamma}_{d,R}$ is located at*

$$E^{\gamma}_{d,l,R} = \frac{\omega^2_{d,l}}{R^2}\left(1 - \frac{2\prod_{k=1}^{\frac{d-3}{2}}(\omega^2_{d,l} - k^2)}{(4\pi)^{\frac{d}{2}}\Gamma(\frac{d}{2})R^{d-2}\gamma\left(R^{-2}\omega^2_{d,l}\right)} + \mathcal{O}(R^{-2d+4+4\nu})\right). \tag{4.82}$$

3. *In the special case where $\gamma(z) = \gamma_0 \in \mathbb{R} \setminus \{0\}$ is constant, the lth shifted pole of $G^{s,\gamma}_{d,R}$ is located at*

$$E^{\gamma}_{d,l,R} = \frac{\omega^2_{d,l}}{R^2}\left(1 - \frac{2\prod_{k=1}^{\frac{d-3}{2}}(\omega^2_{d,l} - k^2)}{(4\pi)^{\frac{d}{2}}\Gamma(\frac{d}{2})R^{d-2}\gamma_0} + \mathcal{O}\big(R^{-2(d-2)}\big)\right). \tag{4.83}$$

   In particular, the first correction to the unperturbed eigenvalues is inversely proportional to the volume of the sphere.

Proof The first case $\gamma(z) \equiv 0$ follows directly from Lemma 4.6. The third case is a special case of the second, and the latter can be derived analogously to lower dimensions. The only complication is that $\gamma(z)$ may vanish at zero, in which case the scaling of the pole shift with R is modified. This is taken into account by the introduction of ν. □

We note that ν can be as large as $\frac{d-3}{2}$, in which case the shift of the eigenvalue is proportional to R^{-3} and the first neglected term is proportional to R^{-4}. That is, the scaling of the unperturbed eigenvalue and the scaling of the shift with R differ only by a single power.

Even dimensions $d \geq 4$. In even dimensions, we considered a family of reference self-energies parametrized by $\varepsilon \in \mathbb{R}$. We look for the zeros of the reference self-energies first.

Lemma 4.8 *Let $d \geq 4$ be an even integer. For large R, the zeros of the family of reference self-energies are located at $\zeta = \mathrm{i}\beta \geq 0$ such that*

$$\zeta^2 = \begin{cases} \frac{1}{R^2}\left(\frac{1}{2} + j\right)^2, & j = 0, \dots, \frac{d-4}{2}, \\ \frac{1}{R^2}\left(\omega^2_{d,l} + \frac{\omega_{d,l}}{\ln(\mathrm{e}^{\varepsilon}R)} + \mathcal{O}\left(\frac{1}{\ln^2(\mathrm{e}^{\varepsilon}R)}\right)\right), & l \in \mathbb{N}_0. \end{cases} \tag{4.84}$$

Proof Let $\beta R = \mathrm{i}t$. The reference self-energies are

$$\Sigma^{s,\varepsilon}_{d,R}\left(\tfrac{t^2}{R^2}\right) = \frac{1}{(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)}\left(\psi\left(\tfrac{d-1}{2} + t\right) + \psi\left(\tfrac{d-1}{2} - t\right) - 2\varepsilon - 2\ln R\right)$$
$$\times R^{2-d}\prod_{j=0}^{\frac{d-4}{2}}\left(t^2 - \left(\tfrac{1}{2} + j\right)^2\right).$$

The zeros of the second line are obvious. If $t = 0$, the whole expression is $\sim R^{2-d} \ln R$, which neither corresponds to a pole nor a zero (but it is an approximate zero for large R).

We look for the zeros of the first line, which correspond to

$$\psi\big(\tfrac{d-1}{2} + t\big) + \psi\big(\tfrac{d-1}{2} - t\big) = 2\varepsilon + 2 \ln R. \tag{4.85}$$

The right-hand side is large for large R, so as for $d = 2$, we denote it $\frac{1}{\epsilon}$. (Note the difference between ϵ and ε.) We perturb the left-hand side around the unperturbed poles by setting $t = \frac{d-1}{2} + l + \delta$ for $l \in \mathbb{N}_0$. We obtain

$$\frac{1}{\delta} + \mathcal{O}(1)_{\delta\to 0} = \frac{1}{\epsilon}, \tag{4.86}$$

so by the implicit function theorem there exists a solution $\delta = \epsilon + \mathcal{O}(\epsilon^2)$. □

Lemma 4.8 allows us to describe the poles of the perturbed Green's functions.

Theorem 4.9 *Let $d \geq 4$ be an even integer.*

1. *Suppose that $\eta(z) \equiv 0$ vanishes identically, then $G^{s,\varepsilon,0}_{d,R}$ has a sequence of isolated poles located at*

$$E^{\varepsilon,0}_{d,l,R} = \frac{1}{R^2}\Big(\omega^2_{d,l} + \frac{\omega_{d,l}}{\ln(\mathrm{e}^{\epsilon} R)} + \mathcal{O}\Big(\frac{1}{\ln^2(\mathrm{e}^{\epsilon} R)}\Big)\Big), \quad l \in \mathbb{N}_0, \tag{4.87}$$

and a finite number of additional poles at

$$E^{\varepsilon,0,\mathrm{exceptional}}_{d,j,R} = \frac{1}{R^2}\Big(j + \frac{1}{2}\Big)^2, \quad j = 0, \ldots, \tfrac{d-4}{2}. \tag{4.88}$$

2. *Suppose that $\eta(z)$ does not vanish identically and let ν be the order of vanishing of $\eta(z)$ at 0 ($\nu = 0$ if $\eta(0) \neq 0$). The lth shifted pole of $G^{s,\gamma}_{d,R}$ is located at*

$$E^{\varepsilon,\eta}_{d,l,R} = \frac{1}{R^2}\left(\omega^2_{d,l} - \frac{2\omega_{d,l}\prod_{j=0}^{\frac{d-4}{2}}\Big(\omega^2_{d,l} - \big(\frac{1}{2} + j\big)^2\Big)}{(4\pi)^{\frac{d}{2}}\Gamma\big(\frac{d}{2}\big)R^{d-2}\eta\big(\frac{\omega^2_{d,l}}{R^2}\big)} + \mathcal{O}\Big(\ln(\mathrm{e}^{\varepsilon} R)R^{-2d+4+4\nu}\Big)\right). \tag{4.89}$$

3. *In the special case where $\eta(z) = \eta_0 \in \mathbb{R} \setminus \{0\}$ is constant, we have*

$$E^{\varepsilon,\eta_0}_{d,l,R} = \frac{1}{R^2}\left(\omega^2_{d,l} - \frac{2\omega_{d,l}\prod_{j=0}^{\frac{d-4}{2}}\Big(\omega^2_{d,l} - \big(\frac{1}{2} + j\big)^2\Big)}{(4\pi)^{\frac{d}{2}}\Gamma\big(\frac{d}{2}\big)R^{d-2}\eta_0} + \mathcal{O}\Big(\ln(\mathrm{e}^{\varepsilon} R)R^{-2(d-2)}\Big)\right). \tag{4.90}$$

In particular, the first correction to the unperturbed eigenvalues is inversely proportional to the volume of the sphere.

Proof The first statement follows from Lemma 4.8. To show the second statement, we need to consider the equation

$$\Big(\psi\big(\tfrac{d-1}{2}+t\big)+\psi\big(\tfrac{d-1}{2}-t\big)\Big)\prod_{j=0}^{\frac{d-4}{2}}\Big(t^2-\big(\tfrac{1}{2}+j\big)^2\Big)$$
$$=-(4\pi)^{\frac{d}{2}}\Gamma\big(\tfrac{d}{2}\big)R^{d-2}\eta\big(\tfrac{t^2}{R^2}\big)+2\ln(\mathrm{e}^{\varepsilon}R)\prod_{j=0}^{\frac{d-4}{2}}\Big(t^2-\big(\tfrac{1}{2}+j\big)^2\Big). \tag{4.91}$$

If $t=j+\frac{1}{2}$ for some $j=0,\dots,\frac{d-4}{2}$, then (4.91) becomes

$$0=R^{d-2}\eta\Big(R^{-2}\big(j+\tfrac{1}{2}\big)^2\Big). \tag{4.92}$$

The right-hand side of (4.92) is a polynomial in R. Therefore, Eq. (4.92) is not satisfied if R is large enough. Hence, $t=j+\frac{1}{2}$ with $j=0,\dots,\frac{d-4}{2}$ does not correspond to a pole of Green's function. We may rewrite (4.91) as

$$\psi\big(\tfrac{d-1}{2}+t\big)+\psi\big(\tfrac{d-1}{2}-t\big)=-\frac{(4\pi)^{\frac{d}{2}}\Gamma\big(\frac{d}{2}\big)R^{d-2}\eta\big(\frac{t^2}{R^2}\big)}{\prod_{j=0}^{\frac{d-4}{2}}\Big(t^2-\big(\frac{1}{2}+j\big)^2\Big)}+2\ln(\mathrm{e}^{\varepsilon}R). \tag{4.93}$$

The first term on the right-hand side is of order $R^{d-2-2\nu}$ and the second term is $\sim\ln R$, so both blow up for $R\to\infty$. Now as before, we denote the right-hand side by $\frac{1}{\epsilon}$ and write $t=\omega_{d,l}+\delta$. This gives (4.89). The third claim is a special case of the second. $\square$

We remark that for simplicity we did not indicate the dependence of error terms in the in higher dimensions on γ and η. Moreover, in all results of this section the error bounds are not uniform in l.

A more precise analysis of the poles of the perturbed Green's function—including a detailed analysis of the dependence of the error terms on the polynomials γ and η and estimates that are uniform in l—is desirable but beyond the scope of the current paper.

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A Generalized Integrals

Generalized integrals go back to ideas of Hadamard [16,17] and Riesz [29]. In a parallel work [10], we revisited this concept in a manner that is well suited for our applications. In the latter reference, the proofs for all generalized integrals appearing in Appendices B and C are displayed in detail.

Definition A1 Let $a \in \mathbb{R}$. We say that a function f on $]a,\infty[$ is *integrable in the generalized sense* if it is integrable on $]a+1,\infty[$ and if there exists a finite set $\Omega \subset \mathbb{C}$ and complex coefficients $(f_k)_{k\in\Omega}$ such that

$$f - \sum_{k\in\Omega} f_k (r-a)^k \tag{A.1}$$

is integrable on $]a, a+1[$. We define

$$\mathrm{gen}\int_a^\infty f(r)\mathrm{d}r := \sum_{k\in\Omega\setminus\{-1\}} \frac{f_k}{k+1} + \int_a^{a+1} \Big(f(r) - \sum_{k\in\Omega} f_k (r-a)^k \Big)\mathrm{d}r + \int_{a+1}^\infty f(r)\mathrm{d}r. \tag{A.2}$$

Note that the set $\{k \in \Omega \,|\, \Re k \le -1\}$ and the corresponding f_k are uniquely determined by f. It is convenient to allow $k \in \Omega$ with $\Re k > -1$. The generalized integral of f does not depend on the choice of Ω.

Clearly

$$\mathrm{gen}\int_a^\infty f(r)\mathrm{d}r = \int_a^\infty f(r)\mathrm{d}r \quad \text{for } f \in L^1[a,\infty[. \tag{A.3}$$

If Φ is any other extension of the integration functional from $L^1[a,\infty[$ to the class of all functions integrable in the generalized sense, then Φ is given by

$$\Phi(f) = \mathrm{gen}\int_a^\infty f(r)\mathrm{d}r + \sum_{\substack{k\in\Omega \\ \Re k \le -1}} f_k \lambda_k \tag{A.4}$$

for some coefficients λ_k. Conversely, for any set of λ_k one may define an extension Φ by (A.4). To some extent, the definition of $\mathrm{gen}\int_a^\infty$ is arbitrary and one could use some other extension instead. $\mathrm{gen}\int_a^\infty$ has several simple properties which make it a useful reference point.

The generalized integral is invariant with respect to translations and taking power of the integration variable,

$$\mathrm{gen}\int_a^\infty f(r)\mathrm{d}r = \mathrm{gen}\int_{a-\alpha}^\infty f(u+\alpha)\mathrm{d}u, \qquad \alpha \in \mathbb{R}, \tag{A.5}$$

$$\mathrm{gen}\int_0^\infty f(r)\mathrm{d}r = \mathrm{gen}\int_0^\infty f(u^\alpha)\,\alpha u^{\alpha-1}\mathrm{d}u, \qquad \alpha > 0. \tag{A.6}$$

Due to the first property, there is no loss in assuming $a = 0$.

Generalized integral behaves in an interesting way under coordinate transformations. Let $g : [0,\infty[\to [0,\infty[$ be a bijection, smooth down to 0, such that $g(0) = 0$ and $g'(0) \neq 0$. The map $f \mapsto (f \circ g)g'$ preserves the class of functions integrable in the generalized sense and (by the change of variables formula) the classical integration functional. Hence, one may define generalized integration in the changed coordinate system as

$$\mathrm{gen}_g\int_0^\infty f(r)\mathrm{d}r = \mathrm{gen}\int_0^\infty f(g(u))g'(u)\mathrm{d}u. \tag{A.7}$$

The corresponding coefficients λ_k in the comparison formula (A.4) have been calculated in [10]:

$$\begin{aligned}\mathrm{gen}_g\int_0^\infty f(r)\mathrm{d}r =& \,\mathrm{gen}\int_0^\infty f(r)\mathrm{d}r + f_{-1}\ln\frac{1}{g'(0)}\\ &+ \sum_{\substack{l=2\\ -l\in\Omega}}^{\infty} f_{-l}\frac{1}{(l-1)(l-1)!}\frac{\mathrm{d}^{l-1}}{\mathrm{d}u^{l-1}}\left(\frac{u}{g(u)}\right)^{l-1}\Bigg|_{u=0}.\end{aligned} \tag{A.8}$$

This involves only coefficients $f_{-1}, f_{-2}, \dots$. Other f_k appear if one considers more general coordinate transformations. For example, if $g^{(n)}(0) = 0$ for $n = 0, \dots, N-1$ but $g^{(N)}(0) \neq 0$, then also nonzero coefficients $f_{-\frac{k}{N}}$, $k \in \mathbb{N}$, will cause anomalous behavior. Note that the sum on the right-hand side of (A.8) is finite and that the number of appearing derivatives of g is governed by the scaling behavior of the integrand.

A particularly important change of variables is the scaling:

$$\mathrm{gen}\int_0^\infty f(\alpha u)\alpha\mathrm{d}u = \mathrm{gen}\int_0^\infty f(r)\mathrm{d}r - f_{-1}\ln\alpha, \quad \alpha > 0. \tag{A.9}$$

One should carefully distinguish this integral from integration with respect to coordinate αu (which amounts to relabeling u to αu):

$$\mathrm{gen}\int_0^\infty f(\alpha u)d\alpha u = \mathrm{gen}\int_0^\infty f(u)du. \tag{A.10}$$

Combining with (A.9), one obtains the formula

$$\mathrm{gen}\int_0^\infty f(u)d\alpha u = \alpha\left(\mathrm{gen}\int_0^\infty f(u)du + f_{-1}\ln\alpha\right). \tag{A.11}$$

As seen from (A.9) and (A.11), the generalized integral is only scale invariant on the class of function with $f_{-1} = 0$. If $f_{-1} \neq 0$, we say that the integral has a *scaling anomaly.* On the grounds of (A.8), integrals with $f_k \neq 0$ for any $k = -1, -2, \dots$ were called *anomalous* in [10].

In quantum field theory jargon, generalized integrals with a scaling anomaly depend on the choice of a renormalization scale. In Definition A1, we set this scale for 1 for mathematical convenience.

The coefficient f_{-1} plays a special role also in computations of generalized integrals by analytic continuation, as in the method of dimensional regularization. More details on this and other properties of generalized integrals can be found in the parallel work [10] and the aforementioned literature [16–18,22,27,29]. We remark that dimensional regularization was used in [10] to compute generalized integrals in Appendices B and C.

B The Bessel Equation

The *modified Bessel equation*,

$$\left(\partial_r^2 + \frac{1}{r}\partial_r - \frac{\alpha^2}{r^2} - 1\right) v(r) = 0, \tag{B.1}$$

has two kinds of standard solutions: the *modified Bessel function*, which can be defined by the power series

$$I_\alpha(r) = \sum_{n=0}^{\infty} \frac{\left(\frac{r}{2}\right)^{2n+\alpha}}{n!\Gamma(\alpha+n+1)}, \tag{B.2}$$

and at zero behaves as $\sim \frac{1}{\Gamma(\alpha+1)}\left(\frac{r}{2}\right)^\alpha$, and the Macdonald function, which for $\Re(r) > 0$ and all α can be defined by the absolutely convergent integral

$$K_{-\alpha}(r) = K_\alpha(r) := \frac{1}{2}\int_0^\infty \exp\left(-\frac{r}{2}(s+s^{-1})\right) s^{\alpha-1}\mathrm{d}s. \tag{B.3}$$

The Macdonald function can be characterized by its asymptotics at infinity: for $|\arg r| < \pi - \epsilon$, $\epsilon > 0$,

$$\lim_{|r|\to\infty} \frac{K_\alpha(r)}{\frac{\mathrm{e}^{-r}\sqrt{\pi}}{\sqrt{2r}}} = 1. \tag{B.4}$$

Note the connection formula

$$K_\alpha(r) = \frac{\pi}{2\sin\pi\alpha}(I_{-\alpha}(r) - I_\alpha(r)), \tag{B.5}$$

and an asymptotic expansion for $r \to \infty$ in the sector $|\arg r| < \pi - \epsilon$, $\epsilon > 0$ [25]:

$$K_\alpha(r) \simeq \sqrt{\pi}\mathrm{e}^{-r} \sum_{n=0}^{\infty} \frac{(\frac{1}{2}+\alpha-n)_{2n}}{n!(2r)^{n+\frac{1}{2}}}. \tag{B.6}$$

In the half-integer case, we have an expression in terms of elementary functions [25]:

$$K_{\pm(\frac{1}{2}+k)}(r) = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} r^{\frac{1}{2}+k}\left(-\frac{1}{r}\partial_r\right)^k \frac{\mathrm{e}^{-r}}{r}. \tag{B.7}$$

We will need the following bilinear integral identities for $\Re b > 0$ [15]:

$$\int_0^\infty K_\alpha(br)^2 2r\mathrm{d}r = \frac{\pi\alpha}{b^2\sin(\pi\alpha)}, \qquad \alpha \neq 0,\ |\Re(\alpha)| < 1, \tag{B.8}$$

$$\int_0^\infty K_0(br)^2 2r\mathrm{d}r = \frac{1}{b^2}. \tag{B.9}$$

If we replace the conditions $|\Re\alpha| < 1$, $\alpha \neq 0$ with $\alpha \notin \mathbb{Z}$ and in the integrals replace $\int$ with $\mathrm{gen}\int$, then (B.8) remains true. (B.8) can also be generalized to $\alpha \in \mathbb{Z}$ using anomalous generalized integrals [10]:

$$\mathrm{gen}\int_0^\infty K_\alpha(br)^2 2r\mathrm{d}r = \frac{(-1)^\alpha}{b^2}\Big(1 + |\alpha|\ln\big(\tfrac{b^2}{4}\big) + 2|\alpha|\big(1 - \psi(1+|\alpha|)\big)\Big). \tag{B.10}$$

The *(standard) Bessel equation* is obtained by setting $r \to \pm ir$ in the modified one:

$$\Big(\partial_r^2 + \frac{1}{r}\partial_r - \frac{\alpha^2}{r^2} + 1\Big) v(r) = 0. \tag{B.11}$$

We have several kinds of standard solutions of (B.11). The most important is the *Bessel function*, defined as

$$J_\alpha(r) = \mathrm{e}^{\pm \mathrm{i}\pi\frac{\alpha}{2}} I_\alpha(\mp \mathrm{i}r). \tag{B.12}$$

The two *Hankel functions* also solve (B.11):

$$H_\alpha^\pm(r) = \frac{2}{\pi}\mathrm{e}^{\mp\mathrm{i}\frac{\pi}{2}(\alpha+1)} K_\alpha(\mp \mathrm{i}r). \tag{B.13}$$

Remark B.1 In the literature, the usual notation for Hankel functions is

$$H_\alpha^{(1)}(r) = H_\alpha^+(r), \quad H_\alpha^{(2)}(r) = H_\alpha^-(r). \tag{B.14}$$

C The Gegenbauer Equation

The Gegenbauer equation is the special case of the hypergeometric equation with the symmetry $w \to -w$ and the singular points put at $-1, 1, \infty$:

$$\Big((1-w^2)\partial_w^2 - 2(1+\alpha)w\partial_w + \lambda^2 - \big(\alpha + \tfrac{1}{2}\big)^2\Big) f(w) = 0. \tag{C.1}$$

The Gegenbauer equation is closely related to the associated Legendre equation, see, for example, [15,25,30]. Moreover, there exist various conventions for the parameters of the Gegenbauer equation (cf. [11,12], [7,8]). The convention as in (C.1) is the most convenient for our purposes.

One of its standard solutions is the function the function $\mathbf{S}_{\alpha,\beta}$ characterized by asymptotics $\sim \frac{1}{\Gamma(\alpha+1)}$ at 1:

$$\mathbf{S}_{\alpha,\pm\lambda}(w) := {}_2\mathbf{F}_1\big(\tfrac{1}{2}+\alpha+\lambda, \tfrac{1}{2}+\alpha-\lambda; 1+\alpha; \tfrac{1-w}{2}\big) \tag{C.2}$$

$$= \Big(\tfrac{2}{w+1}\Big)^\alpha {}_2\mathbf{F}_1\big(\tfrac{1}{2}+\lambda, \tfrac{1}{2}-\lambda; 1+\alpha; \tfrac{1-w}{2}\big) \tag{C.3}$$

$$= \Big(\tfrac{2}{w+1}\Big)^{\frac{1}{2}+\alpha\pm\lambda} {}_2\mathbf{F}_1\big(\tfrac{1}{2}+\alpha\pm\lambda, \tfrac{1}{2}\pm\lambda; 1+\alpha; \tfrac{w-1}{w+1}\big), \tag{C.4}$$

where ${}_2\mathbf{F}_1(a,b;c;z) := \sum_{j=0}^{\infty} \frac{(a)_j(b)_j}{\Gamma(c+j)j!} z^j$ is the Gauß hypergeometric function in Olver's normalization. There is also the solution characterized by asymptotics $\sim \frac{1}{w^{\frac{1}{2}+\alpha+\lambda}\Gamma(\lambda+1)}$ at ∞:

$$\mathbf{Z}_{\alpha,\lambda}(w) := \frac{\Gamma(1+2\lambda)}{\Gamma(1+\lambda)(w\pm 1)^{\frac{1}{2}+\alpha+\lambda}} {}_2\mathbf{F}_1\big(\tfrac{1}{2}+\lambda, \tfrac{1}{2}+\lambda+\alpha; 1+2\lambda; \tfrac{2}{1\pm w}\big) \tag{C.5}$$

$$= \frac{\Gamma(1+2\lambda)}{\Gamma(\lambda+1)(w\pm 1)^{\frac{1}{2}+\lambda}(w\mp 1)^{\alpha}} {}_2\mathbf{F}_1\big(\tfrac{1}{2}+\lambda, \tfrac{1}{2}+\lambda-\alpha; 1+2\lambda; \tfrac{2}{1\pm w}\big). \tag{C.6}$$

The equality of the series representations (C.2), (C.3), (C.4), respectively (C.5), (C.6) follows from Kummer's table of hypergeometric functions (see, e.g., [25]). In fact, only (C.2), (C.3) and the expressions with the +-sign in (C.5), (C.6) are convergent in the whole region of physical interest.

It is convenient to introduce the notation

$$(w^2-1)_\bullet^\alpha := (w-1)^\alpha (w+1)^\alpha, \tag{C.7}$$

where $(w\mp 1)^\alpha$ are the usual principal branches with the domains $\mathbb{C}\backslash]-\infty, \pm 1]$. We note the identities

$$\mathbf{S}_{\alpha,\lambda}(w) = \mathbf{S}_{\alpha,-\lambda}(w), \quad \mathbf{Z}_{\alpha,\lambda}(w) = \frac{\mathbf{Z}_{-\alpha,\lambda}(w)}{(w^2-1)_\bullet^\alpha} \tag{C.8}$$

as well as the slightly more subtle *Whipple transformations*:

$$\mathbf{Z}_{\alpha,\lambda}(w) := (w^2-1)_\bullet^{-\frac{1}{4}-\frac{\alpha}{2}-\frac{\lambda}{2}} \mathbf{S}_{\lambda,\alpha}\left(\frac{w}{(w^2-1)_\bullet^{\frac{1}{2}}}\right), \tag{C.9}$$

$$\mathbf{S}_{\alpha,\lambda}(w) := (w^2-1)_\bullet^{-\frac{1}{4}-\frac{\alpha}{2}-\frac{\lambda}{2}} \mathbf{Z}_{\lambda,\alpha}\left(\frac{w}{(w^2-1)_\bullet^{\frac{1}{2}}}\right), \qquad \Re w > 0. \tag{C.10}$$

For $n = 0, 1, \ldots,$ we define the *Gegenbauer polynomials*:

$$C_n^{\alpha+\frac{1}{2}}(w) := \frac{\Gamma(\alpha+1)(2\alpha+1)_n}{n!} \mathbf{S}_{\alpha,\frac{1}{2}+\alpha+n}(w). \tag{C.11}$$

Here are the connection formulas:

$$\begin{aligned}\mathbf{S}_{\alpha,\lambda}(-w) = &-\frac{\cos(\pi\lambda)}{\sin(\pi\alpha)}\mathbf{S}_{\alpha,\lambda}(w)\\ &+\frac{2^{2\alpha}\pi}{\sin(\pi\alpha)\Gamma(\frac{1}{2}+\alpha+\lambda)\Gamma(\frac{1}{2}+\alpha-\lambda)}\frac{\mathbf{S}_{-\alpha,-\lambda}(w)}{(1-w^2)^\alpha},\end{aligned} \tag{C.12}$$

$$\begin{aligned}\mathbf{Z}_{\alpha,\lambda}(w) = &-\frac{2^{\lambda-\alpha-\frac{1}{2}}\sqrt{\pi}}{\sin(\pi\alpha)\Gamma(\frac{1}{2}-\alpha+\lambda)}\mathbf{S}_{\alpha,\lambda}(w)\\ &+\frac{2^{\lambda+\alpha-\frac{1}{2}}\sqrt{\pi}}{\sin(\pi\alpha)\Gamma(\frac{1}{2}+\alpha+\lambda)}\frac{\mathbf{S}_{-\alpha,-\lambda}(w)}{(w^2-1)_\bullet^\alpha}.\end{aligned} \tag{C.13}$$

For $\Re\alpha > -1$, we can compute the following generalized bilinear integrals of $\mathbf{S}$ functions. For $\alpha \notin \mathbb{Z}$, they are non-anomalous [10]:

$$\mathrm{gen}\int_{-2}^{2} \mathbf{S}_{\alpha,\mathrm{i}\beta}(w)^2(1-w^2)^\alpha \mathrm{d}2w = \frac{2^{2\alpha+1}\mathrm{i}\cosh(\pi\beta)}{\beta\sin\pi\alpha\,\Gamma(\frac12+\alpha-\mathrm{i}\beta)\Gamma(\frac12+\alpha+\mathrm{i}\beta)} \times\Big(\psi\big(\tfrac12+\alpha+\mathrm{i}\beta\big)-\psi\big(\tfrac12+\alpha-\mathrm{i}\beta\big)+\psi\big(\tfrac12-\mathrm{i}\beta\big)-\psi\big(\tfrac12+\mathrm{i}\beta\big)\Big), \quad (C.14)$$

$$\mathrm{gen}\int_{-2}^{2} \mathbf{S}_{\alpha,0}(w)^2(1-w^2)^\alpha \mathrm{d}2w = \frac{2^{2\alpha+1}\big(\pi^2-2\psi'\big(\frac12+\alpha\big)\big)}{\sin(\pi\alpha)\Gamma(\frac12+\alpha)^2}, \quad (C.15)$$

For $|\Re\alpha| < 1$, the integrals (C.14) and (C.15) are standard. For $\alpha \in \mathbb{N}$, we have anomalous generalized integrals:

$$\begin{aligned}&\mathrm{gen}\int_{-2}^{2} \mathbf{S}_{\alpha,\mathrm{i}\beta}(w)^2(1-w^2)^\alpha \mathrm{d}2w\\ &= \frac{(-1)^\alpha 2^{2\alpha+2}\cosh(\pi\beta)}{\pi\Gamma(\frac12+\alpha+\mathrm{i}\beta)\Gamma(\frac12+\alpha-\mathrm{i}\beta)}\Bigg(\frac{\mathrm{i}}{2\beta}\Big(\psi'\big(\tfrac12+\alpha+\mathrm{i}\beta\big)-\psi'\big(\tfrac12+\alpha-\mathrm{i}\beta\big)\Big)\\ &\quad-\frac{\mathrm{i}}{2\beta}\Big(H_\alpha\big(\tfrac12+\mathrm{i}\beta\big)-H_\alpha\big(\tfrac12-\mathrm{i}\beta\big)\Big)\ln 4\\ &\quad+\sum_{k=0}^{\alpha-1}\frac{\psi\big(-\frac12-k+\mathrm{i}\beta\big)+\psi\big(-\frac12-k-\mathrm{i}\beta\big)-\psi(\alpha-k)-\psi(1+k)}{\big(\frac12+k\big)^2+\beta^2}\Bigg),\end{aligned} \quad (C.16)$$

$$\begin{aligned}\mathrm{gen}\int_{-2}^{2} \mathbf{S}_{\alpha,0}(w)^2(1-w^2)^\alpha \mathrm{d}2w &= \frac{2^{2\alpha+2}(-1)^\alpha}{\pi\Gamma\big(\frac12+\alpha\big)^2}\Bigg(-\psi''\big(\tfrac12+\alpha\big)+H'_\alpha\big(\tfrac12\big)\ln 4\\ &+\sum_{k=0}^{\alpha-1}\frac{2\psi\big(-\frac12-k\big)-\psi(\alpha-k)-\psi(1+k)}{\big(\frac12+k\big)^2}\Bigg).\end{aligned} \quad (C.17)$$

If $\alpha = 0$, then (C.16) and (C.17) are still true as standard integrals. Besides, they greatly simplify:

$$\int_{-2}^{2} \mathbf{S}_{0,\mathrm{i}\beta}(w)^2 \mathrm{d}2w = \frac{2\mathrm{i}\cosh^2(\pi\beta)\Big(\psi'(\frac12+\mathrm{i}\beta)-\psi'(\frac12-\mathrm{i}\beta)\Big)}{\beta\pi^2}, \quad (C.18)$$

$$\int_{-2}^{2} \mathbf{S}_{0,0}(w)^2 \mathrm{d}2w = -\frac{4\psi''\big(\frac12\big)}{\pi^2}. \quad (C.19)$$

Here are generalized integrals of squares of $\mathbf{Z}$ functions for $\Re\lambda > 0$, as computed in [10]. For $\alpha \in \mathbb{C}\setminus\mathbb{Z}$, we have non-anomalous integrals.

$$\mathrm{gen}\int_{2}^{\infty} \mathbf{Z}_{\alpha,\lambda}(w)^2(w^2-1)^\alpha \mathrm{d}2w = \frac{2^{2\lambda}(\psi(\frac12+\alpha+\lambda)-\psi(\frac12-\alpha+\lambda))}{\lambda\sin\pi\alpha\Gamma(\frac12-\alpha+\lambda)\Gamma(\frac12+\alpha+\lambda)}. \quad (C.20)$$

For $|\Re\alpha| < 1$, (C.20) is a standard integral.

For $\alpha \in \mathbb{Z}\backslash\{0\}$ and $\Re\lambda > 0$, we have anomalous integrals:

$$\operatorname{gen}\int_2^\infty \mathbf{Z}_{\alpha,\lambda}(w)^2(w^2-1)^\alpha \mathrm{d}2w = \frac{(-1)^\alpha 2^{2\lambda+1}}{\pi\Gamma\big(\frac12-\alpha+\lambda\big)\Gamma\big(\frac12+\alpha+\lambda\big)}$$
$$\times\left(\frac{\psi'(\frac12-\alpha+\lambda)+\psi'(\frac12+\alpha+\lambda)}{2\lambda}+\frac{H_{|\alpha|}(\frac12-\lambda)-H_{|\alpha|}(\frac12+\lambda)}{2\lambda}\ln 4\right.$$
$$\left.+\sum_{k=0}^{|\alpha|-1}\frac{\psi(\frac32+k+\lambda)+\psi(-\frac12-k+\lambda)-\psi(|\alpha|-k)-\psi(1+k)}{\lambda^2-\big(\frac12+k\big)^2}\right). \tag{C.21}$$

If $\alpha = 0$, then (C.21) is still true in the sense of standard integrals. Besides, it greatly simplifies:

$$\int_2^\infty \mathbf{Z}_{0,\lambda}(w)^2 \mathrm{d}2w = \frac{2^{2\lambda+1}\psi'(\frac12+\lambda)}{\pi\lambda\Gamma(\frac12+\lambda)^2}. \tag{C.22}$$

The Gegenbauer functions have the following asymptotics [10] (see also [25, 26]):

Theorem C.1 *Let $\alpha \geq -\frac12$ and $\pi > \delta > 0$ be fixed. Then we have the following estimates:*

1. *Uniformly for $\theta \in [0, \pi-\delta]$ and $\beta \to \infty$,*

$$\frac{\pi \mathrm{e}^{-\pi\beta}(\sin\theta)^{\alpha+\frac12}}{2^\alpha\theta^{\alpha+\frac12}}\mathbf{S}_{\alpha,\pm\mathrm{i}\beta}(-\cos\theta) = (\theta\beta)^{-\alpha}K_\alpha(\beta\theta)\big(1+\mathcal{O}(\beta^{-1})\big). \tag{C.23}$$

2. *Uniformly for $\theta \geq 0$ and $\lambda \to \infty$,*

$$\frac{\sqrt{\pi}\Gamma(\frac12-\alpha+\lambda)(\sinh\theta)^{\alpha+\frac12}}{2^{\lambda+\frac12}\theta^{\alpha+\frac12}}\mathbf{Z}_{\alpha,\lambda}(\cosh\theta) = (\lambda\theta)^{-\alpha}K_\alpha(\lambda\theta)\big(1+\mathcal{O}(\lambda^{-1})\big). \tag{C.24}$$

Correspondingly, the bilinear generalized integrals of Gegenbauer functions that are needed to determine the Green's functions of point-like perturbations have the expected asymptotics [10]:

Theorem C.2 *For $\beta, \lambda \to \infty$, we have*

$$\frac{\pi^2\mathrm{e}^{-2\pi\beta}\beta^{2\alpha}}{2^{2\alpha}}\operatorname{gen}\int_{-2}^2 \mathbf{S}_{\alpha,\mathrm{i}\beta}(w)^2(1-w^2)^\alpha \mathrm{d}2w$$
$$= \Big(1+\mathcal{O}\big(\tfrac1\beta\big)\Big)\operatorname{gen}\int_0^\infty K_\alpha(\beta r)^2 2r\mathrm{d}r, \qquad \Re(\alpha) > -1, \tag{C.25}$$

$$\frac{\pi\Gamma\big(\frac12+\alpha+\lambda\big)^2}{2^{2\lambda+1}\lambda^{2\alpha}}\operatorname{gen}\int_2^\infty \mathbf{Z}_{\alpha,\lambda}(w)^2(w^2-1)^\alpha \mathrm{d}2w$$
$$= \Big(1+\mathcal{O}\big(\tfrac1\lambda\big)\Big)\operatorname{gen}\int_0^\infty K_\alpha(\lambda r)^2 2r\mathrm{d}r, \qquad \alpha \in \mathbb{C}. \tag{C.26}$$

D Pochhammer Symbols and Harmonic Numbers

The *Pochhammer symbol*, as it is usually defined, is a generalization of the factorial:

$$(a)_n := a(a+1)\cdots(a+n-1), \qquad (1)_n = n!. \tag{D.1}$$

We will also use the *harmonic numbers*

$$H_n(a) := \frac{1}{a} + \frac{1}{a+1} + \cdots + \frac{1}{a+n-1}, \qquad H_n{:=}H_n(1). \tag{D.2}$$

Sometimes it is convenient to extend the definitions (D.1) and (D.2) to complex parameters z:

$$(a)_z{:=}\frac{\Gamma(a+z)}{\Gamma(a)}, \quad H_z(a){:=}\psi(a+z)-\psi(a), \quad z \in \mathbb{C}\setminus(-a-\mathbb{N}_0). \tag{D.3}$$

We have

$$\partial_a (a)_z = (a)_z H_z(a). \tag{D.4}$$

For $n \in \mathbb{N}$, we have the useful identities

$$(-1)^n(\tfrac12 - a)_n(\tfrac12 + a)_n = (a-n+\tfrac12)_{2n} = \prod_{j=0}^{n-1}\Big(a^2 - (\tfrac12 + j)^2\Big), \tag{D.5}$$

$$(-1)^n(\tfrac12 - a)_{n+\frac12}(\tfrac12 + a)_{n+\frac12} = \cot(\pi a)(a-n)_{2n+1} = \cot(\pi a) a \prod_{j=1}^{n}\Big(a^2 - j^2\Big). \tag{D.6}$$

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