Confluent functions, Laguerre polynomials and their (generalized) bilinear integrals

Jan Dereziński[1], Christian Gaß[1], and Joonas Mikael Vättö[2]

[1]Department of Mathematical Methods in Physics, Faculty of Physics,
University of Warsaw, Pasteura 5, 02-093 Warszawa, Poland,

email: jan.derezinski@fuw.edu.pl, christian.gass@fuw.edu.pl

[2]Department of Mathematics and Systems Analysis,
Aalto University, Otakaari 1, 02150 Espoo, Finland

email: joonas.vatto@aalto.fi

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Abstract

We review properties of confluent functions and the closely related Laguerre polynomials, and determine their bilinear integrals. As is well-known, these integrals are convergent only for a limited range of parameters. However, when one uses *the generalized integral* they can be computed essentially without restricting the parameters. This gives the (generalized) Gram matrix of Laguerre polynomials. If the parameters are not negative integers, then Laguerre polynomials are orthogonal, or at least pseudo-orthogonal in the case of generalized integrals. For negative integer parameters, the orthogonality relations are more complicated.

Keywords: special functions, orthogonal polynomials, regularization

1 Introduction

Consider the (formal) differential operator

$$\mathcal{F}_\alpha := -z\partial_z^2 - (1 + \alpha - z)\partial_z. \tag{1.1}$$

Kummer's confluent function $F_{\theta,\alpha}(z) := {}_1F_1(a; c; z)$ is an eigenfunction of $\mathcal{F}_\alpha$ with eigenvalue $\frac{1+\theta+\alpha}{2}$, where following [4, 5] instead of a, c we prefer to use the parameters

$$\alpha := c - 1, \qquad \theta := -c + 2a. \tag{1.2}$$

Besides Kummer's function, (1.1) possesses a second type of distinguished eigenfunction with eigenvalues $\frac{1+\theta+\alpha}{2}$, called *Tricomi's confluent functions*, which we denote $U(a; c; z)$ or, in the parameters (1.2), $U_{\theta,\alpha}(z)$. Tricomi's functions usually have a better behavior near infinity than Kummer's functions.

The main topic of this paper is the computation of bilinear integrals of Tricomi functions, more precisely, integrals of the form

$$\int_0^\infty U_{\theta_1,\alpha}(z)U_{\theta_2,\alpha}(z)\mathrm{e}^{-z}z^\alpha\mathrm{d}z. \tag{1.3}$$

It is known that for $|\mathrm{Re}\alpha| < 1$ (1.3) can be computed explicitly, see e.g. [6]. For $|\mathrm{Re}\alpha| \geq 1$, (1.3) is divergent — except for very specific combinations of the parameters — because of the behavior of the integrand near 0. However, when one applies the so-called *generalized integral*, one can compute (1.3) for all values of α. For $\alpha \in \mathbb{Z}$, $\alpha \neq 0$ one needs to apply the anomalous generalized integral, and one obtains a much more complicated expression.

For special values of the parameters, Tricomi's functions coincide up to a coefficient with the well-known Laguerre polynomials. Their classical definition uses the so-called *Rodriguez formula*

$$L_n^\alpha(z) := \frac{1}{n!}\mathrm{e}^z z^{-\alpha}\partial_z^n \mathrm{e}^{-z}z^{n+\alpha}, \quad n \in \mathbb{N}_0, \tag{1.4}$$

and applies to all $\alpha \in \mathbb{C}$. They are, however, orthogonal with respect to the positive definite measure $\mathrm{d}\mu(z) = z^\alpha\mathrm{e}^{-z}\mathrm{d}z$ on $L^2[0,\infty[$ only if $\alpha > -1$.

The orthogonality relations for Laguerre polynomials, valid for $\mathrm{Re}\alpha > -1$,

$$\int_0^\infty L_m^\alpha(z)L_n^\alpha(z)z^\alpha\mathrm{e}^{-z}\mathrm{d}z = \frac{\Gamma(1+n+\alpha)}{n!}\delta_{m,n}, \tag{1.5}$$

belong to the standard knowledge. Again, using the generalized integral, we extend the relations (1.5) to all $\alpha \in \mathbb{C}$. For $-\alpha \notin \mathbb{N}$ we still have the same expression as in (1.5). In particular, L_n^α are (pseudo-)orthogonal to one another for $m \neq n$. The situation is more complicated for $-\alpha \in \mathbb{N}$, where the generalized integral becomes anomalous.

One can derive the bilinear integrals of Laguerre polynomials from those of Tricomi's functions. However, we prefer a direct method, which mimics the usual proof of the orthogonality relations based on the Rodriguez formula and repeated integration by parts, known from standard textbooks. Interestingly, integration by parts works quite well also for generalized integrals. This is not obvious, so we dedicate a subsection in Appendix A to the description of integration by parts in the context of generalized integrals.

For $\alpha \notin [-1,\infty[$ (typically real, preferably integer), Laguerre polynomials are of interest in several fields of mathematics, see for example [24, 25, 26, 14]. Apparently, the value $\alpha = -1$ is especially interesting in applications [2].

In a recent work [7], some of us (with collaborators) used the generalized integral, a concept that goes back to independent considerations of Riesz [23] and Hadamard [9, 10], to give a meaning to integrals over classically non-integrable functions. The generalized integral is a linear continuation of the standard integral. It is closely related to the extension of homogeneous distributions [11, 22] and the so-called barred integral [18]. We find it interesting that replacing the integral by the generalized integral, we are able to compute bilinear integrals of the form (1.3) and (1.5) for arbitrary complex α.

Our paper is not the first to address generalized orthogonality relations of the classical orthogonal polynomials. There is some literature on the subject [12, 15, 16, 17, 19, 13], where generalized orthogonality relations for some $\alpha \in \mathbb{C}$ with $\mathrm{Re}(\alpha) < -1$ are often described by changing the integral measure, typically allowing for distributional measures. The generalized integral is a natural tool that allows us to treat all α simultaneously. We have not seen such a comprehensive description in the literature.

Here is the plan of our paper. Section 2 is devoted to the confluent equation for general parameters. First we recall its basic theory, following mostly [4, 5]. In particular, we review the definitions of four standard solutions of the confluent equation. Three of them typically blow up at infinity, and only one of them, the Tricomi function, can be used in bilinear (generalized) integrals of the form (1.3). The main new result of this section are the expressions for these integrals for all parameters.

Section 3 is devoted to Laguerre polynomials. Again, we start with a concise exposition of their theory. Then we compute their bilinear integrals. We use two methods. Our first method is based on integration by parts, which interestingly works well also in the case of generalized integrals. Then we give an alternative derivation: we show how bilinear integrals of Laguerre polynomials can be obtained as special cases of bilinear integrals of Tricomi functions obtained in Section 2.

In Appendix A we give a resumé of properties of generalized integrals based mostly on [7]. The appendix contains also a new fact not discussed in [7]: the integration by parts property of generalized integrals. Appendix B contains proofs of two identities for the digamma functions and the Pochhammer symbols needed in our analysis.

2 Confluent equation

2.1 Hypergeometric ${}_1F_1$ function

(Kummer's) confluent function, also called the *hypergeometric ${}_1F_1$ function*, is defined by the following series convergent in $\mathbb{C}$:

$$ {}_1F_1(a;c;z) := \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} \frac{z^n}{n!}. \tag{2.1} $$

It is a confluent form of the ${}_2F_1$ function

$$ {}_1F_1(a;c;z) = \lim_{b\to\infty} {}_2F_1\big(a,b;c;\tfrac{z}{b}\big), \tag{2.2} $$

which satisfies the *confluent quation*

$$ (z\partial_z^2 + (c-z)\partial_z - a)\,{}_1F_1(a;c;z) = 0 \tag{2.3} $$

and the *1st Kummer identity*

$$ {}_1F_1(a;c;z) = \mathrm{e}^z\,{}_1F_1(c-a;c;-z). \tag{2.4} $$

It is often preferable to replace it by the so-called *Olver's normalized ${}_1F_1$ function*

$$ {}_1\mathbf{F}_1(a;c;z) := \frac{{}_1F_1(a;c;z)}{\Gamma(c)} = \sum_{n=0}^{\infty} \frac{(a)_n}{\Gamma(c+n)} \frac{z^n}{n!}. \tag{2.5} $$

Here are its integral representations: for all parameters

$$\frac{1}{2\pi \mathrm{i}} \int\limits_{]-\infty,(0,z)^+,-\infty[} t^{a-c}\mathrm{e}^t(t-z)^{-a}\mathrm{d}t \quad = \quad {}_1\mathbf{F}_1(a;c;z), \tag{2.6}$$

for $\mathrm{Re}a > 0,\ \mathrm{Re}(c-a) > 0$,

$$\frac{1}{\Gamma(a)\Gamma(c-a)} \int\limits_{[1,+\infty[} \mathrm{e}^{\frac{z}{t}}t^{-c}(t-1)^{c-a-1}\mathrm{d}t \quad = \quad {}_1\mathbf{F}_1(a;c;z), \tag{2.7}$$

and for $\mathrm{Re}(c-a) > 0$ and $a \notin \mathbb{N}$,

$$\frac{\Gamma(1-a)}{2\pi \mathrm{i}\Gamma(c-a)} \int\limits_{[1,0^+,1]} \mathrm{e}^{\frac{z}{t}}(-t)^{-c}(-t+1)^{c-a-1}\mathrm{d}t \quad = \quad {}_1\mathbf{F}_1(a;c;z). \tag{2.8}$$

The meaning of the contours in (2.6) and (2.8) is the following. In (2.6), we start at $-\infty$, go around both 0 and z in the positive sense and go back to $-\infty$. Similarly, in (2.8), we start from 1, go around 0 in the positive sense and go back to 1.

For integer m we have expressions in terms of elementary functions:

$$ {}_1F_1(c+m;c;z) = \frac{z^{1-c}}{(c)_m}\partial_z^m z^{c-1+m}\mathrm{e}^z, \tag{2.9}$$

$$ {}_1\mathbf{F}_1(c+m;c;z) = \frac{z^{1-c}}{\Gamma(c+m)}\partial_z^m z^{c-1+m}\mathrm{e}^z. \tag{2.10}$$

Here, to define ∂_z^m for negative m we use

$$\partial_z^{-1} f(z) := \mathrm{gen}\int_0^z f(y)\mathrm{d}y, \tag{2.11}$$

provided that (2.11) is non-anomalous and that the function is integrable in the generalized sense often enough.

2.2 Hypergeometric ${}_2F_0$ function

The following function arises as a result of a different confluence of singularities of the hypergeometric function. It is defined for $w \in \mathbb{C}\backslash[0,+\infty[$ by

$$ {}_2F_0(a,b;-;w) := \lim_{c\to\infty} {}_2F_1(a,b;c;cw),$$

where $|\arg c - \pi| < \pi - \epsilon$, $\epsilon > 0$. It extends to an analytic function on the universal cover of $\mathbb{C}\backslash\{0\}$ with a branch point of an infinite order at 0. It has the following asymptotic expansion:

$$ {}_2F_0(a,b;-;w) \sim \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{n!}w^n, \ |\arg w - \pi| < \pi - \epsilon.$$

It satisfies the equation

$$\left(w^2\partial_w^2 + (-1+(1+a+b)w)\partial_w + ab\right){}_2F_0(a,b;-;w) = 0, \tag{2.12}$$

which we will call the ${}_2F_0$ *equation*.

We have an integral representation for $\mathrm{Re}a > 0$,

$$\frac{1}{\Gamma(a)} \int_0^\infty \mathrm{e}^{-\frac{1}{t}} t^{b-a-1} (t-w)^{-b} \mathrm{d}t = F(a,b;-;w), \quad w \notin [0,\infty[,$$

and for $a \notin \mathbb{N}$,

$$\frac{\Gamma(1-a)}{2\pi\mathrm{i}} \int\limits_{[0,w^+,0]} \mathrm{e}^{-\frac{1}{t}} t^{b-a-1} (t-w)^{-b} \mathrm{d}t = F(a,b;-;w), \quad w \notin [0,\infty[.$$

2.3 Equivalence between the ${}_1F_1$ and ${}_2F_0$ equations

It is easy to derive the following identity:

$$(-w)^{a+1}\Big(w^2\partial_w^2 + (-1+(1+a+b)w)\partial_w + ab\Big)(-w)^{-a} = \ z\partial_z^2 + (1+a-b-z)\partial_z - a,$$

where $z = -w^{-1}$, $w = -z^{-1}$. Hence the ${}_2F_0$ equation is equivalent to the ${}_1F_1$ equation with the following relation between the parameters

$$c = 1+a-b, \quad b = 1+a-c.$$

Because of this equivalence, *Tricomi's confluent function*

$$U(a,c,z) := z^{-a}{}_2F_0(a, 1+a-c; -; -z^{-1})$$

is one of solutions of the ${}_1F_1$ equation,

2.4 Confluent equation in Lie-algebraic parameters

Instead of the classical parameters we usually prefer the Lie-algebraic parameters θ, α:

$$\alpha := c-1 = a-b, \qquad \theta := -c+2a = -1+a+b;$$

$$a = \tfrac{1+\alpha+\theta}{2}, \qquad b = \tfrac{1-\alpha+\theta}{2}, \qquad c = 1+\alpha.$$

Here are the ${}_1F_1$ and ${}_2F_0$ equations in the Lie-algebraic parameters:

$$\Big(z\partial_z^2 + (1+\alpha-z)\partial_z - \frac{1}{2}(1+\theta+\alpha)\Big)f(z) = 0, \tag{2.13}$$

$$\Big(w^2\partial_w^2 + \big(-1+(2+\theta)w\big)\partial_w + \big(\tfrac{1+\theta}{2}\big)^2 - \big(\tfrac{\alpha}{2}\big)^2\Big)f(w) = 0. \tag{2.14}$$

We introduce the ${}_1F_1$ *operator* and the ${}_2F_0$ *operator*:

$$\mathcal{F}_\alpha(z,\partial_z) := -z\partial_z^2 - (1+\alpha-z)\partial_z \tag{2.15}$$

$$= -z^{-\alpha}\mathrm{e}^z\partial_z z^{1+\alpha}\mathrm{e}^{-z}\partial_z, \tag{2.16}$$

$$\tilde{\mathcal{F}}_\theta(z,\partial_w) := -w^2\partial_w^2 - \big(-1+(2+\theta)w\big)\partial_w \tag{2.17}$$

$$= -w^{-\theta}\mathrm{e}^{-w^{-1}}\partial_w w^{2+\theta}\mathrm{e}^{w^{-1}}\partial_w. \tag{2.18}$$

The ${}_1F_1$ equation and the ${}_2F_0$ equation can be interpreted as the eigenequation of the corresponding operators:

$$\Big(-\mathcal{F}_\alpha(z,\partial_z)-\frac{1}{2}(1+\theta+\alpha)\Big)f(z)=0, \tag{2.19}$$

$$\Big(-\tilde{\mathcal{F}}_\theta(w,\partial_w)+\big(\tfrac{1+\theta}{2}\big)^2-\big(\tfrac{\alpha}{2}\big)^2\Big)f(w)=0. \tag{2.20}$$

We will treat the ${}_1F_1$ confluent equation as the principal one.

Here are Kummer's and Tricomi's confluent functions in the Lie-algebraic parameters:

$$F_{\theta,\alpha}(z):={}_1F_1\Big(\frac{1+\alpha+\theta}{2};1+\alpha;z\Big), \tag{2.21}$$

$$\mathbf{F}_{\theta,\alpha}(z):={}_1\mathbf{F}_1\Big(\frac{1+\alpha+\theta}{2};1+\alpha;z\Big)=\frac{1}{\Gamma(\alpha+1)}F_{\theta,\alpha}(z), \tag{2.22}$$

$$U_{\theta,\alpha}(z):=U\big(\tfrac{1+\alpha+\theta}{2};1+\alpha;z\big)=z^{\frac{-1-\alpha-\theta}{2}}{}_2F_0\big(\tfrac{1+\alpha+\theta}{2},\tfrac{1-\alpha+\theta}{2};-;-z^{-1}\big). \tag{2.23}$$

We have four standard solutions of the ${}_1F_1$ equation:

$$F_{\theta,\alpha}(z)=\mathrm{e}^zF_{-\theta,\alpha}(-z), \qquad \sim 1 \text{ at } 0; \tag{2.24}$$

$$z^{-\alpha}F_{\theta,-\alpha}(z)=z^{-\alpha}\mathrm{e}^zF_{-\theta,-\alpha}(-z), \qquad \sim z^{-\alpha} \text{ at } 0; \tag{2.25}$$

$$U_{\theta,\alpha}(z)=z^{-\alpha}U_{\theta,-\alpha}(z), \qquad \sim z^{-a} \text{ at } +\infty; \tag{2.26}$$

$$\mathrm{e}^zU_{-\theta,\alpha}(-z)=(-z)^{-\alpha}\mathrm{e}^zU_{-\theta,-\alpha}(-z), \qquad \sim (-z)^{b-1}\mathrm{e}^z \text{ at } -\infty. \tag{2.27}$$

2.5 Connection formulas

The space of solutions of the confluent equation is 2-dimensional. Therefore, the Tricomi function for $\alpha\notin\mathbb{Z}$ can be expressed in terms of Kummer's functions:

$$U_{\theta,\alpha}(z)=\frac{\pi}{\sin\pi\alpha}\left(-\frac{\mathbf{F}_{\theta,\alpha}(z)}{\Gamma\left(\frac{1+\theta-\alpha}{2}\right)}+\frac{z^{-\alpha}\mathbf{F}_{\theta,-\alpha}(z)}{\Gamma\left(\frac{1+\theta+\alpha}{2}\right)}\right), \tag{2.28}$$

$$\mathrm{e}^zU_{-\theta,\alpha}(-z)=\frac{\pi}{\sin\pi\alpha}\left(-\frac{\mathbf{F}_{\theta,\alpha}(z)}{\Gamma\left(\frac{1-\theta-\alpha}{2}\right)}+\frac{(-z)^{-\alpha}\mathbf{F}_{\theta,-\alpha}(z)}{\Gamma\left(\frac{1-\theta+\alpha}{2}\right)}\right). \tag{2.29}$$

2.6 Degenerate case

The case of integer α is called degenerate. For $\alpha=m\in\mathbb{Z}$ we have the identity

$$\big(\tfrac{1-m+\theta}{2}\big)_mF_{\theta,m}(z)=z^{-m}F_{\theta,-m}(z). \tag{2.30}$$

We also have additional integral representations:

$$\frac{1}{2\pi\mathrm{i}}\int\limits_{[(z,0)^+]}\mathrm{e}^t\Big(1-\frac{z}{t}\Big)^{-a}t^{-m-1}\mathrm{d}t \;=\; \mathbf{F}_{-1+2a-m,m}(z), \tag{2.31}$$

$$\frac{1}{2\pi\mathrm{i}}\int\limits_{[(0,1)^+]}\exp\Big(\frac{z}{t}\Big)(1-t)^{-a}t^{-m-1}\mathrm{d}t \;=\; z^{-m}\mathbf{F}_{-1+2a+m,-m}(z). \tag{2.32}$$

There are also corresponding generating functions

$$\mathrm{e}^t\big(1-\tfrac{z}{t}\big)^{-a} \;=\; \sum_{m\in\mathbb{Z}} t^m \mathbf{F}_{-1+2a-m,m}(z), \tag{2.33}$$

$$\exp\big(\tfrac{z}{t}\big)(1-t)^{-a} \;=\; \sum_{m\in\mathbb{Z}} t^m z^{-m}\mathbf{F}_{-1+2a+m,-m}(z). \tag{2.34}$$

The Tricomi function for $\alpha = m \in \mathbb{Z}$ has a logarithmic singularity:

$$\begin{aligned} U_{\theta,m}(z) =& \frac{(-1)^{m+1}}{\Gamma(\frac{1+\theta-m}{2})}\Bigg(\sum_{k=1}^{m}(-1)^{k-1}\frac{(k-1)!\big(\frac{1+m+\theta}{2}\big)_{-k}}{(m-k)!}z^{-k} \\ &+\sum_{j=0}^{\infty}\frac{(\frac{1+m+\theta}{2})_j}{(m+j)!j!}\Big(\ln(z)+\psi\big(\tfrac{1+m+\theta}{2}+j\big)-\psi(j+1)-\psi(j+m+1)\Big)z^j\Bigg). \end{aligned} \tag{2.35}$$

2.7 Relationship to the modified Bessel equation

The confluent equation for $\theta = 0$ is equivalent to the modified Bessel equation (hence also to the Bessel equation):

$$\begin{aligned} &2z^{\nu-1}\mathrm{e}^{-\frac{z}{2}}\Big(z\partial_z^2+(1+2\nu-z)\partial_z-\tfrac{1}{2}-\nu\Big)z^{-\nu}\mathrm{e}^{\frac{z}{2}} \\ &= \partial_r^2+\frac{1}{r}\partial_r-1-\frac{\nu^2}{r^2}, \qquad z=2r, \quad \alpha=2\nu. \end{aligned} \tag{2.36}$$

The standard solutions of the confluent equation can be expressed in terms of standard solutions of the modified Bessel equation, that is, the modified Bessel function and the Macdonald function:

$$\begin{aligned} I_\nu(r) &= \frac{1}{\Gamma(\nu+1)}\Big(\frac{r}{2}\Big)^{\nu}\mathrm{e}^{-r}{}_1F_1\Big(\nu+\frac{1}{2};2\nu+1;2r\Big) \\ &= \frac{\Gamma(2\nu+1)}{\Gamma(\nu+1)}\Big(\frac{r}{2}\Big)^{\nu}\mathrm{e}^{-r}\mathbf{F}_{0,2\nu}(2r) \end{aligned} \tag{2.37}$$

$$\begin{aligned} K_\nu(r) &= \sqrt{\frac{\pi}{2r}}\mathrm{e}^{-r}{}_2F_0\Big(\frac{1}{2}+\nu,\frac{1}{2}-\nu;-;-\frac{1}{2r}\Big) \\ &= \sqrt{\pi}(2r)^{\nu}\mathrm{e}^{-r}U_{0,2\nu}(2r). \end{aligned} \tag{2.38}$$

2.8 Bilinear integrals

Let us start with the usual bilinear integrals of Tricomi's functions. The following identities are known, and can be found e.g. in [6] (where instead of Kummer's and Tricomi's confluent functions the equivalent Whittaker functions of the first and second kind are used).

Theorem 2.1. *For* $|\mathrm{Re}(\alpha)|<1$*, the following identities hold:*

$$\begin{aligned} &\int_0^\infty U_{\theta_1,\alpha}(z)U_{\theta_2,\alpha}(z)\mathrm{e}^{-z}z^{\alpha}\mathrm{d}z \\ &= \frac{2\pi}{(\theta_1-\theta_2)\sin\pi\alpha}\left(\frac{1}{\Gamma(\frac{1+\theta_1-\alpha}{2})\Gamma(\frac{1+\theta_2+\alpha}{2})}-(\theta_1\leftrightarrow\theta_2)\right), \quad \theta_1\neq\theta_2, \end{aligned} \tag{2.39}$$

and

$$\int_0^\infty U_{\theta,\alpha}(z)^2 \mathrm{e}^{-z} z^\alpha \mathrm{d}z = \frac{\pi}{\sin\pi\alpha}\left(\frac{\psi(\frac{1+\theta+\alpha}{2}) - \psi(\frac{1+\theta-\alpha}{2})}{\Gamma(\frac{1+\theta+\alpha}{2})\Gamma(\frac{1+\theta-\alpha}{2})}\right). \tag{2.40}$$

In the special case $\alpha = 0$, we have

$$\int_0^\infty U_{\theta_1,0}(z)U_{\theta_2,0}(z)\mathrm{e}^{-z}\mathrm{d}z = \frac{2}{\theta_1-\theta_2}\frac{\psi\big(\frac{1+\theta_1}{2}\big) - \psi\big(\frac{1+\theta_2}{2}\big)}{\Gamma(\frac{1+\theta_1}{2})\Gamma(\frac{1+\theta_2}{2})}, \quad \theta_1 \neq \theta_2, \tag{2.41}$$

and

$$\int_0^\infty U_{\theta,0}(z)^2\mathrm{e}^{-z}\mathrm{d}z = \frac{\psi'\big(\frac{1+\theta}{2}\big)}{\Gamma\big(\frac{1+\theta}{2}\big)^2}. \tag{2.42}$$

Proof. By (2.19) and (2.16), we have

$$\begin{aligned} &\frac{\theta_1-\theta_2}{2}\int_0^\infty U_{\theta_1,\alpha}(z)U_{\theta_2,\alpha}(z)\mathrm{e}^{-z}z^\alpha\mathrm{d}z \\ &= \int_0^\infty \big(\partial_z z^{1+\alpha}\mathrm{e}^{-z}\partial_z U_{\theta_1,\alpha}(z)\big)U_{\theta_2,\alpha}(z) - U_{\theta_1,\alpha}(z)\partial_z z^{1+\alpha}\mathrm{e}^{-z}\partial_z U_{\theta_2,\alpha}(z)\mathrm{d}z \\ &= \lim_{z\downarrow 0}\left(z^{1+\alpha}\mathrm{e}^{-z}\Big(U_{\theta_1,\alpha}(z)\overset{\leftrightarrow}{\partial}_z U_{\theta_2,\alpha}(z)\Big)\right), \end{aligned} \tag{2.43}$$

where the left-right derivative is defined by $f\overset{\leftrightarrow}{\partial}g := f\partial g - (\partial f)g$. Then, using $|\mathrm{Re}(\alpha)| < 1$, the connection formula (2.28) and $\frac{\pi z}{\sin\pi z} = \Gamma(1+z)\Gamma(1-z)$, we obtain (2.39). The formulas (2.40), (2.41) and (2.42) are obtained by applying the rule of de l'Hôpital. □

The formulas (2.39) and (2.40) remain true for $\alpha \in \mathbb{C}\setminus\mathbb{Z}$ if the integral is replaced by the generalized integral. For $\alpha \in \mathbb{Z}$, the generalized integral is anomalous. It can be computed via dimensional regulatization:

Theorem 2.2. *Let $\alpha \in \mathbb{Z}$. Then, the following identities hold:*

$$\begin{aligned} &\mathrm{gen}\int_0^\infty U_{\theta_1,\alpha}(z)U_{\theta_2,\alpha}(z)\mathrm{e}^{-z}z^\alpha\mathrm{d}z \\ &= \frac{(-1)^\alpha}{\theta_1-\theta_2}\left(\frac{\psi(\frac{1+\theta_1+\alpha}{2}) + \psi(\frac{1+\theta_1-\alpha}{2})}{\Gamma(\frac{1+\theta_1-|\alpha|}{2})\Gamma(\frac{1+\theta_2+|\alpha|}{2})} - (\theta_1 \leftrightarrow \theta_2)\right) \\ &+ \frac{(-1)^\alpha}{\Gamma(\frac{1+\theta_1+|\alpha|}{2})\Gamma(\frac{1+\theta_2+|\alpha|}{2})}\sum_{k=0}^{|\alpha|-1}\big(\tfrac{1+\theta_1-|\alpha|}{2}\big)_k\big(\tfrac{3+\theta_2-|\alpha|}{2}+k\big)_{|\alpha|-1-k} \\ &\times\Big(-\psi(|\alpha|-k) - \psi(k+1) + \tfrac{1}{2}H_k\big(\tfrac{1+\theta_1-|\alpha|}{2}\big) - \tfrac{1}{2}H_{|\alpha|-1-k}\big(\tfrac{3+\theta_2-|\alpha|}{2}+k\big)\Big); \end{aligned} \tag{2.44}$$

$$\begin{aligned} &\mathrm{gen}\int_0^\infty U_{\theta,\alpha}(z)^2\mathrm{e}^{-z}z^\alpha\mathrm{d}z \\ &= \frac{(-1)^\alpha}{2\Gamma\big(\frac{1+\theta+\alpha}{2}\big)\Gamma\big(\frac{1+\theta-\alpha}{2}\big)}\Bigg(\psi\big(\tfrac{1+\theta+|\alpha|}{2}\big)^2 - \psi\big(\tfrac{1+\theta-|\alpha|}{2}\big)^2 + \psi'\big(\tfrac{1+\theta+\alpha}{2}\big) + \psi'\big(\tfrac{1+\theta-\alpha}{2}\big) \\ &\qquad - 2\sum_{k=0}^{|\alpha|-1}\frac{\psi(|\alpha|-k)+\psi(k+1)}{\frac{1+\theta-|\alpha|}{2}+k}\Bigg). \end{aligned} \tag{2.45}$$

Proof. The generalized integral (2.44) can be computed by dimensional regularization as outlined in Appendix A. By (2.26), the integrand is symmetric under $\alpha \leftrightarrow -\alpha$. Hence, it is sufficient to determine determine the generalized integral for $-\alpha \in \mathbb{N}$.

Actually, it is convenient to pull out a prefactor and determine the generalized integral over the auxiliary function

$$f(\alpha, z) := \Gamma\left(\tfrac{1+\theta_1-\alpha}{2}\right)\Gamma\left(\tfrac{1+\theta_2-\alpha}{2}\right)U_{\theta_1,\alpha}(z)U_{\theta_2,\alpha}(z)\mathrm{e}^{-z}z^{\alpha}. \tag{2.46}$$

By (2.39),

$$\int_0^\infty f(\alpha, z)\mathrm{d}z = \frac{2\pi}{(\theta_1-\theta_2)\sin\pi\alpha}\left(\frac{\Gamma(\frac{1+\theta_2-\alpha}{2})}{\Gamma(\frac{1+\theta_2+\alpha}{2})} - \frac{\Gamma(\frac{1+\theta_1-\alpha}{2})}{\Gamma(\frac{1+\theta_1+\alpha}{2})}\right). \tag{2.47}$$

For $m \in \mathbb{N}$, one then finds

$$\mathrm{fp}\int_0^\infty f(\alpha, z)\mathrm{d}z\Big|_{\alpha=-m} \tag{2.48}$$

$$= \frac{(-1)^m}{\theta_1-\theta_2}\left(\left(\tfrac{1+\theta_1-m}{2}\right)_m\left(\psi\left(\tfrac{1+\theta_1+m}{2}\right) + \psi\left(\tfrac{1+\theta_1-m}{2}\right)\right) - (\theta_1 \leftrightarrow \theta_2)\right). \tag{2.49}$$

Let us write

$$f(\alpha, z) := z^{\alpha}\left(\Gamma\left(\tfrac{1+\theta_1-\alpha}{2}\right)U_{\theta_1,\alpha}(z)\right)\left(\Gamma\left(\tfrac{1+\theta_2-\alpha}{2}\right)U_{\theta_2,\alpha}(z)\mathrm{e}^{-z}\right) \tag{2.50}$$

Recalling that $\alpha < 0$ one sees that negative powers in (2.50) come from one source: the two terms in the brackets can be rewritten using the connection formulas (2.28) and (2.29). This yields a sum of four terms, but three of them have at least a power $z^{-\alpha}$. The only term which contains a singularity at $z = 0$ is the product of

$$-\frac{\pi}{\sin\pi\alpha}\mathbf{F}_{\theta_1,\alpha}(z) = \sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{1+\theta_1-\alpha}{2}\right)_k z^k\Gamma(-\alpha-k)}{k!}, \tag{2.51}$$

and

$$-\frac{\pi}{\sin\pi\alpha}\mathbf{F}_{-\theta_2,\alpha}(-z) = \sum_{k=0}^{\infty}\frac{\left(\frac{1-\theta_2-\alpha}{2}\right)_k z^k\Gamma(-\alpha-k)}{k!}. \tag{2.52}$$

Therefore, if $p \le |\alpha| = -\alpha$, the coefficient of $f(\alpha, z)$ at $z^{\alpha+p}$ is

$$f_p(\alpha) = \sum_{k=0}^{p}\frac{(-1)^k\left(\frac{1+\theta_1+\alpha}{2}\right)_k\left(\frac{1-\theta_2+\alpha}{2}\right)_{p-k}\Gamma(-\alpha-k)\Gamma(-\alpha-p+k)}{k!\,(p-k)!} \tag{2.53}$$

$$= \sum_{k=0}^{p}\frac{(-1)^p\left(\frac{1+\theta_1+\alpha}{2}\right)_k\left(\frac{1+\theta_2-\alpha}{2}-p+k\right)_{p-k}\Gamma(-\alpha-k)\Gamma(-\alpha-p+k)}{k!\,(p-k)!}, \tag{2.54}$$

where we used $(z)_k = (-1)^k(1-z-k)_k$ in the last step. Finally, we use

$$\mathrm{gen}\int_0^\infty f(-m, z) = \mathrm{fp}\int_0^\infty f(\alpha, z)\mathrm{d}z\Big|_{\alpha=-m} - \partial_\alpha f_{m-1}(\alpha)\Big|_{\alpha=-m}. \tag{2.55}$$

We next prove (2.45). Applying the rule of de l'Hôpital to the first line of the right hand side of (2.44) we obtain

$$\frac{(-1)^\alpha}{2\Gamma\left(\frac{1+\theta+\alpha}{2}\right)\Gamma\left(\frac{1+\theta-\alpha}{2}\right)}\Bigg(\psi\left(\tfrac{1+\theta+|\alpha|}{2}\right)^2-\psi\left(\tfrac{1+\theta-|\alpha|}{2}\right)^2+\psi'\left(\tfrac{1+\theta+\alpha}{2}\right)+\psi'\left(\tfrac{1+\theta-\alpha}{2}\right)\Bigg). \tag{2.56}$$

Next we use

$$\left(\tfrac{1+\theta-|\alpha|}{2}\right)_k\left(\tfrac{3+\theta-|\alpha|}{2}+k\right)_{|\alpha|-1-k}=\frac{\left(\frac{1+\theta-|\alpha|}{2}\right)_{|\alpha|}}{\frac{1+\theta-|\alpha|}{2}+k} \tag{2.57}$$

to transform the last two lines of the right hand side of (2.44) into

$$\frac{(-1)^\alpha}{\Gamma\left(\frac{1+\theta+\alpha}{2}\right)\Gamma\left(\frac{1+\theta-\alpha}{2}\right)}\sum_{k=0}^{|\alpha|-1}\frac{-\psi(|\alpha|-k)-\psi(k+1)+\frac{1}{2}H_k\left(\frac{1+\theta-|\alpha|}{2}\right)-\frac{1}{2}H_{|\alpha|-1-k}\left(\frac{3+\theta-|\alpha|}{2}+k\right)}{\frac{1+\theta-|\alpha|}{2}+k}.$$

To simplify this sum, we use the following identity that can be proven inductively:

$$\sum_{k=0}^{m-1}\frac{H_k(z)}{z+k}=\frac{1}{2}\left(H_m'(z)+H_m(z)^2\right). \tag{2.58}$$

Namely, using (2.58) we obtain

$$\begin{aligned}&\sum_{k=0}^{|\alpha|-1}\frac{H_k\left(\frac{1+\theta-|\alpha|}{2}\right)-H_{|\alpha|-1-k}\left(\frac{3+\theta-|\alpha|}{2}+k\right)}{\frac{1+\theta-|\alpha|}{2}+k}\\ =&\sum_{k=0}^{|\alpha|-1}\left(\frac{2H_k\left(\frac{1+\theta-|\alpha|}{2}\right)-H_{|\alpha|}\left(\frac{1+\theta-|\alpha|}{2}\right)}{\frac{1+\theta-|\alpha|}{2}+k}+\frac{1}{\left(\frac{1+\theta-|\alpha|}{2}+k\right)^2}\right)\\ =&H'_{|\alpha|}\left(\tfrac{1+\theta-|\alpha|}{2}\right)+H_{|\alpha|}\left(\tfrac{1+\theta-|\alpha|}{2}\right)^2-H_{|\alpha|}\left(\tfrac{1+\theta-|\alpha|}{2}\right)^2-H'_{|\alpha|}\left(\tfrac{1+\theta-|\alpha|}{2}\right)\quad=\quad 0.\end{aligned} \tag{2.59}$$

□

Note that for special choices of the parameters $\theta_1, \theta_2, \theta$, the right-hand sides of (2.39), (2.40), (2.41), (2.42), (2.44) and (2.45) may contain terms of the form 0 divided by 0. These special choices of parameters include the case when the Tricomi functions reduce to (multiples of) Laguerre polynomials. We therefore treat this case separately in the following section. We will show that the singularities are merely apparent and that the right-hand sides of the mentioned equations also make sense for the described special choice of parameters.

3 Laguerre polynomials

3.1 Basic properties

The Laguerre polynomials are traditionally defined by a Rodriguez-type formula:

$$L_n^\alpha(z) := \frac{1}{n!}\mathrm{e}^z z^{-\alpha}\partial_z^n \mathrm{e}^{-z}z^{n+\alpha} \tag{3.1}$$

$$= (-1)^n\sum_{k=0}^{n}\frac{(-\alpha-n)_{n-k}z^k}{k!(n-k)!}. \tag{3.2}$$

which is meaningful for all complex $\alpha \in \mathbb{C}$. For real $\alpha > -1$, the Laguerre polynomials are one of the sets of classical orthogonal polynomials. Their natural interval is $[0, \infty[$ and the weight is $w(z) = \mathrm{e}^{-z} z^\alpha$.

Up to a normalization, Laguerre polynomials are special cases of confluent functions:

$$
\begin{aligned}
L_n^\alpha(z) &= \frac{(1+\alpha)_n}{n!} F(-n; 1+\alpha; z) && (3.3)\\
&= \frac{\Gamma(1+\alpha+n)}{n!} \mathbf{F}_{-1-\alpha-2n,\alpha}(z) && (3.4)\\
&= \frac{(-z)^n}{n!} {}_2F_0(-n, -\alpha-n; -; -z^{-1}) && (3.5)\\
&= \frac{(-1)^n}{n!} U_{-1-\alpha-2n,\alpha}(z). && (3.6)
\end{aligned}
$$

Their generating functions are:

$$
\mathrm{e}^{-tz}(1+t)^\alpha = \sum_{n=0}^\infty t^n L_n^{\alpha-n}(z), \tag{3.7}
$$

$$
(1-t)^{-\alpha-1} \exp \frac{tz}{t-1} = \sum_{n=0}^\infty t^n L_n^\alpha(z). \tag{3.8}
$$

They have the integral representation

$$
L_n^\alpha(z) = \frac{1}{2\pi \mathrm{i}} \int\limits_{[0^+]} \mathrm{e}^{-tz}(1+t)^{\alpha+n} t^{-n-1} \mathrm{d}t. \tag{3.9}
$$

The value at 0 and behavior at ∞:

$$
L_n^\alpha(0) = \frac{(\alpha+1)_n}{n!}, \quad \lim_{z\to\infty} \frac{L_n^\alpha(z)}{z^n} = \frac{(-1)^n}{n!}. \tag{3.10}
$$

For $\alpha \in \mathbb{N}$ with $\alpha \le n$, we have [27]

$$
L_n^{-\alpha}(z) = \frac{(n-\alpha)!}{n!} (-z)^\alpha L_{n-\alpha}^\alpha(z), \quad \alpha \in \mathbb{N},\ \alpha \le n. \tag{3.11}
$$

Note that in this case, $L_n^{-\alpha}(z)$ has a zero of order α at the origin.

3.2 Bilinear integrals

The bilinear identity for standard integrals, which for real $\alpha > -1$ expresses the orthogonality relations, is well-known:

Theorem 3.1. *Let* $\operatorname{Re}(\alpha) > -1$. *Then*

$$
\int_0^\infty L_m^\alpha(z) L_n^\alpha(z) z^\alpha \mathrm{e}^{-z} \mathrm{d}z = \frac{\Gamma(1+n+\alpha)}{n!} \delta_{m,n}. \tag{3.12}
$$

Proof. For completeness we sketch the proof. Assume without loss of generality that $m \geq n$. Applying the Rodriguez formula (3.1) for L_m^α, and then integrating m times by parts and using $\mathrm{Re}(\alpha) > -1$, one obtains

$$\int_0^\infty L_m^\alpha(z)L_n^\alpha(z)z^\alpha \mathrm{e}^{-z}\mathrm{d}z = \int_0^\infty \left(\partial_z \mathrm{e}^{-z}z^{m+\alpha}\right)L_n^\alpha(z)\mathrm{d}z \tag{3.13}$$

$$= \frac{(-1)^m}{m!}\int_0^\infty \mathrm{e}^{-z}z^{m+\alpha}\partial_z^m L_n^\alpha(z)\mathrm{d}z. \tag{3.14}$$

Because $m \geq n$, we find $\partial_z^m L_n^\alpha(z) = (-1)^m\delta_{m,n}$ with the Kronecker delta on the right-hand side. Then the remaining integral is nothing but the integral representation of $\Gamma(1+n+\alpha)$. □

If we replace the integral on the left-hand side of (3.12) with the generalized integral, formula (3.12) remains valid if $\alpha \in \mathbb{C} \setminus -\mathbb{N}$, because then the generalized integral is non-anomalous and can be computed by analytic continuation:

Theorem 3.2. *Let $\alpha \in \mathbb{C} \setminus -\mathbb{N}$. Then (3.12) remains true if we replace the usual integral with the generalized integral.*

The case of negative integer α is more complicated. The bilinear integral reduces to the standard integral also for negative integer α provided that both m and n are not less than $|\alpha|$:

Theorem 3.3. *Let $\alpha \in -\mathbb{N}$ and $m, n \geq |\alpha|$. Then (3.12) remains true (in the sense of usual integrals):*

$$\int_0^\infty L_m^\alpha(z)L_n^\alpha(z)z^\alpha \mathrm{e}^{-z}\mathrm{d}z = \frac{\Gamma(1+n+\alpha)}{n!}\delta_{m,n} = \frac{(n-|\alpha|)!}{n!}\delta_{m,n} \tag{3.15}$$

Proof. By the identity (3.11)

$$\begin{aligned} &\mathrm{gen}\int_0^\infty L_m^\alpha(z)L_n^\alpha(z)z^\alpha \mathrm{e}^{-z}\mathrm{d}z \qquad (3.16)\\ &= \frac{(m-|\alpha|)!(n-|\alpha|)!}{m!n!}\int_0^\infty L_{m-|\alpha|}^{|\alpha|}(z)L_{n-|\alpha|}^{|\alpha|}(z)z^{|\alpha|}\mathrm{e}^{-z}\mathrm{d}z \\ &= \frac{(n-|\alpha|)!}{n!}\delta_{m,n}. \end{aligned}$$

□

For low degree Laguerre polynomials and negative integer α we need anomalous integrals:

Theorem 3.4. *Let $-\alpha \in \mathbb{N}$, $m, n \in \mathbb{N}_0$ such that $|\alpha| > n$ and, without loss of generality, $m \geq n$. Then*

$$\mathrm{gen}\int_0^\infty L_m^\alpha(z)L_n^\alpha(z)z^\alpha \mathrm{e}^{-z}\mathrm{d}z = \frac{(-1)^{\alpha+n}}{n!(|\alpha|-n-1)!(m-n)}, \qquad n < m; \tag{3.17}$$

$$\mathrm{gen}\int_0^\infty L_n^\alpha(z)^2 z^\alpha \mathrm{e}^{-z}\mathrm{d}z = \frac{(-1)^{\alpha+n+1}}{n!(|\alpha|-n-1)!}\psi(|\alpha|-n). \tag{3.18}$$

Proof. Let us first consider the case $m > n$ and $|\alpha| > n$. We find

$$\mathrm{gen}\int_0^\infty L_m^\alpha(z)L_n^\alpha(z)z^\alpha \mathrm{e}^{-z}\mathrm{d}z = \frac{1}{m!}\mathrm{gen}\int_0^\infty \left(\partial^m \mathrm{e}^{-z}z^{m+\alpha}\right)L_n^\alpha(z)\mathrm{d}z. \tag{3.19}$$

We may integrate the right-hand side by parts $n+1$ times, each time collecting the regular value of the remaining integrand. Thus

$$\operatorname{gen}\int_0^\infty L_m^\alpha(z)L_n^\alpha(z)z^\alpha \mathrm{e}^{-z}\mathrm{d}z = \frac{1}{m!}\sum_{k=1}^{n+1}(-1)^k \operatorname{rv}_0\Big(\big(\partial^{m-k}\mathrm{e}^{-z}z^{m+\alpha}\big)\partial^{k-1}L_n^\alpha(z)\Big). \tag{3.20}$$

Inserting the series expansions

$$\partial_z^{m-k}\mathrm{e}^{-z}z^{m+\alpha} = (-1)^{m-k}\left(\sum_{j=0}^{|\alpha|-m-1} + \sum_{j=|\alpha|-k}^{\infty}\right)\frac{(-1)^j}{j!}(|\alpha|-m-j)_{m-k}z^{j+k-|\alpha|}, \tag{3.21}$$

$$\partial_z^{k-1}L_n^\alpha(z) = (-1)^n\sum_{i=0}^{n+1-k}\frac{(-\alpha-n)_{n-i-k+1}z^i}{i!(n-i-k+1)!}, \tag{3.22}$$

we obtain a triple sum:

$$\begin{aligned}&\operatorname{gen}\int_0^\infty L_m^\alpha(z)L_n^\alpha(z)z^\alpha \mathrm{e}^{-z}\mathrm{d}z \\ &= \frac{(-1)^{m+n}}{m!}\sum_{k=1}^{n+1}\operatorname{rv}_0\left(\left(\sum_{j=0}^{|\alpha|-m-1} + \sum_{j=|\alpha|-k}^{\infty}\right)\sum_{i=0}^{n+1-k}\frac{(-1)^j}{j!}(|\alpha|-m-j)_{m-k}\right. \\ &\quad \left.\times\frac{(|\alpha|-n)_{n-k-i+1}}{i!(n-i-k+1)!}z^{i+k-|\alpha|+j}\right).\end{aligned} \tag{3.23}$$

The regular value evaluated at 0 is then the coefficient for $j=|\alpha|-k-i$. One quickly finds that only the term $i=0$ is non-zero, so actually, the regular value is the coefficient corresponding to $j=|\alpha|-k$ and $i=0$. Hence we are left with the single sum

$$\begin{aligned}(3.23) &= \frac{(-1)^{m+n+|\alpha|-k}}{m!}\sum_{k=1}^{n+1}\frac{(k-m)_{m-k}(|\alpha|-n)_{n-k+1}}{(|\alpha|-k)!(n+1-k)!} \\ &= \frac{(-1)^{n+\alpha}}{m!(|\alpha|-n-1)!}\sum_{k=1}^{n+1}\frac{(m-k)!}{(n+1-k)!} \\ &= \frac{(-1)^{n+\alpha}}{n!(|\alpha|-n-1)!(m-n)},\end{aligned} \tag{3.24}$$

where in the last step we used an identity about telescoping series recalled in Lemma B.1.

Now consider $m=n<|\alpha|$. We again use the Rodriguez type formula. However, we can now only integrate by parts n times, so that a simple generalized integral survives:

$$\begin{aligned}&\operatorname{gen}\int_0^\infty \big(L_n^\alpha(z)\big)^2 z^\alpha \mathrm{e}^{-z}\mathrm{d}z \\ &= \frac{1}{n!}\left(\sum_{k=1}^{n}(-1)^k \operatorname{rv}_0\Big(\big(\partial^{n-k}\mathrm{e}^{-z}z^{n+\alpha}\big)\partial^{k-1}L_n^\alpha(z)\Big) + \operatorname{gen}\int_0^\infty \mathrm{e}^{-z}z^{n+\alpha}\mathrm{d}z\right).\end{aligned} \tag{3.25}$$

The remaining generalized integral is the regularized Gamma function, which is treated as an example in [7]:

$$\operatorname{gen}\int_0^\infty \mathrm{e}^{-z} z^{n+\alpha}\mathrm{d}z = -\frac{(-1)^{\alpha+n}}{(|\alpha|-n-1)!}\psi(|\alpha|-n), \quad n<|\alpha|\in\mathbb{N}. \tag{3.26}$$

The sum of regular values can be written as

$$\frac{1}{n!}\sum_{k=1}^{n}\operatorname{rv}_0\left(\left(\sum_{j=0}^{|\alpha|-n-1}+\sum_{j=|\alpha|-k}^{\infty}\right)\sum_{i=0}^{n+1-k}\frac{(-1)^j}{j!}(|\alpha|-n-j)_{n-k}\right. \\ \left.\times\frac{(|\alpha|-n)_{1+n-k-i}}{i!(n-i-k+1)!}z^{i+k-|\alpha|+j}\right). \tag{3.27}$$

In contrast to the case $m>n$, now there are two terms that contribute to rv_0: $(j=|\alpha|-k\wedge i=0)$ and $(j=|\alpha|-n-1\wedge i=n+1-k)$. We obtain

$$(3.27)=\frac{(-1)^{n+\alpha}}{n!(|\alpha|-n-1)!}\sum_{k=1}^{n}\left(\frac{(n-k)!}{(n+1-k)!}-\frac{(n-k)!}{(n+1-k)!}\right)=0 \tag{3.28}$$

We thus proved (3.17). □

We may write down a "generalized Gram matrix" $G(\alpha)$ whose entries are the bilinear generalized integrals over Laguerre polynomials. We have

$$G(\alpha)=\operatorname{diag}\left(\frac{\Gamma(1+\alpha+n)}{n!}\right), \qquad \alpha\in\mathbb{C}\setminus-\mathbb{N}, \tag{3.29}$$

and, for $\alpha\in-\mathbb{N}$:

$$G(\alpha)=\begin{array}{c} \\ 0 \\ 1 \\ \vdots \\ |\alpha|-1 \\ |\alpha|\cdots \end{array}\begin{array}{ccccc} 0 & 1 & \cdots & |\alpha|-1 & |\alpha|\;\cdots \end{array} \hspace{-20em} \left(\begin{array}{ccccc} -\frac{(-1)^{|\alpha|}\psi(|\alpha|)}{(|\alpha|-1)!0!} & \frac{(-1)^{|\alpha|}}{(|\alpha|-1)!} & \cdots & \frac{(-1)^{|\alpha|}}{(|\alpha|-1)(|\alpha|-1)!} & \cdots \\ \frac{(-1)^{|\alpha|}}{(|\alpha|-1)!} & \frac{(-1)^{|\alpha|}\psi(|\alpha|-1)}{(|\alpha|-2)!1!} & \cdots & \frac{(-1)^{1+|\alpha|}}{(|\alpha|-2)(|\alpha|-2)!} & \cdots \\ \vdots & \vdots & \ddots & & \\ \frac{(-1)^{|\alpha|}}{(|\alpha|-1)(|\alpha|-1)!} & \frac{(-1)^{1+|\alpha|}}{(|\alpha|-2)(|\alpha|-2)!} & & \frac{\psi(1)}{0!(|\alpha|-1)!} & \cdots \\ \vdots & \vdots & & \vdots & \operatorname{diag}\left(\frac{(n-|\alpha|)!}{n!}\right) \end{array}\right).$$

In particular, all entries of the rows and columns $0,\dots,|\alpha|-1$ are non-zero and the sign of the entries of these rows and columns oscillates.

Integration by parts seems to be the simplest way to determine bilinear generalized integrals of Laguerre polynomials. Alternatively, we can derive Thms 3.2, 3.3 and 3.4 from Thm 2.2, using the fact that Laguerre polynomials are essentially special cases of Tricomi functions. For Thms 3.2 and 3.3 this is easy. To derive Thm 3.4 we need to analyze certain seemingly singular expressions.

Alternative proof of Thm. 3.4. Let $m, n \in \mathbb{N}_0$ and $\alpha \in \mathbb{Z}$. Let us show that

$$\begin{aligned} &\operatorname{gen} \int_0^\infty L_n^\alpha(z) L_m^\alpha(z) \mathrm{e}^{-z} z^\alpha \mathrm{d}z && \text{(3.30a)} \\ =&\frac{(-1)^{m+n}}{m!n!} \operatorname{gen} \int_0^\infty U_{-1-\alpha-2n,\alpha}(z) U_{-1-\alpha-2m,\alpha}(z) \mathrm{e}^{-z} z^\alpha \mathrm{d}z, \quad m \neq n, && \text{(3.30b)} \end{aligned}$$

and

$$\begin{aligned} &\operatorname{gen} \int_0^\infty L_n^\alpha(z)^2 \mathrm{e}^{-z} z^\alpha \mathrm{d}z && \text{(3.31a)} \\ &= \frac{1}{(n!)^2} \operatorname{gen} \int_0^\infty U_{-1-\alpha-2n,\alpha}(z)^2 \mathrm{e}^{-z} z^\alpha \mathrm{d}z. && \text{(3.31b)} \end{aligned}$$

The generalized integrals of Tricomi functions given by Thm 2.2 are seemingly singular for the parameters in (3.30b) and (3.31b). However, these singularities are merely apparent. To see this, we replace (3.30b) and (3.31b) by the following limits:

$$I_1 := \frac{(-1)^{m+n}}{m!n!} \lim_{\epsilon \to 0} \operatorname{gen} \int_0^\infty U_{-1-\alpha-2n-2\epsilon,\alpha}(z) U_{-1-\alpha-2m-2\epsilon,\alpha}(z) \mathrm{e}^{-z} z^\alpha \mathrm{d}z, \quad m \neq n, \tag{3.32}$$

$$I_2 := \frac{1}{(n!)^2} \lim_{\epsilon \to 0} \operatorname{gen} \int_0^\infty U_{-1-\alpha-2n-2\epsilon,\alpha}(z)^2 \mathrm{e}^{-z} z^\alpha \mathrm{d}z. \tag{3.33}$$

To determine I_1, we may without loss of generality assume that $m > n$. Note that

$$\begin{aligned} &\Bigg| \sum_{k=0}^{|\alpha|-1} \left(\tfrac{1+\theta_1-|\alpha|}{2}\right)_k \left(\tfrac{3+\theta_2-|\alpha|}{2} + k\right)_{|\alpha|-1-k} \\ &\times \Big(-\psi(|\alpha|-k) - \psi(k+1) + \tfrac{1}{2} H_k\left(\tfrac{1+\theta_1-|\alpha|}{2}\right) - \tfrac{1}{2} H_{|\alpha|-1-k}\left(\tfrac{3+\theta_2-|\alpha|}{2} + k\right)\Big) \Bigg| < \infty \end{aligned} \tag{3.34}$$

for any values of the parameters because possible vanishing denominators of the harmonic numbers are balanced by the Pochhammer symbol. Due to the Γ functions in the prefactor, see Thm. 2.2, this sum only contributes if $-\alpha > m$ *and* $-\alpha > n$. In the cases where the sum does not contribute, we have

$$\begin{aligned} I_1 =& \frac{(-1)^{\alpha+m+n}}{2(m-n)n!m!} \\ &\times \lim_{\epsilon \to 0} \begin{cases} \dfrac{\psi(-n-\epsilon) + \psi(-\alpha-n-\epsilon)}{\Gamma(-\alpha-n-\epsilon)\Gamma(-m-\epsilon)} - (m \leftrightarrow n), & \alpha \geq 0; \\[2ex] \dfrac{\psi(-n-\epsilon) + \psi(|\alpha|-n-\epsilon)}{\Gamma(-n-\epsilon)\Gamma(|\alpha|-m-\epsilon)} - (m \leftrightarrow n), & \alpha < 0,\ m \geq |\alpha|. \end{cases} \end{aligned} \tag{3.35}$$

Now we use

$$\frac{\psi(z)}{\Gamma(z)}\bigg|_{z=-k} = (-1)^{k+1} k! \tag{3.36}$$

to obtain

$$I_1 = \begin{cases} 0, & \alpha \geq 0; \\ \dfrac{(-1)^{\alpha+n}}{(m-n)n!\Gamma(|\alpha|-n)}, & \alpha < 0,\ m \geq |\alpha|. \end{cases} \tag{3.37}$$

In particular, if $\alpha < 0$ and both $m, n \geq |\alpha|$, then $I_1 = 0$.

Let us now assume that $-\alpha > m$ and $-\alpha > n$ (and still, $m > n$). Then

$$I_1 = \frac{(-1)^{\alpha+m+n}}{2n!m!\Gamma(|\alpha|-m)\Gamma(|\alpha|-n)} \Bigg(\frac{(-1)^m m!\Gamma(|\alpha|-m) - (-1)^n n!\Gamma(|\alpha|-n)}{m-n} \tag{3.38}$$
$$+ \sum_{k=0}^{|\alpha|-1} (-n)_k (1-m+k)_{|\alpha|-1-k} \Big(H_k(-n) - H_{|\alpha|-1-k}(1-m+k) \Big) \Bigg),$$

where

$$(z)_k H_k(z) = \sum_{j=0}^{k-1} \prod_{l=0, l\neq j}^{k-1} (z+l). \tag{3.39}$$

With similar manipulations as in the proof of Theorem 3.4 and using Lemma B.1, the sum in the last line of (3.38) can be simplified and we obtain the expression from Thm 3.4.

The analysis of (3.33) is similar. We distinguish the three cases $\alpha > 0$, $(\alpha < 0 \wedge n \geq |\alpha|)$ and $-\alpha > n$. The sum in (2.45) only contributes if $-\alpha > n$. To obtain the limit, one needs to evaluate certain combinations of $\psi'(z)$, $\psi(z)$ and $\Gamma(z)$ at negative integers. We derive the respective formulas in Lemma B.2. □

A Generalized integral

A.1 Definition

Let us recall from [7] the definition of the generalized integral where the integrand has a non-integrable homogeneous singularity at 0.

Definition A.1. *We say that a function f on $]0, \infty[$ is* integrable in the generalized sense *if it is integrable on $]1, \infty[$ and there exists a finite set $\Omega \subset \mathbb{C}$ and complex coefficients $(f_k)_{k\in\Omega}$ such that*

$$f - \sum_{k\in\Omega} f_k r^k \tag{A.1}$$

is integrable on $]0, 1[$. The set of such functions is denoted $\mathcal{F}$. For $f \in \mathcal{F}$ we define

$$\mathrm{gen} \int_0^\infty f(r)\mathrm{d}r := \sum_{k\in\Omega\backslash\{-1\}} \frac{f_k}{k+1} + \int_0^1 \Big(f(r) - \sum_{k\in\Omega} f_k r^k \Big) \mathrm{d}r + \int_1^\infty f(r)\mathrm{d}r. \tag{A.2}$$

Note that the set $\{k \in \Omega \,|\, \mathrm{Re}(k) \leq -1\}$ and the corresponding f_k are uniquely determined by f. It is convenient to allow k with $\mathrm{Re}(k) > -1$. The generalized integral of f does not depend on the choice of Ω.

It is clear that the generalized integral extends the standard integral:

$$\mathrm{gen}\int_0^\infty f(r)\mathrm{d}r = \int_0^\infty f(r)\mathrm{d}r \quad \text{for } f \in L^1]0,\infty[. \tag{A.3}$$

The generalized integral is in general coordinate dependent. As was proved in [7], the generalized integral is invariant under scaling if and only if $f_{-1} = 0$, and invariant under a large class of a change of variables if $f_k = 0$ for every negative integer k.

Definition A.2. *The generalized integral* (A.2) *is called* anomalous *if there exists* $n \in \mathbb{N}$ *such that* $f_{-n} \neq 0$.

A.2 Integration by parts

Definition A.3. *We say that* $F \in \mathcal{G}$ *if* F *is a measurable function on* $[0,\infty[$*, bounded on* $[1,\infty[$ *and there exists a finite set* $\Theta \subset \mathbb{C}$ *and complex coefficients* $(F_k)_{k\in\Theta}$ *and* $c \in \mathbb{C}$*, such that*

$$F(r) - \sum_{k\in\Theta} F_k r^k - c\ln r \tag{A.4}$$

has a limit as $r \to 0$*. For* $F \in \mathcal{G}$ *we define the regular value of* F *at* 0*:*

$$\mathrm{rv}_0\, F := \lim_{r\searrow 0}\Big(F(r) - \sum_{k\in\Theta\setminus\{0\}} F_k r^k - c\ln r\Big). \tag{A.5}$$

Note that given F the coefficients F_k for $k \in \Theta$ such that $\mathrm{Re}k \leq 0$, $k \neq 0$, as well as c, are uniquely defined. Hence (A.5) depends only on F.

Proposition A.4. *Let* $f \in \mathcal{F}$ *and* $r > 0$*. Then*

$$F(r) := -\int_r^\infty f(y)\mathrm{d}y \tag{A.6}$$

belongs to $\mathcal{G}$ *and*

$$\mathrm{gen}\int_0^\infty f(y)\mathrm{d}y = -\,\mathrm{rv}_0\, F. \tag{A.7}$$

Proof. It is clear that F is bounded on $[1,\infty[$ because $f \in \mathcal{F}$. Let $0 < r < 1$. Then

$$\int_r^\infty f(y)\mathrm{d}y = \int_r^1 \Big(f(y) - \sum_{k\in\Omega} f_k y^k\Big)\mathrm{d}y + \int_1^\infty f(y)\mathrm{d}y \tag{A.8}$$

$$+ \sum_{k\in\Omega\setminus\{-1\}} \frac{f_k}{k+1}(1 - r^{k+1}) - f_{-1}\ln r$$

$$= \mathrm{gen}\int_0^\infty f(y)\mathrm{d}y - \int_0^r \Big(f(y) - \sum_{k\in\Omega} f_k y^k\Big)\mathrm{d}y \tag{A.9}$$

$$- \sum_{k\in\Omega\setminus\{-1\}} \frac{f_k}{k+1} r^{k+1} - f_{-1}\ln r.$$

The second term in (A.9) goes to zero as $r \searrow 0$. After pulling the last two terms to the left-hand side, we may thus perform the limit $r \searrow 0$. We obtain (A.7). □

Corollary A.5. *Let f, g be measurable functions and suppose that $fg', gf' \in \mathcal{F}$, and that $\lim_{r\to\infty} f(r)g(r) = 0$. Then*

$$\operatorname{gen} \int_0^\infty f(r)g'(r)\mathrm{d}r = -\operatorname{gen} \int_0^\infty f'(r)g(r)\mathrm{d}r - \mathrm{rv}_0(fg). \tag{A.10}$$

Proof. Clearly, $(fg)' = fg' + f'g$, hence $(fg)' \in \mathcal{F}$. Moreover

$$f(r)g(r) = -\int_r^\infty (fg)'(y)\mathrm{d}y. \tag{A.11}$$

Therefore, it is enough to apply (A.7) to $F := fg$. □

A.3 Dimensional regularization

We briefly recall from [7] how the generalized integral can be computed using dimensional regularization.

Let $N \in \mathbb{N}$ and let $f :]0,\infty[\times\{\alpha \in \mathbb{C} \,|\, \mathrm{Re}(\alpha) > -N-1\} \to \mathbb{C}$ be a function such that $f(r,\cdot)$ is holomorphic for each r, $\|f(\cdot,\alpha)\|_{L^1[1,\infty[}$ is bounded locally uniformly in α, and there exist holomorphic functions $f_0,\dots,f_N$ of α such that the $L^1]0,1]$ norm of $f(r,\alpha) - \sum_{n=0}^N r^{\alpha+n} f_n(\alpha)$ is bounded locally uniformly in α. Then $f(\cdot,\alpha)$ is integrable in the generalized sense, and for $-\alpha \notin \{1,\dots,N\}$ one has

$$\begin{aligned}
&\operatorname{gen} \int_0^\infty f(r,\alpha)\mathrm{d}r \\
&= \sum_{n=0}^N \frac{f_n(\alpha)}{\alpha+n+1} + \int_0^1 \Big(f(r,\alpha) - \sum_{n=0}^N r^{\alpha+n} f_n(\alpha) \Big)\mathrm{d}r + \int_1^\infty f(r,\alpha)\mathrm{d}r.
\end{aligned} \tag{A.12}$$

By Morera's theorem, the right hand side is, away from the poles at $-1,\dots,-N$, a holomorphic function of α. Therefore, to obtain (A.12) in the non-anomalous case it is enough to compute (A.12) in the region where the usual integral is convergent and continue analytically.

Let $m \in \{1,\dots,N\}$. The right hand side of (A.12) has a simple pole at $\alpha = -m$ with residue $f_{m-1}(-m)$ (possibly zero). Its finite part is

$$\operatorname*{fp}_{\alpha\to-m} \operatorname{gen} \int_0^\infty f(r,\alpha)\mathrm{d}r = \lim_{\alpha\to-m} \left(\operatorname{gen} \int_0^\infty f(r,\alpha)\mathrm{d}r - \frac{f_{m-1}(-m)}{\alpha+m} \right). \tag{A.13}$$

To recover the generalized integral at $\alpha = -m$, one also needs to subract a finite term:

$$\operatorname{gen} \int_0^\infty f(r,-m)\mathrm{d}r = \operatorname*{fp}_{\alpha\to-m} \operatorname{gen} \int f(r,\alpha)\mathrm{d}r - f'_{m-1}(-m). \tag{A.14}$$

B Two useful lemmas

We prove several identities that have been used in the computation of generalized integrals:

Lemma B.1. *We have*

$$\sum_{k=0}^{n} \frac{(a)_k}{k!} = \frac{(a+1)_n}{n!}, \tag{B.1}$$

which implies for $m, n \in \mathbb{N}$, $m > n$*:*

$$\sum_{k=1}^{n+1} \frac{(m-k)!}{(n+1-k)!} = \frac{m!}{n!} \frac{1}{m-n}. \tag{B.2}$$

Proof. To prove the first identity, we compute

$$\sum_{k=0}^{n} \frac{(a)_k}{k!} = \sum_{k=0}^{n} \frac{(a+1)_{k-1}}{k!}(a+k-k) \tag{B.3}$$

$$= \sum_{k=0}^{n} \frac{(a+1)_k}{k!} - \sum_{k=1}^{n} \frac{(a+1)_{k-1}}{(k-1)!}. \tag{B.4}$$

The second identity follows from the first by setting $a = m - n$. □

Lemma B.2. *For* $n \in \mathbb{N}_0$*, we have*

$$\frac{\psi'(z)}{\Gamma(z)^2}\Big|_{z=-n} = (n!)^2, \quad \textit{and} \quad \frac{\psi'(z) - \psi(z)^2}{\Gamma(z)}\Big|_{z=-n} = (-1)^n 2n!\psi(1+n). \tag{B.5}$$

Proof. Let $z \in \mathbb{C} \setminus -\mathbb{N}_0$. Then

$$\frac{\psi'(z)}{\Gamma(z)^2} = z^2 \frac{\psi'(z+1) + \frac{1}{z^2}}{\Gamma(z+1)^2} = z^2 \frac{\psi'(z+1)}{\Gamma(z+1)^2} + \frac{1}{\Gamma(z+1)^2}. \tag{B.6}$$

Therefore,

$$\frac{\psi'(z)}{\Gamma(z)^2} = \big((z)_n\big)^2 \frac{\psi'(z+n)}{\Gamma(z+n)^2} + \sum_{j=1}^{n} \frac{\big((z)_{j-1}\big)^2}{\Gamma(z+j)^2}. \tag{B.7}$$

Taking the limit $z \to -n$, we obtain

$$\frac{\psi'(z)}{\Gamma(z)^2}\Big|_{z=-n} = (n!)^2 \frac{\psi'(z)}{\Gamma(z)^2}\Big|_{z=0} = (n!)^2. \tag{B.8}$$

The proof of the second identity works similar: Using

$$\lim_{\epsilon \to 0} \frac{\psi'(\epsilon) - \psi(\epsilon)^2}{\Gamma(\epsilon)} = -2\gamma_{\mathrm{E}} = 2\psi(1), \tag{B.9}$$

and

$$\frac{\psi'(z)-\psi(z)^2}{\Gamma(z)} = \frac{\psi'(z+1)+\frac{1}{z^2}-\big(\psi(z+1)-\frac{1}{z}\big)^2}{\frac{1}{z}\Gamma(1+z)} \tag{B.10}$$
$$= z\frac{\psi'(z+1)-\psi(z+1)^2}{\Gamma(z+1)} + 2\frac{\psi(z+1)}{\Gamma(z+1)},$$

we inductively obtain

$$\frac{\psi'(z)-\psi(z)^2}{\Gamma(z)} = (z)_n\frac{\psi'(z+n)-\psi(z+n)^2}{\Gamma(z+n)} + 2\sum_{j=1}^{n}(z)_{j-1}\frac{\psi(z+j)}{\Gamma(z+j)}. \tag{B.11}$$

Taking the limit $z \to -n$ and using (B.9), we obtain

$$\frac{\psi'(z)-\psi(z)^2}{\Gamma(z)}\Big|_{z=-n} = 2(-n)_n\psi(1) + 2\sum_{j=1}^{n}(-n)_{j-1}(-1)^{n-j-1}(n-j)! \tag{B.12}$$
$$= (-1)^n 2n!\Big(\psi(1)+\sum_{j=1}^{n}\frac{1}{n+1-j}\Big)$$
$$= (-1)^n 2n!\psi(1+n).$$

□

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