Journal of Statistical Physics (2024) 191:110
https://doi.org/10.1007/s10955-024-03328-2

Beliaev Damping in Bose Gas

Jan Dereziński[1] · Ben Li[1] · Marcin Napiórkowski[1]

Received: 31 October 2023 / Accepted: 14 August 2024 / Published online: 30 August 2024

Abstract
According to the Bogoliubov theory the low energy behaviour of the Bose gas at zero temperature can be described by non-interacting bosonic quasiparticles called phonons. In this work the damping rate of phonons at low momenta, the so-called Beliaev damping, is explained and computed with simple arguments involving the Fermi Golden Rule and Bogoliubov's quasiparticles.

Keywords Many-body quantum mechanics · Bogoliubov approximation · Fermi Golden Rule · Spectral Theory

1 Introduction

The Bose gas near the zero temperature has curious properties that can be partly explained from the first principles by a beautiful argument that goes back to Bogoliubov [5]. In Bogoliubov's approach the Bose gas at zero temperature can be approximately described by a gas of weakly interacting quasiparticles. The dispersion relation of these quasiparticles, that is, their energy in function of the momentum is described by a function $\mathbf{k} \mapsto e_{\mathbf{k}}$ with an interesting shape. At low momenta these quasiparticles are called phonons and $e_{\mathbf{k}} \approx ck$, where $c > 0$ and $k := |\mathbf{k}|$. Thus the low-energy dispersion relation is very different from the non-interacting, quadratic one. It is responsible for superfluidity of the Bose gas.

It is easy to see that phonons could be metastable, because the energy-momentum conservation may not prohibit them to decay into two or more phonons. This decay rate was first computed in perturbation theory by Beliaev [2], hence the name *Beliaev damping*. According to his computation, the imaginary part of the dispersion relation behaves for small momenta as $-c_{\mathrm{Bel}}k^5$. This implies the exponential decay of phonons with the decay rate $2c_{\mathrm{Bel}}k^5$. The

Communicated by Alessandro Giuliani.

✉ Marcin Napiórkowski
marcin.napiorkowski@fuw.edu.pl

Jan Dereziński
jan.derezinski@fuw.edu.pl

Ben Li
ben.li@fuw.edu.pl

1 Department of Mathematical Methods in Physics, Faculty of Physics, University of Warsaw, Pasteura 5, 02-093 Warszawa, Poland

Beliaev damping has been observed in experiments, and appears to be consistent with its theoretical predictions [28, 31].

In our paper we present a systematic derivation of Beliaev damping. Our presentation differs in several points from similar accounts found in the physics literature. We try to make all the arguments as transparent as possible, without hiding some of less rigorous steps. We avoid using diagrammatic techniques, in favor of a mathematically much clearer picture involving a Bogoliubov transformation and the 2nd order perturbation computation (the so-called Fermi Golden Rule) applied to what we call the effective Friedrichs Hamiltonian. We use the grand-canonical picture instead of the canonical one found in a part of the literature. This is a minor difference; on this level both pictures should lead to the same final result. We believe that the derivation of Beliaev damping is a beautiful illustration of methods many-body quantum physics, which is quite convincing even if not fully rigorous.

In the remaining part of the introduction we provide a brief sketch of the main steps of Beliaev's argument. In the main body of our article we discuss these steps in more detail, indicating which parts can be easily made rigorous.

Let v be a real function satisfying $v(x) = v(-x)$. Later on we will need more assumptions: in particular, we will assume that $v(x)$ is rotationally invariant, both $v(x)$ and its Fourier transform $\hat{v}(\mathbf{k})$ decay sufficiently fast at infinity and that $\hat{v}(\mathbf{k}) \geq 0$. The homogeneous Bose gas of N particles interacting with the pair potential v is described by the Hamiltonian and the total momentum

$$H_N = -\sum_{i=1}^{N} \frac{1}{2m}\Delta_i + \sum_{1\leq i<j\leq N} v(x_i - x_j), \tag{1}$$

$$P_N = \sum_{i=1}^{N} \frac{1}{\mathrm{i}}\partial_{x_i}. \tag{2}$$

These operators act on $L^2_{\mathrm{s}}\big((\mathbb{R}^3)^N\big)$, the space of functions symmetric in the positions of N 3-dimensional particles. Note that H_N commutes with P_N, which expresses the spatial homogeneity of the system. From now on we will set $m = 1$.

We would like to describe a Bose gas of positive density in infinite volume. This is difficult to do in terms of the Hamiltonian acting on the whole space $\mathbb{R}^3$. Therefore we replace (1) and (2) with a system enclosed in a box of size L, and then take the thermodynamic limit. In order to preserve translation symmetry we consider periodic boundary conditions. They are not very physical, but it is believed that they should not affect the overall picture in the thermodynamic limit.

Thus v is replaced by its periodized version adapted to the box of size L. The new Hilbert space is $L^2_{\mathrm{s}}\big(([-L/2, L/2]^3)^N\big)$. We will use the same symbols H_N, P_N to denote the Hamiltonian and total momentum in the box. Note that they still commute with one another.

It is very convenient to consider at the same time all numbers of particles. In order to control the density, that is $\frac{N}{L^3}$, we introduce the chemical potential given by a positive number $\mu > 0$, and we use the grand-canonical formalism. It is also convenient to pass from the position to the momentum representation. Thus we replace H_N, P_N with

$$\begin{aligned} H &:= \bigoplus_{N=0}^{\infty}(H_N - \mu N) = \int a_x^*\Big(-\frac{1}{2}\Delta_x - \mu\Big)a_x\,\mathrm{d}x + \frac{1}{2}\int\int \mathrm{d}x\,\mathrm{d}y\,v(x-y)a_x^*a_y^*a_ya_x \\ &= \sum_{\mathbf{p}}\Big(\frac{1}{2}\mathbf{p}^2 - \mu\Big)a_{\mathbf{p}}^*a_{\mathbf{p}}\,\mathrm{d}\mathbf{p} + \frac{1}{2L^3}\sum_{\mathbf{p}}\sum_{\mathbf{q}}\sum_{\mathbf{k}}\hat{v}(\mathbf{k})a_{\mathbf{p}+\mathbf{k}}^*a_{\mathbf{q}-\mathbf{k}}^*a_{\mathbf{q}}a_{\mathbf{p}}, \end{aligned} \tag{3}$$

$$P := \bigoplus_{N=0}^{\infty} P_N = \int a_x^* \frac{1}{\mathrm{i}} \partial_x a_x \,\mathrm{d}x = \sum_{\mathbf{p}} \mathbf{p} a_{\mathbf{p}}^* a_{\mathbf{p}}. \tag{4}$$

a_x^* and a_x are the usual creation/annihilation operators for $x \in [-L/2, L/2]^3$ in the position representation, commuting to the Dirac delta. $a_{\mathbf{p}}^*, a_{\mathbf{p}}$ are the usual creation/annihilation operators for $\mathbf{p} \in 2\pi\mathbb{Z}^3/L$ in the momentum representation commuting to the Kronecker delta. H, P act on the bosonic Fock space with the one-particle space $L^2\big([-L/2, L/2]^3\big)$ in the position representation, and $l^2\big(2\pi\mathbb{Z}^3/L\big)$ in the momentum representation. H and P still commute with one another.

Now there comes the main idea of the Bogoliubov approach. At zero temperature, one expects complete Bose–Einstein condensation. This is expressed by assuming that the zero mode is populated macroscopically and there are only very few particles in nonzero modes. The zero mode is treated classically, and essentially removed from the picture. One obtains an approximate Hamiltonian, which does not preserve the number of particles. One argues that its most important component is the quadratic part which involves operators of the form $a_{\mathbf{k}}a_{-\mathbf{k}}$, $a_{\mathbf{k}}^* a_{-\mathbf{k}}^*$ and $a_{\mathbf{k}}^* a_{\mathbf{k}}$, $\mathbf{k} \neq 0$. It can be diagonalized by a linear transformation which mixes creation and annihilation operators, called since [5] a *Bogoliubov transformation*, and becomes

$$H_{\mathrm{Bog}} := \sum_{\mathbf{k}\neq 0} e_{\mathbf{k}} b_{\mathbf{k}}^* b_{\mathbf{k}}, \tag{5}$$

$$e_{\mathbf{k}} := \sqrt{\frac{1}{4}|\mathbf{k}|^4 + \frac{\hat{v}(\mathbf{k})}{\hat{v}(0)}\mu|\mathbf{k}|^2}. \tag{6}$$

Thus, the Bogoliubov approximation states that

$$H \approx E_{\mathrm{Bog}} + H_{\mathrm{Bog}} \tag{7}$$

where E_{Bog} is a constant, which will not be relevant for our analysis. The operator $b_{\mathbf{k}}^*$ is the creation operator of the *quasiparticle* with momentum $\mathbf{k}$. It is a linear combination of $a_{\mathbf{k}}^*$, $a_{-\mathbf{k}}$. (5) is sometimes called a *Bogoliubov Hamiltonian*. It describes independent quasiparticles with the *dispersion relation* $e_{\mathbf{k}}$. The *Bogoliubov vacuum*, annihilated by $b_{\mathbf{k}}$ and denoted Ω_{Bog}, is its ground state, and can be treated as an approximate ground state of the many-body system. The Bogoliubov Hamiltonian is still translation invariant: in fact, it commutes with the total momentum, described (without any approximation) by

$$P = \sum_{\mathbf{k}\neq 0} \mathbf{k} b_{\mathbf{k}}^* b_{\mathbf{k}}. \tag{8}$$

It is easy to describe the thermodynamic limit of (5) and (8): we simply replace the summation by integration, without changing the dispersion relation:

$$H_{\mathrm{Bog}} = \int e_{\mathbf{k}} b_{\mathbf{k}}^* b_{\mathbf{k}} \,\mathrm{d}\mathbf{k}, \tag{9}$$

$$P = \int \mathbf{k} b_{\mathbf{k}}^* b_{\mathbf{k}} \,\mathrm{d}\mathbf{k}. \tag{10}$$

It is interesting to visualize possible energy-momentum values predicted by the Bogoliubov approximation or, in a more precise mathematical language, the joint spectrum of the total momentum P and the Bogoliubov Hamiltonian H_{Bog}. On the 1-quasiparticle space this joint spectrum is given by the graph of the function $\mathbf{k} \mapsto e_{\mathbf{k}}$. On Fig. 1 we show a typical form of the dispersion relation in the low momentum region, marked with the black line.

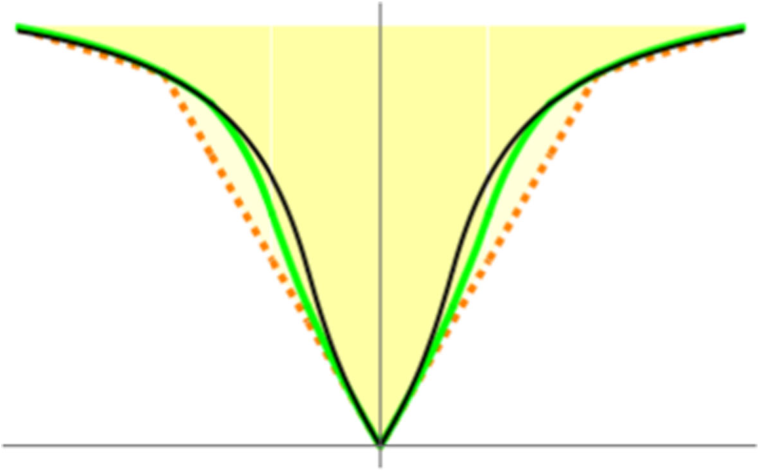

Fig. 1 Joint spectrum of (H_{Bog}, P) for generic potentials

We assume that the potential v satisfies the usual assumptions stated before (1) and the second derivative of $\hat{v}$ near zero is small enough. The green line denotes the bottom of the 2-quasiparticle spectrum, that is the joint spectrum of (H_{Bog}, P) in the 2-quasiparticle sector. The bottom of the full joint spectrum of (H_{Bog}, P) is marked with an orange dashed line.[1] For more details concerning the construction of the excitation spectrum we refer to [12, 18, 19].

One can perform an additional step in the Bogoliubov approach. If the potential v has a very small support, one can argue that $\frac{\hat{v}(\mathbf{k})}{\hat{v}(0)}$ can be approximated by 1. One then usually says that the interaction is given by *contact potentials*, which in the physics literature are often presented in the position representation as $v(x) = 4\pi a\delta(x)$, where a is a constant, called the scattering length. Strictly speaking, this is however not correct. The delta function needs a renormalization to become a well-defined interaction in the two-body case; in the N-body case the situation is even more problematic. In some cases one can justify this approximation in the dilute case using the so-called *Gross-Pitaevski limit*. Anyway, in this approximation we obtain a simpler dispersion relation

$$e_{\mathbf{k}} = \sqrt{\frac{1}{4}|\mathbf{k}|^4 + \mu|\mathbf{k}|^2}. \tag{11}$$

On Fig. 2 we show the energy-momentum spectrum corresponding to (11).

The Hamiltonian H_{Bog}, both with the dispersion relation (6) and (11) has remarkable physical consequences. Note first that the dispersion relation $\mathbf{k} \mapsto e_{\mathbf{k}}$ has a linear cusp at the bottom. It also has a positive critical velocity, that is,

$$c_{\mathrm{crit}} := \sup\{c \mid e_{\mathbf{k}} \geq ck, \quad \mathbf{k} \in \mathrm{R}^3\} > 0. \tag{12}$$

In other words, the graph $\mathbf{k} \mapsto e_{\mathbf{k}}$ is above $\mathbf{k} \mapsto c_{\mathrm{crit}}k$. The full joint spectrum $\sigma(P, H_{\mathrm{Bog}})$ is still above $\mathbf{k} \mapsto c_{\mathrm{crit}}k$. This is interpreted as one of the most important properties of

[1] Strictly speaking, Figures 1 and 2 should be interpreted as follows. We choose coordinates, so that $P = (P_1, P_2, P_3)$. We assume that $P_2 = P_3 = 0$, on the horizontal axis we put P_1, and on the vertical axis H. The full 4-dimensional joint spectrum is rotationally invariant in P, hence easily reconstructed from our pictures.

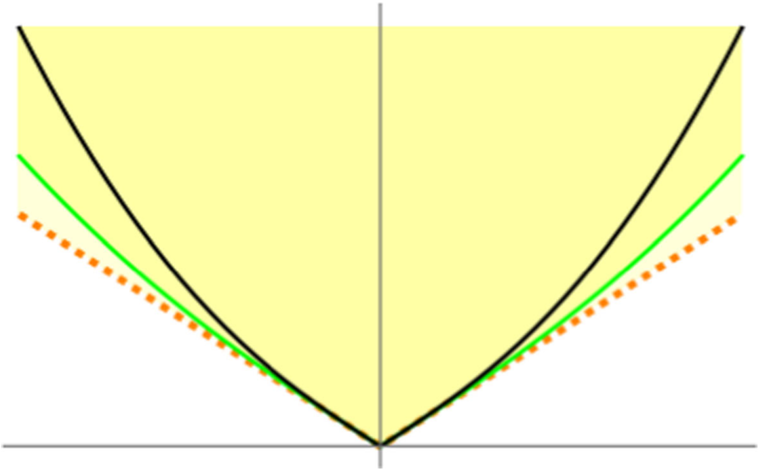

Fig. 2 Joint spectrum of (H_{Bog}, P) for contact potentials

superfluidity: a droplet of the Bose gas travelling with velocity less than $c_{\mathrm{crit}}k$ has negligible friction (see e.g. [12]).

Of course, H_{Bog} yields only an approximate description of the Bose gas. In reality, one cannot treat the quasiparticles given by $b_{\mathbf{k}}^*$, $b_{\mathbf{k}}$ as fully independent. In the derivation of the Bogoliubov Hamiltonian various terms were neglected. In particular, terms of the third and fourth degree in $b_{\mathbf{k}}^*$, $b_{\mathbf{k}}$ were dropped. Replacing v by κv we obtain an (artificial) coupling constant, to be set to 1 at the end. The third order terms are multiplied by $\sqrt{\kappa}$ and the quartic terms by κ. We argue that the quartic terms are of lower order and can be dropped. The third order terms have the form

$$\frac{1}{\sqrt{L^3}} \sum_{\mathbf{k},\mathbf{p},\mathbf{k}+\mathbf{p}\neq 0} u_{\mathbf{k},\mathbf{p}} b_{\mathbf{k}}^* b_{\mathbf{p}}^* b_{\mathbf{k}+\mathbf{p}} + \overline{u_{\mathbf{k},\mathbf{p}}} b_{\mathbf{k}+\mathbf{p}} b_{\mathbf{k}}^* b_{\mathbf{p}}^* \tag{13}$$

$$+\frac{1}{\sqrt{L^3}} \sum_{\mathbf{k},\mathbf{p},\mathbf{k}+\mathbf{p}\neq 0} w_{\mathbf{k},\mathbf{p}} b_{\mathbf{k}}^* b_{\mathbf{p}}^* b_{-\mathbf{k}-\mathbf{p}}^* + \overline{w_{\mathbf{k},\mathbf{p}}} b_{-\mathbf{k}-\mathbf{p}} b_{\mathbf{k}} b_{\mathbf{p}}. \tag{14}$$

We will argue (see Sect. 6) that triple creation and triple annihilation terms do not contribute to the decay of phonons. Thus we drop also (14).

Let us investigate what happens with the quasiparticle state $b_{\mathbf{k}}^*\Omega_{\mathrm{Bog}}$ under the perturbation (13). To this end we first need to check with which states have non-zero matrix elements with $b_{\mathbf{k}}^*\Omega_{\mathrm{Bog}}$. We easily see that it is directly coupled by (13) only to the 2-quasiparticle sector. By taking the thermodynamic limit we can assume that the variable $\mathbf{k}$ is continuous. Thus the perturbed quasiparticle can be described by the space $\mathbb{C} \oplus L^2(\mathbb{R}^3/\mathbb{Z}_2)$ with the Hamiltonian

$$H_{\mathrm{Fried}}(\mathbf{k}) := \begin{bmatrix} e_{\mathbf{k}} & (h_{\mathbf{k}}| \\ |h_{\mathbf{k}}) & e_{\mathbf{p}} + e_{\mathbf{k}-\mathbf{p}} \end{bmatrix}, \tag{15}$$

where $h_{\mathbf{k}}$ can be derived from (13). Here, the action of $\mathbb{Z}_2$ on $\mathbb{R}^3$ is $\mathbf{p} \mapsto \mathbf{k} - \mathbf{p}$, and is related to the Bose symmetry $b_{\mathbf{p}}^* b_{\mathbf{k}-\mathbf{p}}^* \Omega_{\mathrm{Bog}} = b_{\mathbf{k}-\mathbf{p}}^* b_{\mathbf{p}}^* \Omega_{\mathrm{Bog}}$.

Hamiltonians similar to (15) are well understood. They are often used as toy models in quantum physics, and are sometimes called *Friedrichs Hamiltonians*.

It is important to notice that if we set $h_{\mathbf{k}} = 0$, so that the off-diagonal terms in (15) disappear, the unperturbed quasiparticle energy $e_{\mathbf{k}}$ lies inside the continuous spectrum of 2-quasiparticle excitations $\{e_{\mathbf{p}} + e_{\mathbf{k}-\mathbf{p}} \mid \mathbf{p} \in \mathbb{R}^3\}$, at least for small momenta. (To be able to say this we need the thermodynamic limit which makes the momentum continuous.) To see this, note that if $\mathbf{k} \mapsto e_{\mathbf{k}}$ is convex we have a particularly simple expression (cf. Lemma 4) for the infimum of the 2-quasiparticle spectrum:

$$\inf_{\mathbf{p}}\{e_{\mathbf{p}} + e_{\mathbf{k}-\mathbf{p}}\} = 2e_{\mathbf{k}/2}. \tag{16}$$

Now (11) is strictly convex, hence $e_{\mathbf{k}}$ lies inside the continuous spectrum of 2-quasiparticle excitations. If the second derivative of $\hat{v}$ near 0 is small enough, then the generic dispersion relation (6) is convex for small momenta, hence then this property is true at least for small momenta.

Because of that, one can expect that the position of the singularity of the resolvent of (15) becomes complex—it describes a resonance and not a bound state. This is interpreted as the unstability of the quasiparticle: its decay rate is twice the imaginary part of the resonance.

The second order perturbation theory, often called the *Fermi Golden Rule*, says that in order to compute the (complex) energy shift of an eigenvalue we need to find the so-called self-energy $\Sigma_{\mathbf{k}}(z)$, which for $z \notin \mathbb{R}$ in our case is given by the integral

$$\Sigma_{\mathbf{k}}(z) = \frac{1}{2(2\pi)^3} \int \frac{h_{\mathbf{k}}^2(\mathbf{p})\,\mathrm{d}\mathbf{p}}{(z - e_{\mathbf{p}} - e_{\mathbf{k}-\mathbf{p}})}. \tag{17}$$

Then $\Sigma_{\mathbf{k}}(e_{\mathbf{k}} + \mathrm{i}0)$ should give the energy shift of the eigenvalue $e_{\mathbf{k}}$.

The imaginary part of this shift is much easier to compute. In fact, let $\mathcal{P}\frac{1}{x}$ denote the principal value of $\frac{1}{x}$. Applying the Sochocki-Plemelj formula

$$\frac{1}{x + \mathrm{i}0} = \mathcal{P}\frac{1}{x} - \mathrm{i}\pi\delta(x), \tag{18}$$

we obtain

$$\operatorname{Im}\Sigma_{\mathbf{k}}(e_{\mathbf{k}} + \mathrm{i}0) = \frac{-1}{16\pi^2} \int h_{\mathbf{k}}^2(\mathbf{p})\delta(e_{\mathbf{k}} - e_{\mathbf{p}} - e_{\mathbf{k}-\mathbf{p}})\,\mathrm{d}\mathbf{p}. \tag{19}$$

In the main result of our paper we make an assumption which is a compromise between the usual regular case and a contact potential. We assume that $\hat{v}$, the Fourier transform of the potential in a neighborhood of zero is constant, however it decays for large $\mathbf{k}$ sufficiently fast. In Theorem 6 we prove that under these assumptions

$$\operatorname{Im}\Sigma_{\mathbf{k}}(e_k + \mathrm{i}0) = -c_{\mathrm{Bel}}k^5 + O(k^6) \quad \text{as} \quad k \to 0, \quad c_{\mathrm{Bel}} = \frac{3\hat{v}(0)}{640\pi\mu}. \tag{20}$$

Physically (20) means that quasiparticles are almost stable for small k with the lifetime proportional to k^{-5}.

We remark that our analysis is based on the grand-canonical approach where μ is the chemical potential. In the canonical picture the dispersion relation in the thermodynamic limit is conjectured to be

$$e_{\mathbf{k}} = \sqrt{\frac{1}{4}|\mathbf{k}|^4 + 4\pi a\rho|\mathbf{k}|^2}. \tag{21}$$

Comparing (11) with (21) we obtain $4\pi a\rho \approx \mu$. Actually, at positive temperatures ρ should be replaced by the condensate density ρ_0. It is well-known that for weak potentials $\hat{v}(0) \approx$

$4\pi a$. Thus (20) can be rewritten as

$$c_{\mathrm{Bel}} = \frac{3}{640\pi\rho_0}, \tag{22}$$

which is the form of the Beliaev constant usually stated in the physics literature ([14, 25, 34, 43]). In particular, to the leading order the damping rate depends on the potential only through ρ_0.

The Fermi Golden Rule predicts that the real part of the dispersion relation of the interacting system is approximately given by $e_{\mathbf{k}} + \mathrm{Re}\Sigma_{\mathbf{k}}(e_{\mathbf{k}} + \mathrm{i}0)$, where

$$\mathrm{Re}\Sigma_{\mathbf{k}}(e_{\mathbf{k}} + \mathrm{i}0) = \frac{1}{2(2\pi)^3}\mathrm{Re}\int \frac{h_{\mathbf{k}}(\mathbf{p})^2\,\mathrm{d}\mathbf{p}}{(e_{\mathbf{k}} - e_{\mathbf{p}} - e_{\mathbf{k}-\mathbf{p}} + \mathrm{i}0)}. \tag{23}$$

If $\hat{v}(\mathbf{k})$ has a sufficient decay and we use the dispersion relation (6), then (23) is well defined. However if we use the formula (11) for contact potentials, then (23) is divergent for large $\mathbf{k}$. This is related to the fact that constant $\hat{v}(\mathbf{k})$ does not correspond to a well-defined potential (one has to renormalize its coupling constant).

Unfortunately, (23) yields an unphysical prediction for small momenta. Under the same assumptions as in the main theorem, we show that $\lim\limits_{\mathbf{k}\to 0} \Sigma_{\mathbf{k}}(0) = \infty$. Thus the Fermi Golden Rule predicts an infinite energy shift at zero momenta, which is certainly incorrect. This is in agreement with second-order perturbation theory results from physics literature [43].

Similar results about both imaginary and real part of the shift of the dispersion relation can be obtained for more general potentials. We indicate possible generalizations of our result in remarks.

One can conclude that perturbation theory around the Bogoliubov Hamiltonian provides a reasonable method to find the second order imaginary correction to the dispersion relation. However, the computation of its real part seems more dubious, at least for small momenta.

The above problem is an indication of the crudeness of the Bogoliubov approximation. Throwing out the zero mode from the picture (or, which is essentially the same, treating it as a classical quantity), as well as throwing out higher order terms, is a very violent act and we should not be surprised by a punishment. By the way, one expects that the true dispersion relation of phonons goes to zero as $\mathbf{k} \to 0$. This is the content of the so called "Hugenholtz-Pines Theorem" [30], which is a (non-rigorous) argument based on the gauge invariance. Perturbation theory around the Bogoliubov Hamiltonian is compatible with this theorem where it comes to the imaginary part. For the real part it fails.

Better results of computations of the imaginary part over the real part based on the Fermi Golden Rule are not very surprising. It is a general property of Friedrichs Hamiltonians with singular off-diagonal terms: the imaginary part of the perturbed eigenvalue can be computed much more reliably than its real part. We describe this phenomenon briefly in Sects. 2 and 3.

Readers who like clean mathematical results illustrating physical phenomena (which includes the authors) may be somewhat dissatisfied with a relatively long chain of arguments presented in this paper. One of its aspects is the use of a finite system in a box at some of the steps (e.g. Bogoliubov approximation and removal of the zeroth mode), and of the thermodynamical limit in others (computation of the resonance, which requires continuous spectrum, hence, infinite volume). Unfortunately, we do not know a better description. We are just trying to follow the usual physicist's reasoning, without hiding its non-rigorous steps.

Let us now make a few remarks about the literature. The theory of metastable states and their exponential decay goes back to the work of Dirac [21]. The concept of a resonance as a pole on the "unphysical sheet of the complex plane" is usually attributed to Wigner-

Weisskopf [44]. It is discussed, including its historical background, in Chap. XII.6 of [41]. In his lecture notes [23] Fermi formulated two "golden rules" that describe 2nd order theory for eigenvalues and their decay rate. The Friedrichs Hamiltonian is a useful pedagogical toy model, which nicely ilustrates the Fermi Golden Rule. It goes back to [24], see also [17]. An elegant rigorous description of exponential decay expressing the Fermi Golden Rule was given by Davies in his theorem about the weak coupling limit [13], see also [15, 16].

The original paper of Bogoliubov [5] was heuristic, however in recent years there have been many rigorous papers justifying Bogoliubov's approximation in several cases. The first result justifying (7) has been obtained in the mean-field scaling by Seiringer in [42] (see also [19, 26, 32, 38] for related results). Recently, corresponding results have been obtained in the Gross-Pitaevskii regime [3, 10, 39] and even beyond [9]. A time-dependent version of Bogoliubov theory has been successful in describing the dynamics of Bose-Einstein condensates and excitations thereof (see [36, 40] for reviews).

As explained above, to describe damping one has to go beyond Bogoliubov theory. In the mean-field regime this has been done for the ground state energy expansion in [8, 37], including singular interactions [6], and for the dynamics in [7]. Very recently, the results beyond Bogoliubov theory have been obtained in the Gross-Pitaevskii regime [11].

None of the above rigorous papers, with exception of [19], addressed the energy-momentum spectrum. In fact, it is very difficult to study rigorously the dispersion relation in the thermodynamic limit—which is essentially necessary to analyze phonon damping.

The quasiparticle picture of the Bose gas at low temperatures has been confirmed in experiments. The dispersion relation of ^{4}He can be observed in neutron scattering experiments, and is remarkably sharp. It has been measured within a large range of wave numbers covering not only phonons, but also the so-called maxons and rotons, see e.g. [27]. In particular, one can see that the dispersion relation is slightly higher than the 2-quasiparticle spectrum for low wave numbers. The quasiparticle picture has also been confirmed by experiments on Bose Einstein condensates involving alkali atoms. The Beliaev damping has been observed in experiments on Bose Einstein condensates. The results are consistent with theoretical predictions [28, 31]. Note, however, that the precise prediction (19) is difficult to verify experimentally. Bose-Einstein condensates created in labs are not very large, so it is difficult to probe the large wavelength region.

Let us mention that there exists another related phenomenon found in Bose-Einstein condensates, the so-called Landau damping, which involves instability of quasiparticles due to thermal excitations. The Landau damping is absent at zero temperature and becomes dominant at higher temperatures. The Beliaev damping occurs at zero temperature, and for very small temperatures it is still stronger than the Landau damping.

In the physics literature, the damping of phonons was first computed by Beliaev [2]. Landau damping has been for the first time computed by Hohenberg and Martin in [29] (see also [35]). Both these results have been reproduced in [43], also using the formalism of Feynman diagrams and many-body Green's functions. In [34] the damping rate was derived starting from an effective action in the spirit of Popov's hydrodynamical approach. [25] repeated the same computation in the time-dependent mean-field approach. In [14] the mean-field and hydrodynamic approaches were applied to the 2D case. Our derivation is consistent with the above works, however, in our opinion, avoids some unnecessary elements obscuring the simple mechanism of the Beliaev damping.

The plan of the paper is as follows. Sections 2 and 3 concern general well-known facts about about 2nd order perturbation theory of embedded eigenvalues. In Sect. 4 we define the Bose gas Hamiltonian and describe the Bogoliubov approach in the grand-canonical setting. In Sect. 5 we derive heuristically the effective model that we consider. Then, in Sect. 6 we discuss the shape of the energy-momentum spectrum and explain why the contribution from term (14) is irrelevant for the damping rate computation, which is the main result of the paper is proven in Sect. 8 as Theorem 6. The analysis of the real part of the self-energy, and of its (unphysical) behavior at small momenta by the method of this paper is described in Sect. 9.

2 Friedrichs Hamiltonian

Suppose that $\mathcal{H}$ is a Hilbert space with a self-adjoint operator H. Let $\Psi \in \mathcal{H}$ be a normalized vector. We can write $\mathcal{H} \simeq \mathbb{C} \oplus \mathcal{K}$, where $\mathbb{C} \simeq \mathbb{C}\Psi$ and $\mathcal{K} := \{\Psi\}^\perp$. First assume that Ψ belongs to the domain of H and set

$$E_0 := (\Psi|H\Psi), \quad h := H\Psi - E_0\Psi. \tag{24}$$

Note that $h \in \mathcal{K}$. Let K denote H compressed to $\mathcal{K}$. That means, if $I : \mathcal{K} \to \mathcal{H}$ is the embedding, then $K := I^* H I$. Then in terms of $\mathbb{C} \oplus \mathcal{K}$ we can write

$$H = \begin{bmatrix} E_0 & (h| \\ |h) & K \end{bmatrix}. \tag{25}$$

Operators of this form were studied by Friedrichs in [24]. Therefore, sometimes they are referred to as *Friedrichs Hamiltonians*, e.g. in [15, 17]:

Let $z \in \mathbb{C}$. The following identity is a special case of the so-called *Feshbach-Schur formula*:

$$(\Psi|(H-z)^{-1}\Psi) = \frac{1}{E_0 + \Sigma(z) - z}, \tag{26}$$

$$\Sigma(z) := -(h|(K-z)^{-1}h). \tag{27}$$

Following a part of the physics literature, we will call $\Sigma(z)$ the *self-energy*. For further reference let us rewrite (26) as

$$\Sigma(z) = \frac{1}{(\Psi|(H-z)^{-1}\Psi)} + z - E_0. \tag{28}$$

Note that the full resolvent of H can be computed, see e.g. [17], or Equation (1.2) of [20]:

$$\begin{aligned}(H-z)^{-1} = &\begin{bmatrix} 0 & 0 \\ 0 & (K-z)^{-1} \end{bmatrix} \\ &+ \begin{bmatrix} 1 \\ (K-z)^{-1}|h) \end{bmatrix} \frac{1}{E_0 + \Sigma(z) - z} \begin{bmatrix} 1 & (h|(K-z)^{-1} \end{bmatrix}.\end{aligned} \tag{29}$$

If K has continuous spectrum, it often happens that $\Sigma(z)$ can be continued analytically from the upper complex halfplane across the spectrum to the *non-physical sheet of the complex plane*. Then $(\Psi|(H-z)^{-1}\Psi)$ may have a singularity for $z = E = E_\mathrm{R} - \mathrm{i}\frac{\Gamma}{2}$ with $\Gamma > 0$. This singularity E is called a *resonance*. Suppose that Γ is small. A well-known non-rigorous argument, involving a change of the contour of integration and described e.g. in Chap. XII.6 of [41] (see also [22]), shows that over a long period of time (not too small and not too large) we have

$$(\Psi|\mathrm{e}^{-\mathrm{i}tH}\Psi) \simeq C\mathrm{e}^{-\mathrm{i}E_\mathrm{R}t - \frac{\Gamma}{2}t}. \tag{30}$$

This is interpreted as exponential decay of the state Ψ with the decay rate Γ.

We can apply the formulas (26)- (29) also if Ψ does not belong to the domain of H, but belongs to its form domain, so that $(\Psi|H\Psi)$ is well defined. Note that E_0 and $\Sigma(z)$ are then uniquely defined by (24) and (28).

If Ψ does not belong to the form domain of H, then strictly speaking the self-energy is ill defined. In practice in such situations one often introduces a cutoff Hamiltonian H^Λ, which in some sense approximates H. Then, setting $h^\Lambda := H^\Lambda\Psi$, $E_0^\Lambda := (\Psi|H^\Lambda\Psi)$, and denoting by K^Λ the operator H^Λ compressed to $\mathcal{K}$, one can use the cutoff version of the Feshbach-Schur formula:

$$(\Psi|(H^\Lambda - z)^{-1}\Psi) = \frac{1}{E_0^\Lambda + \Sigma^\Lambda(z) - z}, \tag{31}$$

$$\Sigma^\Lambda(z) = -(h^\Lambda|(K^\Lambda - z)^{-1}h^\Lambda). \tag{32}$$

The resolvent of the original Hamiltonian H can be retrieved [17] in the limit $\Lambda \to \infty$:

$$(H - z)^{-1} = \lim_{\Lambda\to\infty}(H^\Lambda - z)^{-1}. \tag{33}$$

Note that E_0^Λ is a sequence of real numbers, typically converging to ∞. They can be treated as *counterterms* renormalizing the self-energy $\Sigma^\Lambda(z)$.

3 Fermi Golden Rule

The meaning of the self-energy is especially clear in perturbation theory. Again, let Ψ be a normalized vector in $\mathcal{H}$. Consider a family of self-adjoint operators $H_\lambda = H_0 + \lambda V$ such that $H_0\Psi = E_0\Psi$. In order to avoid discussing 1st order perturbation theory we assume that $(\Psi|V\Psi) = 0$. Let $h := V\Psi - \frac{1}{\lambda}E_0\Psi$ and K_λ be H_λ compressed to $\mathcal{K}$. Thus we rewrite (25) as

$$H_\lambda = \begin{bmatrix} E_0 & \lambda(h| \\ \lambda|h) & K_\lambda \end{bmatrix}. \tag{34}$$

We extract λ^2 from the definition of the self-energy, so that (26) and (27) are rewritten as

$$(\Psi|(H_\lambda - z)^{-1}\Psi) = \big(E_0 + \lambda^2\Sigma_\lambda(z) - z\big)^{-1}, \tag{35}$$

$$\Sigma_\lambda(z) := -(h|(K_\lambda - z)^{-1}h) = \Sigma_0(z) + O(\lambda). \tag{36}$$

Now (35) has a pole at

$$E_0 + \lambda^2\Sigma_0(E_0 + \mathrm{i}0) + O(\lambda^3). \tag{37}$$

This is often formulated as the *Fermi Golden Rule*: the pole of the resolvent, originally at an eigenvalue E_0, is shifted in the second order by $\lambda^2\Sigma_0(E_0+\mathrm{i}0)$. This shift can have a negative imaginary part, and then the eigenvalue disappears, and instead we have a resonance.

For small couplings λ a rigorous meaning of the decay property (30) is provided by the following version of the *weak coupling limit* ([13], see also [15, 16])

$$\lim_{\lambda\to 0}\big(\Psi\big|\exp\big(-\mathrm{i}\tfrac{t}{\lambda^2}(H_\lambda - E_0)\big)\Psi\big) = \mathrm{e}^{-\mathrm{i}t\Sigma_0(E_0+i0)}. \tag{38}$$

If the perturbation is singular, so that Ψ does not belong to the domain of V, then $\Sigma_0(z)$ is in general ill defined and (37) may lose its meaning. Strictly speaking, one then needs to introduce a cutoff on the perturbation and a counterterm, and only then to apply the appropriately modified Fermi Golden Rule.

Note that it is enough to consider real counterterms. Therefore, if we know that the renormalized energy is close to E_0, then we can still expect that (37) gives a correct prediction for the imaginary part of the resonance. In other words, the imaginary part of the singularity of the resolvent $(H_\lambda - z)^{-1}$ is

$$\lambda^2 \mathrm{Im}\Sigma_0(E_0 + \mathrm{i}0) + O(\lambda^3), \tag{39}$$

where we do not need to cut off the perturbation.

In practice, we start from a singular expression of the form (34). To make it well-defined we need to choose a cutoff and counterterms. These choices will not affect the imaginary part of the resonance, however in principle, one can add an arbitrary real constant to a counterterm, which will affect the real part of the resonance. Therefore, for singular perturbations it may be more difficult to predict the real part of the resonance.

4 Bose Gas and Bogoliubov Ansatz

We consider a homogeneous Bose gas of N particles with a two-body potential described by a function $v : \mathbb{R}^3 \to \mathbb{R}$ with the Fourier transform $\hat{v}(\mathbf{k}) = \int_{\mathbb{R}^3} v(x)\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{x}}\,\mathrm{d}\mathbf{x}$. In the grand canonical setting and the momentum representation such a system is governed by the (second quantized) Hamiltonian

$$H = \int \left(\frac{\mathbf{k}^2}{2} - \mu\right) a_{\mathbf{k}}^* a_{\mathbf{k}}\,\mathrm{d}\mathbf{k} + \frac{\kappa}{2(2\pi)^3}\int \mathrm{d}\mathbf{p}\int \mathrm{d}\mathbf{q}\int \mathrm{d}\mathbf{k}\,\hat{v}(\mathbf{k}) a_{\mathbf{p}-\mathbf{k}}^* a_{\mathbf{q}+\mathbf{k}}^* a_{\mathbf{p}} a_{\mathbf{q}}, \tag{40}$$

where $\mu \geq 0$ is the chemical potential and $a_{\mathbf{k}}^*/a_{\mathbf{k}}$ the creation/annihilation operators for particles of mode $\mathbf{k}$. It acts on the bosonic Fock space $\mathcal{F} = \Gamma_{\mathrm{s}}\big(L^2(\mathbb{R}^3)\big)$, and for each N it leaves invariant its N-particle sector $L^2_{\mathrm{s}}\big((\mathbb{R}^3)^N\big)$. Recall that the creation and annihilation operators satisfy the canonical commutation relation (CCR):

$$[a_{\mathbf{p}}, a_{\mathbf{q}}] = 0 = [a_{\mathbf{p}}^*, a_{\mathbf{q}}^*], \quad [a_{\mathbf{p}}, a_{\mathbf{q}}^*] = \delta(\mathbf{p} - \mathbf{q}), \tag{41}$$

where [ ] is the usual commutator. We introduce the coupling constant $\kappa > 0$ mostly for bookkeeping purposes; note that in the introduction we set $\kappa = 1$.

In most of the paper we will make the following assumption on the potentials:

$$v \in L^1(\mathbb{R}^3), \quad \text{so that } \hat{v} \text{ is a continuous function;} \tag{42a}$$

$$\hat{v}(\mathbf{k}) \geq 0, \quad \mathbf{k} \in \mathbb{R}^3; \tag{42b}$$

$$\hat{v}(\mathbf{k}) = \hat{v}(0) > 0, \quad \text{for } |\mathbf{k}| < \Lambda, \quad \Lambda > 0; \tag{42c}$$

$$|\hat{v}(\mathbf{k})| \leq C(1 + |\mathbf{k}|)^{-\frac{1}{2}-\varepsilon}, \qquad \text{for some } \varepsilon > 0; \tag{42d}$$

$$v \text{ is rotationally invariant} \qquad . \tag{42e}$$

Remark 1 One can relax the condition (42c) to allow for generic potentials. One could also consider potentials which for some constant ν satisfy

$$\hat{v}(\mathbf{k}) = \hat{v}(0) + \frac{\nu}{2}|\mathbf{k}|^2 + O(|\mathbf{k}|^{2+\varepsilon}), \qquad \varepsilon > 0. \tag{43}$$

We will comment about possible extensions of our results to potentials satisfying (43) instead of (42c).

For the reasons explained in the introduction, we replace the infinite space $\mathbb{R}^3$ by the torus $[-L/2, L/2]^3$ with periodic boundary conditions. In the momentum representation the Hamiltonian becomes

$$H = \sum_{\mathbf{k}\in 2\pi\mathbb{Z}^3/L} \left(\frac{\mathbf{k}^2}{2} - \mu\right) a_{\mathbf{k}}^* a_{\mathbf{k}} + \frac{\kappa}{2L^3} \sum_{\mathbf{p},\mathbf{q},\mathbf{k}\in 2\pi\mathbb{Z}^3/L} \hat{v}(\mathbf{k}) a_{\mathbf{p}-\mathbf{k}}^* a_{\mathbf{q}+\mathbf{k}}^* a_{\mathbf{p}} a_{\mathbf{q}}. \tag{44}$$

Note that $\hat{v}$ is the same function as in (40), however it is now sampled only on the lattice $2\pi\mathbb{Z}^3/L$. The commutation relations involve now the Kronecker delta:

$$[a_{\mathbf{p}}, a_{\mathbf{q}}] = 0 = [a_{\mathbf{p}}^*, a_{\mathbf{q}}^*], \quad [a_{\mathbf{p}}, a_{\mathbf{q}}^*] = \delta_{\mathbf{p},\mathbf{q}}. \tag{45}$$

Let us now pass to the quasiparticle representation. To this end we follow the well-known grand-canonical version of the Bogoliubov approach (see e.g. [12]). It involves two unitary transformations.

The first one is a Weyl transformation that introduces a macroscopic occupation of the zero-momentum mode, the Bose-Einstein condensate. (In the canonical version Bogoliubov approach this corresponds to the c-number substitution [33].) To this end, for $\alpha \in \mathbb{C}$, we introduce the Weyl operator of the mode $\mathbf{k} = 0$

$$W_\alpha = \exp(-\alpha a_0^* + \bar{\alpha} a_0). \tag{46}$$

Then

$$W_\alpha^* a_{\mathbf{k}}^* W_\alpha = a_{\mathbf{k}}^* - \bar{\alpha}\delta_{\mathbf{k},0} =: \tilde{a}_{\mathbf{k}}^*.$$

The new annihilation operators with tildes kill the "new vacuum" $\Omega_\alpha = W_\alpha^* \Omega$. We express our Hamiltonian in terms of $\tilde{a}_{\mathbf{k}}^*, \tilde{a}_{\mathbf{k}}$. To simplify the notation, in what follows we drop the tildes and we obtain

$$\begin{aligned} H = &-\mu|\alpha|^2 + \frac{\kappa\hat{v}(0)}{2L^3}|\alpha|^4 + \left(\frac{\kappa\hat{v}(0)}{L^3}|\alpha|^2 - \mu\right)(\alpha a_0^* + \bar{\alpha} a_0) \\ &+ \sum_{\mathbf{k}} \left(\frac{\mathbf{k}^2}{2} - \mu + \frac{\kappa(\hat{v}(\mathbf{k}) + \hat{v}(0))}{L^3}|\alpha|^2\right) a_{\mathbf{k}}^* a_{\mathbf{k}} + \sum_{\mathbf{k}} \frac{\kappa\hat{v}(\mathbf{k})}{2L^3}\left(\alpha^2 a_{\mathbf{k}}^* a_{-\mathbf{k}}^* + \bar{\alpha}^2 a_{\mathbf{k}} a_{-\mathbf{k}}\right) \\ &+ \frac{\kappa}{L^3} \sum_{\mathbf{k}_1,\mathbf{k}_2} \hat{v}(\mathbf{k}_1)\left(\bar{\alpha} a_{\mathbf{k}_1+\mathbf{k}_2}^* a_{\mathbf{k}_1} a_{\mathbf{k}_2} + \alpha a_{\mathbf{k}_1}^* a_{\mathbf{k}_2}^* a_{\mathbf{k}_1+\mathbf{k}_2}\right) \\ &+ \frac{\kappa}{2L^3} \sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4} \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4)\hat{v}(\mathbf{k}_2 - \mathbf{k}_3) a_{\mathbf{k}_1}^* a_{\mathbf{k}_2}^* a_{\mathbf{k}_3} a_{\mathbf{k}_4}. \end{aligned}$$

Note that we have

$$(\Omega_\alpha | H\Omega_\alpha) = -\mu|\alpha|^2 + \frac{\kappa\hat{v}(0)}{2L^3}|\alpha|^4,$$

and we choose $\alpha = \sqrt{\frac{\mu L^3}{\kappa\hat{v}(0)}}$, so that Ω_α minimizes this expectation value. This leads to

$$\begin{aligned} H &= \kappa^{-1} H_0 + H_2 + \sqrt{\kappa} H_3 + \kappa H_4, \qquad (47) \\ H_0 &:= -\frac{\mu^2 L^3}{2\hat{v}(0)}, \\ H_2 &:= \sum_{\mathbf{k}} \left(\frac{\mathbf{k}^2}{2} + \frac{\mu\hat{v}(\mathbf{k})}{\hat{v}(0)}\right) a_{\mathbf{k}}^* a_{\mathbf{k}} + \sum_{\mathbf{k}} \frac{\mu\hat{v}(\mathbf{k})}{2\hat{v}(0)}\left(a_{\mathbf{k}}^* a_{-\mathbf{k}}^* + a_{\mathbf{k}} a_{-\mathbf{k}}\right), \end{aligned}$$

$$H_3 := \frac{1}{L^{3/2}} \sum_{\mathbf{k}_1,\mathbf{k}_2} \frac{\hat{v}(\mathbf{k}_1)\sqrt{\mu}}{\sqrt{\hat{v}(0)}} \left(a^*_{\mathbf{k}_1+\mathbf{k}_2} a_{\mathbf{k}_1} a_{\mathbf{k}_2} + a^*_{\mathbf{k}_1} a^*_{\mathbf{k}_2} a_{\mathbf{k}_1+\mathbf{k}_2}\right),$$

$$H_4 := \frac{1}{2L^3} \sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4} \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4)\hat{v}(\mathbf{k}_2 - \mathbf{k}_3) a^*_{\mathbf{k}_1} a^*_{\mathbf{k}_2} a_{\mathbf{k}_3} a_{\mathbf{k}_4}.$$

We extract from the above Hamiltonian all terms containing only non-zero modes:

$$H_2 = \frac{\mu}{2}(a_0^{*2} + a_0^2 + 2a_0^* a_0) + H_2^{\text{exc}},$$

$$H_2^{\text{exc}} := \sum_{\mathbf{k}\neq 0} \left(\frac{\mathbf{k}^2}{2} + \frac{\mu\hat{v}(\mathbf{k})}{\hat{v}(0)}\right) a^*_{\mathbf{k}} a_{\mathbf{k}} + \sum_{\mathbf{k}\neq 0} \frac{\mu\hat{v}(\mathbf{k})}{2\hat{v}(0)} \left(a^*_{\mathbf{k}} a^*_{-\mathbf{k}} + a_{\mathbf{k}} a_{-\mathbf{k}}\right); \tag{48}$$

$$H_3 = \frac{1}{L^{3/2}} \sum_{\mathbf{k}} \sqrt{\mu\hat{v}(0)}(a_0^* a^*_{\mathbf{k}} a_{\mathbf{k}} + a^*_{\mathbf{k}} a_{\mathbf{k}} a_0)$$
$$+ \frac{1}{L^{3/2}} \sum_{\mathbf{k}\neq 0} \frac{\sqrt{\mu}\hat{v}(\mathbf{k})}{\sqrt{\hat{v}(0)}} \left((a_0^* + a_0) a^*_{\mathbf{k}} a_{\mathbf{k}} + a_0 a^*_{\mathbf{k}} a^*_{-\mathbf{k}} + a_0^* a_{\mathbf{k}} a_{-\mathbf{k}}\right) + H_3^{\text{exc}},$$

$$H_3^{\text{exc}} := \frac{1}{L^{3/2}} \sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_1+\mathbf{k}_2\neq 0} \frac{\hat{v}(\mathbf{k}_1)\sqrt{\mu}}{\sqrt{\hat{v}(0)}} \left(a^*_{\mathbf{k}_1+\mathbf{k}_2} a_{\mathbf{k}_1} a_{\mathbf{k}_2} + a^*_{\mathbf{k}_1} a^*_{\mathbf{k}_2} a_{\mathbf{k}_1+\mathbf{k}_2}\right); \tag{49}$$

$$H_4 = \frac{1}{2L^3}\hat{v}(0)\Big(a_0^* a_0^* a_0 a_0 + 2\sum_{\mathbf{k}\neq 0} a_0^* a_0 a^*_{\mathbf{k}} a_{\mathbf{k}}\Big)$$
$$+ \frac{1}{2L^3} \sum_{\mathbf{k}\neq 0} \hat{v}(\mathbf{k})(a_0^* a_0^* a_{\mathbf{k}} a_{-\mathbf{k}} + a_0 a_0 a^*_{\mathbf{k}} a^*_{-\mathbf{k}} + 2a_0^* a_0 a^*_{\mathbf{k}} a_{\mathbf{k}})$$
$$+ \frac{1}{L^3} \sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_1+\mathbf{k}_2\neq 0} \hat{v}(k_1)\Big(a_0^* a^*_{\mathbf{k}_1+\mathbf{k}_2} a_{\mathbf{k}_1} a_{\mathbf{k}_2} + a_0 a^*_{\mathbf{k}_1} a^*_{\mathbf{k}_2} a_{\mathbf{k}_1+\mathbf{k}_2}\Big) + H_4^{\text{exc}},$$

$$H_4^{\text{exc}} := \frac{1}{2L^3} \sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4\neq 0} \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4)\hat{v}(\mathbf{k}_2 - \mathbf{k}_3) a^*_{\mathbf{k}_1} a^*_{\mathbf{k}_2} a_{\mathbf{k}_3} a_{\mathbf{k}_4}. \tag{50}$$

Let

$$\sigma_{\mathbf{k}} = \frac{\sqrt{\sqrt{e_{\mathbf{k}}^2 + B_{\mathbf{k}}^2} + e_{\mathbf{k}}}}{\sqrt{2e_{\mathbf{k}}}}, \quad \gamma_{\mathbf{k}} = \frac{\sqrt{\sqrt{e_{\mathbf{k}}^2 + B_{\mathbf{k}}^2} - e_{\mathbf{k}}}}{\sqrt{2e_{\mathbf{k}}}},$$
$$\beta_{\mathbf{k}} = \cosh^{-1}(\sigma_{\mathbf{k}}) = \sinh^{-1}(\gamma_{\mathbf{k}}),$$
$$e_{\mathbf{k}} := \sqrt{\frac{1}{4}|\mathbf{k}|^4 + B_{\mathbf{k}}|\mathbf{k}|^2}, \quad B_{\mathbf{k}} := \frac{\hat{v}(\mathbf{k})}{\hat{v}(0)}\mu. \tag{51}$$

Sometimes we will write e_k, σ_k, γ_k, instead of $e_{\mathbf{k}}, \sigma_{\mathbf{k}}, \gamma_{\mathbf{k}}$. We are going to apply a Bogoliubov transformation

$$U_{\text{Bog}} := \exp\left(\sum_{\mathbf{k}\neq 0} \beta_{\mathbf{k}}(a^*_{\mathbf{k}} a^*_{-\mathbf{k}} - a_{\mathbf{k}} a_{-\mathbf{k}})\right), \tag{52}$$

which transforms non-zero mode operators $a^*_{\mathbf{k}}, a_{\mathbf{k}}$ into quasi-particle operators $b^*_{\mathbf{k}}, b_{\mathbf{k}}$:

$$b_{\mathbf{k}} := U_{\text{Bog}} a_{\mathbf{k}} U^*_{\text{Bog}} = \sigma_{\mathbf{k}} a_{\mathbf{k}} + \gamma_{\mathbf{k}} a^*_{-\mathbf{k}},$$
$$b^*_{\mathbf{k}} := U_{\text{Bog}} a^*_{\mathbf{k}} U^*_{\text{Bog}} = \sigma_{\mathbf{k}} a^*_{\mathbf{k}} + \gamma_{\mathbf{k}} a_{-\mathbf{k}}, \tag{53}$$

Let us also note the relation inverse to (53):

$$a_{\mathbf{k}} = \sigma_{\mathbf{k}} b_{\mathbf{k}} - \gamma_{\mathbf{k}} b^*_{-\mathbf{k}},$$
$$a^*_{\mathbf{k}} = \sigma_{\mathbf{k}} b^*_{\mathbf{k}} - \gamma_{\mathbf{k}} b_{-\mathbf{k}}.$$

It is well known that (53) diagonalizes H_2^{exc} in terms of the quasi-particle operators:

$$H_2^{\mathrm{exc}} = E_{\mathrm{Bog}} + H_{\mathrm{Bog}}, \tag{54}$$

where

$$E_{\mathrm{Bog}} := -\frac{1}{2}\sum_{\mathbf{k}\neq 0}\left(\frac{1}{2}|\mathbf{k}|^2 + \frac{\hat{v}(\mathbf{k})}{\hat{v}(0)}\mu - e_{\mathbf{k}}\right), \tag{55}$$

$$H_{\mathrm{Bog}} := \sum_{\mathbf{k}\neq 0} e_{\mathbf{k}} b^*_{\mathbf{k}} b_{\mathbf{k}}. \tag{56}$$

We also express H_3^{exc} in terms of quasiparticles:

$$H_3^{\mathrm{exc}} = \frac{1}{L^{3/2}} \sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_1+\mathbf{k}_2\neq 0} \frac{\sqrt{\mu}\hat{v}(\mathbf{k}_1)}{\sqrt{\hat{v}(0)}} \tag{57}$$
$$\Big(\big(\sigma_{\mathbf{k}_1+\mathbf{k}_2} b^*_{\mathbf{k}_1+\mathbf{k}_2} - \gamma_{-\mathbf{k}_1-\mathbf{k}_2} b_{-\mathbf{k}_1-\mathbf{k}_2}\big)\big(\sigma_{\mathbf{k}_1} b_{\mathbf{k}_1} - \gamma_{-\mathbf{k}_1} b^*_{-\mathbf{k}_1}\big)\big(\sigma_{\mathbf{k}_2} b_{\mathbf{k}_2} - \gamma_{-\mathbf{k}_2} b^*_{-\mathbf{k}_2}\big)$$
$$+ \big(\sigma_{\mathbf{k}_1} b^*_{\mathbf{k}_1} - \gamma_{-\mathbf{k}_1} b_{-\mathbf{k}_1}\big)\big(\sigma_{\mathbf{k}_2} b^*_{\mathbf{k}_2} - \gamma_{-\mathbf{k}_2} b_{-\mathbf{k}_2}\big)\big(\sigma_{\mathbf{k}_1+\mathbf{k}_2} b_{\mathbf{k}_1+\mathbf{k}_2} - \gamma_{-\mathbf{k}_1-\mathbf{k}_2} b^*_{-\mathbf{k}_1-\mathbf{k}_2}\big)\Big).$$

After opening the brackets and using $\sigma_{\mathbf{k}} = \sigma_{-\mathbf{k}}$ and $\gamma_{\mathbf{k}} = \gamma_{-\mathbf{k}}$, we transform this into

$$H_3^{\mathrm{exc}} = H_{3,1}^{\mathrm{exc}} + H_{3,2}^{\mathrm{exc}}, \tag{58}$$

$$H_{3,1}^{\mathrm{exc}} = \sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_1+\mathbf{k}_2\neq 0} \big(b^*_{\mathbf{k}_1+\mathbf{k}_2} b_{\mathbf{k}_1} b_{\mathbf{k}_2} + b^*_{\mathbf{k}_1} b^*_{\mathbf{k}_2} b_{\mathbf{k}_1+\mathbf{k}_2}\big) \tag{59}$$
$$\Bigg(\frac{\sqrt{\mu}\hat{v}(\mathbf{k}_1)}{L^{3/2}\sqrt{\hat{v}(0)}}\big(\sigma_{\mathbf{k}_1+\mathbf{k}_2}\sigma_{\mathbf{k}_1}\sigma_{\mathbf{k}_2} - \gamma_{\mathbf{k}_1+\mathbf{k}_2}\gamma_{\mathbf{k}_1}\gamma_{\mathbf{k}_2} + \gamma_{\mathbf{k}_1+\mathbf{k}_2}\sigma_{\mathbf{k}_1}\gamma_{\mathbf{k}_2} - \sigma_{\mathbf{k}_1+\mathbf{k}_2}\gamma_{\mathbf{k}_1}\sigma_{\mathbf{k}_2}\big)$$
$$+ \frac{\sqrt{\mu}\hat{v}(\mathbf{k}_1+\mathbf{k}_2)}{L^{3/2}\sqrt{\hat{v}(0)}}\big(\gamma_{\mathbf{k}_1+\mathbf{k}_2} - \sigma_{\mathbf{k}_1+\mathbf{k}_2}\big)\gamma_{\mathbf{k}_1}\sigma_{\mathbf{k}_2}\Bigg),$$

$$H_{3,2}^{\mathrm{exc}} = \sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_1+\mathbf{k}_2\neq 0} \big(b^*_{-\mathbf{k}_1-\mathbf{k}_2} b^*_{\mathbf{k}_1} b^*_{\mathbf{k}_2} + b_{-\mathbf{k}_1-\mathbf{k}_2} b_{\mathbf{k}_1} b_{\mathbf{k}_2}\big) \tag{60}$$
$$\frac{\sqrt{\mu}\hat{v}(\mathbf{k}_1)}{L^{3/2}\sqrt{\hat{v}(0)}}\big(\gamma_{\mathbf{k}_1}\gamma_{\mathbf{k}_2}\sigma_{\mathbf{k}_1+\mathbf{k}_2} - \sigma_{\mathbf{k}_1}\sigma_{\mathbf{k}_2}\gamma_{\mathbf{k}_1+\mathbf{k}_2}\big).$$

We could also compute H_4, but we will not need it.

5 Effective Friedrichs Hamiltonian

Recall that $\Omega_\alpha = W_\alpha^*\Omega$. Let $\Omega_{\mathrm{Bog}} := U^*_{\mathrm{Bog}}\Omega_\alpha$ be the quasiparticle vacuum. Let $\mathrm{Span}^{\mathrm{cl}}(K)$ denote the closure of the span of the set $K \subset \mathcal{F}$. Introduce the space $\mathcal{F}^{\mathrm{exc}}$ consisting of the Bogoliubov vacuum and quasiparticle excitations, and its n-quasiparticle sector:

$$\mathcal{F}^{\mathrm{exc}} := \mathrm{Span}^{\mathrm{cl}}\{b^*_{\mathbf{k}_1}\cdots b^*_{\mathbf{k}_n}\Omega_{\mathrm{Bog}} \mid \mathbf{k}_1,\dots,\mathbf{k}_n \neq 0, \quad n = 0,1,\dots\},$$
$$\mathcal{F}^{\mathrm{exc}}_n := \mathrm{Span}^{\mathrm{cl}}\{b^*_{\mathbf{k}_1}\cdots b^*_{\mathbf{k}_n}\Omega_{\mathrm{Bog}} \mid \mathbf{k}_1,\dots,\mathbf{k}_n \neq 0\}.$$

The most "violent" approximation that we are going to make is compressing the Hamiltonian H into the space $\mathcal{F}^{\rm exc}$. We also drop the uninteresting constant $\kappa^{-1}H_0$ and the (somewhat more interesting) constant $E_{\rm Bog}$. Thus we introduce the *excitation Hamiltonian*

$$H^{\rm exc} := I^{\rm exc*}\big(H - \kappa^{-1}H_0 - E_{\rm Bog}\big)I^{\rm exc},$$

where $I^{\rm exc}$ denotes the embedding of $\mathcal{F}^{\rm exc}$ in $\mathcal{F}$. Thus $H^{\rm exc}$ is an operator on $\mathcal{F}^{\rm exc}$ and

$$H^{\rm exc} = H_{\rm Bog} + \sqrt{\kappa}H_3^{\rm exc} + \kappa H_4^{\rm exc}, \tag{61}$$

where $H_3^{\rm exc}$ and $H_4^{\rm exc}$ are defined in (49) and (50).

Remark 2 Let us make some remarks concerning the algebraic meaning of the above construction. Our physical space (in finite volume and in the momentum representation) is the bosonic Fock space over the 1-particle space $l^2\big(\frac{2\pi}{L}\mathbb{Z}\big)$. By the exponential property of Fock spaces (see e.g. [20]) we have the following identification:

$$\Gamma_{\rm s}\Big(l^2\big(\tfrac{2\pi}{L}\mathbb{Z}\big)\Big) \simeq \Gamma_{\rm s}(\mathbb{C}) \otimes \Gamma_{\rm s}\Big(l^2\big(\tfrac{2\pi}{L}\mathbb{Z}\setminus\{0\}\big)\Big), \tag{62}$$

where the first factor describes the "zeroth mode" treated as the "condensate" and the second "excitations outside of the condensate". We will denote by U the (unitary and canonical) identification described in (62). Note that creation and annihilation operators of non-zero modes, $a_{\mathbf{k}}^*, a_{\mathbf{k}}$, $\mathbf{k} \neq 0$, as well as of quasiparticles $b_{\mathbf{k}}^*, b_{\mathbf{k}}$ act only in the second factor. The translations also act only in the second factor. The coherent vector $\Omega_\alpha = W_\alpha^*\Omega$ is translation invariant and can be understood as an element of the first factor. Thus U identifies $\mathcal{F}^{\rm exc}$ with

$$\Omega_\alpha \otimes \Gamma_{\rm s}\Big(l^2\big(\tfrac{2\pi}{L}\mathbb{Z}\setminus\{0\}\big)\Big), \tag{63}$$

The compressed Hamiltonian $H^{\rm exc}$ can be then interpreted as

$$H^{\rm exc} = (\Omega_\alpha|\, UHU^*|\Omega_\alpha), \tag{64}$$

which is an operator on $\Gamma_{\rm s}\Big(l^2\big(\frac{2\pi}{L}\mathbb{Z}\setminus\{0\}\big)\Big)$.

The idea of decomposing the Fock space as in (62), where the first factor describes the "codensate", is common in the literature. It is e.g. used in [19] and (implicitly) in the paper by Lewin-Nam-Serfaty-Solovej [32].

Our compression construction is essentially the most direct interpretation of the "replacing the zeroth mode by a c-number", which is a very common procedure in the physics literature. Physicists expect that this procedure yields physically relevant results. And so do we, at least concerning the imaginary part of the dispersion relation.

However, of course, compression produces an operator which is not unitarily equivalent to the initial operator. Therefore, it is certainly a rather fishy step in our analysis: rigorously it is not clear how much the analysis of $H^{\rm exc}$ will say about H.

We make two more approximations. We drop κH_4, which is of higher order in κ than $\sqrt{\kappa}H_3$. We also drop $H_{3,2}$, which involves 3-quasiparticle creation/annihilation operators, and does not contribute to the damping rate (see Sect. 6 for a justification). Thus $H^{\rm exc}$ is replaced with

$$H^{\rm eff} := H_{\rm Bog} + \sqrt{\kappa}H_{3,1}^{\rm exc}. \tag{65}$$

To make our following discussion consistent with Sect. 3 about the Fermi Golden Rule, we introduce a new coupling constant

$$\lambda := \sqrt{\kappa}. \tag{66}$$

Let $\mathbf{k} \neq 0$. Clearly, $b_{\mathbf{k}}^{*}\Omega_{\mathrm{Bog}}$ is an eigenstate of H^{eff} for $\lambda = 0$. We would like to compute the self-energy for the vector $b_{\mathbf{k}}^{*}\Omega_{\mathrm{Bog}}$ and the Hamiltonian H^{eff}:

$$\lambda^2 \Sigma_{\mathbf{k}}^{\mathrm{eff}}(z) := \frac{-1}{(b_{\mathbf{k}}^{*}\Omega_{\mathrm{Bog}}|(z - H^{\mathrm{eff}})^{-1} b_{\mathbf{k}}^{*}\Omega_{\mathrm{Bog}})} + z - e_{\mathbf{k}}. \tag{67}$$

Introduce the subspaces of $\mathcal{F}^{\mathrm{exc}}$ and $\mathcal{F}_n^{\mathrm{exc}}$ with the total momentum $\mathbf{k}$:

$$\mathcal{F}^{\mathrm{exc}}(\mathbf{k}) := \mathrm{Span}^{\mathrm{cl}}\{b_{\mathbf{k}_1}^{*}\cdots b_{\mathbf{k}_n}^{*}\Omega_{\mathrm{Bog}}, \quad \mathbf{k}_1 + \cdots \mathbf{k}_n = \mathbf{k},\ \mathbf{k}_1, \dots, \mathbf{k}_n \neq 0, \quad n = 0, 1, \dots\},$$
$$\mathcal{F}_n^{\mathrm{exc}}(\mathbf{k}) := \mathrm{Span}^{\mathrm{cl}}\{b_{\mathbf{k}_1}^{*}\cdots b_{\mathbf{k}_n}^{*}\Omega_{\mathrm{Bog}}, \quad \mathbf{k}_1 + \cdots \mathbf{k}_n = \mathbf{k},\ \mathbf{k}_1, \dots, \mathbf{k}_n \neq 0\}.$$

$b_{\mathbf{k}}^{*}\Omega_{\mathrm{Bog}}$ is contained in the space $\mathcal{F}^{\mathrm{exc}}(\mathbf{k})$, which is preserved by H^{eff}. Let $H^{\mathrm{eff}}(\mathbf{k})$ denote the operator H^{eff} restricted to $\mathcal{F}^{\mathrm{exc}}(\mathbf{k})$. Thus we can restrict ourselves to the fiber space $\mathcal{F}^{\mathrm{exc}}(\mathbf{k})$ and the fiber Hamiltonian $H^{\mathrm{eff}}(\mathbf{k})$. In particular, in (67) we can replace H^{eff} with $H^{\mathrm{eff}}(\mathbf{k})$.

For simplicity, we will assume that $\frac{1}{2}\mathbf{k} \notin \frac{2\pi}{L}\mathbb{Z}$, so that at least one coordinate of $\mathbf{k}$ is odd. This guarantees that $\mathbf{p} \neq \mathbf{k} - \mathbf{p}$. Let $Z_{\mathbf{k}}^{L}$ denote the set of (unordered) pairs $\{\mathbf{p}, \mathbf{k} - \mathbf{p}\} \subset \frac{2\pi}{L}\mathbb{Z}^3 \setminus \{0, \mathbf{k}\}$. Then $\mathcal{F}_2^{\mathrm{exc}}(\mathbf{k})$ can be identified with $l^2(Z_{\mathbf{k}}^{L})$.

For our analysis it is enough to know only H^{eff} (or $H^{\mathrm{eff}}(\mathbf{k})$) compressed to $\mathcal{F}_1^{\mathrm{exc}}(\mathbf{k}) \oplus \mathcal{F}_2^{\mathrm{exc}}(\mathbf{k})$. Note that the one-quasiparticle state $b_{\mathbf{k}}^{*}|\Omega_{\mathrm{Bog}}\rangle$ spans $\mathcal{F}_1^{\mathrm{exc}}(\mathbf{k})$, and $\mathcal{F}_2^{\mathrm{exc}}(\mathbf{k})$ is spanned by $b_{\mathbf{p}}^{*}b_{\mathbf{k}-\mathbf{p}}^{*}\Omega_{\mathrm{Bog}}$ with $\{\mathbf{p}, \mathbf{k} - \mathbf{p}\} \in Z_{\mathbf{k}}^{L}$. We compute:

Theorem 3

$$(b_{\mathbf{k}}^{*}\Omega_{\mathrm{Bog}}|H^{\mathrm{eff}} b_{\mathbf{k}}^{*}\Omega_{\mathrm{Bog}}) = e_{\mathbf{k}}, \tag{68}$$
$$(b_{\mathbf{p}}^{*}b_{\mathbf{k}-\mathbf{p}}^{*}\Omega_{\mathrm{Bog}}|H^{\mathrm{eff}} b_{\mathbf{p}}^{*}b_{\mathbf{k}-\mathbf{p}}^{*}\Omega_{\mathrm{Bog}}) = e_{\mathbf{p}} + e_{\mathbf{k}-\mathbf{p}}, \tag{69}$$
$$(b_{\mathbf{p}}^{*}b_{\mathbf{k}-\mathbf{p}}^{*}\Omega_{\mathrm{Bog}}|H^{\mathrm{eff}} b_{\mathbf{k}}^{*}\Omega_{\mathrm{Bog}}) = \frac{\lambda}{L^{3/2}} h_{\mathbf{k}}(\mathbf{p}), \tag{70}$$
$$(b_{\mathbf{k}}^{*}\Omega_{\mathrm{Bog}}|H^{\mathrm{eff}} b_{\mathbf{p}}^{*}b_{\mathbf{k}-\mathbf{p}}^{*}\Omega_{\mathrm{Bog}}) = \frac{\lambda}{L^{3/2}} h_{\mathbf{k}}(\mathbf{p}) \tag{71}$$

with

$$\begin{aligned} h_{\mathbf{k}}(\mathbf{p}) = & \sqrt{\frac{\mu \hat{v}^2(\mathbf{k})}{\hat{v}(0)}} (\gamma_{\mathbf{k}} - \sigma_{\mathbf{k}})(\gamma_{\mathbf{p}}\sigma_{\mathbf{k}-\mathbf{p}} + \sigma_{\mathbf{p}}\gamma_{\mathbf{k}-\mathbf{p}}) \\ & + \sqrt{\frac{\mu \hat{v}^2(\mathbf{p})}{\hat{v}(0)}} (\sigma_{\mathbf{k}}\sigma_{\mathbf{p}}\sigma_{\mathbf{k}-\mathbf{p}} - \gamma_{\mathbf{k}}\gamma_{\mathbf{p}}\gamma_{\mathbf{k}-\mathbf{p}} + \gamma_{\mathbf{k}}\sigma_{\mathbf{p}}\gamma_{\mathbf{k}-\mathbf{p}} - \sigma_{\mathbf{k}}\gamma_{\mathbf{p}}\sigma_{\mathbf{k}-\mathbf{p}}) \\ & + \sqrt{\frac{\mu \hat{v}^2(\mathbf{k}-\mathbf{p})}{\hat{v}(0)}} (\sigma_{\mathbf{k}}\sigma_{\mathbf{p}}\sigma_{\mathbf{k}-\mathbf{p}} - \gamma_{\mathbf{k}}\gamma_{\mathbf{p}}\gamma_{\mathbf{k}-\mathbf{p}} + \gamma_{\mathbf{k}}\gamma_{\mathbf{p}}\sigma_{\mathbf{k}-\mathbf{p}} - \sigma_{\mathbf{k}}\sigma_{\mathbf{p}}\gamma_{\mathbf{k}-\mathbf{p}}). \end{aligned} \tag{72}$$

Proof (68) and (69) are straightforward. Let us prove (70). We have

$$(b_{\mathbf{p}}^{*}b_{\mathbf{k}-\mathbf{p}}^{*}\Omega_{\mathrm{Bog}}|H^{\mathrm{eff}} b_{\mathbf{k}}^{*}\Omega_{\mathrm{Bog}}) \tag{73}$$
$$= (b_{\mathbf{p}}^{*}b_{\mathbf{k}-\mathbf{p}}^{*}\Omega_{\mathrm{Bog}}|H_{3,1}^{\mathrm{exc}} b_{\mathbf{k}}^{*}\Omega_{\mathrm{Bog}}). \tag{74}$$

Remembering that we have $\mathbf{p} \neq \mathbf{k} - \mathbf{p}$, we see that the only terms in $H_{3,1}^{\mathrm{exc}}$ which contribute to (74) are

$$b_{\mathbf{p}}^{*}b_{\mathbf{k}-\mathbf{p}}^{*}b_{\mathbf{k}} \Bigg(\frac{\sqrt{\mu}\hat{v}(\mathbf{p})}{L^{3/2}\sqrt{\hat{v}(0)}} (\sigma_{\mathbf{k}}\sigma_{\mathbf{p}}\sigma_{\mathbf{k}-\mathbf{p}} - \gamma_{\mathbf{k}}\gamma_{\mathbf{p}}\gamma_{\mathbf{k}-\mathbf{p}} + \gamma_{\mathbf{k}}\sigma_{\mathbf{p}}\gamma_{\mathbf{k}-\mathbf{p}} - \sigma_{\mathbf{k}}\gamma_{\mathbf{p}}\sigma_{\mathbf{k}-\mathbf{p}})$$

$$+\frac{\sqrt{\mu}\hat{v}(\mathbf{k})}{L^{3/2}\sqrt{\hat{v}(0)}}\big(\gamma_{\mathbf{k}}-\sigma_{\mathbf{k}}\big)\gamma_{\mathbf{p}}\sigma_{\mathbf{k}-\mathbf{p}} \tag{75}$$

$$+\frac{\sqrt{\mu}\hat{v}(\mathbf{k}-\mathbf{p})}{L^{3/2}\sqrt{\hat{v}(0)}}\big(\sigma_{\mathbf{k}}\sigma_{\mathbf{p}}\sigma_{\mathbf{k}-\mathbf{p}}-\gamma_{\mathbf{k}}\gamma_{\mathbf{p}}\gamma_{\mathbf{k}-\mathbf{p}}+\gamma_{\mathbf{k}}\gamma_{\mathbf{p}}\sigma_{\mathbf{k}-\mathbf{p}}-\sigma_{\mathbf{k}}\sigma_{\mathbf{p}}\gamma_{\mathbf{k}-\mathbf{p}}\big)$$

$$+\frac{\sqrt{\mu}\hat{v}(\mathbf{k})}{L^{3/2}\sqrt{\hat{v}(0)}}\big(\gamma_{\mathbf{k}}-\sigma_{\mathbf{k}}\big)\sigma_{\mathbf{p}}\gamma_{\mathbf{k}-\mathbf{p}}\Big) \tag{76}$$

This yields (70). □

The Hamiltonian H^{eff} compressed to $\mathcal{F}_1^{\mathrm{exc}}(\mathbf{k})\oplus\mathcal{F}_2^{\mathrm{exc}}(\mathbf{k})$ will be called the *effective Friedrichs Hamiltonian* (for volume L^3 and momentum $\mathbf{k}$). It is denoted $H^L_{\mathrm{Fried}}(\mathbf{k})$ and given by

$$H^L_{\mathrm{Fried}}(\mathbf{k}) := \begin{bmatrix} e_{\mathbf{k}} & \frac{\lambda}{L^{3/2}}(h_{\mathbf{k}}| \\ \frac{\lambda}{L^{3/2}}|h_{\mathbf{k}}) & e_{\mathbf{p}}+e_{\mathbf{k}-\mathbf{p}}\end{bmatrix}, \tag{77}$$

$$\text{on}\quad \mathcal{F}_1^{\mathrm{exc}}(\mathbf{k})\oplus\mathcal{F}_2^{\mathrm{exc}}(\mathbf{k})\simeq\mathbb{C}\oplus l^2(Z^L_{\mathbf{k}}), \tag{78}$$

where we explicitly introduced a reference to the volume L^3 in the notation. Thus we end up in a situation described in Sect. 3, with $b^*_{\mathbf{k}}\Omega_{\mathrm{Bog}}$, resp. $l^2(Z^L_{\mathbf{k}})$ corresponding to Ψ, resp. $\mathcal{K}$. According to the Fermi Golden Rule (37) the self-energy of $H^L_{\mathrm{Fried}}(\mathbf{k})$ is

$$\Sigma^L_{\mathbf{k}}(z)=\frac{1}{2L^3}\sum_{\mathbf{p},\mathbf{k}-\mathbf{p}\neq 0}\frac{h^2_{\mathbf{k}}(\mathbf{p})}{(z-e_{\mathbf{p}}-e_{\mathbf{k}-\mathbf{p}})}, \tag{79}$$

where $\frac{1}{2}$ in front of the sum accounts for double counting.

The function $\mathbf{p}\mapsto e_{\mathbf{p}}$ is well defined for all $\mathbf{p}\in\mathbb{R}^3$, and not only for $\mathbf{p}\in\frac{2\pi}{L}\mathbb{Z}^3\backslash\{0\}$. Similarly, $h_{\mathbf{k}}(\mathbf{p})$ are well defined for all $\mathbf{p}\in\mathbb{R}^3\setminus\{0,\mathbf{k}\}$, and not only for $\frac{2\pi}{L}\mathbb{Z}^3\setminus\{0,\mathbf{k}\}$. The expression (79) can be interpreted as the Riemann sum converging as $L\to\infty$ to the integral

$$\Sigma_{\mathbf{k}}(z)=\frac{1}{2(2\pi)^3}\int\frac{h_{\mathbf{k}}(\mathbf{p})^2\,\mathrm{d}\mathbf{p}}{(z-e_{\mathbf{p}}-e_{\mathbf{k}-\mathbf{p}})}. \tag{80}$$

We can also introduce the *infinite volume effective Friedrichs Hamiltonian*

$$H_{\mathrm{Fried}}(\mathbf{k}) := \begin{bmatrix} e_{\mathbf{k}} & \lambda(h_{\mathbf{k}}| \\ \lambda|h_{\mathbf{k}}) & e_{\mathbf{p}}+e_{\mathbf{k}-\mathbf{p}}\end{bmatrix}, \quad \text{on}\quad \mathbb{C}\oplus L^2(\mathbb{R}^3/\mathbb{Z}_2), \tag{81}$$

where $\mathbb{Z}_2$ is the two-element group generated by $\mathbf{p}\mapsto\mathbf{k}-\mathbf{p}$. The Fermi Golden Rule predicts that $\Sigma_{\mathbf{k}}(e_{\mathbf{k}}+\mathrm{i}0)$ describes the energy shift of the eigenvalue of the infinite volume Hamiltonian $H_{\mathrm{Fried}}(\mathbf{k})$.

It is maybe worth mentioning that all the steps that lead to $H^L_{\mathrm{Fried}}(\mathbf{k})$ and $H_{\mathrm{Fried}}(\mathbf{k})$ are translation invariant.

6 The Shape of the Quasiparticle Spectrum

If $\mathbf{k}\mapsto e_{\mathbf{k}}$ is a dispersion relation of quasiparticles, then the infimum of the n-quasiparticle spectrum is

$$\inf\{e_{\mathbf{p}_1}+\cdots e_{\mathbf{p}_n}\mid \mathbf{p}_1+\cdots+\mathbf{p}_n=\mathbf{k}\}. \tag{82}$$

Sometimes, it is possible to compute (82) exactly, as shown in the following lemma.

Lemma 4 *Let* $\mathbf{k} \mapsto e_{\mathbf{k}}$ *be a convex function. Then*

$$\inf_{\mathbf{p}}\{e_{\mathbf{p}} + e_{\mathbf{k}-\mathbf{p}}\} = 2e_{\mathbf{k}/2}. \tag{83}$$

In particular,

$$\inf_{\mathbf{p}}\{e_{\mathbf{p}} + e_{\mathbf{k}-\mathbf{p}}\} \leq e_{\mathbf{k}}. \tag{84}$$

If in addition $\mathbf{k} \mapsto e_{\mathbf{k}}$ *is a strictly convex function, then*

$$\inf_{\mathbf{p}}\{e_{\mathbf{p}} + e_{\mathbf{k}-\mathbf{p}}\} < e_{\mathbf{k}}, \quad \mathbf{k} \neq 0. \tag{85}$$

Proof The left hand side of (83) is called infimal involution and is often denoted as

$$e\Box e(\mathbf{k}) := \inf_{\mathbf{p}}\{e_{\mathbf{p}} + e_{\mathbf{k}-\mathbf{p}}\}. \tag{86}$$

Since $e_{\mathbf{k}}$ is a convex function so is $e\Box e(\mathbf{k})$ [1, Chapter 12] and it satisfies

$$(e\Box e)^* = e^* + e^* = 2e^* \tag{87}$$

where e^* denotes the Legendre-Fenchel transform of e. Hence

$$\inf_{\mathbf{p}}\{e_{\mathbf{p}} + e_{\mathbf{k}-\mathbf{p}}\} = e\Box e(\mathbf{k}) = (e\Box e)^{**}(\mathbf{k}) = (2e^*)^*(\mathbf{k}) = 2e_{\mathbf{k}/2}$$

which proves (83). Now (84) follows from convexity. Indeed,

$$2e_{\mathbf{p}/2} = 2e_{\mathbf{p}/2+0/2} \leq e_{\mathbf{p}}.$$

□

Now $e_{\mathbf{k}}$ in (11), that is

$$e_{\mathbf{k}} = \sqrt{\frac{1}{4}|\mathbf{k}|^4 + \mu|\mathbf{k}|^2}, \tag{88}$$

is strictly convex. Therefore, (85) is true, and so the dispersion relation is embedded inside the 2-quasiparticle spectrum.

Remark 5 If we replace Assumption (42c) with Assumption (43), and suppose

$$1 + 2\mu\frac{\nu}{\hat{v}(0)} > 0, \tag{89}$$

then the dispersion relation is still embedded inside the unperturbed 2-quasiparticle spectrum, at least for small momenta. The same is true for the effective Friedrichs Hamiltonian $H_{\text{Fried}}(\mathbf{k})$ for small $\mathbf{k}$.

The Hamiltonian H^{exc} couples $b_{\mathbf{k}}^*\Omega_{\text{Bog}}$ with 4-quasiparticle states through $H_{3,2}^{\text{exc}}$. The bottom of 4-quasiparticle spectrum lies below the dispersion relation (in fact, if it is given by (11), it is equal to $4e_{\mathbf{k}/4} < e_{\mathbf{k}}$). However, $H_{3,2}^{\text{exc}}$ does not couple $b_{\mathbf{k}}^*\Omega_{\text{Bog}}$ to all possible 4-quasiparticle states with the total momentum $\mathbf{k}$, but only to states of the form $b_{\mathbf{p}_1}b_{\mathbf{p}_2}b_{\mathbf{p}_3}b_{\mathbf{k}}\Omega_{\text{Bog}}$ with $\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 = 0$. Their energy is

$$e_{\mathbf{k}} + e_{\mathbf{p}_1} + e_{\mathbf{p}_2} + e_{\mathbf{p}_3} \geq e_{\mathbf{k}}. \tag{90}$$

Thus the state $b_{\mathbf{k}}^*\Omega_{\text{Bog}}$ is situated at the boundary of the energy-momentum spectrum and the only coupling is through $\mathbf{p}_1 = \mathbf{p}_2 = \mathbf{p}_3 = 0$. Before going to the thermodynamic limit this is excluded, because on the excited space all momenta are different from zero. Assuming that this effect survives the thermodynamic limit, we expect that the term $H^{\text{exc}}_{3,2}$ does not lead to damping and we therefore drop it from H_{Fried}, even though in terms of the coupling parameter κ this term is of the same order as $H^{\text{exc}}_{3,1}$, which we keep in our analysis.

Two-quasiparticle states are coupled to three-quasiparticle states through H^{exc}_{31} and to five-quasiparticle states through H^{exc}_{32}. These couplings, however, do not contribute to our Fermi Golden Rule computation—they affect the damping rate in a higher order of the coupling constant. Therefore, we do not include these states in our Hilbert space $\mathcal{F}^{\text{exc}}_1(\mathbf{k}) \oplus \mathcal{F}^{\text{exc}}_2(\mathbf{k})$ on which our effective Friedrichs Hamiltonian acts.

7 Computing the Self-Energy

In the remaining part of our paper, the main goal will be to compute approximately the 3-dimensional integral (80). To do this efficiently it is important to choose a convenient coordinate system.

Let us introduce the notation $k = |\mathbf{k}|$, $p = |\mathbf{p}|$, $l = |\mathbf{l}|$, where $\mathbf{l} = \mathbf{k} - \mathbf{p}$. One could try to compute (80) using the spherical coordinates for $\mathbf{p}$ with respect to the axis determined by $\mathbf{k}$. This means using $p = |\mathbf{p}|, w = \cos\theta, \phi$, so that $\mathbf{p} = (p\sqrt{1-w^2}\cos\phi, p\sqrt{1-w^2}\sin\phi, pw)$. The self-energy in these coordinates is

$$\Sigma_{\mathbf{k}}(z) = \frac{1}{2(2\pi)^3}\int_0^\infty \int_{-1}^1 \int_0^{2\pi} \frac{h_{\mathbf{k}}(p,w)^2 p^2\, \mathrm{d}p\, \mathrm{d}w\, \mathrm{d}\phi}{(z - e_p - e_{l(p,w)})} \tag{91}$$

where, with abuse of notation, $h_{\mathbf{k}}(p,w)$ is the function $h_{\mathbf{k}}(\mathbf{p})$ in the variables p, w, ϕ. The variable ϕ can be easily integrated out. $h_{\mathbf{k}}(\mathbf{p})$ depends only on k, p, l and (91) can be rewritten as

$$\Sigma_{\mathbf{k}}(z) = \frac{1}{2(2\pi)^2}\int_0^\infty \int_{-1}^1 \frac{(h_k(p,l(p,w)))^2 p^2\, \mathrm{d}p\, \mathrm{d}w}{(z - e_p - e_{l(p,w)})},$$

The coordinates p, w are not convenient because they break the natural symmetry $\mathbf{p} \to \mathbf{k} - \mathbf{p}$ of the system. Instead of p, w it is much better to use the variables p, l. Note the constraints

$$|p - l| \le k, \tag{92}$$

$$k \le p + l, \tag{93}$$

that follow from the triangle inequality. We have $w = \frac{k^2+p^2-l^2}{2kp}$. The Jacobian is easily computed:

$$p^2\, \mathrm{d}p\, \mathrm{d}w = \frac{pl}{k}\, \mathrm{d}p\, \mathrm{d}l = \frac{1}{4k}\, \mathrm{d}p^2\, \mathrm{d}l^2. \tag{94}$$

Let us make another change of variables:

$$t = p + l, \quad s = p - l; \qquad p = \frac{t+s}{2}, \quad l = \frac{t-s}{2}; \tag{95}$$

$$\mathrm{d}p^2\, \mathrm{d}l^2 = \frac{t^2 - s^2}{2}\, \mathrm{d}t\, \mathrm{d}s. \tag{96}$$

The limits of integration following from the constraints (92) and (93) are very easy to impose:

$$\Sigma_{\mathbf{k}}(z) = \frac{1}{2(2\pi)^2} \int_k^\infty \mathrm{d}t \int_{-k}^k \mathrm{d}s \frac{h_k(t,s)^2(t^2-s^2)}{8k(z - e_{\frac{t+s}{2}} - e_{\frac{t-s}{2}})}, \tag{97}$$

Another choice of variables can also be useful. If $k \mapsto e_k$ is an increasing function, which is always the case for small k, but also for the important case of constant $\frac{\hat{v}(\mathbf{k})}{\hat{v}(0)}$, we can use the variables $u := e_p$ and $w := e_l$. Set

$$f(e_k) := \frac{\mathrm{d}k^2}{\mathrm{d}e_k^2}. \tag{98}$$

Thus we change the variables

$$\frac{1}{4k} \mathrm{d}p^2 \, \mathrm{d}l^2 = \frac{1}{4k} f(u) f(w) \, \mathrm{d}u^2 \, \mathrm{d}w^2. \tag{99}$$

$$\Sigma_{\mathbf{k}}(z) = \frac{1}{2(2\pi)^2} \int \frac{h_k(u,w)^2 f(u) f(w) \, \mathrm{d}u^2 \, \mathrm{d}w^2}{4k(z-u-w)},$$

We then perform a further change of variable

$$x = u + w, \quad y = u - w; \qquad u = \frac{x+y}{2}, \quad w = \frac{x-y}{2}; \tag{100}$$

$$\mathrm{d}u^2 \, \mathrm{d}w^2 = \frac{x^2 - y^2}{2} \mathrm{d}x \, \mathrm{d}y. \tag{101}$$

Now we can write

$$\Sigma_{\mathbf{k}}(z) = \frac{1}{16\pi^2 k} \iint \frac{h_k(x,y)^2 f(\frac{x+y}{2}) f(\frac{x-y}{2})(x^2-y^2) \, \mathrm{d}y \, \mathrm{d}x}{4(z-x)},$$

where the limits of integration are somewhat more difficult to describe.

When $\frac{\hat{v}(\mathbf{k})}{\hat{v}(0)}$ is a constant, so that

$$e_k = k\sqrt{\mu + \frac{k^2}{4}}, \qquad k^2 = 2\left(\sqrt{e_k^2 + \mu^2} - \mu\right), \tag{102}$$

we can compute the function f:

$$f(u) = \frac{1}{\sqrt{u^2 + \mu^2}}. \tag{103}$$

We also have

$$\sigma_k = \sqrt{\frac{\frac{k^2}{2} + \mu + \sqrt{\frac{k^4}{4} + \mu k^2}}{2\sqrt{\frac{k^4}{4} + \mu k^2}}}, \quad \gamma_k = \sqrt{\frac{\frac{k^2}{2} + \mu - \sqrt{\frac{k^4}{4} + \mu k^2}}{2\sqrt{\frac{k^4}{4} + \mu k^2}}}. \tag{104}$$

8 Damping Rate

The following theorem is the main result of this paper.

Theorem 6 *Suppose that the potential satisfies Assumption* (42). *Then*

$$\Sigma_{\mathbf{k}}(e_k + \mathrm{i}0) = -c_{\mathrm{Bel}} k^5 + O(k^6) \quad as \quad k \to 0, \quad c_{\mathrm{Bel}} = \frac{3\hat{v}(0)}{640\pi\mu}. \tag{105}$$

Remark 7 If we replace Assumption (42c) with Assumption (43) with $\nu = 0$, then Theorem 6 remains true.

Proof of Theorem 6 We will use the variables x, y:

$$\Sigma_{\mathbf{k}}(e_k + \mathrm{i}0) = \frac{1}{16\pi^2 k} \iint \frac{h_{\mathbf{k}}(x,y)^2 (x^2 - y^2)\, \mathrm{d}y\, \mathrm{d}x}{(e_k - x + \mathrm{i}0)\sqrt{(x+y)^2 + 4\mu^2}\sqrt{(x-y)^2 + 4\mu^2}}. \tag{106}$$

It follows from (106) and the Sochocki-Plemelj formula (18) that

$$\Sigma_{\mathbf{k}}(e_k + \mathrm{i}0) = \mathrm{Re}\Sigma_{\mathbf{k}}(e_k + \mathrm{i}0) + \mathrm{i}\mathrm{Im}\Sigma_{\mathbf{k}}(e_k + \mathrm{i}0),$$

$$\mathrm{Re}\Sigma_{\mathbf{k}}(e_k + \mathrm{i}0) = \frac{1}{16\pi^2 k} \iint \frac{h_{\mathbf{k}}(x,y)^2 (x^2 - y^2)\, \mathrm{d}y\, \mathrm{d}x}{(e_k - x)\sqrt{(x+y)^2 + 4\mu^2}\sqrt{(x-y)^2 + 4\mu^2}} \tag{107}$$

$$\mathrm{Im}\Sigma_{\mathbf{k}}(e_k + \mathrm{i}0) = -\frac{\pi}{16\pi^2 k} \iint \frac{h_{\mathbf{k}}(x,y)^2 (x^2 - y^2)\delta(e_k - x)\, \mathrm{d}y\, \mathrm{d}x}{\sqrt{(x+y)^2 + 4\mu^2}\sqrt{(x-y)^2 + 4\mu^2}} \tag{108}$$

$$= -\frac{\pi}{16\pi^2 k} \int \frac{h_{\mathbf{k}}(e_k,y)^2 (e_k^2 - y^2)\, \mathrm{d}y}{\sqrt{(e_k+y)^2 + 4\mu^2}\sqrt{(e_k-y)^2 + 4\mu^2}}. \tag{109}$$

Our starting point is the expression (109). Obviously, we first need to establish the integration limits in y. Recall that $y = e_p - e_l$ but under the additional constraint that $e_k = e_p + e_l$ which comes from the constraint $\delta(x - e_k)$ in (108). It follows immediately that $-e_k \le y \le e_k$. Thus, for $|\mathbf{k}|$ small enough we can replace $\hat{v}$ with $\hat{v}(0)$, so that

$$\begin{aligned} h_{\mathbf{k}}(\mathbf{p}) = 2\sqrt{\mu\hat{v}(0)}\Big(&\sigma_{\mathbf{p}}\gamma_{-\mathbf{k}}\gamma_{\mathbf{p}-\mathbf{k}} + \sigma_{\mathbf{k}-\mathbf{p}}\gamma_{-\mathbf{k}}\gamma_{\mathbf{p}} + \sigma_{\mathbf{p}}\sigma_{\mathbf{k}-\mathbf{p}}\sigma_{\mathbf{k}} \\ &- \gamma_{\mathbf{p}}\sigma_{-\mathbf{k}}\sigma_{\mathbf{p}-\mathbf{k}} - \gamma_{\mathbf{k}-\mathbf{p}}\sigma_{-\mathbf{k}}\sigma_{\mathbf{p}} - \gamma_{\mathbf{p}}\gamma_{\mathbf{k}-\mathbf{p}}\gamma_{\mathbf{k}}\Big). \end{aligned} \tag{110}$$

Hence

$$\begin{aligned} \frac{h_{\mathbf{k}}(\mathbf{p})}{2\sqrt{\mu\hat{v}(0)}} &= \sigma_k(\sigma_p\sigma_l - \sigma_l\gamma_p - \sigma_p\gamma_l) + \gamma_k(\sigma_p\gamma_l + \sigma_l\gamma_p - \gamma_p\gamma_l). \\ &= \frac{\sigma_k}{2\sqrt{uw}}\Bigg(\sqrt{\sqrt{u^2+\mu^2}+u}\sqrt{\sqrt{w^2+\mu^2}+w} - \sqrt{\sqrt{w^2+\mu^2}+w}\sqrt{\sqrt{u^2+\mu^2}-u} \\ &\quad - \sqrt{\sqrt{u^2+\mu^2}+u}\sqrt{\sqrt{w^2+\mu^2}-w}\Bigg) \\ &\quad + \frac{\gamma_k}{2\sqrt{uw}}\Bigg(\sqrt{\sqrt{u^2+\mu^2}+u}\sqrt{\sqrt{w^2+\mu^2}-w} + \sqrt{\sqrt{w^2+\mu^2}+w}\sqrt{\sqrt{u^2+\mu^2}-u} \\ &\quad - \sqrt{\sqrt{u^2+\mu^2}-u}\sqrt{\sqrt{w^2+\mu^2}-w}\Bigg) \end{aligned} \tag{111}$$

$$= \frac{1}{2\sqrt{x^2-y^2}}\Big(\sigma_k\sqrt{(A_1+x+y)(A_2+x-y)} - \gamma_k\sqrt{(A_1-x-y)(A_2-x+y)}$$

$$+ (\gamma_k - \sigma_k)\sqrt{(A_1 - x - y)(A_2 + x - y))} + (\gamma_k - \sigma_k)\sqrt{(A_1 + x + y)(A_2 - x + y))}\Big), \tag{112}$$

where

$$A_1 := A_1(x, y) = \sqrt{(x + y)^2 + 4\mu^2}, \qquad A_2 := A_2(x, y) = \sqrt{(x - y)^2 + 4\mu^2}. \tag{113}$$

Therefore the integrand in (109) becomes

$$\frac{(h_{\mathbf{k}}(x, y))^2(x^2 - y^2)}{\sqrt{(x + y)^2 + 4\mu^2}\sqrt{(x - y)^2 + 4\mu^2}} \tag{114}$$

$$= \frac{\mu\hat{v}(0)}{A_1A_2}\Big(\sigma_k\sqrt{(A_1 + x + y)(A_2 + x - y))} - \gamma_k\sqrt{(A_1 - x - y)(A_2 - x + y)}$$
$$+ (\gamma_k - \sigma_k)\sqrt{(A_1 - x - y)(A_2 + x - y))} + (\gamma_k - \sigma_k)\sqrt{(A_1 + x + y)(A_2 - x + y))}\Big)^2.$$
$$= \frac{\mu\hat{v}(0)}{A_1A_2}\Big(\sigma_k^2\left(3A_1A_2 + (x + y)A_2 + (x - y)A_1 - (x^2 - y^2) - 4\mu(A_1 + A_2 + 2x) + 8\mu^2\right)$$
$$+ \gamma_k^2\left(3A_1A_2 - (x + y)A_2 - (x - y)A_1 - (x^2 - y^2) - 4\mu(A_1 + A_2 - 2x) + 8\mu^2\right)$$
$$+ 2\sigma_k\gamma_k\left(4\mu A_1 + 4\mu A_2 - 2A_1A_2 + 2(x^2 - y^2) - 12\mu^2\right)\Big). \tag{115}$$

Thus the equation in (109) becomes

$$-\frac{1}{16\pi k}\int_{-e_k}^{e_k} \mathrm{d}y\frac{h_{\mathbf{k}}^2(x, y)(x^2 - y^2)}{\sqrt{(x + y)^2 + 4\mu^2}\sqrt{(x - y)^2 + 4\mu^2}} \tag{116}$$

$$= \left(-\frac{\mu\hat{v}(0)}{16\pi k}\right)\int_{-e_k}^{e_k} \mathrm{d}y\Big(\left(3\sigma_k^2 + 3\gamma_k^2 - 4\sigma_k\gamma_k\right) + (\sigma_k^2 - \gamma_k^2)\left(\frac{x - y}{A_2} + \frac{x + y}{A_1} - \frac{8\mu x}{A_1A_2}\right)$$
$$+ (-\sigma_k^2 - \gamma_k^2 + 4\sigma_k\gamma_k)\frac{x^2 - y^2}{A_1A_2} - 4\mu(\sigma_k - \gamma_k)^2\frac{A_1 + A_2}{A_1A_2} + 8\mu^2(\sigma_k^2 + \gamma_k^2 - 3\sigma_k\gamma_k)\frac{1}{A_1A_2}\Big). \tag{117}$$

The integrals involving $\frac{x\pm y}{A_j}$ and $\frac{1}{A_j}$ (where $j = 1, 2$) can be computed explicitly. In particular, setting $x = e_k$, it follows that for $j = 1, 2$

$$\int_{-e_k}^{e_k} \mathrm{d}y\frac{e_k \pm y}{A_j(e_k, y)} = \int_{-e_k}^{e_k} \mathrm{d}y\left(\frac{e_k \pm y}{\sqrt{(e_k \pm y)^2 + 4\mu^2}}\right) = 2\sqrt{\mu^2 + e_k^2} - 2\mu, \tag{118}$$

$$\int_{-e_k}^{e_k} \mathrm{d}y\frac{1}{A_j(e_k, y)} = \int_{-e_k}^{e_k} \mathrm{d}y\left(\frac{1}{\sqrt{(e_k \pm y)^2 + 4\mu^2}}\right) = \log\left(\frac{e_k}{\mu} + \sqrt{1 + \frac{e_k^2}{\mu^2}}\right). \tag{119}$$

This yields

$$\left(-\frac{1}{16\pi k}\right)\int_{-e_k}^{e_k} \mathrm{d}y\left(\frac{h_{\mathbf{k}}^2(e_k, y)(e_k^2 - y^2)}{\sqrt{(e_k + y)^2 + 4\mu^2}\sqrt{(e_k - y)^2 + 4\mu^2}}\right) \tag{120}$$

$$= \left(-\frac{\mu\hat{v}(0)}{16\pi k}\right)\left(2\left(3\sigma_k^2 + 3\gamma_k^2 - 4\sigma_k\gamma_k\right)e_k + 4\sqrt{\mu^2 + e_k^2} - 4\mu - 8\mu(\sigma_k - \gamma_k)^2\log\left(\frac{e_k}{\mu} + \sqrt{1 + \frac{e_k^2}{\mu^2}}\right)\right)$$

$$+\left(-\frac{\mu\hat{v}(0)}{16\pi k}\right)\int_{-e_k}^{e_k}\mathrm{d}y\left(\frac{-(\sigma_k^2-4\sigma_k\gamma_k+\gamma_k^2)(e_k^2-y^2)-8\mu e_k+8\mu^2(\sigma_k^2+\gamma_k^2-3\sigma_k\gamma_k)}{A_1A_2}\right). \tag{121}$$

where two types of integrals, namely

$$\int\left(\frac{-y^2}{A_1A_2}\right)\mathrm{d}y \quad\text{and}\quad \int\left(\frac{1}{A_1A_2}\right)\mathrm{d}y, \tag{122}$$

still appear as they cannot be computed explicitly. We will approximate them by expansions in e_k(which is small, as k is small). To this end, we recall

$$\sigma_k=\sqrt{\frac{\sqrt{e_k^2+\mu^2}+e_k}{2e_k}},\qquad \gamma_k=\sqrt{\frac{\sqrt{e_k^2+\mu^2}-e_k}{2e_k}}, \tag{123}$$

which gives

$$\sigma_k^2+\gamma_k^2=\frac{\sqrt{e_k^2+\mu^2}}{e_k},\qquad \sigma_k\gamma_k=\frac{\mu}{2e_k}. \tag{124}$$

Then (121) equals to

$$\begin{aligned}
&\left(-\frac{\mu\hat{v}(0)}{16\pi k}\right)\left(2\left(3\sigma_k^2+3\gamma_k^2-4\sigma_k\gamma_k\right)e_k+4\sqrt{\mu^2+e_k^2}-4\mu-8\mu(\sigma_k-\gamma_k)^2\times\log\left(\frac{e_k}{\mu}+\sqrt{1+\frac{e_k^2}{\mu^2}}\right)\right)\\
&+\left(-\frac{\mu\hat{v}(0)}{16\pi k}\right)\int_{-e_k}^{e_k}\mathrm{d}y\left(\frac{-(\sigma_k^2-4\sigma_k\gamma_k+\gamma_k^2)(e_k^2-y^2)-8\mu e_k+8\mu^2(\sigma_k^2+\gamma_k^2-3\sigma_k\gamma_k)}{A_1A_2}\right)\\
&=\left(-\frac{\mu\hat{v}(0)}{16\pi k}\right)\left(2(3\sqrt{e_k^2+\mu^2}-2\mu)+2(2\sqrt{\mu^2+e_k^2}-2\mu)-8\mu\frac{\sqrt{e_k^2+\mu^2}-\mu}{e_k}\times\log\left(\frac{e_k}{\mu}+\sqrt{1+\frac{e_k^2}{\mu^2}}\right)\right)\\
&+\left(\frac{\mu\hat{v}(0)}{16\pi k}\right)\frac{\sqrt{e_k^2+\mu^2}-2\mu}{e_k}\int_{-e_k}^{e_k}\mathrm{d}y\left(\frac{e_k^2-y^2}{A_1A_2}\right)\\
&+\left(\frac{1}{16\pi k}\right)\left(8\mu^2\hat{v}(0)e_k-8\mu^3\hat{v}(0)\frac{2\sqrt{e_k^2+\mu^2}-3\mu}{2e_k}\right)\int_{-e_k}^{e_k}\mathrm{d}y\left(\frac{1}{A_1A_2}\right)
\end{aligned}$$

$$=\left(-\frac{\mu\hat{v}(0)}{16\pi k}\right)\left(10\mu\sqrt{(e_k/\mu)^2+1}-8\mu-8\mu\frac{\sqrt{(e_k/\mu)^2+1}-1}{e_k/\mu}\log\left(\frac{e_k}{\mu}+\sqrt{1+\frac{e_k^2}{\mu^2}}\right)\right) \tag{125}$$

$$+\left(\frac{\mu\hat{v}(0)}{16\pi k}\right)\frac{\sqrt{(e_k/\mu)^2+1}-2}{e_k/\mu}\int_{-e_k}^{e_k}\mathrm{d}y\left(\frac{e_k^2-y^2}{A_1A_2}\right) \tag{126}$$

$$+\left(\frac{\mu\hat{v}(0)}{16\pi k}\right)\left(\frac{8\mu e_k^2-4\mu^3(2\sqrt{(e_k/\mu)^2+1}-3)}{e_k}\right)\int_{-e_k}^{e_k}\mathrm{d}y\left(\frac{1}{A_1A_2}\right). \tag{127}$$

We expand (125) up to order $O(e_k^8)$. A tedious computation yields

$$(125)=\left(-\frac{\mu\hat{v}(0)}{16\pi k}\right)\left(2\mu+\frac{e_k^2}{\mu}+\frac{5e_k^4}{12\mu^3}-\frac{41e_k^6}{120\mu^5}+O(e_k^8)\right). \tag{128}$$

We shall now deal with the terms (126) and (127). To this end we write

$$A_1A_2=\sqrt{4\mu^2+(e_k+y)^2}\sqrt{4\mu^2+(e_k-y)^2} \tag{129}$$

$$=4\mu^2\sqrt{1+\left(\frac{e_k+y}{2\mu}\right)^2}\sqrt{1+\left(\frac{e_k-y}{2\mu}\right)^2} \tag{130}$$

$$= 4\mu^2\sqrt{1+\frac{e_k^2+y^2}{2\mu^2}+\left(\frac{e_k^2-y^2}{4\mu^2}\right)^2} \tag{131}$$

$$= 4\mu^2\sqrt{1+Q_1} \tag{132}$$

$$= 4\mu^2\left(1+\frac{1}{2}Q_1-\frac{1}{8}Q_1^2+\frac{1}{16}Q_1^3\right)+O(Q_1^4). \tag{133}$$

where

$$Q_1 := \frac{e_k^2+y^2}{2\mu^2}+\left(\frac{e_k^2-y^2}{4\mu^2}\right)^2 \tag{134}$$

Then

$$\frac{1}{A_1A_2}=\frac{1}{4\mu^2(1+Q_2)}=\frac{1}{4\mu^2}(1-Q_2+Q_2^2-Q_2^3)+O(Q_2^4) \tag{135}$$

where

$$Q_2 := \frac{1}{2}Q_1-\frac{1}{8}Q_1^2+\frac{1}{16}Q_1^3. \tag{136}$$

This leads to

$$\frac{1}{A_1A_2}=\frac{1}{4\mu^2}-\frac{e_k^2}{16\mu^4}+\frac{e_k^4}{64\mu^6}-\frac{e_k^6}{256\mu^8}-\frac{y^2}{16\mu^4}+\frac{e_k^2y^2}{16\mu^6}-\frac{9e_k^4y^2}{256\mu^8}$$
$$+\frac{y^4}{64\mu^6}-\frac{9e_k^2y^4}{256\mu^8}-\frac{y^6}{256\mu^8}+O(e_k^{\iota_1}y^{\iota_2}) \tag{137}$$

where $\iota_1+\iota_2=7$. In turn

$$\int_{-e_k}^{e_k}\frac{1}{A_1A_2}\,\mathrm{d}y=\frac{e_k}{2\mu^2}-\frac{e_k^3}{6\mu^4}+\frac{19e_k^5}{240\mu^6}-\frac{13e_k^7}{280\mu^8}+O(e_k^8) \tag{138}$$

and

$$\int_{-e_k}^{e_k}\frac{e_k^2-y^2}{A_1A_2}\,\mathrm{d}y=\frac{e_k^3}{3\mu^2}-\frac{e_k^5}{10\mu^4}+\frac{11e_k^7}{280\mu^6}+O(e_k^8). \tag{139}$$

This implies

$$(126)=\left(\frac{\mu\hat{v}(0)}{16\pi k}\right)\left(-\frac{e_k^2}{3\mu}+\frac{4e_k^4}{15\mu^3}-\frac{11e_k^6}{84\mu^5}\right)+O(e_k^8), \tag{140}$$

and

$$(127)=\left(\frac{\mu\hat{v}(0)}{16\pi k}\right)\left(2\mu+\frac{4e_k^2}{3\mu}+\frac{3e_k^4}{20\mu^3}-\frac{2e_k^6}{7\mu^5}\right)+O(e_k^8). \tag{141}$$

Combining (140), (141) and (128) we obtain

$$-\frac{1}{16\pi k}\int_{-e^k}^{e_k}\frac{(h_{\mathbf{k}}^{\Lambda}(e_k,y))^2(e_k^2-y^2)}{\sqrt{(e_k+y)^2+4\mu^2}\sqrt{(e_k-y)^2+4\mu^2}}\,\mathrm{d}y$$
$$=\left(-\frac{\mu\hat{v}(0)}{16\pi k}\right)\left(\frac{5}{12}-\frac{41}{120}\right)\frac{e_k^6}{\mu^5}=-\frac{3\hat{v}(0)}{640\pi\mu^4}\frac{e_k^6}{k}. \tag{142}$$

This yields (105). □

9 Full Self-Energy

Recall that the self-energy is given by

$$\Sigma_{\mathbf{k}}(z) = \frac{1}{2(2\pi)^2} \int_k^\infty \mathrm{d}t \int_{-k}^{k} \mathrm{d}s \frac{h_k(t,s)^2(t^2-s^2)}{8k(z - e_{\frac{t+s}{2}} - e_{\frac{t-s}{2}})}, \tag{143}$$

For contact potentials $h_k(\mathbf{p})$, that is, if $\hat{v}(\mathbf{k}) = \hat{v}(0)$ for all $\mathbf{k}$, then $h(\mathbf{k})$ is given by

$$\frac{h_{\mathbf{k}}(\mathbf{p})}{2\sqrt{\mu\hat{v}(0)}} = \frac{1}{2}(\sigma_k + \gamma_k)(\sigma_p\sigma_l - \gamma_p\gamma_l) + \frac{1}{2}(\sigma_k - \gamma_k)(\sigma_p\sigma_l + \gamma_p\gamma_l - 2\sigma_p\gamma_l - 2\gamma_p\sigma_l). \tag{144}$$

We easily see that the self-energy is then divergent in the ultraviolet regime. Indeed, for large t, $h_{\mathbf{k}}(t,s)$ is asymptotic to a nonzero constant, $e_{\frac{t+s}{2}}$ and $e_{\frac{t-s}{2}}$ behave as t^2 and we have t^2 in the numerator. Therefore, (143) is linearly divergent at large t. We should not be surprised—contact potentials are not true potentials, they need a renormalization of the coupling constant, therefore they may lead to problems.

For generic potentials h_k is given by (72):

$$\begin{aligned} h_{\mathbf{k}}(\mathbf{p}) = &\sqrt{\frac{\mu\hat{v}^2(\mathbf{k})}{\hat{v}(0)}}(\gamma_{\mathbf{k}} - \sigma_{\mathbf{k}})\big(\gamma_{\mathbf{p}}\sigma_{\mathbf{k}-\mathbf{p}} + \sigma_{\mathbf{p}}\gamma_{\mathbf{k}-\mathbf{p}}\big) \\ &+ \sqrt{\frac{\mu\hat{v}^2(\mathbf{p})}{\hat{v}(0)}}\big(\sigma_{\mathbf{k}}\sigma_{\mathbf{p}}\sigma_{\mathbf{k}-\mathbf{p}} - \gamma_{\mathbf{k}}\gamma_{\mathbf{p}}\gamma_{\mathbf{k}-\mathbf{p}} + \gamma_{\mathbf{k}}\sigma_{\mathbf{p}}\gamma_{\mathbf{k}-\mathbf{p}} - \sigma_{\mathbf{k}}\gamma_{\mathbf{p}}\sigma_{\mathbf{k}-\mathbf{p}}\big) \\ &+ \sqrt{\frac{\mu\hat{v}^2(\mathbf{k}-\mathbf{p})}{\hat{v}(0)}}\big(\sigma_{\mathbf{k}}\sigma_{\mathbf{p}}\sigma_{\mathbf{k}-\mathbf{p}} - \gamma_{\mathbf{k}}\gamma_{\mathbf{p}}\gamma_{\mathbf{k}-\mathbf{p}} + \gamma_{\mathbf{k}}\gamma_{\mathbf{p}}\sigma_{\mathbf{k}-\mathbf{p}} - \sigma_{\mathbf{k}}\sigma_{\mathbf{p}}\gamma_{\mathbf{k}-\mathbf{p}}\big). \end{aligned} \tag{145}$$

If we assume (42d), then $\hat{v}$ decays sufficiently fast and provides a natural cutoff, so that the self-energy is well-defined. We formulate this as a theorem:

Theorem 8 *Suppose that Assumption (42) holds. Then for $\mathbf{k} \neq 0$, the self-energy $\Sigma_{\mathbf{k}}(z)$ for* $\mathrm{Im} z > 0$ *is finite. One can also take its limit on the real line:*

$$\Sigma_{\mathbf{k}}(e_{\mathbf{k}} + \mathrm{i}0) := \lim_{\varepsilon \searrow 0} \Sigma_{\mathbf{k}}(e_{\mathbf{k}} + \mathrm{i}\varepsilon). \tag{146}$$

The same is true the cutoff self-energy $\Sigma_{\mathbf{k}}^{\Lambda}(z)$ involving contact potentials.

Proof Let us sketch a proof of the first statement. For large $|\mathbf{k}|$ we have

$$\sigma_k \simeq 1, \qquad \gamma_k \simeq \frac{2\hat{v}(\mathbf{k})}{k^2}. \tag{147}$$

$h_k(t,s)^2$ contains several terms. Those containing $\gamma_{\frac{t\pm s}{2}}$ are integrable because of (147) and (42d). The only dangerous terms in $h_k(t,s)^2$ are

$$\frac{\mu\hat{v}(\frac{t\pm s}{2})^2}{\hat{v}(0)}\sigma_k^2\sigma_{\frac{t+s}{2}}^2\sigma_{\frac{t-s}{2}}^2. \tag{148}$$

They are integrable by (42d). □

Unfortunately, there are also bad news. The energy shift has a non-physical feature: it diverges as $\mathbf{k} \to 0$, as follows from the theorem below. Therefore, we cannot treat seriously the results obtained from the Fermi Golden Rule concerning the real part of the excitation spectrum.

Theorem 9 *Suppose that Assumption (42) holds. Then*

$$\lim_{k\to 0} \Sigma_{\mathbf{k}}(0) = -\infty. \tag{149}$$

Remark 10 The same statement could be obtained for more general potentials, e.g. satisfying Assumption (43) instead of (42c)

First note that under Assumption (42c), for $|\mathbf{k}|, |\mathbf{p}|, |\mathbf{k}-\mathbf{p}| < \Lambda$ we have

$$e_k = (88), \qquad h_k(\mathbf{p}) = (144), \tag{150}$$

as for contact potentials.

Let $|\mathbf{k}| < \Lambda/2$ and let us split the integral for the self-energy as

$$(143) = \int_0^{\Lambda/2} \mathrm{d}t + \int_{\Lambda/2}^{\infty} \mathrm{d}t. \tag{151}$$

Taking into account $|s| < |\mathbf{k}|$ we see that the first integral involves only $|\mathbf{k}|, |\mathbf{p}|, |\mathbf{k}-\mathbf{p}| < \Lambda$. Therefore, in this integral all quantities such as e_k, σ_k, γ_k, e_p, σ_p, γ_p, e_l, σ_l, γ_l, are as for contact potentials. Let us prove some lemmas about these quantities.

Lemma 11 *For small p, l, we have*

$$\frac{e_{\frac{t}{2}}}{e_p + e_l} - \frac{1}{2} = O(s^2), \tag{152}$$

$$\frac{pl}{e_p e_l} - \frac{t^2}{4e_{\frac{t}{2}}^2} = O(s^2), \tag{153}$$

$$\sigma_p \sigma_l \sqrt{e_p e_l} - \sigma_{\frac{t}{2}}^2 e_{\frac{t}{2}} = O(s^2), \tag{154}$$

$$\gamma_p \gamma_l \sqrt{e_p e_l} - \gamma_{\frac{t}{2}}^2 e_{\frac{t}{2}} = O(s^2). \tag{155}$$

Proof e can assume that $s \geq 0$.

$$e_p' = \left(\tfrac{p^2}{2} + \mu\right)\left(\tfrac{p^2}{4} + \mu\right)^{-\frac{1}{2}}, \quad e_p'' = p\left(\tfrac{p^2}{8} + \tfrac{3\mu}{4}\right)\left(\tfrac{p^2}{4} + \mu\right)^{-\frac{3}{2}} = O(p). \tag{156}$$

Therefore,

$$2e_{\frac{t}{2}} - e_p - e_l = -\int_{-\frac{s}{2}}^{\frac{s}{2}} \left(\tfrac{s}{2} - |v|\right) e''_{\frac{t}{2}+v}\, \mathrm{d}v = O(ts^2),$$

and hence

$$\frac{e_{\frac{t}{2}}}{e_p + e_l} - \frac{1}{2} = \frac{2e_{\frac{t}{2}} - e_p - e_l}{2(e_p + e_l)}$$

is $O(s^2)$, which proves (152).

Next, set $f(p) := \frac{p}{e_p}$. We have

$$\frac{\mathrm{d}}{\mathrm{d}p} f(p) = \frac{-2p}{(p^2 + 4\mu)^{\frac{3}{2}}} = O(p), \qquad \frac{\mathrm{d}^2}{\mathrm{d}p^2} f(p) = \frac{4(p^2 - 2\mu)}{(p^2 + 4\mu)^{\frac{5}{2}}} = O(1). \tag{157}$$

Hence

$$\frac{pl}{e_p e_l} - \frac{t^2}{4e_{\frac{t}{2}}^2} = f(p)f(l) - f\big(\tfrac{t}{2}\big)^2 \tag{158}$$

$$= \int_0^{\frac{s}{2}} \big(\tfrac{s}{2} - v\big)\Big(f''\big(\tfrac{t}{2}+v\big)f\big(\tfrac{t}{2}-v\big) - 2f'\big(\tfrac{t}{2}+v\big)f'\big(\tfrac{t}{2}-v\big) + f\big(\tfrac{t}{2}+v\big)f''\big(\tfrac{t}{2}-v\big)\Big)\,\mathrm{d}v,$$

which is $O(s^2)$, which proves (153).

We check that the 0th, 1st and 2nd derivatives of

$$\sigma_p\sqrt{e_p} = \frac{1}{\sqrt{2}}\sqrt{\frac{p^2}{2} + \mu + \sqrt{\frac{p^4}{4} + \mu p^2}}, \tag{159}$$

$$\gamma_p\sqrt{e_p} = \frac{1}{\sqrt{2}}\sqrt{\frac{p^2}{2} + \mu - \sqrt{\frac{p^4}{4} + \mu p^2}} \tag{160}$$

are bounded. Then we argue as in (158), proving (154) and (155). □

Lemma 12

$$\lim_{k\to 0}\int_k^{\Lambda}\mathrm{d}t\int_{-k}^{k}\mathrm{d}s\,\frac{(\sigma_p\sigma_l - \gamma_p\gamma_l)^2 pl}{8k(e_p+e_l)} = \int_0^{\Lambda}\mathrm{d}t\,\frac{t^2}{64e_{\frac{t}{2}}}, \tag{161}$$

where the right hand side is a finite positive number.

Proof We have

$$\frac{(\sigma_p\sigma_l - \gamma_p\gamma_l)^2 pl}{8k(e_p+e_l)} - \frac{t^2}{8\cdot 8ke_{\frac{t}{2}}} \tag{162}$$

$$= \frac{\big((\sigma_p\sigma_l - \gamma_p\gamma_l)\sqrt{e_pe_l} + e_{\frac{t}{2}}\big)pl}{8k(e_p+e_l)e_pe_l}\Big((\sigma_p\sigma_l - \gamma_p\gamma_l)\sqrt{e_pe_l} - e_{\frac{t}{2}}\Big) \tag{163}$$

$$+ \frac{e_{\frac{t}{2}}^2}{8k(e_p+e_l)}\Big(\frac{pl}{e_pe_l} - \frac{t^2}{4e_{\frac{t}{2}}^2}\Big) \tag{164}$$

$$+ \frac{t^2}{32ke_{\frac{t}{2}}}\Big(\frac{e_{\frac{t}{2}}}{e_p+e_l} - \frac{1}{2}\Big). \tag{165}$$

By Lemma 11 the terms in the big brackets on the right of (163), (164) and (165) are $O(s^2)$. The terms in (164), (165) on the left are all $\frac{1}{k}O(t)$. The most singular in t term is the one on the left of (163) and it is of order $\frac{1}{k}O(t^{-1})$. Therefore,

$$\int_k^{\Lambda}\mathrm{d}t\int_{-k}^{k}\mathrm{d}s\Big(\frac{(\sigma_p\sigma_l - \gamma_p\gamma_l)^2 pl}{8k(e_p+e_l)} - \frac{t^2}{64e_{\frac{t}{2}}}\Big) \tag{166}$$

$$= \int_k^{\Lambda}\mathrm{d}t\int_{-k}^{k}\mathrm{d}s\,O(t^{-1})\frac{O(s^2)}{k} = \int_k^{\Lambda}\mathrm{d}t\,O(t^{-1}k^2) = O(k^2\ln k) \to 0. \tag{167}$$

□

Proof of Theorem 9 The second integral on the right of (151) is convergent as $k \to 0$. Let us consider the first integral:

$$\frac{1}{2(2\pi)^2}\int_k^{\Lambda/2} \mathrm{d}t \int_{-k}^{k} \mathrm{d}s \frac{h_k(t,s)^2(t^2-s^2)}{8k(z-e_{\frac{t+s}{2}}-e_{\frac{t-s}{2}})} \tag{168}$$

$$= (\sigma_k+\gamma_k)^2 \int_k^{\Lambda/2} \mathrm{d}t \int_{-k}^{k} \mathrm{d}s \frac{(\sigma_p\sigma_l-\gamma_p\gamma_l)^2 pl}{2k(e_p+e_l)} \tag{169}$$

$$+2\int_k^{\Lambda/2} \mathrm{d}t \int_{-k}^{k} \mathrm{d}s \frac{(\sigma_p\sigma_l-\gamma_p\gamma_l)(\sigma_p\sigma_l+\gamma_p\gamma_l-2\sigma_p\gamma_l-2\gamma_p\sigma_l)pl}{2k(e_p+e_l)} \tag{170}$$

$$+(\sigma_k-\gamma_k)^2 \int_k^{\Lambda/2} \mathrm{d}t \int_{-k}^{k} \mathrm{d}s \frac{(\sigma_p\sigma_l+\gamma_p\gamma_l-2\sigma_p\gamma_l-2\gamma_p\sigma_l)^2 pl}{2k(e_p+e_l)} \tag{171}$$

where we used that $\sigma_k^2-\gamma_k^2=1$. Since $\Lambda/2$ is fixed we are only interested in the small t region. Since k is small too, this implies also p and l are small. For such we have

$$(\sigma_k+\gamma_k)^2 \geq Ck^{-1}, \quad C>0 \tag{172}$$

$$(\sigma_k-\gamma_k)^2 = O(k), \tag{173}$$

$$(\sigma_p\sigma_l-\gamma_p\gamma_l)\sqrt{pl} = O(p)+O(l)=O(t), \tag{174}$$

$$(\sigma_p\sigma_l+\gamma_p\gamma_l-2\sigma_p\gamma_l-2\gamma_p\sigma_l)\sqrt{pl} = O(1), \tag{175}$$

$$\frac{1}{e_p+e_l} = O(t^{-1}). \tag{176}$$

By Lemma 12 and (172),

$$|(169)| \geq C_1 k^{-1} \to +\infty. \tag{177}$$

By (174), (175) and (176),

$$|(170)| \leq C\int_k^{\Lambda/2} \mathrm{d}t \int_{-k}^{k} \mathrm{d}s \frac{1}{k} \to C_\Lambda \qquad \text{as } k \to 0. \tag{178}$$

By (173), (175) and (176),

$$|(171)| \leq Ck\int_k^{\Lambda} \mathrm{d}t \int_{-k}^{k} \mathrm{d}s \frac{1}{kt} \leq Ck|\ln(k)| \to 0, \tag{179}$$

Hence (143) converges to $-\infty$. □

Acknowledgements The work of all authors was supported by the Polish-German NCN-DFG grant Beethoven Classic 3 (project no. 2018/31/G/ST1/01166). We thank the anonymous referees for many helpful remarks.

Data Availability This manuscript has no associated data.

Declarations

Conflict of interest This manuscript has no conflict of interest.

References

1. Bauschke, H.H., Combettes, P.L.: Convex Analysis and Monotone Operator Theory in Hilbert Spaces. CMS Books in Mathematics, Springer, Berlin (2017)
2. Beliaev, S.T.: Energy spectrum of a non-ideal Bose gas. Sov. Phys. JETP **34**(7), 299 (1958)
3. Boccato, C., Brennecke, C., Cenatiempo, S., Schlein, B.: Bogoliubov theory in the Gross–Pitaevskii limit. Acta Math. **222**(2), 219–335 (2019)
4. Boccato, C., Brennecke, C., Cenatiempo, S., Schlein, B.: The excitation spectrum of Bose gases interacting through singular potentials. J. Eur. Math. Soc. **22**(7), 2331–2403 (2020)
5. Bogoliubov, N.N.: On the theory of superfluidity. J. Phys. (USSR) **11**, 23 (1947)
6. Bossmann, L., leopold, N., Petrat, S., Rademacher, S.: Ground state of Bose gases interacting through singular potentials. arXiv:2309.12233 [math-ph]
7. Bossmann, L., Petrat, S., Pickl, P., Soffer, A.: Beyond Bogoliubov dynamics. Pure Appl. Anal. **3**(4), 677–726 (2022)
8. Bossmann, L., Petrat, S., Seiringer, R.: Asymptotic expansion of the low-energy excitation spectrum for weakly interacting bosons. Forum Math. Sigma **9**, e28 (2021)
9. Brennecke, C., Caporaletti, M., Schlein, B.: Excitation spectrum of Bose gases beyond the Gross–Pitaevskii regime. Rev. Math. Phys. **34**(9), 2250027 (2022)
10. Brennecke, C., Schlein, B., Schraven, S.: Bogoliubov theory for trapped bosons in the Gross–Pitaevskii regime. Ann. Henri Poincare **23**, 1583 (2022)
11. Caraci, C., Olgiati, A., Saint-Aubin, D., Schlein, B.: Third order corrections to the ground state energy of a Bose gas in the Gross–Pitaevskii regime. arXiv preprint arXiv:2311.07433
12. Cornean, H.D., Dereziński, J., Ziń, P.: On the infimum of the energy-momentum spectrum of a homogeneous Bose gas. J. Math. Phys. **50**, 062103 (2009)
13. Davies, E.B.: Markovian master equations. Commun. Math. Phys. **39**, 91–110 (1974)
14. Chung, M.-C., Bhattacherjee, A.B.: Damping in 2d and 3d dilute Bose gases. New J. Phys. **11**(12), 123012 (2009)
15. Dereziński, J., de Roeck, W.: Extended weak coupling limit for Friedrichs Hamiltonians. J. Math. Phys. **48**, 012103 (2007)
16. Dereziński, J., de Roeck, W.: Extended weak coupling limit for Pauli–Fierz Hamiltonians. Commun. Math. Phys. **279**, 1–30 (2008)
17. Dereziński, J., Früboes, R.: Renormalization of Friedrichs Hamiltonians. Rep. Math. Phys. **50**, 433–438 (2002)
18. Dereziński, J., Meissner, K.A., Napiórkowski, M.: On the energy-momentum spectrum of a homogeneous Fermi gas. Ann. Henri Poincaré **14**(1), 1–36 (2013)
19. Dereziński, J., Napiórkowski, M.: Excitation spectrum of interacting bosons in the mean-field infinite-volume limit, Ann. Henri Poincaré, **15**, 2409–2439 (2014). Erratum: Ann. Henri Poincaré **16,** 1709–1711 (2015)
20. Dereziński, J., Gérard, C.: Mathematics of Quantization and Quantum Fields. Cambridge Monographs in Mathematical Physics, Cambridge University Press, Cambridge (2013)
21. Dirac, P.A.M.: The quantum theory of emission and absorption of radiation. Proc. R. Soc. A. **114**, 243–265 (1927)
22. Fetter, A.L., Walecka, J.D.: Quantum Theory of Many-Particle Systems. Dover, New York (2003)
23. Fermi, E.: Nuclear Physics. University of Chicago Press, Chicago (1950)
24. Friedrichs, K.O.: Perturbation of Spectra in Hilbert Space. American Mathematical Society, Providence (1965)
25. Giorgini, S.: Damping in dilute Bose gases: a mean-field approach. Phys. Rev. A **57**, 2949 (1998)
26. Grech, P., Seiringer, R.: The excitation spectrum for weakly interacting bosons in a trap. Commun. Math. Phys. **322**, 559–591 (2013)
27. Godfrin, H., Beauvois, K., Sultan, A., Krotscheck, E., Dawidowski, J., Fak, B., Ollivier, J.: Dispersion relation of Landau elementary excitations and thermodynamic properties of superfluid 4He. Phys. Rev. B **103**, 104516 (2021)
28. Hodby, E., Marago, O.M., Hechenblaikner, G., Foot, C.J.: Experimental observation of Beliaev coupling in a Bose–Einstein condensate. Phys. Rev. Lett. **86**, 2196 (2001)

29. Hohenberg, P.C., Martin, P.C.: Microscopic theory of superfluid helium. Ann. Phys. **34**(2), 291–359 (1965)
30. Hugenholtz, N.M., Pines, D.: Ground-state energy and excitation spectrum of a system of interacting bosons. Phys. Rev. **116**, 489 (1959)
31. Katz, N., Steinhauer, J., Ozeri, R., Davidson, N.: Beliaev damping of quasiparticles in a Bose–Einstein condensate. Phys. Rev. Lett. **89**, 220401 (2002)
32. Lewin, M., Nam, P.T., Serfaty, S., Solovej, J.P.: Bogoliubov spectrum of interacting Bose gases. Commun. Pure Appl. Math. **68**, 413–471 (2015)
33. Lieb, E.H., Seiringer, R., Yngvason, J.: Justification of c-number substitutions in Bosonic Hamiltonians. Phys. Rev. Lett. **94**, 080401 (2005)
34. Liu, W.V.: Theoretical study of the damping of collective excitations in a Bose–Einstein condensate. Phys. Rev. Lett. **79**, 4056–4059 (1997)
35. Mohling, F., Morita, M.: Temperature dependence of the low-momentum excitations in a Bose gas of hard spheres. Phys. Rev. **120**(3), 681–688 (1960)
36. Nam, P.T., Napiórkowski, M.: Norm approximation for many-body quantum dynamics and Bogoliubov theory. In: Advances in Quantum Mechanics: Contemporary Trends and Open Problems. Springer, Berlin (2017)
37. Nam, P.T., Napiórkowski, M.: Two-term expansion of the ground state one-body density matrix of a mean-field Bose gas. Calc. Var. PDE **60**(3), 1–30 (2021)
38. Nam, P.T., Seiringer, R.: Collective excitations of Bose gases in the mean-field regime. Arch. Ration. Mech. Anal. **215**(2), 381–417 (2015)
39. Nam, P.T., Triay, A.: Bogoliubov excitation spectrum of trapped Bose gases in the Gross–Pitaevskii regime. J. Math. Pures Appl. **176**, 18–101 (2023)
40. Napiórkowski, M.: Dynamics of interacting bosons: a compact review. In: Density Functionals for Many-Particle Systems - Mathematical Theory and Physical Applications of Effective Equations. World Scientific, Singapore (2023)
41. Reed, M., Simon, B.: Methods of Modern Mathematical Physics. IV: Analysis of Operators. Academic Press, New York (1978)
42. Seiringer, R.: The excitation spectrum for weakly interacting bosons. Commun. Math. Phys. **306**, 565–578 (2011)
43. Shi, H., Griffin, A.: Finite-temperature excitations in a dilute Bose-condensed gas. Phys. Rep. **304**, 1–87 (1998)
44. Weisskopf, V., Wigner, E.P.: Berechnung der natürlichen Linienbreite auf Grund der Diracschen Lichttheorie. Z. Phys. **63**, 54–73 (1930)