Exactly solvable Schrödinger operators related to the confluent equation

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Abstract

Our paper investigates one-dimensional Schrödinger operators defined as closed operators on $L^2(\mathbb{R})$ or $L^2(\mathbb{R}_+)$ that are exactly solvable in terms of confluent functions (or, equivalently, Whittaker functions). We allow the potentials to be complex. They fall into three families: Whittaker operators (or radial Coulomb Hamiltonians), Schrödinger operators with Morse potentials and isotonic oscillators. For each of them, we discuss the corresponding basic holomorphic family of closed operators and the integral kernel of their resolvents. We also describe transmutation identities that relate these resolvents. These identities interchange spectral parameters with coupling constants across different operator families.

A similar analysis is performed for one-dimensional Schrödinger operators solvable in terms of Bessel functions (which are reducible to special cases of Whittaker functions). They fall into two families: Bessel operators and Schrödinger operators with exponential potentials.

To make our presentation self-contained, we include a short summary of the theory of closed one-dimensional Schrödinger operators with singular boundary conditions. We also provide a concise review of special functions that we use.

1 Introduction

One-dimensional Schrödinger operators are operators of the form

$$L := -\partial_x^2 + V(x), \tag{1.1}$$

where $V(x)$ is the *potential*, which in this paper is allowed to be a complex-valued function. In some rare cases, (not necessarily square integrable) eigenfunctions of (1.1) can be computed in terms of standard special functions. We call such operators *exactly solvable*.

Our paper is devoted to several families of operators of the form (1.1), interpreted as *closed operators* on $L^2(]a, b[)$ for appropriate $-\infty \le a < b \le +\infty$, which are exactly solvable in terms of confluent functions, or equivalently, Whittaker functions.

We study three categories of one-dimensional Schrödinger operators:

(1) those reducible to the Bessel equation, having two families: the Bessel operator and the Schrödinger operator with an exponential potential [8, 14, 17, 19, 26, 31];

(2) those associated with the Whittaker equation, which includes three families: the Whittaker operator, the isotonic oscillator, and the Schrödinger operator with Morse potential [14, 29];

(3) finally, for completeness, we include the (well-known) harmonic oscillator, reducible to the Weber equation.

Note that Bessel and Weber functions can be reduced to subclasses of Whittaker functions. One can also remark that there exist several classes of one-dimensional Schrödinger equations exactly solved in terms of the Gauss hypergeometric function [2, 14, 22, 39], which are not considered in this paper.

On the algebraic level, all these families are discussed in many references, e.g., [4, 5, 14, 18]. They are usually treated as formal differential expressions without a functional-analytic setting. In this paper, we consider them as closed operators on an appropriate Hilbert space. For exactly solvable Schrödinger operators, we are able to express their resolvent in terms of special functions. We can do the same with eigenprojections and the spectral measure. Sometimes we can also compute other related operators, such as their exponential or Møller (wave) operators.

Exactly solvable Schrödinger operators, interpreted as closed (usually self-adjoint) operators, are essential in applications, serving as reference models for various perturbation and scattering problems. They are widely used by theoretical physicists to model real quantum systems and for instructional purposes in quantum mechanics (see, e.g., [21, 25]).

Sturm-Liouville operators can be reduced to one-dimensional Schrödinger operators by the so-called Liouville transformation. Therefore, our paper is also related to many works on Sturm-Liouville operators, such as [20, 23], which however are usually restricted to real potentials.

We always choose the Hilbert space to be $L^2(]a, b[)$, where a and b are singular points of the corresponding eigenvalue equation, with the possibility that $a = -\infty$ and $b = +\infty$. With these endpoints, for most parameters, there is no need to choose boundary conditions (b.c.). There are, however, exceptional parameter ranges where b.c. must be selected. (In the case of real potentials, this scenario is often called the "limit circle" case, as opposed to the "limit point" case, where the b.c. are unique.)

The operators that we study depend on (complex) parameters. We try to organize them into *holomorphic families of closed operators*, as advocated, e.g., in [6, 15]. To find such families, first, we identify a subset of parameters that uniquely determine a closed extension. In the terminology we use, L has index zero at both endpoints. In all cases that we consider, this subset forms a large set of parameters with a non-empty interior. We obtain a holomorphic family of closed realizations of L, which is then extended to its largest possible domain of holomorphy. The holomorphic family obtained in this way will be called *basic*. In Table 1, we describe all basic holomorphic families of Schrödinger operators considered in this paper.

Schrödinger operators with potentials from Table 1 can have realizations that do not belong to their basic holomorphic families. This occurs, in particular, when the index of an endpoint is 2 for a given parameter. In such cases, there is a whole family of closed realizations of a single formal expression, of which at most two are basic. The remaining ones involve *mixed b.c.* In particular, Bessel and Whittaker operators, as well as the isotonic oscillator, can have mixed b.c. at 0 for $-1 < \mathrm{Re}(m) < 1$. These cases are not discussed in this paper. For further details, see [10] for Bessel operators and [12] for Whittaker operators.

The domain of holomorphy, by definition, is an open set. Schrödinger operators with exponential and Morse potentials, as well as the harmonic and isotonic potentials, have a domain of holomorphy with a boundary at $\mathrm{Re}(k) = 0$. The basic holomorphic family can be extended by continuity to this boundary. In all cases, operators with parameters on this boundary must be treated separately. We discuss them all, with the exception of Morse potentials.

If A is an operator on $L^2(]a, b[)$, then the integral (or possibly distributional) kernel of the operator A will be denoted $A(x, y)$, $x, y \in]a, b[$. I.e.,

$$(f|Ag) = \int \overline{f(x)} A(x, y) g(y) \, \mathrm{d}x \mathrm{d}y. \tag{1.2}$$

As mentioned above, for all operators from Table 1, we will compute the integral kernel of their resolvent. We will see that Hamiltonians with very different properties can be solved in terms of the same class of special functions. It is curious to observe that they are linked by somewhat mysterious identities involving their resolvents. In these identities, we can observe a "transmutation" of spectral parameters into coupling constants.

Table 1: Families of operators covered in this paper

Operator or Potential	Symbol and Formula	Spectral Parameter	Domain of Holomorphy	$L^2(]a,b[)$
Bessel equation:				
Bessel operator	$H_m := -\partial_r^2 + \left(m^2 - \frac{1}{4}\right)\frac{1}{r^2}$	k^2	$\mathrm{Re}(m) > -1$	$L^2(\mathbb{R}_+)$
Exponential potential	$M_k := -\partial_x^2 + k^2 \mathrm{e}^{2x}$	m^2	$\mathrm{Re}(k) > 0$	$L^2(\mathbb{R})$
Whittaker equation:				
Whittaker operator	$H_{\beta,m} := -\partial_r^2 + \left(m^2 - \frac{1}{4}\right)\frac{1}{r^2} - \frac{\beta}{r}$	k^2	$\mathrm{Re}(m) > -1$, $(\beta, m) \neq (0, -\frac{1}{2})$	$L^2(\mathbb{R}_+)$
Morse potential	$M_{\beta,k} := -\partial_x^2 + k^2 \mathrm{e}^{2x} - \beta \mathrm{e}^x$	m^2	$\mathrm{Re}(k) > 0$ $\beta \in \mathbb{C} \setminus \mathbb{R}_+$	$L^2(\mathbb{R})$
Isotonic oscillator	$N_{k,m} := -\partial_v^2 + \left(m^2 - \frac{1}{4}\right)\frac{1}{v^2} + k^2 v^2$	-2β	$\mathrm{Re}(m) > -1$, $\mathrm{Re}(k) > 0$,	$L^2(\mathbb{R}_+)$
Weber equation:				
Harmonic oscillator	$N_k := -\partial_r^2 + k^2 r^2$	-2β	$\mathrm{Re}(k) > 0$	$L^2(\mathbb{R})$

Here is the transmutation identity involving the resolvent of the Bessel operator and of the Schrödinger operator with the exponential potential:

$$\frac{1}{(M_k + m^2)}(x, y) = \mathrm{e}^{-\frac{x}{2}} \frac{1}{(H_m + k^2)}(\mathrm{e}^x, \mathrm{e}^y)\, \mathrm{e}^{-\frac{y}{2}}. \tag{1.3}$$

Note that on the left m^2 is the spectral parameter, and on the right it is the coupling constant. On the left k^2 is the coupling constant, on the right it is the spectral parameter. Thus, their roles are curiously interchanged.

For the three families solved in terms of Whittaker functions, we have transmutations involving three parameters: m, k, and β:

$$\frac{1}{(N_{k,m} - 2\beta)}(u, v) = 2\left(\frac{u^2}{2}\right)^{-\frac{1}{4}} \frac{1}{(H_{\beta,\frac{m}{2}} + k^2)}\left(\frac{u^2}{2}, \frac{v^2}{2}\right)\left(\frac{v^2}{2}\right)^{-\frac{1}{4}}, \tag{1.4}$$

$$\frac{1}{(M_{\beta,k} + m^2)}(x, y) = \mathrm{e}^{-\frac{x}{2}} \frac{1}{(H_{\beta,m} + k^2)}(\mathrm{e}^x, \mathrm{e}^y)\mathrm{e}^{-\frac{y}{2}}, \tag{1.5}$$

$$\frac{1}{(N_{k,m} - 2\beta)}(u, v) = 2\left(\frac{u^2}{2}\right)^{\frac{1}{4}} \frac{1}{(M_{\beta,k} + (\frac{m}{2})^2)}\left(\log \frac{u^2}{2}, \log \frac{v^2}{2}\right)\left(\frac{v^2}{2}\right)^{\frac{1}{4}}. \tag{1.6}$$

The case $\mathrm{Re}(k) = 0$ of the exponential potential is quite curious and analyzed in our paper. Setting $k = \mathrm{i}\ell$, we can rewrite it as:

$$M_\ell^\gamma := -\partial_x^2 - \ell^2 \mathrm{e}^{2x}. \tag{1.7}$$

In this operator, we need to fix a b.c. at ∞, which can be naturally parametrized by $\gamma \in \mathbb{C} \cup \{\infty\}$. The parameter $\ell > 0$ is not very interesting — in fact, the translation by $\ln \ell$ yields unitary equivalence with the case $\ell = 1$. Note that $\gamma \mapsto M_\ell^\gamma$ is a family of operators holomorphic away from 0 or ∞. We prove that M_ℓ^γ with $\gamma = 0, \infty$ can be reached as limits of the family M_k. The transmutation of the resolvents for M_ℓ^γ into

resolvents of H_m is quite interesting:

$$\frac{1}{(M_\ell^\gamma + m^2)}(x,y) = \frac{\mathrm{e}^{\mathrm{i}m\pi}}{(\mathrm{e}^{\mathrm{i}m\pi} - \gamma)}\mathrm{e}^{-\frac{x}{2}}\frac{\ell}{(-\ell^2 + \mathrm{i}0 + H_m)}(\mathrm{e}^x, \mathrm{e}^y)\mathrm{e}^{-\frac{y}{2}}$$
$$+ \frac{\gamma}{(\mathrm{e}^{\mathrm{i}m\pi} - \gamma)}\mathrm{e}^{-\frac{x}{2}}\frac{\ell}{(-\ell^2 - \mathrm{i}0 + H_m)}(\mathrm{e}^x, \mathrm{e}^y)\mathrm{e}^{-\frac{y}{2}}. \tag{1.8}$$

Note also that M_ℓ^γ for $\gamma \neq 0, \infty$ possess point spectrum, which was recently described independently in [37].

Let us briefly describe the structure of the paper. We begin with a concise overview of the basic theory of 1-dimensional Schrödinger operators, presented in Section 2. This is a classic subject covered in various textbooks. We primarily follow the presentation in [13]. We explain how to determine when boundary conditions are needed to define a closed realization with a non-empty resolvent set. We also demonstrate how to fix b.c. with the help of the Wronskian. Finally, we describe the integral kernel for a candidate of the resolvent.

In Section 3, we discuss families of closed operators solved in terms of Bessel functions; in Section 4, the families solved in terms of Whittaker functions; and in Section 5, the harmonic oscillator, solved in terms of Weber functions.

In the appendices, we concisely describe elements of the theory of special functions used in our paper. In Appendix A, we discuss various kinds of Bessel functions, as well as the $_0F_1$ functions. First of all, one can distinguish between the trigonometric and hyperbolic Bessel equations. The former has oscillating solutions, while the solutions of the latter behave exponentially. One can pass from the former to the latter by rotating the complex plane by the angle $\pm\frac{\pi}{2}$. Secondly, both the trigonometric and hyperbolic Bessel equations have forms adapted to all dimensions. In most of the literature, the standard Bessel equation, which is 2-dimensional and trigonometric, and the modified Bessel equation, which is 2-dimensional and hyperbolic, are considered. In our paper, it is also convenient to use functions that solve the 1-dimensional Bessel equation, both trigonometric and hyperbolic. All these variations of the Bessel equation are equivalent to one another, and they are also equivalent to the so-called $_0F_1$ equation. They can also be reduced to a subclass of the confluent equation.

The Whittaker equation is equivalent to the confluent equation, which exists in two equivalent variants: $_1F_1$ and $_2F_0$. Therefore, the *confluent equation*, which we put in the title of this manuscript, can be replaced by the *Whittaker equation*. There are also varieties of the Whittaker equation for any dimension. In our paper, we use those corresponding to $d = 1$ and $d = 2$. We also introduce modifications of Whittaker functions adapted to the isotonic oscillator. All of this is described in Appendix B.

Finally, in Appendix C, we briefly describe the Weber equation, which is equivalent to the Hermite equation and to a subclass of the Whittaker equation.

2   Basic theory of 1-dimensional Schrödinger operators

2.1   Boundary conditions

Let us sketch the theory of closed realizations of 1-dimensional Schrödinger operators with complex potentials. This is a classic subject, discussed in various textbooks and presented in several forms. Our presentation follows mostly [13]. Note that we avoid using the so-called boundary triplets and Krein-type formulas.

Consider an open interval $]a, b[$, where $-\infty \leq a < b \leq +\infty$. (In our paper, we will consider only $\mathbb{R}$ and $\mathbb{R}_+$; however, the theory in this section applies to an arbitrary interval $]a, b[$). Consider a complex function $V \in L^1_{\mathrm{loc}}(]a, b[)$. Suppose L is formally given by

$$L := -\partial_x^2 + V(x). \tag{2.1}$$

We would like to describe closed realizations of L on $L^2(]a, b[)$ possessing a non-empty resolvent set and compute its resolvent. (A linear operator possessing a non-empty resolvent set is sometimes called *well-posed*).

Note that in most of the literature only real V are considered, and the authors are interested only in self-adjoint realizations of L. If $L_\bullet$ is such a realization of L, its spectrum, denoted $\sigma(L_\bullet)$, is contained in $\mathbb{R}$. Thus its resolvent set is automatically non-empty. In our paper, however, we will consider also complex potentials and non-self-adjoint realizations.

Let $z \in \mathbb{C}$. Let $AC^1(]a, b[)$ denote the set of functions on the open interval $]a, b[$ whose derivative is absolutely continuous. The space

$$\mathcal{N}(L - z) := \left\{ \Psi \in AC^1(]a, b[) \mid (L - z)\Psi(x) = 0 \right\} \tag{2.2}$$

is 2-dimensional. Let $\mathcal{U}_a(z)$ denote the subspace of $\mathcal{N}(L - z)$ solutions square integrable near a. One can show that one of the following holds:

$$\text{either } \dim \mathcal{U}_a(z) = 2 \text{ for all } z \in \mathbb{C}; \tag{2.3}$$

$$\text{or } \dim \mathcal{U}_a(z) \leq 1 \text{ for all } z \in \mathbb{C}. \tag{2.4}$$

In the first case we say that the index of the endpoint a, denoted $\nu_a(L)$, is 2, and in the second case it is 0. There are analogous definitions associated to the second endpoint b.

Remark 2.1. In the context of real potentials, the case $\nu_a(L) = 2$ is called "limit circle" and $\nu_a(L) = 0$ "limit point". As explained, e.g., in [13], these names are not fully appropriate for complex V.

Let us define

$$\mathcal{D}(L^{\max}) := \left\{ f \in L^2(]a, b[) \cap AC^1(]0, \infty[) \mid Lf \in L^2(]a, b[) \right\}, \tag{2.5}$$

$$\mathcal{D}(L^{\min}) := \text{the closure in the graph norm of } \{ f \in \mathcal{D}(L^{\max}) \mid f = 0 \text{ near } a \text{ and } b \}. \tag{2.6}$$

(The graph norm is $\|f\|_A := \sqrt{\|Lf\|^2 + \|f\|^2}$.) One can show that

$$\dim \mathcal{D}(L^{\max})/\mathcal{D}(L^{\min}) = \nu_a(L) + \nu_b(L). \tag{2.7}$$

We define the maximal and minimal realization of L:

$$L^{\max} := L\Big|_{\mathcal{D}(L^{\max})}, \quad L^{\min} := L\Big|_{\mathcal{D}(L^{\min})}. \tag{2.8}$$

In what follows, we will need the Wronskian for two functions Φ and Ξ, defined as

$$\mathcal{W}(\Phi, \Xi)(x) := \Phi(x)\Xi'(x) - \Phi'(x)\Xi(x). \tag{2.9}$$

Note that if $\Phi, \Xi \in \mathcal{N}(L - z)$, then $\mathcal{W}(\Phi, \Xi)(x)$ is a constant. In this case, we will simply write $\mathcal{W}(\Phi, \Xi)$ without specifying the argument. Besides, if $\Phi, \Xi \in \mathcal{D}(L^{\max})$, then the Wronskian, a priori defined in $]a, b[$, can be extended to the endpoints:

$$\mathcal{W}(\Phi, \Xi)(a) := \lim_{x \searrow a} \mathcal{W}(\Phi, \Xi)(x), \quad \mathcal{W}(\Phi, \Xi)(b) := \lim_{x \nearrow b} \mathcal{W}(\Phi, \Xi)(x). \tag{2.10}$$

We also can equip $\mathcal{D}(L^{\max})/\mathcal{D}(L^{\min})$ with a non-degenerate bilinear antisymmetric form:

$$\mathcal{W}(\Phi, \Xi)(a) - \mathcal{W}(\Phi, \Xi)(b) = -(\overline{L\Phi}|\Xi) + (\overline{\Phi}|L\Psi). \tag{2.11}$$

(The identity in (2.11) is sometimes called the *Lagrange identity* or then *integrated Green identity*.)

If (2.7)$= 0$, then there exists a unique closed realization of L, which coincides with $L^{\max} = L^{\min}$. Otherwise $\sigma(L^{\max}) = \sigma(L^{\min}) = \mathbb{C}$, and therefore the operators $L^{\max}$ and $L^{\min}$ are not well-posed. In order to define operators that may have a non-empty resolvent set, one needs to select a space $\mathcal{D}(L_\bullet)$ such that

$$\mathcal{D}(L^{\min}) \subset \mathcal{D}(L_\bullet) \subset \mathcal{D}(L^{\max}), \tag{2.12}$$

$$\dim \mathcal{D}(L^{\max})/\mathcal{D}(L_\bullet) = \dim \mathcal{D}(L_\bullet)/\mathcal{D}(L^{\min}) = \frac{1}{2}(\nu_a + \nu_b). \tag{2.13}$$

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Then we set $L_\bullet := L^{\max}\big|_{\mathcal{D}(L_\bullet)}$.

To do this it is convenient to introduce the *boundary space*

$$\mathcal{B} := \big(\mathcal{D}(L^{\max})/\mathcal{D}(L^{\min})\big)', \tag{2.14}$$

where the prime denotes the dual. Clearly $\dim \mathcal{B} = 2\nu_a + 2\nu_b$.

If $\dim \mathcal{B} = 2$, in order to define $L_\bullet$ satisfying (2.13) we need to fix a single nonzero functional $\phi_\bullet \in \mathcal{B}$ to define the domain of an operator

$$\mathcal{D}(L_\bullet) = \{\Xi \in \mathcal{D}(L^{\max}) \mid \phi_\bullet(\Xi) = 0\}. \tag{2.15}$$

This corresponds to two possibilities. If $\nu_a(L) = 2$ and $\nu_b(L) = 0$, a convenient way to define a functional $\phi_\bullet$ is to choose $\Phi_a \neq 0$, which near a belongs to $\mathcal{D}(L^{\max})$ but does not belong to $\mathcal{D}(L^{\min})$, and then to set

$$\phi(\Xi) := \mathcal{W}(\Phi_a, \Xi)(a) = 0. \tag{2.16}$$

A good choice for Φ_a is an element of $\mathcal{U}_a(z)$ for some $z \in \mathbb{C}$. (Usually, $z = 0$ is most convenient). The condition (2.16) will be called *the boundary condition (b.c.) at a set by Φ_a*.

Note that what is important in (2.16) is a nonzero functional on $\mathcal{D}(L^{\max})$ vanishing on $\mathcal{D}(L^{\min})$, which depends only on the behavior of Ξ near a. Sometimes for this end it is convenient to use a well chosen Φ_a, which does not belong to $\mathcal{D}(L^{\max})$. We will see an example of this in the definition of the Whittaker operator.

Similarly we proceed if $\nu_a(L) = 0$ and $\nu_b(L) = 2$. We select $\Phi_b \neq 0$ which near b belongs to $\mathcal{D}(L^{\max})$ but not $\mathcal{D}(L^{\min})$, and set

$$\phi_\bullet(\Xi) := \mathcal{W}(\Phi_b, \Xi)(b). \tag{2.17}$$

In our paper, we will not consider operators with $\dim \mathcal{B} = 4$, that is, $\nu_a(L) = \nu_b(L) = 2$. Nevertheless, for completeness let us discuss briefly this case. To define a realization satisfying (2.13) we need to fix two linearly independent functionals $\phi_{\bullet 1}, \phi_{\bullet 2} \in \mathcal{B}$ and set

$$\mathcal{D}(L_\bullet) = \big\{\Xi \in \mathcal{D}(L^{\max}) \mid \phi_{\bullet 1}(\Xi) = \phi_{\bullet 2}(\Xi) = 0\big\}. \tag{2.18}$$

In particular, often one considers the so-called *separated boundary conditions*, where $\phi_{\bullet 1}$ is a b.c. at a and $\phi_{\bullet 1}$ is a b.c. at b.

2.2 Candidate for resolvent

Suppose now that we want to describe the spectrum and resolvent of $L_\bullet$. First consider the case $\nu_a(L) = \nu_b(L) = 0$. Suppose $z \in \mathbb{C}$ satisfies $\dim \mathcal{U}_a(z) \geq 1$ and $\dim \mathcal{U}_b(z) \geq 1$. We select

$$\Psi_a(z, \cdot) \in \mathcal{U}_a(z) \backslash \{0\}, \quad \Psi_b(z, \cdot) \in \mathcal{U}_b(z) \backslash \{0\}, \quad \text{and set} \quad \mathcal{W}(z) := \mathcal{W}\big(\Psi_b(z, \cdot), \Psi_a(z, \cdot)\big).$$

Then, we define

$$R_\bullet(z; x, y) := \frac{1}{\mathcal{W}(z)} \begin{cases} \Psi_a(z, x)\, \Psi_b(z, y) & \text{if } a < x < y < b, \\ \Psi_a(z, y)\, \Psi_b(z, x) & \text{if } a < y < x < b. \end{cases} \tag{2.19}$$

If $\nu_a(L) = 2$, $\nu_b(L) = 0$, then we need to select Φ_a, setting the b.c. at a and define

$$\mathcal{D}(L_\bullet) = \{\Xi \in \mathcal{D}(L^{\max}) \mid \mathcal{W}(\Xi, \Phi_a)(a) = 0\}. \tag{2.20}$$

Similarly, if $\nu_a(L) = 0$, $\nu_b(L) = 2$, then we need to select Φ_b, setting the b.c. at b, so that

$$\mathcal{D}(L_\bullet) = \{\Xi \in \mathcal{D}(L^{\max}) \mid \mathcal{W}(\Xi, \Phi_b)(b) = 0\}. \tag{2.21}$$

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If $\nu_a(L) = 2$, $\nu_b(L) = 2$ and we use separated b.c, then

$$\mathcal{D}(L_\bullet) = \{\Xi \in \mathcal{D}(L^{\max}) \mid \mathcal{W}(\Xi, \Phi_a)(a) = \mathcal{W}(\Xi, \Phi_b)(b) = 0\}. \tag{2.22}$$

Note that Ψ_a and Ψ_b above are defined uniquely up to a multiplicative constant. $R_\bullet(z; x, y)$ does not depend on this choice. The operator $R_\bullet(z)$ defined by the kernel $R_\bullet(z; x, y)$ sends $C_c(]a, b[)$ into functions in $L^2(]a, b[)$ satisfying the b.c.. Besides, we have

$$\left(-\partial_x^2 + V(x) - z\right)R_\bullet(z; x, y) = \left(-\partial_y^2 + V(y) - z\right)R_\bullet(z; x, y) = \delta(x - y). \tag{2.23}$$

The following theorem is proven in [13, Propositions 7.8 and 7.9]:

Theorem 2.2. *The following conditions are equivalent:*

1. $\dim \mathcal{U}_a(z) \geq 1$, $\mathcal{U}_b(z) \geq 1$ *and* $R_\bullet(z)$ *is bounded;*

2. $z \in \mathbb{C}\backslash\sigma(L_\bullet)$.

If the above conditions hold, then

$$(L_\bullet - z)^{-1}(x, y) = R_\bullet(z; x, y). \tag{2.24}$$

Hence, the strategy for studying 1-dimensional Schrödinger operators on $L^2(]a, b[)$ given by the expression (2.1) is the following:

1. For each $z \in \mathbb{C}$, determine $\mathcal{N}(L - z)$.

2. Determine $\mathcal{U}_a(z)$ and $\mathcal{U}_b(z)$.

3. If $\nu_a(L) = 2$ or $\nu_b(L) = 2$, fix b.c defining $L_\bullet$.

4. For these b.c. (if needed), and if $\dim \mathcal{U}_a(z) \geq 1$ and $\mathcal{U}_b(z) \geq 1$, write down $R_\bullet(z)$, using equation (2.19), which is a candidate for the resolvent $(L_\bullet - z)^{-1}$.

5. Check whether $R_\bullet(z)$ is bounded. If so, $z \in \mathbb{C}\backslash\sigma(L_\bullet)$ and $R_\bullet = (L_\bullet - z)^{-1}$.

2.3 Boundedness of resolvent

Let us quote two useful lemmas for proving the boundedness of an operator K on $L^2(]a, b[)$ given by the integral kernel $K(x, y)$.

Lemma 2.3. *The Hilbert-Schmidt norm of K is given by*

$$\|K\|_2 := \sqrt{\text{Tr} K^* K} = \left(\int |K(x, y)|^2 \, dxdy\right)^{\frac{1}{2}}, \tag{2.25}$$

and we have $\|K\| \leq \|K\|_2$.

Lemma 2.4 (Schur's Criterion). *Suppose that*

$$\sup_{x \in]a,b[} \int_a^b K(x, y) \, dy = c_1, \quad \sup_{y \in]a,b[} \int_a^b K(x, y) \, dx = c_2. \tag{2.26}$$

Then $\|K\| \leq \sqrt{c_1 c_2}$.

Note that, if applicable, Lemma 2.3 is superior to Lemma 2.4, because it estimates a stronger norm. For translation-invariant operators, we can also use a special case of the Young inequality, which actually follows from Schur's criterion:

Lemma 2.5. *If $K(x, y) = f(x - y)$, then*

$$\|K\| \le \int |f(x)| \, dx. \tag{2.27}$$

Let us make some remarks on how to check the boundedness of an operator of the form (2.19). More precisely, suppose that R is an operator with the integral kernel

$$R(x, y) := \begin{cases} \Psi_a(x) \, \Psi_b(y) & \text{if } a < x < y < b, \\ \Psi_a(y) \, \Psi_b(x) & \text{if } a < y < x < b; \end{cases} \tag{2.28}$$

where Ψ_a, resp. Ψ_b is square integrable near a, resp. b. Choose c such that $a < c < b$. Then we can split the operator R into the sum of four operators

$$R = R_{--} + R_{-+} + R_{+-} + R_{++} \tag{2.29}$$

with kernels

$$R_{--}(x, y) := \theta(c - x)\theta(c - y)R(x, y), \tag{2.30a}$$
$$R_{+-}(x, y) := \theta(x - c)\theta(c - y)R(x, y), \tag{2.30b}$$
$$R_{-+}(x, y) := \theta(c - x)\theta(y - c)R(x, y), \tag{2.30c}$$
$$R_{++}(x, y) := \theta(x - c)\theta(y - c)R(x, y). \tag{2.30d}$$

Now R_{+-} and R_{-+} are bounded because both $\|R_{+-}\|_2^2$ and $\|R_{-+}\|_2^2$ can be estimated by

$$\int_a^c |\Psi_a(x)|^2 \, dx \int_c^b |\Psi_b(y)|^2 \, dy. \tag{2.31}$$

Thus, we obtain the following criterion:

Lemma 2.6. *R is bounded iff R_{--} and R_{++} are bounded.*

2.4 Krein formula

We will always use equation (2.19) to construct the resolvent of a Schrödinger operator. In the literature, in similar situations, one often uses the so-called Krein formula and the formalism of *boundary triplets*. In this subsection, we compare the two approaches. We will not use boundary triplets in this paper; however, we discuss them briefly for the convenience of the reader.

Suppose, for definiteness, that $\nu_a(L) = 2$ and $\nu_b(L) = 0$. Suppose we fix a basis $\{\phi^0, \phi^1\}$ of the boundary space $\mathcal{B}$ defined in (2.14). (This is equivalent to fixing two distinct b.c. at a.) Then, for any $\kappa \in \mathbb{C} \cup \{\infty\}$, we can define a closed realization L_κ of L by setting

$$\mathcal{D}(L_\kappa) := \{\Xi \in \mathcal{D}(L^{\max}) \mid \phi^0(\Xi) + \kappa\phi^1(\Xi) = 0\}, \quad \kappa \in \mathbb{C}; \tag{2.32}$$
$$\mathcal{D}(L_\infty) := \{\Xi \in \mathcal{D}(L^{\max}) \mid \phi^1(\Xi) = 0\}, \quad \kappa = \infty. \tag{2.33}$$

Let $z \in \mathbb{C}$ and $0 \ne \Psi_b(z, \cdot) \in \mathcal{U}_b(z)$. Note that under our assumptions $\Psi_b(z, \cdot) \in L^2(]a, b[)$.

Let $0 \ne \Psi_a^i(z, \cdot) \in \mathcal{U}_a(z) = \mathcal{N}(L - z)$ with $\phi_a^i(\Psi_a^i(z, \cdot)) = 0$, $i = 0, 1$. Set

$$\mathcal{W}^i(z) := \mathcal{W}(\Psi_b(z, \cdot), \Psi_a^i(z, \cdot)), \quad i = 0, 1. \tag{2.34}$$

It follows from (2.19) that

$$R_\kappa(z; x, y) := \frac{1}{(\mathcal{W}^0(z) + \kappa\mathcal{W}^1(z))} \begin{cases} (\Psi_a^0(z, x) + \kappa\Psi_a^1(z, x)) \, \Psi_b(z, y) & \text{if } a < x < y < b, \\ (\Psi_a^0(z, y) + \kappa\Psi_a^1(z, y)) \, \Psi_b(z, x) & \text{if } a < y < x < b. \end{cases} \tag{2.35}$$

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is a candidate for the kernel of $R_\kappa(z) := (L_\kappa - z)^{-1}$. It is easy to check that

$$\mathcal{W}\Big(\Psi_a^1(z,\cdot) - \frac{\mathcal{W}^1(z)}{\mathcal{W}^0(z)}\Psi_a^0(z,x), \Psi_b(z,\cdot)\Big) = 0. \tag{2.36}$$

Therefore, changing if needed the normalization of $\Psi_b(z,\cdot)$, we can assume that

$$\Psi_a^1(z,\cdot) - \frac{\mathcal{W}^1(z)}{\mathcal{W}^0(z)}\Psi_a^0(z,\cdot) = \Psi_b(z,\cdot). \tag{2.37}$$

Using this we can rewrite (2.35) as

$$R_\kappa(z;x,y) = R_0(z;x,y) + \frac{\Psi_b(z,x)\Psi_b(z,y)}{\kappa^{-1}\mathcal{W}^0(z) + \mathcal{W}^1(z)}. \tag{2.38}$$

This is often called *Krein formula*, which is the basis of the boundary triplet approach. It expresses the resolvent with mixed b.c. by the resolvent with unperturbed b.c. plus a rank one perturbation. Note that if we check the boundedness of $R_0(z)$, then $R_\kappa(z)$ is well-defined and bounded unless $\kappa^{-1}\mathcal{W}^0(z) + \mathcal{W}^1(z) = 0$.

Thus, in the above approach, one introduces three objects: the space $\mathcal{B}$ and a pair of its distinguished linearly independent elements ϕ^0 and ϕ^1. Jointly, they are often called a *boundary triplet* [16, 1]. If V is integrable near $a \in \mathbb{R}$, one usually chooses the Dirichlet and Neumann b.c., that is, $\phi^0(\Xi) = \Xi(a)$ and $\phi^1(\Xi) = \Xi'(a)$. More generally, in most (but not all) Hamiltonians considered in this paper, we have such a distinguished pair. For example, for the Bessel operator with $|\mathrm{Re}(m)| < 1$, $m \neq 0$, these are the b.c. set by $r^{\frac{1}{2}+m}$ and $r^{\frac{1}{2}-m}$. For $m = 0$, one can choose $r^{\frac{1}{2}}$ and $r^{\frac{1}{2}}\ln r$; however, one can argue that only the first is truly distinguished.

3 Schrödinger operators related to the Bessel equation

3.1 Bessel operator

Various elements of the material of this subsection can be found in the literature, e.g., in [24, 27, 31]. We treat [6, 10] as the main references.

Let $m \in \mathbb{C}$. The *Bessel operator* is formally given by

$$H_m := -\partial_r^2 + \Big(m^2 - \frac{1}{4}\Big)\frac{1}{r^2}. \tag{3.1}$$

We would like to interpret it as a closed operator on $L^2(\mathbb{R}_+)$.

Note that when we deal with (3.1), it is more convenient to use 1d Bessel functions rather than the (usual) 2d Bessel functions. Depending on the circumstances, one may prefer to use hyperbolic or elliptic Bessel functions — therefore, for some quantities, we provide expressions in terms of both. For more detail about 1d Bessel functions, see Appendix A, especially Subsections A.3 and A.5.

In the following table, for each parameter and for each eigenvalue, we provide a few functions that span the space of eigenfunctions of (3.1) (usually, but not always, a basis of this space).

eigenvalue	parameters	eigenfunctions
$-k^2$ with $k \neq 0$		$\mathcal{I}_{\pm m}(kx), \quad \mathcal{K}_m(kx)$
0	$m \neq 0$	$x^{\frac{1}{2}\pm m}$
0	$m = 0$	$x^{\frac{1}{2}}, \quad x^{\frac{1}{2}}\ln x$

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After checking the square integrability of these functions near the endpoints, we see that the endpoints have the following types:

endpoint	parameters	index
0,	$|\mathrm{Re}(m)| < 1$	2
0,	$|\mathrm{Re}(m)| \geq 1$	0
$+\infty,$		0

The following theorem describes the basic holomorphic family of Bessel operators:

Theorem 3.1. *For $\mathrm{Re}(m) \geq 1$, there exists a unique closed operator H_m in the sense of $L^2(\mathbb{R}_+)$, which on $C_{\mathrm{c}}^\infty(]0,\infty[)$ is given by (3.1). In particular, for real $m \geq 1$, H_m is essentially self-adjoint on $C_{\mathrm{c}}^\infty(]0,\infty[)$. The family $m \mapsto H_m$ is holomorphic and possesses a unique holomorphic extension to $\mathrm{Re}(m) > -1$. The spectrum of H_m is*

$$\sigma(H_m) = [0, \infty[\, , \tag{3.2}$$

and its resolvent is

$$\frac{1}{(H_m + k^2)}(x, y) = \frac{1}{k} \begin{cases} \mathcal{I}_m(kx)\,\mathcal{K}_m(ky) & \text{if } 0 < x < y, \\ \mathcal{I}_m(ky)\,\mathcal{K}_m(kx) & \text{if } 0 < y < x, \end{cases} \qquad \mathrm{Re}(k) > 0; \tag{3.3}$$

$$\frac{1}{(H_m - k^2)}(x, y) = \pm\frac{\mathrm{i}}{k} \begin{cases} \mathcal{J}_m(kx)\,\mathcal{H}_m^\pm(ky) & \text{if } 0 < x < y, \\ \mathcal{J}_m(ky)\,\mathcal{H}_m^\pm(kx) & \text{if } 0 < y < x, \end{cases} \qquad \pm\mathrm{Im}(k) > 0. \tag{3.4}$$

Proof. For $\mathrm{Re}(m) > -1$, we define H_m to be the closed realization of (3.1) with the b.c. at 0 given by $x^{\frac{1}{2}+m}$. From the table above, we see that for $\mathrm{Re}(m) \geq 1$, it is the unique realization of (3.1).

We check that, for such $\mathrm{Re}(m) > -1$,

$$\lim_{x \to 0} \mathcal{W}(\mathcal{I}_m(kx), x^{\frac{1}{2}+m}) = 0, \tag{3.5}$$

and $\mathcal{I}_m(kx)$ is square integrable near zero. Moreover, $\mathcal{K}_m(kx)$ is square integrable near $+\infty$ if and only if $\mathrm{Re}(k) > 0$. We also find

$$\mathcal{W}\big(\mathcal{I}_m(kx), \mathcal{K}_m(kx)\big) = k. \tag{3.6}$$

Next we apply (2.19), which yields the kernel on the rhs of (3.3) as a candidate for the resolvent of H_m. We check that it is bounded. Hence it equals $(H_m + k^2)^{-1}$. We also verify that it depends holomorphically on m on a larger domain $\{\mathrm{Re}(m) > -1\}$. Therefore, $\{\mathrm{Re}(m) > -1\} \ni m \mapsto H_m$ is an holomorphic family.

See [6] for the details. $\square$

The Bessel operator is exactly solvable in a very strong sense. Besides the resolvent, one can also compute the integral kernel of the holomorphic semigroup generated by H_m, see, e.g., [27]. It can also be written in two equivalent ways.

$$\mathrm{e}^{-\frac{t}{2}H_m}(x, y) = \sqrt{\frac{2}{\pi t}}\, \mathcal{I}_m\!\left(\frac{xy}{t}\right) \mathrm{e}^{-\frac{x^2+y^2}{2t}}, \qquad \mathrm{Re}(t) \geq 0; \tag{3.7}$$

$$\mathrm{e}^{\pm\frac{\mathrm{i}t}{2}H_m}(x, y) = \mathrm{e}^{\pm\mathrm{i}\frac{\pi}{2}(m+1)}\sqrt{\frac{2}{\pi t}}\, \mathcal{J}_m\!\left(\frac{xy}{t}\right) \mathrm{e}^{\frac{\mp\mathrm{i}x^2\mp\mathrm{i}y^2}{2t}}, \qquad \pm\mathrm{Im}(t) \geq 0. \tag{3.8}$$

As noted in [6, 31], we have the identity

$$H_m = \Xi_m^{-1}(A)K^{-1}\Xi_m(A), \tag{3.9}$$

where

$$\Xi_m(t) := \mathrm{e}^{\mathrm{i}\ln(2)t}\frac{\Gamma\!\left(\frac{m+1+\mathrm{i}t}{2}\right)}{\Gamma\!\left(\frac{m+1-\mathrm{i}t}{2}\right)}, \quad A := \frac{1}{2\mathrm{i}}(x\partial_x + \partial_x x), \quad \text{and} \quad K := x^2. \tag{3.10}$$

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The operator $\Xi_m(A)$ is called the *Hankel transformation*. Its integral kernel can be also computed:

$$\Xi_m(A)(x,y) = \sqrt{\frac{2}{\pi}} \mathcal{J}_m(xy). \tag{3.11}$$

Note that (3.9) describes diagonalization of the Bessel operator — transforming it into the multiplication operator $\frac{1}{K}$.

Remark 3.2. For $-1 < \mathrm{Re}(m) < 1$, we can also consider mixed b.c.. For more details, see e.g. [10].

3.2 Exponential potentials

Let $\mathrm{Re}(k) \geq 0$. The *Schrödinger operator with the exponential potential* is formally given by

$$M_k := -\partial_x^2 + k^2 \mathrm{e}^{2x}. \tag{3.12}$$

We will interpret it as a closed operator on $L^2(\mathbb{R})$. Without restricting the generality we can assume that $\mathrm{Re}(k) \geq 0$.

Note that M_k is very common in the literature. For example, it has important applications in Liouville CFT, see, e.g., [36, Chapter 4], and on hyperbolic manifolds.

For $r = \mathrm{e}^x$, we have the following formal identity:

$$r^2\Big(-\partial_r^2 + \big(m^2 - \tfrac{1}{4}\big)\frac{1}{r^2} + k^2\Big) = \mathrm{e}^{\frac{x}{2}}\Big(-\partial_x^2 + k^2\mathrm{e}^{2x} + m^2\Big)\mathrm{e}^{-\frac{x}{2}}. \tag{3.13}$$

Using this, we can express eigenfunctions of M_k in terms of Bessel functions. In this case, 2-dimensional Bessel functions are more convenient than 1-dimensional ones, see Appendix, Section A.2.

Eigenvalues and corresponding eigenfunctions of (3.12) are given by:

eigenvalue	parameters	eigenfunctions
$-m^2$	$\mathrm{Re}(k) \geq 0, \quad k \neq 0$	$I_{\pm m}(k\mathrm{e}^x), \quad K_m(k\mathrm{e}^x)$
$-m^2, \quad m \neq 0$	$k = 0$	$\mathrm{e}^{\pm mx},$
$0,$	$k = 0$	$1, \quad x$

After checking the square integrability of these functions near the endpoints, we see that the endpoints have the following indices:

endpoint	parameters	index
$-\infty,$	$\mathrm{Re}(k) \geq 0$	0
$+\infty,$	$\mathrm{Re}(k) > 0$ or $k = 0,$	0
$+\infty,$	$\mathrm{Re}(k) = 0, \quad k \neq 0,$	2

The following theorem describes the basic holomorphic family of Schrödinger operators with the exponential potential:

Theorem 3.3. *For $\mathrm{Re}(k) > 0$ and $k = 0$, the expression (3.12) defines a unique closed operator on $L^2(\mathbb{R})$, which will be denoted M_k. The spectrum is $\sigma(M_k) = [0, \infty[$.*

Moreover, $\{\mathrm{Re}(k) > 0\} \ni k \mapsto M_k$ is a holomorphic family of closed operators, and for $\mathrm{Re}(m) > 0$, the resolvent is given by

$$\frac{1}{(M_k + m^2)}(x,y) = \begin{cases} I_m(k\mathrm{e}^x)\,K_m(k\mathrm{e}^y) & \text{if } x < y, \\ I_m(k\mathrm{e}^y)\,K_m(k\mathrm{e}^x) & \text{if } y < x. \end{cases} \tag{3.14}$$

Proof. The case $k = 0$ is well known, let us restrict ourselves to $\mathrm{Re}(k) > 0$. We check that for $\mathrm{Re}(m) > 0$ $I_m(k\mathrm{e}^x)$ is up to a multiplicative constant the only eigensolution square integrable close to $-\infty$. Similarly for $K_m(k\mathrm{e}^x)$ close to $+\infty$. We check that

$$\mathcal{W}\big(I_m(k\mathrm{e}^x), K_m(k\mathrm{e}^x)\big) = 1. \tag{3.15}$$

Next, we apply (2.19), which yields the kernel on the rhs of (3.14) as a candidate for the resolvent of M_k. Let us denote it by $R_k(-m^2; x, y)$. In Lemma 3.4, following the strategy given in Section 2.3, we check the boundedness of $R_k(-m^2)$ and its analyticity wrt k. This ends the proof of the theorem. $\qquad\square$

Lemma 3.4 (Boundedness of Kernel). *Let* $\mathrm{Re}(m) > 0$ *and* $\mathrm{Re}(k) > 0$. *Let* $R_k(-m^2)$ *be the operator with kernel* (3.14). *Then* $R_k(-m^2)$ *is a bounded operator and the map* $k \mapsto R_k(-m^2)$ *is a holomorphic family of bounded operators, which does not have a holomorphic extention to a larger subset of the complex plane.*

Proof. Since (a) both modified Bessel functions I_m and K_m are analytic for fixed m with $\mathrm{Re}(m) > 0$ and (b) the function $k\mathrm{e}^x$ is analytic in k, the kernel $R_k(-m^2; x, y)$ is an analytic function of parameter k. It is easy to see that for any $f, g \in C_\mathrm{c}^\infty(\mathbb{R})$, the quantity

$$(f | R_k(-m^2)\, g) = \int \overline{f(x)} R_k(-m^2; x, y) g(y) \mathrm{d}x \mathrm{d}y$$

is analytic in k. Since $C_\mathrm{c}^\infty(\mathbb{R})$ is a dense subset of $L^2(\mathbb{R})$, it remains to prove that $R_k(-m^2; x, y)$ is locally bounded in k.

To proceed further, we use the method of Section 2.3. We split the resolvent as in (2.29) and (2.30) with $c = 0$. By Lemma 2.6, we need to prove the boundedness of R_{--} and R_{++}.

We have, for $x \to 0$,

$$I_m(x) \sim \frac{1}{\Gamma(m+1)} \left(\frac{x}{2}\right)^m, \quad m \neq -1, -2, \ldots,$$

$$K_m(x) \sim \begin{cases} \mathrm{Re}(\Gamma(m)\left(\frac{2}{x}\right)^m) & \text{if } \mathrm{Re}(m) = 0, m \neq 0, \\ -\ln\left(\frac{x}{2}\right) - \gamma & \text{if } m = 0, \\ \frac{\Gamma(m)}{2}\left(\frac{2}{x}\right)^m & \text{if } \mathrm{Re}(m) > 0, \\ \frac{\Gamma(-m)}{2}\left(\frac{2}{x}\right)^m & \text{if } \mathrm{Re}(m) < 0, \end{cases}$$

and, for $x \to \infty$,

$$I_m(x) \sim \frac{1}{\sqrt{2\pi x}}\mathrm{e}^x \quad \text{and} \quad K_m(x) \sim \sqrt{\frac{\pi}{2x}}\mathrm{e}^{-x}.$$

We check

$$\left| R_k^{--}(-m^2; x, y) \right|$$
$$\leq C_{m,k} \left(\mathrm{e}^{-\mathrm{Re}(m)x}\mathrm{e}^{\mathrm{Re}(m)y}\mathbb{1}_{-\infty < y < x < 0}(x, y) + \mathrm{e}^{\mathrm{Re}(m)x}\mathrm{e}^{-\mathrm{Re}(m)y}\mathbb{1}_{-\infty < x < y < 0}(x, y) \right).$$

Since

$$\sup_{x \in\,]-\infty, 0]} \int \mathrm{e}^{-\mathrm{Re}(m)x}\mathrm{e}^{\mathrm{Re}(m)y}\mathbb{1}_{-\infty < y < x < 0}(x, y)\,\mathrm{d}y$$

$$= \sup_{x \in\,]-\infty, 0]} \mathrm{e}^{-\mathrm{Re}(m)x} \int_{-\infty}^x \mathrm{e}^{\mathrm{Re}(m)y}\,\mathrm{d}y = \sup_{x \in\,]-\infty, 0]} \mathrm{e}^{-\mathrm{Re}(m)x} \frac{1}{\mathrm{Re}(m)}\mathrm{e}^{\mathrm{Re}(m)x} = \frac{1}{\mathrm{Re}(m)},$$

$$\sup_{y \in\,]-\infty, 0]} \int \mathrm{e}^{-\mathrm{Re}(m)x}\mathrm{e}^{\mathrm{Re}(m)y}\mathbb{1}_{-\infty < y < x < 0}(x, y)\,\mathrm{d}x$$

$$= \sup_{y \in\,]-\infty, 0]} \mathrm{e}^{\mathrm{Re}(m)y} \int_y^0 \mathrm{e}^{-\mathrm{Re}(m)x}\,\mathrm{d}x = \sup_{y \in\,]-\infty, 0]} \mathrm{e}^{\mathrm{Re}(m)y} \frac{1}{\mathrm{Re}(m)}\left(-\int_0^{-y} \mathrm{e}^{\mathrm{Re}(m)x}\,\mathrm{d}x\right) = \frac{1}{\mathrm{Re}(m)},$$

by Schur's criterion we obtain

$$\left\| R_k^{--}(-m^2) \right\| \leq 2C_{m,k}.$$

For the $R_k^{++}(-m^2)$, we have the following: Since $\mathrm{Re}(k) > 0$, we obtain

$$\left| R_k^{++}(-m^2; x, y) \right| \leq C_{m,k} \mathrm{e}^{-\frac{x+y}{2}} \mathrm{e}^{-\mathrm{Re}(k)|\mathrm{e}^x - \mathrm{e}^y|} \leq C_{m,k} \mathrm{e}^{-\frac{x+y}{2}}.$$

Therefore, the Hilbert-Schmidt norm of $R_k^{++}(-m^2)$ is finite.

Now, we prove that $R_k(-m^2)$ cannot be extended to a holomorphic family of bounded operators beyond the axis $\mathrm{Re}(k) = 0$. Let us fix $g \in C_\mathrm{c}^\infty(\mathbb{R})$. For $k \mapsto R_k(-m^2)g$, with values in $L^2_\mathrm{loc}(\mathbb{R})$ is entire analytic. If $R_k(-m^2)$ could be extended to a holomorphic family of bounded operators, when applied to the function g this extension should coincde with $R_k(-m^2)g$. Then, for x below the support of g, we have

$$R_k(-m^2)g(x) = \int I_m(k\mathrm{e}^x) K_m(k\mathrm{e}^y) g(y)\,\mathrm{d}y = I_m(k\mathrm{e}^x) \int K_m(k\mathrm{e}^y) g(y)\,\mathrm{d}y = C_{m,k} I_m(k\mathrm{e}^x)$$

and

$$R_k(-m^2)g(x) = \int K_m(k\mathrm{e}^x) I_m(k\mathrm{e}^y) g(y)\,\mathrm{d}y = K_m(k\mathrm{e}^x) \int I_m(k\mathrm{e}^y) g(y)\,\mathrm{d}y = C_{m,k} I_m(k\mathrm{e}^x)$$

for some constant $C_{m,k}$.

If $\mathrm{Re}(k) < 0$, then $I_m(k\mathrm{e}^x) \notin L^2(\mathbb{R})$ because $|I_m(x)|$ diverges as $|\mathrm{Re}(x)| \to \infty$. Hence, the map cannot be extend to $\mathrm{Re}(k) < 0$. $\qquad\square$

Remark 3.5. One of applications of perturbed Bessel operators to quantum physics is the concept of *Regge poles* [33]. They are defined as poles of the holomorphic function $m \mapsto (H_m + V + k^2)^{-1}$, where V is typically a short-range potential. Substituting $r = \mathrm{e}^x$ similarly to (3.13), we obtain

$$r^2(H_m + V(r) + k^2) = \mathrm{e}^{\frac{x}{2}}\left(-\partial_x^2 + k\mathrm{e}^x + \mathrm{e}^{2x}V(\mathrm{e}^x) + m^2 \right)\mathrm{e}^{-\frac{x}{2}}.$$

This substitution is quite useful in many cases and sometimes called the *Langer substitution* [28, Eq. (22)]. Set $W(x) := \mathrm{e}^{2x}V(\mathrm{e}^x)$. Then, the transmutation property (1.3) can be generalized to include a potential:

$$\frac{1}{(M_k + W + m^2)}(x, y) = \mathrm{e}^{-\frac{x}{2}} \frac{1}{(H_m + V + k^2)}(\mathrm{e}^x, \mathrm{e}^y)\,\mathrm{e}^{-\frac{y}{2}}. \tag{3.16}$$

Thus Regge poles coincide with the poles of $m \mapsto (M_k + W + m^2)^{-1}$. Therefore, we have another equivalent definition of Regge poles, which is used e.g. in [3].

3.3 Negative exponential potential

The previous subsection covered the case $\mathrm{Re}(k) > 0$. In this subsection, we consider the case $\mathrm{Re}(k) = 0$, $k \neq 0$, that is, the *Schrödinger operator with a negative exponential potential*. Surprisingly, it appears in interesting physical applications, e.g., it is the main ingredient of the Feynman propagator on the Poincaré patch of the de Sitter and anti-de Sitter space. Clearly, it defines a symmetric operator, which possesses a 1-parameter family of closed realizations on $L^2(\mathbb{R})$.

In this section, for convenience, we introduce the parameter $\ell > 0$, so that $k = \mathrm{i}\ell$, and we consider the formal expression

$$M_{\mathrm{i}\ell} := -\partial_x^2 - \ell^2 \mathrm{e}^{2x}. \tag{3.17}$$

The corresponding maximal and minimal operators of $M_{\mathrm{i}\ell}$ in $L^2(\mathbb{R})$ are denoted $M_{\mathrm{i}\ell}^{\mathrm{max}}$ and $M_{\mathrm{i}\ell}^{\mathrm{min}}$. The domain of $M_{\mathrm{i}\ell}^{\mathrm{max}}$ is given by

$$\mathcal{D}(M_{\mathrm{i}\ell}^{\mathrm{max}}) = \{ f \in L^2(\mathbb{R}) \,|\, (-\partial_x^2 - \ell^2 \mathrm{e}^{2x})f \in L^2(\mathbb{R}) \},$$

(a) For $M_k := -\partial_x^2 + k^2 e^{2x}$ with $k \in \mathbb{C}$ (b) For $-\partial_x^2 + c\,e^{2x}$ with $c \in \mathbb{C}$
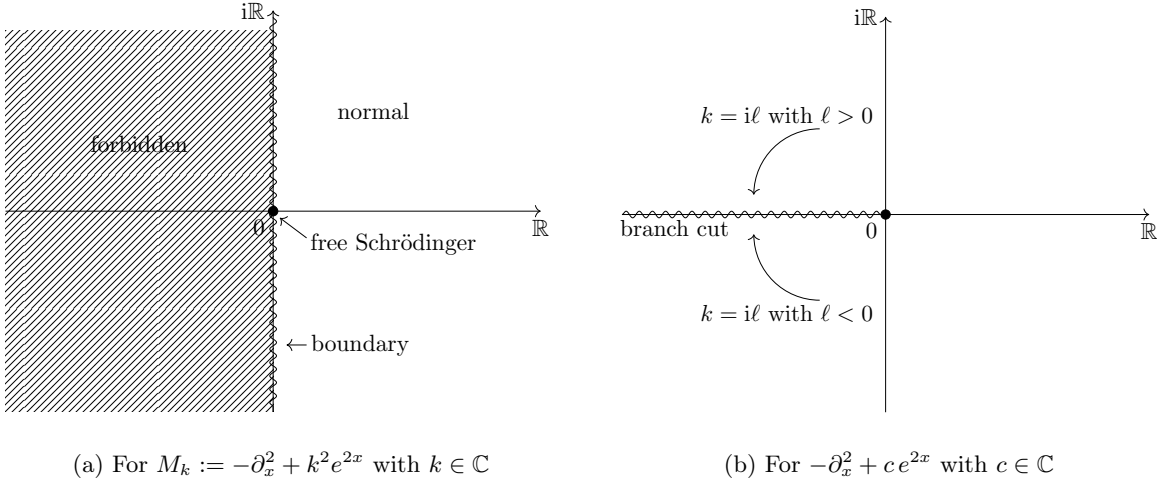
Figure 1: Domains of the parameters k and $c = k^2$

and $M_{\mathrm{i}\ell}^{\min}$ is the closure of the restriction of (3.17) to $C_{\mathrm{c}}^\infty(\mathbb{R})$.

In order to set b.c., we will use the Hankel functions

$$H_{\frac{1}{2}}^{\pm}(r) = \mp\mathrm{i}\Big(\frac{2}{\pi r}\Big)^{\frac{1}{2}} \mathrm{e}^{\pm\mathrm{i}r}.$$

(We could use $H_m^{\pm}(r)$ with other m, but the parameter $\frac{1}{2}$ gives especially simple elementary functions).

First, we describe the two distinguished realizations:

Theorem 3.6. *Let $\ell > 0$ and $\gamma \in \mathbb{C} \cup \{\infty\}$. There exists a unique closed operator in the sense of $L^2(\mathbb{R})$ that on $C_{\mathrm{c}}^\infty(]0,\infty[)$ is given by (3.17) and satisfies the following b.c. at $+\infty$:*

$$\mathcal{D}(M_{\mathrm{i}\ell}^0) = \Big\{\Xi \in \mathcal{D}(M_{\mathrm{i}\ell}^{\max}) \mid \lim_{x\to\infty} \mathcal{W}\big(H_{\frac{1}{2}}^+(\ell\mathrm{e}^x), \Xi(x)\big) = 0\Big\}, \tag{3.18}$$

$$\mathcal{D}(M_{\mathrm{i}\ell}^\infty) = \Big\{\Xi \in \mathcal{D}(M_{\mathrm{i}\ell}^{\max}) \mid \lim_{x\to\infty} \mathcal{W}\big(H_{\frac{1}{2}}^-(\ell\mathrm{e}^x), \Xi(x)\big) = 0\Big\}. \tag{3.19}$$

Both $M_{\mathrm{i}\ell}^0$ and $M_{\mathrm{i}\ell}^\infty$ do not have point spectrum, more precisely,

$$\sigma(M_{\mathrm{i}\ell}^0) = \sigma(M_{\mathrm{i}\ell}^\infty) = [0,\infty[\,, \tag{3.20}$$

and, for $\mathrm{Re}(m) > 0$,

$$\frac{1}{(M_{\mathrm{i}\ell}^0 + m^2)}(x,y) = \frac{\pi\mathrm{i}}{2}\begin{cases} J_m(\ell\mathrm{e}^x)\,H_m^+(\ell\mathrm{e}^y) & \text{if } x < y, \\[2mm] J_m(\ell\mathrm{e}^y)\,H_m^+(\ell\mathrm{e}^x) & \text{if } y < x; \end{cases} \tag{3.21}$$

$$\frac{1}{(M_{\mathrm{i}\ell}^\infty + m^2)}(x,y) = -\frac{\pi\mathrm{i}}{2}\begin{cases} J_m(\ell\mathrm{e}^x)\,H_m^-(\ell\mathrm{e}^y) & \text{if } x < y, \\[2mm] J_m(\ell\mathrm{e}^y)\,H_m^-(\ell\mathrm{e}^x) & \text{if } y < x. \end{cases} \tag{3.22}$$

Proof. First we check that

$$\lim_{x\to\infty} \mathcal{W}\big(H_{\frac{1}{2}}^{\pm}(\ell\mathrm{e}^x), H_m^{\pm}(\ell\mathrm{e}^x)\big) = 0. \tag{3.23}$$

Therefore, the rhs of (3.21), resp. (3.22) are good candidates for the inverses of $M_{\mathrm{i}\ell}^\infty + m^2$, resp. $M_{\mathrm{i}\ell}^0 + m^2$. But the rhs of (3.21), resp. (3.22) coincide with $R_k(-m^2)$ for $k = \mathrm{i}\ell$, resp. $k = -\mathrm{i}\ell$, in the notation of Lemma 3.4 (see the proof of Thm 3.7). And the proof of this lemma applies. $\qquad\square$

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In the following theorem, we prove that the realizations described above are limiting cases of the basic holomorphic family from the previous subsection.

Theorem 3.7. *We have the following weak convergence:*

1. We have

$$\mathrm{w}-\lim_{\varepsilon\searrow 0}(M_\varepsilon + m^2)^{-1} = (-\partial_x^2 + m^2)^{-1}. \tag{3.24}$$

2. For $\ell > 0$ and $\mathrm{Re}(m) > 0$, we have

$$\mathrm{w}-\lim_{\varepsilon\searrow 0}(M_{\varepsilon+\mathrm{i}\ell} + m^2)^{-1} = (M_{\mathrm{i}\ell}^\infty + m^2)^{-1}, \tag{3.25}$$

$$\mathrm{w}-\lim_{\varepsilon\searrow 0}(M_{\varepsilon-\mathrm{i}\ell} + m^2)^{-1} = (M_{\mathrm{i}\ell}^0 + m^2)^{-1}. \tag{3.26}$$

Proof. Note that

$$J_m(\ell\mathrm{e}^x) = \mathrm{e}^{\pm\mathrm{i}\frac{\pi}{2}m}I_m(\mp\mathrm{i}\ell\mathrm{e}^x) \quad\text{and}\quad H_m^\pm(\ell\mathrm{e}^y) = \frac{2}{\pi}\mathrm{e}^{\mp\mathrm{i}\frac{\pi}{2}(m+1)}K_m(\mp\ell\mathrm{e}^y).$$

Then, for all $x < y$, either

$$I_m\left((\varepsilon+\mathrm{i}\ell)\mathrm{e}^x\right)K_m\left((\varepsilon+\mathrm{i}\ell)\mathrm{e}^y\right) \to \frac{\pi}{2}J_m(\ell\mathrm{e}^x)H_m^-(\ell\mathrm{e}^y) = I_m(\mathrm{i}\ell\mathrm{e}^x)K_m(\mathrm{i}\ell\mathrm{e}^y)$$

or

$$I_m\left((\varepsilon+\mathrm{i}\ell)\mathrm{e}^x\right)K_m\left((\varepsilon+\mathrm{i}\ell)\mathrm{e}^y\right) \to \frac{\pi}{2}J_m(-\ell\mathrm{e}^x)H_m^+(-\ell\mathrm{e}^y) = I_m(-\mathrm{i}\ell\mathrm{e}^x)K_m(-\mathrm{i}\ell\mathrm{e}^y)$$

as $\varepsilon \to 0$. Similar convergence can be obtained for $y < x$. Then as $\ell \neq 0$, with uniform boundness of I_m and K_m and Lebesgue dominate convergence theorem gives the norm convergence. $\square$

From Theorem 3.7, we see that by setting

$$M_0 := -\partial_x^2, \quad M_{\mathrm{i}\ell} := M_{\mathrm{i}\ell}^\infty, \quad M_{-\mathrm{i}\ell} := M_{\mathrm{i}\ell}^0; \quad \ell > 0, \tag{3.27}$$

we extend the basic holomorphic family $\{\mathrm{Re}(k) > 0\} \ni k \mapsto M_k$ to a continuous family $\{\mathrm{Re}(k) \geq 0\} \ni k \mapsto M_k$.

Before we continue, let us note that $J_m(\ell\mathrm{e}^x)$ belong to $L^2(\mathbb{R})$ for $\mathrm{Re}(m) > 0$. Indeed,

$$\int_\mathbb{R} |J_m(\ell\mathrm{e}^x)|^2\,\mathrm{d}x = \int_\mathbb{R} \frac{|J_m(y)|^2}{y}\,\mathrm{d}y = \frac{1}{2m}, \tag{3.28}$$

where the last identity is e.g. in [30, Eq. (10.22.57)] with $a = 1$, $\mu = \nu = m$, and $\lambda = 1$.

Now, we are ready describe the remaining mixed realizations (see [37] for a similar result).

Theorem 3.8. *Let $\ell > 0$ and $\gamma \in \mathbb{C}\setminus\{0\}$. There exists a unique closed operator in the sense of $L^2(\mathbb{R})$ that on $C_\mathrm{c}^\infty(]0,\infty[)$ is given by (3.17) and satisfies the following b.c. at $+\infty$:*

$$\mathcal{D}(M_{\mathrm{i}\ell}^\gamma) = \left\{\Xi \in \mathcal{D}(M_{\mathrm{i}\ell}^{\max}) \mid \lim_{x\to\infty}\mathcal{W}\left(H_{\frac{1}{2}}^+(\ell\mathrm{e}^x) + \gamma H_{\frac{1}{2}}^-(\ell\mathrm{e}^x),\, \Xi(x)\right) = 0\right\}. \tag{3.29}$$

$\mathbb{C}\setminus\{0\} \ni \gamma \mapsto M_{\mathrm{i}\ell}^\gamma$ is a holomorphic family of closed operators. If $\gamma = \mathrm{e}^{\mathrm{i}\alpha}$, the eigenvalues and eigenfunctions of $M_{\mathrm{i}\ell}^\gamma$ are

$$-(\alpha+n)^2, \quad J_{\alpha+n}(\mathrm{e}^x), \quad n \in \mathbb{Z}, \quad \mathrm{Re}(\alpha+n) > 0. \tag{3.30}$$

its spectrum and resolvent are

$$\sigma(M_{\mathrm{i}\ell}^\gamma) = [0,\infty[\,\cup\,\{-(n+\alpha)^2 \mid \mathrm{Re}(n+\alpha) > 0, \quad n \in \mathbb{Z}\}, \tag{3.31}$$

$$\frac{1}{(M_{\mathrm{i}\ell}^\gamma + m^2)}(x,y) = \frac{\pi\mathrm{i}}{2(\mathrm{e}^{\mathrm{i}m\pi} - \gamma)} \begin{cases} J_m(\ell\mathrm{e}^x)\left(\mathrm{e}^{\mathrm{i}m\pi}H_m^+(\ell\mathrm{e}^y) + \gamma H_m^-(\ell\mathrm{e}^y)\right) & \text{if } x < y, \\ J_m(\ell\mathrm{e}^y)\left(\mathrm{e}^{\mathrm{i}m\pi}H_m^+(\ell\mathrm{e}^x) + \gamma H_m^-(\ell\mathrm{e}^x)\right) & \text{if } y < x. \end{cases} \tag{3.32}$$

4 Schrödinger operators related to the Whittaker equation

4.1 Whittaker operator

This is another classic problem, described in many sources. We treat [11, 12] as the main references for this subsection.

Let $m, \beta \in \mathbb{C}$. The *Whittaker operator* is formally defined by

$$H_{\beta,m} := -\partial_r^2 + \left(m^2 - \tfrac{1}{4}\right)\frac{1}{r^2} - \frac{\beta}{r}. \tag{4.1}$$

It is the radial part of the Schrödinger operator with the Coulomb potential, in dimension 3 used to describe the Hydrogen atom. We will interpret it as a closed operator on $L^2(\mathbb{R}_+)$.

We will use 1d Whittaker functions with conventions described in Appendix, Subsect. B.3.

We first find eigenvalues and corresponding eigenfunctions of (4.1):

eigenvalue	parameters	eigenfunctions
$-k^2$ with $k \neq 0$		$\mathcal{I}_{\frac{\beta}{2k},\pm m}(2kr), \quad \mathcal{K}_{\frac{\beta}{2k},m}(2kr)$
0	$\beta \neq 0$	$r^{\frac{1}{4}}\mathcal{J}_{\pm 2m}(2\sqrt{\beta r}), \quad r^{\frac{1}{4}}\mathcal{H}_{2m}^{\pm}(2\sqrt{\beta r})$
0	$\beta = 0, \quad m \neq 0$	$r^{\frac{1}{2}\pm m}$
0	$\beta = 0, \quad m = 0$	$r^{\frac{1}{2}}, \quad r^{\frac{1}{2}}\ln(r)$

After checking the square integrability of these functions near the endpoints, we see that the endpoints have the following indices:

endpoint	parameters	index
$0,$	$|\mathrm{Re}(m)| < 1$	2
$0,$	$|\mathrm{Re}(m)| \geq 1$	0
$+\infty,$		0

Let us describe the basic holomorphic family of Whittaker operators:

Theorem 4.1. *For $\mathrm{Re}(m) \geq 1$, there exists a unique closed operator $H_{\beta,m}$ in the sense of $L^2(\mathbb{R}_+)$, which on $C_c^\infty(]0,\infty[)$ is given by (4.1). It depends holomorphically on β, m. It can be uniquely extended to a holomorphic family of closed operators on $L^2(\mathbb{R}_+)$ defined for $\mathrm{Re}(m) > -1$, $\beta \in \mathbb{C}$, $(\beta, m) \neq (0, -\frac{1}{2})$. Its spectrum is*

$$\sigma(H_{\beta,m}) = [0,\infty[\cup \left\{ -\frac{\beta^2}{4(n+m+\frac{1}{2})^2} \;\middle|\; n+m+\frac{1}{2} \neq 0, \; \mathrm{Re}\left(\frac{\beta}{n+m+\frac{1}{2}}\right) > 0, \; n \in \mathbb{N} \right\}. \tag{4.2}$$

Outside of the spectrum, the kernel of the resolvent of $H_{\beta,m}$ is

$$\frac{1}{(H_{\beta,m}+k^2)}(x,y) := \tfrac{1}{2k}\,\Gamma\left(\tfrac{1}{2}+m-\tfrac{\beta}{2k}\right)\begin{cases} \mathcal{I}_{\frac{\beta}{2k},m}(2kx)\mathcal{K}_{\frac{\beta}{2k},m}(2ky) & \text{for } 0 < x < y, \\ \mathcal{I}_{\frac{\beta}{2k},m}(2ky)\mathcal{K}_{\frac{\beta}{2k},m}(2kx) & \text{for } 0 < y < x. \end{cases} \tag{4.3}$$

Proof of Theorem 4.1. We define $H_{\beta,m}$ for $\mathrm{Re}(m) > -1$, $(\beta, m) \neq (0, -\frac{1}{2})$ by the b.c. at 0 given by $x^{\frac{1}{4}}\mathcal{J}_{2m}(2\sqrt{\beta x})$. We check that for $\mathrm{Re}(m) > -1$

$$\lim_{x \to 0} \mathcal{W}\left(\mathcal{I}_{\frac{\beta}{2k},\pm m}(2kx), x^{\frac{1}{4}}\mathcal{J}_{2m}(2\sqrt{\beta x})\right) = 0. \tag{4.4}$$

Moreover, $\mathcal{K}_{\frac{\beta}{2k},m}(2kx)$ is square integrable near $+\infty$. We also find

$$\mathcal{W}\Big(\mathcal{I}_{\frac{\beta}{2k},m}(2kx),\mathcal{K}_{\frac{\beta}{2k},m}(2kx)\Big) = \frac{2k}{\Gamma\big(\frac{1}{2}+m-\frac{\beta}{2k}\big)}. \tag{4.5}$$

Next, we apply (2.19), which yields the kernel on the right-hand side of (3.3) as a candidate for the resolvent of $H_{\beta,m}$. We check that it is bounded. Hence, it equals $(H_{\beta,m}+k^2)^{-1}$. We also verify that it depends analytically on β and m. Therefore,

$$\mathbb{C}\times\{\operatorname{Re}(m)>-1\}\backslash(0,-\tfrac{1}{2})\ni(\beta,m)\mapsto H_{\beta,m} \tag{4.6}$$

is an analytic family. For $\operatorname{Re}(m)\geq 1$, the b.c. is not needed, hence $H_{\beta,m}$ is then uniquely defined.

See [11] for details. $\qquad\square$

The operator $H_{\beta,m}$ is sometimes called the *Whittaker operator with pure b.c.*. Note that for $m>-\frac{1}{2}$, we can simplify the b.c. — we can set it by $x^{\frac{1}{2}+m}$. For $-1<\operatorname{Re}(m)\leq\frac{1}{2}$, interestingly, this does not work. Instead, we can use $x^{\frac{1}{2}+m}(1-\frac{\beta}{1+2m})$ to set the b.c.. Details can be found in [11].

Note that $(0,-\frac{1}{2})$ is a singularity of the holomorphic function $(\beta,m)\mapsto H_{\beta,m}$. Additionally, we set $H_{0,-\frac{1}{2}}:=H_{-\frac{1}{2}}$. Then, the basic family of Whittaker operators extends the basic family of Bessel operators:

$$H_{0,m}=H_m,\quad\operatorname{Re}(m)>-1. \tag{4.7}$$

Remark 4.2. For $-1<\operatorname{Re}(m)<1$, $\beta\in\mathbb{C}$, we can also consider mixed b.c.. We do not study them here, see e.g. [12].

4.2 Morse potentials

The *Schrödinger operator with the Morse potential* is formally given by

$$M_{\beta,k}:=-\partial_x^2+k^2\mathrm{e}^{2x}-\beta\mathrm{e}^x. \tag{4.8}$$

We will interpret it as a closed operator on $L^2(\mathbb{R})$. Without restricting the generality we can assume that $\operatorname{Re}(k)\geq 0$.

For $r=\mathrm{e}^x$, we have the following formal identity:

$$r^2\Big(-\partial_r^2+\big(m^2-\tfrac{1}{4}\big)\frac{1}{r^2}-\frac{\beta}{r}+k^2\Big)=\mathrm{e}^{\frac{x}{2}}\Big(-\partial_x^2+k^2\mathrm{e}^{2x}-\beta\mathrm{e}^x+m^2\Big)\mathrm{e}^{-\frac{x}{2}}. \tag{4.9}$$

Therefore, eigenfunctions of $M_{\beta,k}$ can be expressed in terms of Whittaker functions. In this subsection instead of the standard (1-dimensional) Whittaker functions $\mathcal{I}_{\beta,m},\mathcal{K}_{\beta,m}$ it is more convenient to use 2d Whittaker functions $I_{\beta,m},K_{\beta,m}$, see Appendix B.4.

We first find eigenvalues and corresponding eigenfunctions of (4.15):

eigenvalue	parameters	eigenfunctions
$-m^2$	$\operatorname{Re}(k)\geq 0,\quad k\neq 0$	$I_{\frac{\beta}{2k},\pm m}(2k\mathrm{e}^x),\quad K_{\frac{\beta}{2k},m}(2k\mathrm{e}^x)$
$-m^2$	$k=0,\quad\beta\neq 0$	$I_{\pm 2m}(\sqrt{-4\beta}\,\mathrm{e}^{\frac{x}{2}}),\quad K_{2m}(\sqrt{-4\beta}\,\mathrm{e}^{\frac{x}{2}})$
$-m^2$ with $m\neq 0$	$k=0,\quad\beta=0$	$\mathrm{e}^{\pm mx}$
0	$k=0,\quad\beta=0$	$1,\quad x$

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After checking the square integrability of these functions near the endpoints, we see that the endpoints have the following indices:

endpoint	parameters	index
$-\infty$,		0
$+\infty$,	$\mathrm{Re}(k) > 0$	0
$+\infty$,	$k = 0, \beta \in \mathbb{C} \setminus \mathbb{R}_{>0}$	0
$+\infty$,	$\mathrm{Re}(k) = 0, k \neq 0$	2
$+\infty$,	$k = 0, \beta > 0$	2

Here is a description of the basic holomorphic family of Schrödinger operators with Morse potentials:

Theorem 4.3. *For $\beta, k \in \mathbb{C}$ with $\mathrm{Re}(k) > 0$ there exists a unique closed operator in the sense of $L^2(\mathbb{R})$, denoted $M_{\beta,k}$ which on $C_c^\infty(\mathbb{R})$ it is given by (4.8). It forms a holomorphic family of closed operators.*
Its spectrum is

$$\sigma(M_{\beta,k}) = \left\{ -m^2 \,\Big|\, m = \frac{\beta}{2k} - n - \frac{1}{2}, \quad \mathrm{Re}(m) > 0, \quad n \in \mathbb{N} \right\} \cup [0, \infty[. \tag{4.10}$$

Outside of the spectrum, its resolvent is given by

$$\frac{1}{(M_{\beta,k} + m^2)}(x, y) := \Gamma\big(\tfrac{1}{2} + m - \tfrac{\beta}{2k}\big) \begin{cases} I_{\frac{\beta}{2k}, m}(2ke^x) K_{\frac{\beta}{2k}, m}(2ke^y), & \text{if } x < y, \\ I_{\frac{\beta}{2k}, m}(2ke^y) K_{\frac{\beta}{2k}, m}(2ke^x), & \text{if } y < x. \end{cases} \tag{4.11}$$

Proof. The uniqueness of a closed realization of $N_{k,m}$ for $\mathrm{Re}(k) > 0$ follows from the table above. $I_{\frac{\beta}{2k}, m}(2ke^x)$ is then square integrable near $-\infty$ and $K_{\frac{\beta}{2k}, m}(2ke^x)$ is square integrable near $+\infty$. We compute the Wronskian:

$$\mathcal{W}(I_{\frac{\beta}{2k}, m}(2ke^x), K_{\frac{\beta}{2k}, m}(2ke^x)) = \frac{1}{\Gamma\big(\tfrac{1}{2} + m - \tfrac{\beta}{2k}\big)}. \tag{4.12}$$

Now, by (2.19) the kernel on the rhs of (4.11) is a candidate of the resolvent of $M_{\beta,k}$. The boundedness is obtained with Lemma 4.4, similarly as in the previous (sub)section. $\qquad\square$

Lemma 4.4. *Let $k, \beta \in \mathbb{C}$ with $\mathrm{Re}(k) > 0$ and fix m with $\mathrm{Re}(m) > 0$ and $-m^2$ outside of (4.10). Let $R_{\beta,k}(-m^2)$ be the operator with the kernel on the right hand side of (4.11). Then $R_{\beta,k}(-m^2)$ is bounded and depends analytically on β, k.*

Proof. The proof is very similar to that of Lemma 3.4. We use the method of Section 2.3. We split the resolvent as in (2.29) and (2.30) with $c = 0$. Then we prove the boundedness of R_{--} and R_{++}. $\qquad\square$

Remark 4.5. Clearly, the family of Morse potentials extends the family of exponential potentials:

$$M_{0,k} = M_k, \tag{4.13}$$

i.e., if $\beta = 0$, then the Morse potential is the exponential potential covered in Section 3.2.

Remark 4.6. If $k = 0$ and $\beta \in \mathbb{C} \setminus \mathbb{R}_+$, then after scaling, the Morse potential is the exponential potential covered in Section 3.2. This case is covered in Figure 1b with $\beta = -c$. If $k = 0$ and $\beta > 0$, then after scaling, Morse potentials is the negative exponential covered in Section 3.3.

Let $A = \frac{xp + px}{2}$ where $p = -i\partial_x$. Let $U_\tau := e^{i\tau A}$ be the *dilation operator*. It acts on functions as follows:

$$(U_\tau f)(x) = e^{\frac{\tau}{2}} f(e^\tau x).$$

Then with U_τ, we have the following identity:

$$U_{\ln 2}^{-1} M_{\beta,0} U_{\ln 2} = U_{\ln 2}^{-1}\Big(-\partial_x^2 - \beta e^x\Big)U_{\ln 2} = \frac{1}{4}\Big(-\partial_x^2 + (-4\beta)e^{2x}\Big) = \frac{1}{4}M_{\sqrt{-4\beta}}. \tag{4.14}$$

Remark 4.7. We have not analyzed the case $\mathrm{Re}(k) = 0$, except for $k = 0$, which is discussed above. We leave it for future research.

4.3 Isotonic oscillator

The *isotonic harmonic oscillator* is formally defined by

$$N_{k,m} := -\partial_v^2 + \big(m^2 - \tfrac{1}{4}\big)\frac{1}{v^2} + k^2 v^2. \tag{4.15}$$

It appears in physics as the radial part of the radially symmetric harmonic oscillator in any dimension > 1. The name "isotonic" indicates that the frequencies in all directions are the same [38]. We will interpret (4.15) as a closed operator on $L^2(\mathbb{R}_+)$. Without restricting the generality, we can assume that $\mathrm{Re}(k) \geq 0$.

Consider the change of variables $r = \frac{v^2}{2}$. We have the formal identity, which connects the Whittaker operator with the isotonic oscillator:

$$-\partial_r^2 + \big(\tfrac{m^2}{4} - \tfrac{1}{4}\big)\frac{1}{r^2} - \frac{\beta}{r} + k^2 = v^{-\frac{3}{2}}\Big(-\partial_v^2 + \big(m^2 - \tfrac{1}{4}\big)\frac{1}{v^2} + k^2 v^2 - 2\beta\Big)v^{-\frac{1}{2}}. \tag{4.16}$$

Therefore, eigenfunctions of the isotonic oscillator can be expressed in terms of the Whittaker functions. The details are described in Appendix B.5, where the functions $\mathbb{I}$ and $\mathbb{K}$ are introduced. We will use them in the following table describing eigenvalues and corresponding eigenfunctions of (4.15):

eigenvalue	parameters	eigenfunctions
2β	$\mathrm{Re}(k) \geq 0, \quad k \neq 0$	$\mathbb{I}_{\frac{\beta}{k},\pm m}(\sqrt{k}v), \quad \mathbb{K}_{\frac{\beta}{k},m}(\sqrt{k}v)$
$-p^2$ with $p \neq 0$	$k = 0$	$\mathcal{I}_{\pm m}(pv), \quad \mathcal{K}_m(pv)$
0	$k = 0, \quad m \neq 0$	$v^{\frac{1}{2}\pm m}$
0	$k = 0, \quad m = 0$	$v^{\frac{1}{2}}, \quad v^{\frac{1}{2}}\ln v$

After checking the square integrability of these functions near the endpoints, we see that the endpoints have the following indices:

endpoint	parameters	index
$0,$	$|\mathrm{Re}(m)| < 1,$	2
$0,$	$|\mathrm{Re}(m)| \geq 1,$	0
$+\infty,$		0

Let us describe the basic family of isotonic harmonic oscillators:

Theorem 4.8. *For $\mathrm{Re}(k) \geq 0$ and $\mathrm{Re}(m) \geq 1$ there exists a unique closed operator $N_{k,m}$ in the sense of $L^2(\mathbb{R}_+)$ given on $C_c^\infty(]0,\infty[)$ by (4.15). It uniquely extends by analyticity in m to $\mathrm{Re}(k) \geq 0$ and $\mathrm{Re}(m) > -1$.*

1. For $\mathrm{Re}(k) > 0$, $\mathrm{Re}(m) > -1$ we have a holomorphic family with the spectrum

$$\sigma(N_{k,m}) = \{2\beta = 2k(2n + m + 1) \mid n \in \mathbb{N}_0\}, \quad \mathrm{Re}(k) > 0. \tag{4.17}$$

Outside of its spectrum, its resolvent is given by

$$\frac{1}{(N_{k,m} - 2\beta)}(u,v) := \frac{1}{2}\,\Gamma\Big(\frac{1}{2} + \frac{m}{2} - \frac{\beta}{2k}\Big)\begin{cases}\mathbb{I}_{\frac{\beta}{k},m}(\sqrt{k}u)\,\mathbb{K}_{\frac{\beta}{k},m}(\sqrt{k}v) & \text{if } 0 < u < v, \\ \mathbb{I}_{\frac{\beta}{k},m}(\sqrt{k}v)\,\mathbb{K}_{\frac{\beta}{k},m}(\sqrt{k}u) & \text{if } 0 < v < u.\end{cases} \tag{4.18}$$

2. *Let* $\mathrm{Re}(k) = 0$, $k \neq 0$ *with* $k = \mathrm{i}\ell$, $\ell > 0$. *Then, we have* $\sigma(N_{k,m}) = \mathbb{R}$ *and, for* $\pm\mathrm{Im}2\beta > 0$

$$\frac{1}{(N_{k,m} - 2\beta)}(u,v) := \tfrac{1}{2}\,\Gamma\Big(\tfrac{1}{2} + \tfrac{m}{2} - \tfrac{\beta}{\mp 2k}\Big) \begin{cases} \mathbb{I}_{\frac{\beta}{\mp k},m}(\sqrt{\mp k}u)\,\mathbb{K}_{\frac{\beta}{\mp k},m}(\sqrt{\mp k}v) & \textit{if } 0 < u < v, \\ \mathbb{I}_{\frac{\beta}{\mp k},m}(\sqrt{\mp k}v)\,\mathbb{K}_{\frac{\beta}{\mp k},m}(\sqrt{\mp k}u) & \textit{if } 0 < v < u. \end{cases} \tag{4.19}$$

For $\mathrm{Re}(k) = 0$, $k \neq 0$ *with* $k = -\mathrm{i}\ell$, $\ell > 0$, *we set* $N_{k,m} = N_{-k,m}$.

3. *The case* $k = 0$ *coincides with the Bessel operators:* $N_{0,m} = H_m$.

Proof. For $\mathrm{Re}(k) \geq 0$ and $\mathrm{Re}(m) > -1$, we define $N_{k,m}$ by setting the b.c. at zero with $v^{\frac{1}{2}+m}$. We check that

$$\mathcal{W}\big(v^{\frac{1}{2}+m}, \mathbb{I}_{\frac{\beta}{k},m}(\sqrt{k}v)\big) = 0 \tag{4.20}$$

and $\mathbb{I}_{\frac{\beta}{k},m}(\sqrt{k}v)$ is square integrable near 0.

Let us now consider Cases 1,2,3 separately.

For $\mathrm{Re}(k) > 0$, $\mathbb{K}_{\frac{\beta}{k},m}(\sqrt{k}v)$ is square integrable near $+\infty$. We check that

$$\mathcal{W}\big(\mathbb{I}_{\frac{\beta}{k},m}(\sqrt{k}v), \mathbb{K}_{\frac{\beta}{k},m}(\sqrt{k}v)\big) = \frac{2}{\Gamma\big(\tfrac{1}{2} + \tfrac{m}{2} - \tfrac{\beta}{2k}\big)}. \tag{4.21}$$

Now, (2.19) yields the kernel on the rhs of (4.18) as a candidate of the resolvent of $N_{k,m}$. In Lemma 4.9.1., we check that it is bounded and depends analytically on parameters k, m. This proves Case 1.

For $k = \mathrm{i}\ell$, $\ell > 0$, and $\pm\mathrm{Im}(2\beta) > 0$, we have

$$\mathrm{Re}\Big(\frac{\beta}{\mp k}\Big) = \mp\frac{\mathrm{Im}\beta}{\ell} < 0, \tag{4.22}$$

thus, $\mathbb{K}_{\frac{\beta}{\mp k}}(\sqrt{\mp k}v)$, which can be estimated by $Cv^{-\frac{1}{2}+\mathrm{Re}\frac{\beta}{\mp k}}$, is square integrable near $+\infty$. Therefore, using (2.19) the rhs of (4.19) is a candidate of the kernel of the resolvent of $N_{k,m}$. We check that $\pm\mathrm{Im}\beta < 0$ and $\mathrm{Re}(m) > -1$, $\ell > 0$ implies that there are no solutions of

$$2\beta = 2(\mp\mathrm{i}\ell)(2n + m + 1). \tag{4.23}$$

Therefore, besides $\mathbb{R}$ there is no spectrum of $N_{k,m}$. In Lemma 4.9.2. we check that (4.19) is bounded and depends analytically on m. This proves Case 2.

Finally, Case 3 was treated before. $\qquad\square$

Lemma 4.9. *For* $k \neq 0$, *let* $R_{k,m}(2\beta)$ *denote the operator with the integral kernel* (4.18). *Let* $\mathrm{Re}(m) > -1$.

1. *If* $\mathrm{Re}(k) > 0$ *and* 2β *is outside of the spectrum, then* $R_{k,m}(2\beta)$ *is bounded and depends analytically on* k, m.

2. *If* $k = \mp\mathrm{i}\ell$, $\ell > 0$, *and* $\pm\mathrm{Im}(2\beta) > 0$, *then* $R_{k,m}(2\beta)$ *is bounded and depends analytically on* m.

Proof. We use Lemma 2.6 with, say, $c = \frac{1}{\sqrt{|k|}}$, to split $R_{k,m}$. We have the estimates

$$|R_{k,m}^{--}(-2\beta; u,v)| \leq \begin{cases} Cu^{\frac{1}{2}+\mathrm{Re}(m)}v^{\frac{1}{2}-\mathrm{Re}(m)} & \text{if } 0 < u < v < c, \\ Cv^{\frac{1}{2}+\mathrm{Re}(m)}u^{\frac{1}{2}-\mathrm{Re}(m)} & \text{if } 0 < v < u < c, \end{cases} \quad m \neq 0; \tag{4.24}$$

$$|R_{k,0}^{--}(-2\beta; u,v)| \leq \begin{cases} Cu^{\frac{1}{2}}v^{\frac{1}{2}}|\ln(v)| & \text{if } 0 < u < v < c, \\ Cv^{\frac{1}{2}}u^{\frac{1}{2}}|\ln(u)| & \text{if } 0 < v < u < c, \end{cases} \tag{4.25}$$

$$|R_{k,m}^{++}(-2\beta; u,v)| \leq \begin{cases} Cu^{-\frac{1}{2}-\mathrm{Re}\frac{\beta}{k}}v^{-\frac{1}{2}+\mathrm{Re}\frac{\beta}{k}}\mathrm{e}^{(u^2-v^2)\mathrm{Re}(k)} & \text{if } c < u < v, \\ Cv^{-\frac{1}{2}-\mathrm{Re}\frac{\beta}{k}}u^{-\frac{1}{2}+\mathrm{Re}\frac{\beta}{k}}\mathrm{e}^{(v^2-u^2)\mathrm{Re}(k)} & \text{if } c < v < u. \end{cases} \tag{4.26}$$

Now, we use (4.24), (4.25) and the Hilbert-Schmidt estimate to prove the boundedness of $R_{k,m}^{--}(2\beta)$, both in Case 1 and 2. Then, we use (4.26) to prove the boundedness of $R_{k,m}^{++}(2\beta)$. We treat separately for Case 1 and Case 2.

Let $\mathrm{Re}(k) > 0$. Set $b := -\mathrm{Re}(\frac{\beta}{k})$. Using by (4.26) we see that that

$$R_{k,m}^{++}(-2\beta; u, v) \leq C_1 \mathrm{e}^{-|u-v|(\frac{(\ln u - \ln v)}{(u-v)}b + (u+v)\mathrm{Re}(k))}. \tag{4.27}$$

The function

$$[c, \infty[\times [c, \infty[\ni (u, v) \mapsto \frac{\ln u - \ln v}{u - v} b + (u + v)\mathrm{Re}(k)$$

is continuous and goes to $+\infty$ as $u \to +\infty$ or $v \to +\infty$. Therefore, away from a bounded set, it can be bounded from below by some $\epsilon > 0$. Hence, (4.27) can be estimated from above by $\leq C_1 \mathrm{e}^{-|u-v|\epsilon}$. Now (4.27) is the kernel of a bounded operator by Young's inequality or Schur's criterion. This proves Case 1.

Now assume $\mathrm{Re}(k) = 0$. Let $b := -\mathrm{Re}\left(\frac{\beta}{k}\right)$, as before. We now have $b > 0$. We have

$$|R_{k,m}^{++}(-2\beta; u, v)| \leq \begin{cases} Cu^{-\frac{1}{2}+b}v^{-\frac{1}{2}-b} & \text{if } c < u < v, \\ Cv^{-\frac{1}{2}+b}u^{-\frac{1}{2}-b} & \text{if } c < v < u. \end{cases} \tag{4.28}$$

Without loss of generality, one may let $c = 0$. Making substitutions $u = \mathrm{e}^t$, $v = \mathrm{e}^s$, we obtain

$$(f|R_{k,m}^{++}(-2\beta; u, v)g) = \int_0^\infty \int_u^\infty |f(u)||g(v)|u^{-\frac{1}{2}+b}v^{-\frac{1}{2}-b} + (u \leftrightarrow v)\,\mathrm{d}v\mathrm{d}u \tag{4.29}$$

$$= \int \int |f(\mathrm{e}^t)||g(\mathrm{e}^s)|\mathrm{e}^{\frac{t}{2}+\frac{s}{2}-b|t-s|}\,\mathrm{d}s\mathrm{d}t$$

$$\leq \frac{2}{b}\left(\int |f(\mathrm{e}^t)|^2\mathrm{e}^t\,\mathrm{d}t\right)^{\frac{1}{2}}\left(\int |g(\mathrm{e}^s)|^2\mathrm{e}^s\,\mathrm{d}s\right)^{\frac{1}{2}}$$

$$= \frac{2}{b}\left(\int |f(u)|^2\,\mathrm{d}u\right)^{\frac{1}{2}}\left(\int |g(v)|^2\,\mathrm{d}v\right)^{\frac{1}{2}}. \tag{4.30}$$

This shows Case 2. $\qquad\qquad\square$

Remark 4.10. The operators $N_{k,\mp\frac{1}{2}}$ are the harmonic oscillators on $\mathbb{R}_+$ with the Neumann, respectively. Dirichlet conditions. We can denote them as follows:

$$N_{k,-\frac{1}{2}} =: N_k^{\mathrm{N}}, \quad N_{k,\frac{1}{2}} =: N_k^{\mathrm{D}}. \tag{4.31}$$

Here are their spectra:

$$\sigma(N_k^{\mathrm{N}}) = \{k(4n+1) \mid n \in \mathbb{N}_0\}, \tag{4.32}$$

$$\sigma(N_k^{\mathrm{D}}) = \{k(4n+3) \mid n \in \mathbb{N}_0\}. \tag{4.33}$$

In the next section we will consider them again.

Remark 4.11. For $-1 < \mathrm{Re}(m) < 1$, we can also consider mixed b.c.. They can be analyzed similarly as for Bessel operators.

5   Harmonic Oscillator

Let $\mathrm{Re}(k) \geq 0$. This section is devoted to the *harmonic oscillator*, formally defined on $L^2(\mathbb{R})$ by

$$N_k := -\partial_v^2 + k^2 v^2. \tag{5.1}$$

We will consider its closed realizations on $L^2(\mathbb{R})$. Without loss of generality, we assume that $\mathrm{Re}(k) \geq 0$.

For real k, the self-adjoint realization of N_k is one of the best-known operators in Quantum Mechanics. For complex k, its closed realization is an interesting example of an operator with sometimes surprising properties, and has been studied, e.g., in [7, 32].

To describe the eigenfunctions of (4.15), we will use Weber functions, defined in Appendix C.

eigenvalue	parameters	eigenfunctions
2β	$\mathrm{Re}(k) \geq 0, \quad k \neq 0$	$\mathbb{I}_{\frac{\beta}{k}, \pm}(\sqrt{k}v), \quad \mathbb{K}_{\frac{\beta}{k}}(\pm\sqrt{k}v)$
$-p^2$ with $p \neq 0$	$k = 0,$	$\mathrm{e}^{pv}, \quad \mathrm{e}^{-pv}$
0	$k = 0,$	$v, \quad 1$

The endpoints have always index 0:

endpoint	parameters	index
$-\infty,$	$\mathrm{Re}(k) \geq 0$	0
$+\infty,$	$\mathrm{Re}(k) \geq 0$	0

For the harmonic oscillator, we have a unique closed realization for all parameters:

Theorem 5.1. *Let $\mathrm{Re}(k) \geq 0$. There exist a unique closed operator in the sense of $L^2(\mathbb{R})$, which on $C_\mathrm{c}^\infty(\mathbb{R})$ coincides with (5.1). It will be denoted N_k.*

1. *$\{\mathrm{Re}(k) > 0\} \ni k \mapsto N_k$ is a holomorphic family of closed operators with the following spectrum:*

$$\sigma(N_k) = \{k(2n+1) \mid n \in \mathbb{N}_0\}. \tag{5.2}$$

For 2β away of its spectrum, its resolvent is given by

$$\frac{1}{(N_k - 2\beta)}(u,v) = \frac{1}{2\sqrt{2\pi}}\, 2^{\frac{\beta}{k}}\, \Gamma(\tfrac{1}{2} - \tfrac{\beta}{k}) \begin{cases} \mathbb{K}_{\frac{\beta}{k}}(-\sqrt{k}u)\, \mathbb{K}_{\frac{\beta}{k}}(\sqrt{k}v) & \text{if } u < v, \\ \mathbb{K}_{\frac{\beta}{k}}(-\sqrt{k}v)\, \mathbb{K}_{\frac{\beta}{k}}(\sqrt{k}u) & \text{if } v < u. \end{cases} \tag{5.3}$$

2. *Let $\mathrm{Re}(k) = 0$, $k \neq 0$ with $k = \mathrm{i}\ell$, $\ell > 0$. Then, we have $\sigma(N_k) = \mathbb{R}$ and, for $\pm\mathrm{Im}2\beta > 0$,*

$$\frac{1}{(N_k - 2\beta)}(u,v) = \frac{1}{2\sqrt{2\pi}}\, 2^{\frac{\beta}{\mp k}}\, \Gamma(\tfrac{1}{2} - \tfrac{\beta}{\mp\mathrm{i}\ell}) \begin{cases} \mathbb{K}_{\frac{\beta}{\mp k}}(-\sqrt{\mp k}u)\, \mathbb{K}_{\frac{\beta}{\mp k}}(\sqrt{\mp k}v) & \text{if } u < v, \\ \mathbb{K}_{\frac{\beta}{\mp k}}(-\sqrt{\mp k}v)\, \mathbb{K}_{\frac{\beta}{\mp k}}(\sqrt{\mp k}u) & \text{if } v < u. \end{cases} \tag{5.4}$$

For $\mathrm{Re}(k) = 0$, $k \neq 0$ with $k = -\mathrm{i}\ell$, $\ell > 0$, we set $N_k = N_{-k}$.

3. *Finally, $\sigma(N_0) = [0, \infty[$ and, for $\mathrm{Re}(p) > 0$,*

$$\frac{1}{(N_0 + p^2)}(u,v) = \frac{\mathrm{e}^{-p|u-v|}}{2p}. \tag{5.5}$$

Proof. The uniqueness of a closed realization follows immediately from the table.

For $\mathrm{Re}(k) > 0$, $\mathbb{K}_{\frac{\beta}{k}}(-\sqrt{k}v)$ is square integrable near $-\infty$ and $\mathbb{K}_{\frac{\beta}{k}}(\sqrt{k}v)$ is square integrable near $-\infty$. Using (C.5) and (C.6), and then Legendre's duplication formula, we get the Wronskian:

$$\mathcal{W}(\mathbb{K}_{\frac{\beta}{k}}(-\sqrt{k}u), \mathbb{K}_{\frac{\beta}{k}}(\sqrt{k}u)) = \frac{4\pi}{\Gamma(\frac{1}{4} - \frac{\beta}{2k})\,\Gamma(\frac{3}{4} - \frac{\beta}{2k})} = \frac{2\sqrt{2\pi}}{2^{\frac{\beta}{k}}\,\Gamma(\frac{1}{2} - \frac{\beta}{k})}. \tag{5.6}$$

Now, we apply (2.19) to have the kernel on the rhs of (5.3) as a candidate of the resolvent of N_k, denoting it by $R_k(-2\beta)$. The boundedness of $R_k(-2\beta)$ is immediate from the subsection on the isotonic oscillator.

For $k = \mathrm{i}\ell$, $\ell > 0$, and $\pm\mathrm{Im}(2\beta) > 0$ we have

$$\mathrm{Re}\Big(\frac{\beta}{\mp\mathrm{i}\ell}\Big) = \mp\frac{\mathrm{Im}\beta}{\ell} < 0, \tag{5.7}$$

thus, $\mathbb{K}_{\frac{\beta}{\mp\mathrm{i}\ell}}(\sqrt{\mp\mathrm{i}\ell}v)$ is square integrable near $+\infty$ and $\mathbb{K}_{\frac{\beta}{\mp\mathrm{i}\ell}}(-\sqrt{\mp\mathrm{i}\ell}v)$ near $-\infty$. The boundedness of $R_k(-2\beta)$ is proven as for the isotonic oscillator.

The case $k = 0$ is just the well-known free 1d Laplacian. $\qquad\square$

As is well-known, the propagator for N_k can be expressed in elementary functions. It is given by the so-called *Mehler's formula*

$$\mathrm{e}^{-\mathrm{i}t\frac{1}{2}N_1}(u, v) = \frac{1}{\sqrt{\pi}}\sqrt{\frac{\rho}{1 - \rho^2}}\exp\Big(-\frac{(1 + \rho^2)(u^2 + v^2) - 2\rho u v}{2(1 - \rho^2)}\Big), \tag{5.8}$$

where $\rho = \mathrm{e}^{-\mathrm{i}t}$.

Remark 5.2. Recall that in (4.31) we introduced the harmonic oscillators with the Neumann and Dirichlet boundary conditions $N_k^{\mathrm{N/D}} = N_{k,\mp\frac{1}{2}}$ on $\mathbb{R}_+$, as special cases of the isotonic harmonic oscillator. They are closely related to the harmonic oscillator N_k on $\mathbb{R}$.

Let $L^2_\pm(\mathbb{R})$ denote the subspace of $L^2(\mathbb{R})$ consisting of even, resp. odd functions. Let us define the unitary operators

$$U_\pm : L^2(\mathbb{R}_+) \to L^2_\pm(\mathbb{R}), \tag{5.9}$$

$$(U_+ g)(v) := \frac{1}{\sqrt{2}}g(|v|), \quad v \in \mathbb{R}; \tag{5.10}$$

$$(U_- g)(v) := \frac{\mathrm{sgn}(v)}{\sqrt{2}}g(|v|), \quad v \in \mathbb{R}; \tag{5.11}$$

so that

$$(U_\pm^* f)(v) = \sqrt{2}f(v), \quad v \in \mathbb{R}_+. \tag{5.12}$$

Then

$$N_k = U_+ N_k^{\mathrm{N}} U_+^* + U_- N_k^{\mathrm{D}} U_-^*, \tag{5.13}$$

where "+" can be replaced by "$\oplus$" in the sense of the direct sum $L^2(\mathbb{R}) = L^2_+(\mathbb{R}) \oplus L^2_-(\mathbb{R})$. Consequently,

$$\sigma(N_k) = \sigma(N_k^{\mathrm{D}}) \cup \sigma(N_k^{\mathrm{N}}) \tag{5.14}$$

where the spectra of N_k^{D} and N_k^{N} were computed in (4.32).

A   Bessel equation

There are several kinds of Bessel equation, all equivalent to one another. Their main application is the Helmholtz equation in d dimensions:

$$(-\Delta_d + E)f = 0. \tag{A.1}$$

If $E > 0$, (A.1) can be simplified to $E = 1$. This case is often referred to as *hyperbolic*. Conversely, the case with $E < 0$, which can similarly be simplified to $E = -1$, is sometimes known as *trigonometric*. The radial part of (A.1) on spherical harmonics of order ℓ is

$$\left(-\partial_r^2 - \frac{(d-1)}{r}\partial_r + \left(\left(\ell + \tfrac{d}{2} - 1\right)^2 - \left(\tfrac{d}{2} - 1\right)^2\right)\frac{1}{r^2} + E\right)f = 0. \tag{A.2}$$

The versions of (A.2) for various dimensions are equivalent by gauging (or conjugating) the operator with a power of r.[1]

For (A.6) and (B.10), we also gauge with the exponential of r.

The standard Bessel equation corresponds to the 2d trigonometric case. The so-called modified Bessel equation is the 2d hyperbolic case. However, sometimes it is convenient to use versions of the Bessel equation for other dimensions.

In our paper, in some cases 1d Bessel functions are more convenient; in others, 2d Bessel functions. Therefore, we will discuss both.

All forms of the Bessel equation are equivalent to the so-called $_0F_1$ equation, which is not as well-known. Actually, one could argue that the $_0F_1$ equation and its standard solutions $\mathbf{F}_\alpha$ and U_α have a simpler theory than the usual Bessel equation and functions.

In this section, we first discuss the $_0F_1$ equation and its solutions. Then we describe hyperbolic and trigonometric 1d and 2d Bessel equations and functions.

A.1 $_0F_1$ equation

Let $c \in \mathbb{C}$. The standard solution of the *the $_0F_1$ equation*

$$(w\,\partial_w^2 + c\,\partial_w - 1)f(w) = 0 \tag{A.3}$$

is the *hypergeometric $_0F_1$ function*

$$_0F_1(c; z) := \sum_{n=0}^{\infty} \frac{w^n}{(c)_n n!}. \tag{A.4}$$

If $c \neq 0, -1, -2, \ldots$, then it is the only solution of the $_0F_1$ equation ~ 1 at 0. It is convenient to normalize it differently:

$$_0\mathbf{F}_1(c; w) := \frac{_0F_1(c; w)}{\Gamma(c)} = \sum_{n=0}^{\infty} \frac{w^n}{n!\Gamma(c+n)}, \tag{A.5}$$

so that it is defined for all c.

The $_0F_1$ equation can be reduced to a special class of the confluent equation by the so-called *Kummer's 2nd transformation*:

$$z\,\partial_z^2 + c\,\partial_z - 1 = \frac{4}{w}e^{-\frac{w}{2}}\left(z\,\partial_w^2 + (2c - 1 - w)\,\partial_w - c + \frac{1}{2}\right)e^{\frac{w}{2}}, \tag{A.6}$$

where $w = \pm 4\sqrt{z}$, $z = \frac{1}{16}w^2$. Using this, we can derive an expression for the $_0F_1$ function in terms of the confluent function (B.3):

$$_0F_1(c; z) = e^{\mp 2\sqrt{z}}\,_1F_1\left(\frac{2c-1}{2}; 2c - 1; \pm 4\sqrt{z}\right).$$

---

[1]By gauging (or conjugating) the operator A with a function $f(r)$ we mean replacing it with $f(r)Af(r)^{-1}$. See, for example, (A.6), (A.12), (A.19), (B.10), or (B.23), for gauging with a power of r, and (A.6) and (B.10) for gauging with an exponential.

Instead of c, it is often more natural to use $\alpha := c - 1$, and rewrite (A.3) as

$$(z\,\partial_z^2 + (\alpha + 1)\,\partial_z - 1)v(z) = 0, \tag{A.7}$$

and set

$$F_\alpha(z) := {}_0F_1(\alpha + 1; z), \quad \mathbf{F}_\alpha(z) := {}_0\mathbf{F}_1(\alpha + 1; z). \tag{A.8}$$

The following function is also a solution of the ${}_0F_1$ equation (A.7):

$$U_\alpha(z) := \mathrm{e}^{-2\sqrt{z}} z^{-\frac{\alpha}{2} - \frac{1}{4}} {}_2F_0\left(\frac{1}{2} + \alpha, \frac{1}{2} - \alpha; -; -\frac{1}{4\sqrt{z}}\right),$$

where we used the ${}_2F_0$ function (B.7). Obviously,

$$U_\alpha(z) = z^{-\alpha} U_{-\alpha}(z).$$

As $|z| \to \infty$ and $|\arg z| < \frac{\pi}{2} - \varepsilon$, we have

$$U_\alpha(z) \sim \exp(-2z^{\frac{1}{2}}) z^{-\frac{\alpha}{2} - \frac{1}{4}}. \tag{A.9}$$

U_α is a unique solution of (A.7) with this property.

We can express U_α in terms of the solutions of with a simple behavior at zero

$$U_\alpha(z) = \frac{\sqrt{\pi}}{\sin \pi(-\alpha)} \mathbf{F}_\alpha(z) + \frac{\sqrt{\pi}}{\sin \pi\alpha} z^{-\alpha} \mathbf{F}_{-\alpha}(z). \tag{A.10}$$

A.2 Hyperbolic 2d Bessel equation

The usual *modified Bessel* equation has the form

$$\left(-\partial_r^2 - \frac{1}{r}\partial_r + \frac{m^2}{r^2} + 1\right)g = 0. \tag{A.11}$$

We use the name the *hyperbolic 2d Bessel equation* for (A.11). It is equivalent to the ${}_0F_1$ equation:

$$w^{\frac{m}{2}}\left(w\,\partial_w^2 + (1 + m)\,\partial_w - 1\right)w^{-\frac{m}{2}} = \partial_r^2 + \frac{1}{r}\partial_r - 1 - \frac{m^2}{r^2}, \tag{A.12}$$

where $w = \frac{r^2}{4}$, $r = \pm 2\sqrt{w}$.

The *hyperbolic 2d Bessel function* I_m is defined by

$$I_m(r) := \left(\frac{r}{2}\right)^m {}_0\mathbf{F}_1\left(m + 1; \frac{r^2}{4}\right). \tag{A.13}$$

We have the Wronskian

$$\mathcal{W}(I_m, I_{-m}) = -\frac{2\sin(\pi m)}{\pi r}, \tag{A.14}$$

and for $m \in \mathbb{Z}$

$$I_m(r) = I_{-m}(r). \tag{A.15}$$

The *2d Macdonald function* K_m is defined by

$$K_{-m}(r) = K_m(r) := \frac{\sqrt{\pi}}{2}\left(\frac{r}{2}\right)^m U_m\left(\frac{r^2}{4}\right) \tag{A.16}$$

$$= \frac{\pi}{2\sin(\pi m)}\left(I_{-m}(r) - I_m(r)\right). \tag{A.17}$$

A.3 Hyperbolic 1d Bessel equation

The *hyperbolic 1d Bessel equation*:

$$\left(-\partial_r^2 + \left(m^2 - \frac{1}{4} \right) \frac{1}{r^2} + 1 \right) g = 0. \tag{A.18}$$

is equivalent to the hyperbolic 2d Bessel equation by a simple gauge transformation:

$$r^{-\frac{1}{2}} \left(-\partial_r^2 + \left(m^2 - \frac{1}{4} \right) \frac{1}{r^2} + 1 \right) r^{\frac{1}{2}} = -\partial_r^2 - \frac{1}{r} \partial_r + \frac{m^2}{r^2} + 1. \tag{A.19}$$

The *hyperbolic 1d Bessel function* $\mathcal{I}_m$ is defined by

$$\mathcal{I}_m(r) := \sqrt{\pi} \left(\frac{r}{2} \right)^{\frac{1}{2}+m} {}_0\mathbf{F}_1 \left(m+1; \frac{r^2}{4} \right) = \sqrt{\frac{\pi r}{2}} I_m(r). \tag{A.20}$$

We have the Wronskian:

$$\mathcal{W}(\mathcal{I}_m, \mathcal{I}_{-m}) = -\sin(\pi m). \tag{A.21}$$

and for $m \in \mathbb{Z}$,

$$\mathcal{I}_m(r) = \mathcal{I}_{-m}(r), \tag{A.22}$$

The *1d Macdonald function* $\mathcal{K}_m$ is defined by

$$\mathcal{K}_{-m}(r) := \mathcal{K}_m(r) = \left(\frac{r}{2} \right)^{\frac{1}{2}+m} U_m \left(\frac{r^2}{4} \right) = \sqrt{\frac{2r}{\pi}} K_m(r) \tag{A.23}$$

$$= \frac{1}{\sin(\pi m)} \left(\mathcal{I}_{-m}(r) - \mathcal{I}_m(r) \right). \tag{A.24}$$

A.4 Trigonometric 2d Bessel equation

The usual *Bessel equation*, which can be called the *trigonometric 2d Bessel equation*, has the form

$$\left(-\partial_r^2 - \frac{1}{r} \partial_r + \frac{m^2}{r^2} - 1 \right) g = 0. \tag{A.25}$$

We can pass from the hyperbolic 2d to the trigonometric 2d Bessel equation by the substitution $r \to \mathrm{i}r$.

The (usual) *Bessel function* (or the *trigonometric 2d Bessel function*) is

$$J_m(r) = \mathrm{e}^{\pm \mathrm{i}\frac{\pi}{2} m} I_m(\mathrm{e}^{\mp \mathrm{i}\frac{\pi}{2}} r). \tag{A.26}$$

We also have two *Hankel functions*:

$$H_m^{\pm}(r) = \frac{2}{\pi} \mathrm{e}^{\mp \mathrm{i}\frac{\pi}{2}(m+1)} K_m(\mathrm{e}^{\mp \mathrm{i}\frac{\pi}{2}} r). \tag{A.27}$$

Note that the traditional notation for $H_m^{\pm}$ is $H_m^{(1)}$ and $H_m^{(2)}$. The authors believe that their notation is more handy.

A.5 Trigonometric 1d Bessel equation

We also have the trigonometric 1d Bessel equation:

$$\left(-\partial_r^2 + \left(m^2 - \frac{1}{4} \right) \frac{1}{r^2} - 1 \right) g = 0. \tag{A.28}$$

We can pass from hyperbolic 1d to trigonometric Bessel equations by the substitution $r \to \mathrm{i}r$.

We can introduce various kinds of solutions of the 1d trigonometric Bessel equation: the *1d Bessel function*

$$\mathcal{J}_m(r) := e^{\pm i \frac{\pi}{2}(m+\frac{1}{2})} \mathcal{I}_m(e^{\mp i \frac{\pi}{2}} r) = \sqrt{\frac{\pi r}{2}} J_m(r), \tag{A.29}$$

and the *1d Hankel function of the 1st/2nd kind*

$$\mathcal{H}_m^{\pm}(r) := e^{\mp i \frac{\pi}{2}(m+\frac{1}{2})} \mathcal{K}_m(e^{\mp i \frac{\pi}{2}} r) = \sqrt{\frac{\pi r}{2}} H_m^{\pm}(r). \tag{A.30}$$

B Whittaker equation

The $_1F_1$ equation, the $_2F_0$ equation, and the Whittaker equation are equivalent to one another by certain substitutions and gauge transformations. In this section, we briefly describe conventions and properties of solutions to these equations.

Note that our definitions of Whittaker functions differ slightly from some of the literature, e.g., [30]. In particular, we use what is sometimes called *Olver's normalization*, which is advantageous because it avoids singularities for the parameters under consideration. We follow the conventions of [9, 11].

The Whittaker equation is equivalent to the radial part of the Schrödinger equation with the Coulomb potential in any dimension:

$$\left(-\Delta_d - \frac{\beta}{r} + \frac{1}{4} \right) f = 0. \tag{B.1}$$

Therefore, there exists a variant of the Whittaker equation for any dimension. The standard one corresponds to $d = 1$. We will also find it convenient to consider the Whittaker equation for $d = 2$.

In the following subsections, we review several equations, equivalent to one another: the $_1F_1$ equation, the $_2F_0$ equation, the 1d Whittaker equation, the 2d Whittaker equation, and the eigenequation of the isotonic oscillator.

B.1 $_1F_1$ equation

The $_1F_1$ *hypergeometric equation*, also called the *confluent equation* has the form

$$\left(r \partial_r^2 + (c - r) \partial_r - a \right) f(r) = 0. \tag{B.2}$$

Its standard solution is *Kummer's confluent hypergeometric function* $_1F_1(a; c; \cdot)$

$$_1F_1(a; c; r) := \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} \frac{r^k}{k!}. \tag{B.3}$$

It is the only solution of (B.2) behaving as 1 in the vicinity of $r = 0$. It is often convenient to normalize it differently:

$$_1\mathbf{F}_1(a; c; r) := \sum_{k=0}^{\infty} \frac{(a)_k}{\Gamma(c+k)} \frac{r^k}{k!} = \frac{_1F_1(a; c; r)}{\Gamma(c)}. \tag{B.4}$$

It satisfies 1st Kummer's identity

$$_1F_1(a;\, c;\, r) = e^r \,_1F_1(c - a;\, c;\, -r). \tag{B.5}$$

B.2 $_2F_0$ equation

The $_2F_0$ *hypergeometric equation* has the form

$$\left(w^2 \partial_w^2 + (-1 + (1 + a + b)w) \partial_w + ab \right) v(w) = 0. \tag{B.6}$$

The $_2F_0$ equation has a distinguished solution, which can be expressed as a limit of the Gauss hypergeometric function:

$$_2F_0(a, b; -; w) := \lim_{c \to \infty} {}_2F_1(a, b; c; cw), \tag{B.7}$$

where we take the limit over $|\arg(c) - \pi| < \pi - \varepsilon$ with $\varepsilon > 0$, and the above definition is valid for $w \in \mathbb{C}\backslash[0, +\infty[$. Obviously one has

$$_2F_0(a, b; -; w) = {}_2F_0(b, a; -; w). \tag{B.8}$$

The function extends to an analytic function on the universal cover of $\mathbb{C}\backslash\{0\}$ with a branch point of an infinite order at 0, and the following asymptotic expansion holds:

$$_2F_0(a, b; -; w) \sim \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n!} w^n, \quad |\arg(w)| < \pi - \varepsilon.$$

B.3   The 1d Whittaker equation

The usual *Whittaker equation* corresponds to dimension 1 and has the following form:

$$\left(-\partial_r^2 + \left(m^2 - \frac{1}{4}\right)\frac{1}{r^2} - \frac{\beta}{r} + \frac{1}{4} \right) v(r) = 0. \tag{B.9}$$

It can be reduced to the $_1F_1$-equation,

$$-r^{\frac{1}{2}\mp m} e^{\frac{r}{2}} \left(-\partial_r^2 + \left(m^2 - \frac{1}{4}\right)\frac{1}{r^2} - \frac{\beta}{r} + \frac{1}{4} \right) r^{\frac{1}{2}\pm m} e^{-\frac{r}{2}} = r\partial_r^2 + (c - r)\partial_r - a \tag{B.10}$$

for the parameters $c = 1 \pm 2m$ and $a = \frac{1}{2} \pm m - \beta$. Here the sign $\pm$ has to be understood as two possible choices. The following function solves the Whittaker equation (B.9):

$$\mathcal{I}_{\beta,m}(r) := r^{\frac{1}{2}+m} e^{\mp\frac{r}{2}} {}_1\mathbf{F}_1\left(\frac{1}{2} + m \mp \beta; 1 + 2m; \pm r \right). \tag{B.11}$$

Note that the sign independence comes from the 1st Kummer's identity. We have the Wronskian

$$\mathcal{W}(\mathcal{I}_{\beta,m}, \mathcal{I}_{\beta,-m}) = -\frac{\sin(2\pi m)}{\pi}. \tag{B.12}$$

The Whittaker equation is also equivalent to the $_2F_0$ equation. Indeed by setting $w = -r^{-1}$ we obtain

$$-r^{2-\beta} e^{\frac{r}{2}} \left(-\partial_r^2 + \left(m^2 - \frac{1}{4}\right)\frac{1}{r^2} - \frac{\beta}{r} + \frac{1}{4} \right) r^\beta e^{-\frac{r}{2}}$$
$$= w^2\partial_w^2 + (-1 + (1 + a + b)w)\,\partial_w + ab.$$

for the parameters $a = \frac{1}{2} + m - \beta$ and $b = \frac{1}{2} - m - \beta$.

We define

$$\mathcal{K}_{\beta,m}(r) := r^\beta e^{-\frac{r}{2}} {}_2F_0\left(\frac{1}{2} + m - \beta, \frac{1}{2} - m - \beta; -; -r^{-1} \right),$$

which is thus a solution of the Whittaker equation (B.9). The symmetry relation (B.8) implies that

$$\mathcal{K}_{\beta,m}(r) = \mathcal{K}_{\beta,-m}(r). \tag{B.13}$$

The following connection formulas hold for $2m \notin \mathbb{Z}$:

$$\mathcal{K}_{\beta,m}(r) = \frac{\pi}{\sin(2\pi m)} \left(-\frac{\mathcal{I}_{\beta,m}(r)}{\Gamma(\frac{1}{2} - m - \beta)} + \frac{\mathcal{I}_{\beta,-m}(r)}{\Gamma(\frac{1}{2} + m - \beta)} \right), \tag{B.14}$$

$$\mathcal{I}_{\beta,m}(r) = \frac{\Gamma(\frac{1}{2} - m + \beta)}{2\pi} \left(e^{i\pi m}\mathcal{K}_{-\beta,m}(e^{i\pi}r) + e^{-i\pi m}\mathcal{K}_{-\beta,m}(e^{-i\pi}r) \right). \tag{B.15}$$

Here is an estimate for small r:

$$\mathcal{I}_{\beta,m}(r) = \frac{r^{\frac{1}{2}+m}}{\Gamma(1+2m)}(1+O(r)). \tag{B.16}$$

If $\mathrm{Re}(m) > 0$, then

$$\mathcal{K}_{\beta,m}(r) = \frac{\Gamma(2m)}{\Gamma(\frac{1}{2}+m-\beta)}\left(r^{\frac{1}{2}-m} + o(|r|^{\frac{1}{2}-m})\right). \tag{B.17}$$

For large r, if $\varepsilon > 0$, then for $|\arg r| < \pi - \varepsilon$,

$$\mathcal{K}_{\beta,m}(r) = r^\beta e^{-\frac{r}{2}}\left(1 + O(r^{-1})\right). \tag{B.18}$$

This together with (B.15) implies the estimates

$$\mathcal{I}_{\beta,m}(r) = \frac{1}{\Gamma(\frac{1}{2}+m-\beta)}r^{-\beta}e^{\frac{r}{2}}(1+O(r^{-1})), \quad |\arg r| < \frac{\pi}{2} - \epsilon; \tag{B.19}$$

$$|\mathcal{I}_{\beta,m}(r)| \le \frac{1}{|\Gamma(\frac{1}{2}+m-\beta)|}r^{-\mathrm{Re}(\beta)}(1+O(r^{-1})), \quad |\arg r| = \frac{\pi}{2}. \tag{B.20}$$

The relation of the functions $\mathcal{I}_{0,m}$ and $\mathcal{K}_{0,m}$ with $\mathcal{I}_m$ and $\mathcal{K}_m$ reads

$$\mathcal{I}_{0,m}(r) = \frac{2}{\Gamma(\frac{1}{2}+m)}\mathcal{I}_m\left(\frac{r}{2}\right), \quad \mathcal{K}_{0,m}(r) = \mathcal{K}_m\left(\frac{r}{2}\right). \tag{B.21}$$

B.4   The 2d Whittaker equation

The 2d Whittaker equation has the form

$$\left(-\partial_r^2 - \frac{1}{r}\partial_r + \frac{m^2}{r^2} - \frac{\beta}{r} + \frac{1}{4}\right)v = 0. \tag{B.22}$$

It is equivalent to the 1d Whittaker equation by a simple gauge transformation

$$r^{-\frac{1}{2}}\left(-\partial_r^2 + \left(m^2 - \frac{1}{4}\right)\frac{1}{r^2} - \frac{\beta}{r} + \frac{1}{4}\right)r^{\frac{1}{2}} = -\partial_r^2 - \frac{1}{r}\partial_r + \frac{m^2}{r^2} - \frac{\beta}{r} + \frac{1}{4}. \tag{B.23}$$

In order to make our presentation more transparent, similarly as in the case of Bessel functions, beside the usual (1d) Whittaker functions, we define *2d Whittaker functions*, which solve the 2d Whittaker equation:

$$I_{\beta,m}(r) := \sqrt{\frac{2}{\pi r}}\,\mathcal{I}_{\beta,m}(r) \qquad K_{\beta,m}(r) := \sqrt{\frac{\pi}{2r}}\,\mathcal{K}_{\beta,m}(r). \tag{B.24}$$

B.5   Eigenequation of isotonic oscillator

The eigenequation of the isotonic harmonic oscillator (4.15) (with $k=1$) has the form

$$\left(-\partial_v^2 + \left(m^2 - \frac{1}{4}\right)\frac{1}{v^2} + v^2 - 2\beta\right)f(v) = 0. \tag{B.25}$$

Let us remind the following identity (4.16) involving the change of variables $r = \frac{v^2}{2}$:

$$-\partial_r^2 + \left(\frac{m^2}{4} - \frac{1}{4}\right)\frac{1}{r^2} - \frac{\beta}{r} + k^2 = v^{-\frac{3}{2}}\left(-\partial_v^2 + \left(m^2 - \frac{1}{4}\right)\frac{1}{v^2} + k^2 v^2 - 2\beta\right)v^{-\frac{1}{2}}. \tag{B.26}$$

$\mathcal{I}_{\frac{\beta}{2k},\frac{m}{2}}(2r)$, $\mathcal{K}_{\frac{\beta}{2k},\frac{m}{2}}(2r)$ are annihilated by the left hand side of (B.26). Therefore, (B.25) is solved in terms of the following functions:

$$\mathbb{I}_{\beta,m}(v) := v^{\frac{1}{2}+m}e^{\mp\frac{1}{2}v^2}{}_1\mathbf{F}_1\Big(\frac{1+m\mp\beta}{2}; 1+m; \pm v^2\Big), \tag{B.27}$$

$$= v^{-\frac{1}{2}}\mathcal{I}_{\frac{\beta}{2},\frac{m}{2}}(v^2) \tag{B.28}$$

$$\mathbb{K}_{\beta,m}(v) := v^{\beta-\frac{1}{2}}e^{-\frac{1}{2}v^2}{}_2F_0\Big(\frac{1+m-\beta}{2}, \frac{1-m-\beta}{2}; -; -v^{-2}\Big) \tag{B.29}$$

$$= v^{-\frac{1}{2}}\mathcal{K}_{\frac{\beta}{2},\frac{m}{2}}(v^2). \tag{B.30}$$

Note from (B.13) that $\mathbb{K}_{\beta,m}(r) = \mathbb{K}_{\beta,-m}(r)$,

$$\mathcal{W}(\mathbb{I}_{\beta,m}, \mathbb{I}_{\beta,-m}) = -\frac{2\sin\pi m}{\pi}, \tag{B.31}$$

$$\mathbb{K}_{\beta,m}(v) = \frac{\pi}{\sin(\pi m)}\Big(-\frac{\mathbb{I}_{\beta,m}(v)}{\Gamma(\frac{1-m-\beta}{2})} + \frac{\mathbb{I}_{\beta,-m}(v)}{\Gamma(\frac{1+m-\beta}{2})}\Big), \tag{B.32}$$

$$\mathbb{I}_{\beta,m}(v) = \frac{\Gamma(\frac{1-m+\beta}{2})}{2\pi}\Big(e^{i\frac{\pi}{2}m}\mathbb{K}_{-\beta,m}(e^{i\frac{\pi}{2}}v) + e^{-i\frac{\pi}{2}m}\mathbb{K}_{-\beta,m}(e^{-i\frac{\pi}{2}}v)\Big). \tag{B.33}$$

For small v, we have

$$\mathbb{I}_{\beta,m}(v) = \frac{v^{\frac{1}{2}+m}}{\Gamma(1+m)}(1+O(v^2)). \tag{B.34}$$

Hence,

$$\mathbb{K}_{\beta,m}(v) = \frac{\Gamma(m)}{\Gamma(\frac{1+m-\beta}{2})}\Big(v^{\frac{1}{2}-m} + o(|v|^{\frac{1}{2}-m})\Big). \tag{B.35}$$

For large v we have the asymptotics

$$\mathbb{K}_{\beta,m}(v) = v^{\beta-\frac{1}{2}}e^{-\frac{v^2}{2}}(1+O(v^{-2})), \quad |\arg v| < \frac{\pi}{2} - \epsilon, \quad \epsilon > 0. \tag{B.36}$$

Together with (B.33), this implies

$$\mathbb{I}_{\beta,m}(v) = \frac{1}{\Gamma(\frac{1+m-\beta}{2})}v^{-\beta-\frac{1}{2}}e^{\frac{v^2}{2}}(1+O(v^{-2})), \quad |\arg v| < \frac{\pi}{4} - \epsilon; \tag{B.37}$$

$$|\mathbb{I}_{\beta,m}(v)| \le \frac{1}{|\Gamma(\frac{1+m-\beta}{2})|}v^{-\mathrm{Re}(\beta)-\frac{1}{2}}(1+O(v^{-2})), \quad |\arg v| = \frac{\pi}{4}. \tag{B.38}$$

C   Weber equation

The *Weber equation* (also called the *parabolic cylinder equation*) has the form

$$\big(-\partial_v^2 + v^2 - 2\beta\big)f(v) = 0. \tag{C.1}$$

It is the eigenequation of the harmonic oscillator and a special case of (B.25), the eigenequation of the isotonic oscillator, with $m = \pm\frac{1}{2}$.

Note that (C.1), unlike (B.25), does not have a singularity at $v = 0$. Therefore, its solutions are analytic at $v = 0$.

Let us introduce notation for distinguished solutions of (C.1):

$$\mathbb{I}_{\beta,\pm}(v) := \mathbb{I}_{\beta,\mp\frac{1}{2}}(v), \tag{C.2}$$

$$\mathbb{K}_\beta(v) := \mathbb{K}_{\beta,\pm\frac{1}{2}}(v). \tag{C.3}$$

(In the literature, they are called *Weber(-Hermite) functions* or *parabolic cylinder functions*). Note that (C.2) and (C.3) should be understood as follows: we first define the functions on $[0, \infty[$ as in (B.27) and (B.28); then we extend them analytically to the whole complex plane.

The equation (C.1) is invariant with respect to the mirror symmetry. Therefore, it is spanned by even and odd solutions. It is easy to see that $\mathbb{I}_{\beta,+}$ is even and $\mathbb{I}_{\beta,-}$ is odd:

$$\mathbb{I}_{\beta,\pm}(-v) = \pm\mathbb{I}_{\beta,\pm}(v). \tag{C.4}$$

The function $\mathbb{K}_\beta(v)$ has the decaying asymptotics (B.36) in the positive direction. In the negative direction, it usually blows up. $\mathbb{K}_\beta(-v)$ is also a solution of the Weber equation. It decays in the negative direction.

We have

$$\mathbb{K}_\beta(\pm v) = \pm\frac{\mathbb{I}_{\beta,-}(v)}{\Gamma(\frac{3}{4} - \frac{\beta}{2})} - \frac{\mathbb{I}_{\beta,+}(v)}{\Gamma(\frac{1}{4} - \frac{\beta}{2})}, \tag{C.5}$$

$$\mathcal{W}(\mathbb{I}_{\beta,+}, \mathbb{I}_{\beta,-}) = \frac{2}{\pi}. \tag{C.6}$$

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