AXIAL ANOMALY IN THE PRESENCE OF ARBITRARY SPINOR INTERACTIONS

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We consider N Dirac fermions on a 4-dimensional Euclidean space with a quadratic interaction given by arbitrary external Clifford-valued fields. The divergence of the axial current satisfies on the classical level a relation that is violated after quantization. Using the Pauli–Villars method to regularize the fields, we find the conditions that guarantee the finiteness of the anomaly. We also find this anomaly. Our result generalizes the well-known computation of axial anomaly of Dirac fermions interacting with an external Yang–Mills field.

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1. Introduction

We shall consider a set of classical anti-commuting Dirac spinor fields $\psi_I, \bar{\psi}_I$, $I = 1, \dots, N$, in a 4-dimensional Euclidean space (with positive signature). We also introduce a quadratic Hermitian action for these fields of the form

$$\mathcal{S}[\psi] = \int \mathrm{d}^4x \sum_{IJ} \bar{\psi}_I(x) \left((\gamma^\mu \partial_\mu + m)\, \delta^{IJ} + \Phi^{IJ}(x) \right) \psi_J(x)\,. \tag{1}$$

Thus, all fields have the same mass m and are coupled to external Clifford-algebra-valued fields $\Phi^{IJ}(x)$.

Conserved or approximately conserved currents play an important role in QFT. For instance, the vector current

$$\mathcal{J}^\mu(x) := \sum_I \bar{\psi}_I(x) \gamma^\mu \psi_I(x) \tag{2}$$

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satisfies formally

$$\partial_\mu \mathcal{J}^\mu(x) = 0\,. \tag{3}$$

Relation (3) is true also inside correlation functions

$$\langle \partial_\mu \mathcal{J}^\mu(x)\rangle = 0\,. \tag{4}$$

To be precise, (3) is not fully rigorous, since it involves putting two fields at coinciding points. Relation (4) is exact, since we only consider such regularizations that keep this condition true.

Let us also introduce the following currents:

$$\mathcal{J}^{5\mu}(x) = \sum_I \bar{\psi}_I(x)\gamma^5\gamma^\mu\psi_I(x)\,, \qquad \mathcal{J}^5(x) = \sum_I \bar{\psi}_I(x)\gamma^5\psi_I(x)\,. \tag{5}$$

If the Dirac field is coupled only to an external Yang–Mills field, that is, the field Φ has just one component $\Phi(x) = \mathrm{i}A_\mu(x)\gamma^\mu(x)$, and fields ψ satisfy the classical equations of motion, then formally on the classical level, (5) satisfy the equality

$$\partial_\mu \mathcal{J}^{5\mu}(x) + 2m\mathcal{J}^5(x) = 0\,. \tag{6}$$

However, on the quantum level, after appropriate renormalization, we get

$$\left\langle \partial_\mu \mathcal{J}^{5\mu}(x)\right\rangle_{\mathrm{ren}} + 2m\left\langle \mathcal{J}^5(x)\right\rangle_{\mathrm{ren}} = \frac{1}{16\pi^2}\,\mathrm{Tr}\left(\epsilon^{\mu\nu\rho\sigma}A_{\mu\nu}(x)A_{\rho\sigma}(x)\right) =: \mathcal{A}(x)\,, \tag{7}$$

where $A_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \mathrm{i}[A_\mu, A_\nu]$ and Tr denotes the trace over the space enumerating different fields ψ_I. $\mathcal{A}(x)$ is called the axial anomaly. This is described in essentially every modern textbook on Quantum Field Theory such as [1, 2], see also [3, 4] for a more specialized treatment.

For more general external fields, the classical relation analogous to (6) is

$$\partial_\mu \mathcal{J}^{5\mu}(x) + 2m\mathcal{J}^5(x) + \sum_{IJ} \bar{\psi}_I(x)\left(\gamma^5\Phi^{IJ}(x) + \Phi^{IJ}(x)\gamma^5\right)\psi_J(x) = 0\,, \tag{8}$$

which holds if the fields ψ satisfy the classical equations of motion. Since this is a generalization of the simpler case, we still expect an anomaly to be present on the quantum level

$$\begin{aligned}&\left\langle \partial_\mu \mathcal{J}^{5\mu}(x)\right\rangle_{\mathrm{ren}} + 2m\left\langle \mathcal{J}^5(x)\right\rangle_{\mathrm{ren}} \\ &+ \sum_{IJ}\langle \bar{\psi}_I(x)\left(\gamma^5\Phi^{IJ}(x) + \Phi^{IJ}(x)\gamma^5\right)\psi_J(x)\rangle_{\mathrm{ren}} =: \mathcal{A}(x)\,.\end{aligned} \tag{9}$$

As in the case of the vector current, $\mathcal{J}^{5\mu}(x)$ and equation (9) are problematic because they involve two fields at coinciding points. In order to give sense to them, we shall use the Pauli–Villars regularization. We will show that with this regularization the renormalized vector current $\mathcal{J}^{\mu}(x)$ is conserved. We will find conditions on the external fields $\Phi^{IJ}(x)$ which guarantee that the axial anomaly is finite. These conditions say that certain local polynomials in external fields and their derivatives vanish. There are three such conditions: one involves a polynomial of degree 1, another of degree 2, and the third of degree 3. We will compute the anomaly — again, given by a local polynomial in external fields. This is a rather complicated polynomial of degree four. One should note that there are no anomalies of degree five and more.

Lagrangians of the form (1) may appear in phenomenological description of various physical systems. Our results show that the usual approximate conservation of the axial current can often be generalized to a more general setting.

2. Euclidean Dirac field

Consider the Clifford algebra generated by matrices γ_μ such that

$$\{\gamma_\mu, \gamma_\nu\} = \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu}\,, \qquad \gamma_\mu^\dagger = \gamma_\mu\,, \qquad \mu,\nu = 1,\ldots,4\,, \tag{10}$$

where $g_{\mu\nu}$ is the Euclidean metric tensor $g = \mathrm{diag}(1,1,1,1)$. We will use the conventions $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$, $\gamma_{\mu\nu} := \frac{1}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)$, and $\bar{\psi}_I = (\psi_I)^\dagger\gamma_5$. Thus (10) can be extended to $\mu,\nu = 5$, and $\gamma_{\mu\nu} = -\gamma_{\mu\nu}^\dagger = -\gamma_{\nu\mu}$ for $\mu,\nu = 1,\ldots,5$. Still, whenever contraction of such indices appears in the text, the summation it denotes is only over the range $\{1\ldots4\}$.

The action of (1) can be written as

$$\mathcal{S}[\psi] = \int \mathrm{d}^4x \sum_{IJ} \bar{\psi}_I(x) D(m,\Phi)^{IJ}\psi_J(x)\,, \tag{11}$$

where

$$D(m,\Phi)^{IJ} = (\gamma^\mu\partial_\mu + m)\delta^{IJ} + \Phi^{IJ} = D_0(m)\delta^{IJ} + \Phi^{IJ}\,. \tag{12}$$

With the chosen conventions, the free part of the action is Hermitian. For the whole action to be Hermitian, we need

$$\left(\Phi^{IJ}\right)^\dagger = \gamma^5\Phi^{JI}\gamma^5\,. \tag{13}$$

Note that with this assumption, $\gamma^5 D(m,\Phi)$ is a Hermitian operator.

It can be noted that a rotation of fields $\psi^I(x) \to U^{IJ}(x)\psi^J(x)$, with U being unitary matrices, is equivalent to an appropriate transformation of the external fields Φ

$$\mathcal{S}[U\psi] = \int \mathrm{d}^4x \sum_{IJ} \bar{\psi}_I(x) \left(U^{-1}D(m,\Phi)U\right)^{IJ} \psi_J(x)\,, \tag{14}$$

$$U^{-1}D(m,\Phi)U = D\left(m, U^{-1}\Phi U + \gamma^\mu U^{-1}(\partial_\mu U)\right)\,. \tag{15}$$

Given a basis of the Clifford algebra, the external fields Φ^{IJ} can be decomposed in this basis. We will use a specific basis

$$\Gamma^a \in \left(\mathbf{1}, \mathrm{i}\gamma^\mu, \mathrm{i}\gamma^{\mu\nu}, \gamma^\mu\gamma^5, \gamma^5\right) \tag{16}$$

and introduce varying symbols for different components of Φ^{IJ}

$$\Phi^{IJ} = \Phi^{IJ}_a \Gamma^a = \kappa^{IJ}\mathbf{1} + \mathrm{i}A^{IJ}_\mu \gamma^\mu + \mathrm{i}B^{IJ}_{\mu\nu}\gamma^{\mu\nu} + C^{IJ}_\mu \gamma^\mu\gamma^5 + \lambda^{IJ}\gamma^5\,. \tag{17}$$

In some formulas, we will also use $\tilde{B}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}B^{\rho\sigma}$. With this choice, $(\Gamma^a)^\dagger = \gamma^5\Gamma^a\gamma^5$, which further implies that the interaction part of the action $\mathcal{S}$ is Hermitian iff every component field $\Phi^{IJ}_a(x)$ is a Hermitian $N\times N$ matrix.

Treating $\psi_I(x)$ and $\bar{\psi}_J(y)$ as (independent) Grassmann variables, and using the Berezin integral $\int \mathcal{D}[\psi]$, we can compute correlation functions of the fields from the formula

$$\begin{aligned}&\left\langle \psi_{I_1}(x_1)\cdots\psi_{I_n}(x_n)\bar{\psi}_{J_m}(y_m)\cdots\bar{\psi}_{J_1}(y_1)\right\rangle\\ &:= \frac{1}{Z}\int \mathrm{e}^{-\mathcal{S}[\psi]}\psi_{I_1}(x_1)\cdots\psi_{I_n}(x_n)\bar{\psi}_{J_m}(y_m)\cdots\bar{\psi}_{J_1}(y_1)\mathcal{D}[\psi]\,,\end{aligned} \tag{18}$$

where

$$Z = \int \mathrm{e}^{-\mathcal{S}[\psi]}\mathcal{D}[\psi] \tag{19}$$

is the normalization factor. Since these integrals tend to be infinite, some regularization is usually necessary to obtain physically meaningful results. Since the integrals are Gaussian (in the Grassmannian sense), all correlation functions can be reduced to the correlation function for a pair of fields

$$\left\langle \psi_I(x)\bar{\psi}_J(y)\right\rangle = -\left(D^{-1}\right)_{IJ}(x,y;m,\Phi)\,, \tag{20}$$

where $D^{-1}(x,y;m,\Phi)$ is the integral kernel of the inverse of the operator $D(m,\Phi)$. Note the identity

$$\left\langle \bar{\psi}_I(x)\Gamma^a\psi_J(y)\right\rangle = \mathrm{tr}\left(\Gamma^a\left(D^{-1}\right)_{JI}(y,x;m,\Phi)\right)\,, \tag{21}$$

where the symbol tr denotes the trace over the spinor indices only, and the minus sign from (20) has disappeared due to the anticommution relations of Grassmann variables.

3. Axial anomaly

In Introduction, we presented the vector current $\mathcal{J}^\mu(x)$, and also the axial currents $\mathcal{J}^5(x)$ and $\mathcal{J}^{5\mu}(x)$. Furthermore, we defined the anomaly $\mathcal{A}(x)$.

We have (naively)

$$\left\langle \partial_\mu \mathcal{J}^{5\mu}(x) \right\rangle = \sum_I \operatorname{tr}\left(\gamma^5 \gamma^\mu \frac{\partial \left(D^{-1}\right)_{II}(x,x;m,\Phi)}{\partial x^\mu} \right), \tag{22}$$

$$\left\langle \mathcal{J}^5(x) \right\rangle = \sum_I \operatorname{tr}\left(\gamma^5 \left(D^{-1}\right)_{II}(x,x;m,\Phi) \right), \tag{23}$$

$$\left\langle \bar{\psi}_I(x) \Gamma^a \psi_J(x) \right\rangle = \operatorname{tr}\left(\Gamma^a \left(D^{-1}\right)_{JI}(x,x;m,\Phi) \right). \tag{24}$$

These expressions however have a problem because the integral kernel $D^{-1}(x,y;m)$ is divergent for $y = x$.

To resolve this problem, we shall use the Pauli–Villars regularization. Namely, we define regularized propagators (free and full) as

$$\left(D_0^{-1}\right)_\Lambda(x,y) = \sum_i \mathcal{C}_i D_0^{-1}(x,y;M_i(\Lambda)), \tag{25}$$

$$\left(D^{-1}\right)_\Lambda(x,y;\Phi) = \sum_i \mathcal{C}_i D^{-1}(x,y;M_i(\Lambda),\Phi), \tag{26}$$

where Λ is a regularization parameter (eventually $\Lambda \to +\infty$), masses $M_i(\Lambda)$ and coefficients $\mathcal{C}_i$ are chosen such that

$$\mathcal{C}_0 = 1, \qquad M_0(\Lambda) = m, \qquad \lim_{\Lambda \to +\infty} M_i(\Lambda) = +\infty \;\text{ for }\; i \neq 0,$$

and $(D_0^{-1})_\Lambda(x,y)$ has no divergence for $y \to x$. These conditions mean that the integral

$$\left(D_0^{-1}\right)_\Lambda(x,x) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \sum_i \mathcal{C}_i \frac{-\mathrm{i}\not{p} + M_i}{p^2 + M_i^2} \tag{27}$$

has to be finite. By (A.5) and (A.6), this is guaranteed by the conditions

$$\sum_i \mathcal{C}_i = 0, \qquad \sum_i \mathcal{C}_i M_i = 0, \qquad \sum_i \mathcal{C}_i M_i^2 = 0, \qquad \sum_i \mathcal{C}_i M_i^3 = 0. \tag{28}$$

A possible set of solutions for these equations is

$$i \in \{0, \dots n\}, n \geq 4\,, \qquad \mathcal{C}_i = (-1)^i \binom{n}{i}\,, \qquad M_i = m + i\Lambda\,. \tag{29}$$

A proof that this is indeed a solution is given in Appendix B. Other solutions can be created by linear combinations of these solutions.

Using this regularized propagator, we define

$$\langle \partial_\mu J^\mu(x) \rangle_\Lambda = \sum_I \mathrm{tr}\left(\gamma^\mu \frac{\partial \left(D^{-1}\right)_{II,\Lambda}(x, x; \Phi)}{\partial x^\mu} \right), \tag{30}$$

$$\left\langle \partial_\mu J^{5\mu}(x) \right\rangle_\Lambda = \sum_I \mathrm{tr}\left(\gamma^5 \gamma^\mu \frac{\partial \left(D^{-1}\right)_{II,\Lambda}(x, x; \Phi)}{\partial x^\mu} \right), \tag{31}$$

$$\left\langle J^5(x) \right\rangle_\Lambda = \sum_I \mathrm{tr}\left(\gamma^5 \left(D^{-1}\right)_{II,\Lambda}(x, x; \Phi) \right), \tag{32}$$

$$\left\langle \bar{\psi}^I(x) \Gamma^a \psi^J(x) \right\rangle_\Lambda = \mathrm{tr}\left(\Gamma^a \left(D^{-1}\right)_{JI,\Lambda}(x, x; \Phi) \right), \tag{33}$$

and

$$\begin{aligned} \mathcal{A}_\Lambda(x; \Phi) = {} & \left\langle \partial_\mu J^{5\mu}(x) \right\rangle_\Lambda + 2m \left\langle J^5(x) \right\rangle_\Lambda \\ & + \sum_{IJ} \left\langle \bar{\psi}_I(x) \left(\gamma^5 \Phi^{IJ}(x) + \Phi^{IJ}(x) \gamma^5 \right) \psi_J(x) \right\rangle_\Lambda . \end{aligned} \tag{34}$$

We shall see that this regularization keeps the vector current conserved on the quantum level, *i.e.*

$$\lim_{\Lambda \to \infty} \langle \partial_\mu J^\mu(x) \rangle_\Lambda = 0\,. \tag{35}$$

This relation is actually satisfied even without the limit. We shall also see that the axial currents, in general, produce infinite axial anomalies linear, quadratic, and cubic in fields Φ. If the anomaly has a finite limit for $\Lambda \to \infty$, it is okay; it is analogous to the standard textbook axial anomaly. However, if in this limit it diverges, we find it problematic. We consider the disappearance of these infinite anomalies a necessary condition for the consistency of the quantization; for that to happen, fields Φ need to satisfy some specific conditions, which we will derive in this paper. Under these conditions, we also compute the resulting anomaly. Our result can be summarized as follows.

Theorem 3.1. *Suppose that the following conditions are fullfilled:*

$$\mathrm{Tr}\,\lambda = 0\,, \tag{36}$$

$$\mathrm{Tr}\left(\partial_\mu C^\mu + 2\kappa\lambda + 2B^{\mu\nu}\tilde{B}_{\mu\nu}\right) = 0\,, \tag{37}$$

$$\begin{aligned}\mathrm{Tr}\Big(&-\kappa^2\lambda + \mathrm{i}\kappa[A^\mu, C_\mu] + 2\kappa B^{\mu\nu}\tilde{B}_{\mu\nu} - \mathrm{i}[A^\mu, A^\nu]\tilde{B}_{\mu\nu} \\ &+\mathrm{i}\tilde{B}_{\mu\nu}[C^\mu, C^\nu] + 2B^{\mu\nu}B_{\mu\nu}\lambda + 2C^\mu C_\mu\lambda + \lambda^3\Big) = 0\,. \end{aligned}\tag{38}$$

Then there exists the limit

$$\mathcal{A}(x) := \lim_{\Lambda\to\infty}\mathcal{A}_\Lambda(x)\,, \tag{39}$$

the relation in (9) *is satisfied, and the renormalized anomaly is*

$$\mathcal{A} = \mathcal{A}^{(1)} + \mathcal{A}^{(2)} + \mathcal{A}^{(3)} + \mathcal{A}^{(4)}\,, \tag{40}$$

where

$$\mathcal{A}^{(1)} = \frac{1}{12\pi^2}\mathrm{Tr}\left(\partial^\mu\partial_\mu\partial_\nu C^\nu\right)\,, \tag{41}$$

$$\begin{aligned}\mathcal{A}^{(2)} = \frac{1}{12\pi^2}\mathrm{Tr}\Big(&4\kappa\partial^\mu\partial_\mu\lambda + 6\partial_\mu\kappa\partial^\mu\lambda \\ &+3\epsilon^{\mu\nu\alpha\beta}\partial_\mu A_\nu\partial_\alpha A_\beta + 2A_\nu\partial^\mu\partial_\mu C^\nu - 2C^\nu\partial^\mu\partial_\mu A_\nu \\ &+4\tilde{B}^{\rho\sigma}\partial^\mu\partial_\mu B_{\rho\sigma} + 8\tilde{B}^{\rho\mu}\partial_\mu\partial^\nu B_{\rho\nu} - 8B^{\rho\mu}\partial_\mu\partial^\nu\tilde{B}_{\rho\nu} \\ &+6g^{\mu\nu}\partial_\mu\tilde{B}^{\rho\sigma}\partial_\nu B_{\rho\sigma} + 4\partial_\mu\tilde{B}^{\rho\mu}\partial^\nu B_{\rho\nu} - 4\partial^\mu\tilde{B}^{\rho\nu}\partial_\nu B_{\rho\mu} \\ &-12\,\partial^\mu B_{\mu\nu}\partial^\nu\lambda + \epsilon^{\mu\alpha\nu\beta}\partial_\mu C_\alpha\partial_\nu C_\beta\Big)\,, \end{aligned}\tag{42}$$

$$\begin{aligned}\mathcal{A}^{(3)} = \frac{1}{12\pi^2}\mathrm{Tr}\Big(&(\partial_\mu\kappa)\left(10\{\kappa, C^\mu\} + 4\mathrm{i}[\lambda, A^\mu] + 4\left\{A_\nu, \tilde{B}^{\mu\nu}\right\}\right) \\ &+(\partial_\mu A_\nu)\Big(2\mathrm{i}g^{\mu\nu}[\kappa,\lambda] + 3\mathrm{i}\epsilon^{\mu\nu\alpha\beta}[A_\alpha, A_\beta] - 3\mathrm{i}\epsilon^{\mu\nu\alpha\beta}[C_\alpha, C_\beta] \\ &+4\{A^\mu, C^\nu\} - 4\{A^\nu, C^\mu\} - 2g^{\mu\nu}\{A^\alpha, C_\alpha\} \\ &+12\left\{\kappa, \tilde{B}^{\mu\nu}\right\} - 12\{\lambda, B^{\mu\nu}\} - 12\mathrm{i}g_{\alpha\beta}\left(B^{\mu\alpha}\tilde{B}^{\nu\beta} - B^{\nu\beta}\tilde{B}^{\mu\alpha}\right)\Big) \\ &+(\partial_\mu B_{\alpha\beta})\Big(3\epsilon^{\mu\nu\alpha\beta}\{\kappa, A_\nu\} - 4\mathrm{i}g^{\mu\beta}[\kappa, C^\alpha] + 4\mathrm{i}\epsilon^{\mu\nu\alpha\beta}[\lambda, C_\nu] \\ &-12\mathrm{i}g^{\mu\beta}\left[A_\nu, \tilde{B}^{\alpha\nu}\right] - 12\mathrm{i}\left[A^\alpha, \tilde{B}^{\mu\beta}\right] + 4\mathrm{i}\left[A^\mu, \tilde{B}^{\alpha\beta}\right] \\ &+16g^{\mu\beta}\{C_\nu, B^{\alpha\nu}\} + 8\left\{C^\alpha, B^{\mu\beta}\right\} - 4\left\{C^\mu, B^{\alpha\beta}\right\}\Big)\end{aligned}$$

$$
\begin{aligned}
&+(\partial_\mu C_\nu)\Big(10g^{\mu\nu}\kappa^2-6g^{\mu\nu}\lambda^2\\
&-4g^{\mu\nu}A^\alpha A_\alpha-2A^\mu A^\nu-2A^\nu A^\mu\\
&-8g^{\mu\nu}C^\alpha C_\alpha-2C^\mu C^\nu-2C^\nu C^\mu\\
&+4\mathrm{i}[\kappa,B^{\mu\nu}]-4\mathrm{i}\left[\lambda,\tilde{B}^{\mu\nu}\right]-4g^{\mu\nu}B^{\rho\sigma}B_{\rho\sigma}\Big)\\
&+(\partial_\mu\lambda)\left(-6\mathrm{i}[\kappa,A^\mu]-12\mathrm{i}\left[\tilde{B}^{\mu\nu},C_\nu\right]\right)\Bigg), \qquad (43)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{A}^{(4)} = &\frac{1}{96\pi^2}\mathrm{Tr}\Big(-32\kappa^3\lambda+32\mathrm{i}\kappa^2[A^\mu,C_\mu]+16[A^\alpha,\kappa][A_\alpha,\lambda]\\
&+48\mathrm{i}\lambda^2[A^\mu,C_\mu]+48C^\alpha C_\alpha(\kappa\lambda+\lambda\kappa)+96\kappa C^\mu\lambda C_\mu\\
&+64\kappa^2B^{\mu\nu}\tilde{B}_{\mu\nu}+32\kappa B^{\mu\nu}\kappa\tilde{B}_{\mu\nu}\\
&+48B^{\alpha\beta}B_{\alpha\beta}(\kappa\lambda+\lambda\kappa)+64\kappa B^{\mu\nu}\lambda B_{\mu\nu}\\
&-192\lambda^2B^{\mu\nu}\tilde{B}_{\mu\nu}-96\lambda B^{\mu\nu}\lambda\tilde{B}_{\mu\nu}\\
&-48\mathrm{i}\left(\tilde{B}_{\mu\nu}\kappa+\kappa\tilde{B}_{\mu\nu}\right)[A^\mu,A^\nu]+48\mathrm{i}(B_{\mu\nu}\lambda+\lambda B_{\mu\nu})[A^\mu,A^\nu]\\
&+16\mathrm{i}\left(\tilde{B}_{\mu\nu}\kappa+\kappa\tilde{B}_{\mu\nu}\right)[C^\mu,C^\nu]+128\mathrm{i}\tilde{B}_{\mu\nu}C^\mu\kappa C^\nu\\
&-16\mathrm{i}(B_{\mu\nu}\lambda+\lambda B_{\mu\nu})[C^\mu,C^\nu]-96\mathrm{i}B_{\mu\nu}C^\mu\lambda C^\nu\\
&+32(B_{\mu\nu}\kappa+\kappa B_{\mu\nu})[A^\mu,C^\nu]+32\left(\tilde{B}_{\mu\nu}\lambda+\lambda\tilde{B}_{\mu\nu}\right)[A^\mu,C^\nu]\\
&-64B_{\mu\nu}(A^\mu\kappa C^\nu+C^\nu\kappa A^\mu)-64\tilde{B}_{\mu\nu}(\lambda A^\mu C^\nu+C^\nu A^\mu\lambda)\\
&+64\tilde{B}_{\mu\nu}(A^\mu\lambda C^\nu+C^\nu\lambda A^\mu)+64\tilde{B}_{\mu\nu}(\lambda C^\nu A^\mu+A^\mu C^\nu\lambda)\\
&-256\mathrm{i}g_{\mu\nu}\kappa B^{\mu\alpha}\tilde{B}_{\alpha\beta}B^{\beta\nu}-512\mathrm{i}g_{\mu\nu}\lambda B^{\mu\alpha}B_{\alpha\beta}B^{\beta\nu}\\
&-6\epsilon^{\mu\nu\alpha\beta}[A_\mu,A_\nu][A_\alpha,A_\beta]+2\epsilon^{\mu\nu\alpha\beta}[C_\mu,C_\nu][C_\alpha,C_\beta]\\
&-4\epsilon^{\mu\nu\alpha\beta}[A_\mu,A_\nu][C_\alpha,C_\beta]+8\epsilon^{\mu\nu\alpha\beta}[A_\mu,C_\nu][A_\alpha,C_\beta]\\
&-16\mathrm{i}[A_\mu,A_\nu][A^\mu,C^\nu]+32\mathrm{i}A^\mu A_\mu[A^\nu,C_\nu]\\
&+32A^\alpha A_\alpha B^{\mu\nu}\tilde{B}_{\mu\nu}-32A^\alpha B^{\mu\nu}A_\alpha\tilde{B}_{\mu\nu}-128g_{\alpha\beta}B^{\mu\alpha}\tilde{B}^{\nu\beta}[A_\mu,A_\nu]\\
&-96C^\alpha C_\alpha B^{\mu\nu}\tilde{B}_{\mu\nu}+32C^\alpha B^{\mu\nu}C_\alpha\tilde{B}_{\mu\nu}+64\mathrm{i}g_{\alpha\beta}B^{\mu\alpha}\tilde{B}^{\nu\beta}[C_\mu,C_\nu]\\
&+192\mathrm{i}(A_\mu B_{\nu\alpha}C^\nu B^{\mu\alpha}-A_\mu B^{\mu\alpha}C^\nu B_{\nu\alpha})+64\mathrm{i}B^{\alpha\beta}B_{\alpha\beta}[A^\mu,C_\mu]\\
&+64\mathrm{i}(A_\mu C^\nu B_{\nu\alpha}B^{\mu\alpha}-A_\mu B^{\mu\alpha}B_{\nu\alpha}C^\nu)\\
&-64B^{\alpha\beta}B_{\alpha\beta}B^{\mu\nu}\tilde{B}_{\mu\nu}+64[B^{\alpha\mu},B_{\beta\mu}]\left[B_{\alpha\nu},\tilde{B}^{\beta\nu}\right]\Big). \qquad (44)
\end{aligned}
$$

Among many terms in the above theorem, we can find the terms that reproduce the standard axial anomaly of the Yang–Mills field

$$\begin{aligned}\mathcal{A}(x)_{\kappa,B,C,\lambda=0} &= \frac{1}{4\pi^2}\epsilon^{\mu\nu\alpha\beta}\,\mathrm{Tr}\left(\partial_\mu A_\nu \partial_\alpha A_\beta\right) + \frac{\mathrm{i}}{2\pi^2}\epsilon^{\mu\nu\alpha\beta}\,\mathrm{Tr}\left(\partial_\mu A_\nu A_\alpha A_\beta\right) \\ &\quad + \frac{1}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}\,\mathrm{Tr}\left([A_\mu, A_\nu][A_\alpha, A_\beta]\right) \\ &= \frac{1}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}\,\mathrm{Tr}\left(A_{\mu\nu}A_{\alpha\beta}\right)\,, \end{aligned} \tag{45}$$

where $A_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \mathrm{i}[A_\mu, A_\nu]$.

Another part that we can extract from the full result are the terms that depend only on scalar fields κ and λ, that is when $A_\mu = 0$, $B_{\mu\nu} = 0$, $C_\mu = 0$

$$\mathcal{A}(x)_{A,B,C=0} = \frac{1}{12\pi^2}\,\mathrm{Tr}\left(4\kappa\partial^\mu\partial_\mu\lambda + 6\partial^\mu\kappa\partial_\mu\lambda - 4\kappa^3\lambda\right)\,, \tag{46}$$

with the conditions for the vanishing of divergent part being

$$\mathrm{tr}(\lambda) = 0\,, \tag{47}$$

$$\mathrm{tr}(\kappa\lambda) = 0\,, \tag{48}$$

$$\mathrm{tr}\left(-\kappa^2\lambda + \lambda^3\right) = 0\,. \tag{49}$$

One more special case would be when the configuration of external fields has $A_\mu = 0$, $B_{\mu\nu} = 0$, $\lambda = 0$

$$\begin{aligned}\mathcal{A}(x)_{A,B,\lambda=0} &= \frac{1}{12\pi^2}\,\mathrm{Tr}\left(\epsilon^{\mu\nu\rho\sigma}\partial_\mu C_\nu\partial_\rho C_\sigma + 10\partial_\mu\left(\kappa^2 C^\mu\right)\right) \\ &= \frac{1}{12\pi^2}\partial_\mu\,\mathrm{Tr}\left(\epsilon^{\mu\nu\rho\sigma}C_\nu\partial_\rho C_\sigma + 10\kappa^2 C^\mu\right)\,,\end{aligned} \tag{50}$$

with the only condition for the vanishing of the divergent part being

$$\mathrm{tr}(\partial_\mu C^\mu) = 0\,. \tag{51}$$

4. First steps of proof

The remainder of the paper is devoted to a proof of the above theorem. From the definition of $D^{-1}(x,y;M,\Phi)$, we have

$$\left(\gamma^\mu\frac{\partial}{\partial x^\mu} + M + \Phi(x)\right)D^{-1}(x,y;M,\Phi) = \delta^4(x-y)\,, \tag{52}$$

$$D^{-1}(x,y;M,\Phi)\left(-\gamma^\mu\frac{\overleftarrow{\partial}}{\partial y^\mu} + M + \Phi(y)\right) = \delta^4(x-y)\,, \tag{53}$$

therefore,

$$\begin{aligned}&(\gamma^\mu\partial_\mu+m+\Phi(x))\left(D^{-1}\right)_\Lambda(x,y;\Phi)\\&=\sum_i\mathcal{C}_i\left(\gamma^\mu\partial_\mu+m+\Phi(x)\right)D^{-1}(x,y;M_i(\Lambda),\Phi)\\&=\sum_{iJ}\mathcal{C}_i\left((m-M_i(\Lambda))\,D^{-1}(x,y;M_i(\Lambda),\Phi)+\delta^4(x-y)\right)\\&=\sum_i\mathcal{C}_i\left(m-M_i(\Lambda)\right)D^{-1}(x,y;M_i(\Lambda),\Phi)\,.\end{aligned}\tag{54}$$

Let us first check the conservation of the vector current

$$\begin{aligned}&\langle\partial_\mu J^\mu(x)\rangle_\Lambda\\&=\lim_{y\to x}\mathrm{Tr}\,\mathrm{tr}\left(\gamma^\mu\frac{\partial}{\partial x^\mu}\left(D^{-1}\right)_\Lambda(x,y;\Phi)+\left(D^{-1}\right)_\Lambda(x,y;\Phi)\gamma^\mu\frac{\overleftarrow{\partial}}{\partial y^\mu}\right)\\&=\lim_{y\to x}\left(\mathrm{Tr}\,\mathrm{tr}\left(\sum_i\mathcal{C}_i\left(-M_i(\Lambda)-\Phi(x)\right)D^{-1}(x,y;M_i(\Lambda),\Phi)+\delta^4(x-y)\right)\right.\\&\left.+\,\mathrm{Tr}\,\mathrm{tr}\left(\sum_i\mathcal{C}_iD^{-1}(x,y;M_i(\Lambda),\Phi)\left(M_i(\Lambda)+\Phi(y)\right)-\delta^4(x-y)\right)\right)\\&=0\,.\end{aligned}\tag{55}$$

We can express the regularized anomaly as

$$\begin{aligned}\mathcal{A}_\Lambda(x;\Phi)&=\lim_{y\to x}\mathrm{Tr}\,\mathrm{tr}\left(\gamma^5\left(\gamma^\mu\frac{\partial}{\partial x^\mu}+m+\Phi(x)\right)\left(D^{-1}\right)_\Lambda(x,y;\Phi)\right)\\&\qquad+\mathrm{tr}\left(\gamma^5\left(D^{-1}\right)_\Lambda(x,y;\Phi)\left(-\gamma^\mu\frac{\overleftarrow{\partial}}{\partial y^\mu}+m\right)+\Phi(y)\right)\\&=\lim_{y\to x}\sum_i2\mathcal{C}_i\left(m-M_i(\Lambda)\right)\mathrm{Tr}\,\mathrm{tr}\left(\gamma^5D^{-1}(x,y;M_i(\Lambda),\Phi)\right)\,.\end{aligned}\tag{56}$$

Let us interrupt for a moment the proof of Theorem 3.1 to comment on whether the anomalies we compute are gauge invariant. Unfortunately, while operator $D(m,\Phi)$ satisfies relation (15), and thus we have also

$$U^{-1}D^{-1}(M_i(\Lambda),\Phi)U = D\left(M_i(\Lambda),U^{-1}\Phi U+\gamma^\mu U^{-1}(\partial_\mu U)\right)\,,\tag{57}$$

$$U^{-1}\left(D^{-1}\right)_\Lambda(\Phi)U = \left(D^{-1}\right)_\Lambda\left(U^{-1}\Phi U+\gamma^\mu U^{-1}(\partial_\mu U)\right)\,,\tag{58}$$

no similar relation will be satisfied by $\mathcal{A}_\Lambda(x;\Phi)$ as we defined it.

In fact, from (56) we have

$$
\begin{aligned}
&\mathcal{A}_\Lambda\left(x;U^{-1}\Phi U+\gamma^\mu U^{-1}(\partial_\mu U)\right)=\lim_{y\to x}\sum_i 2\mathcal{C}_i\left(m-M_i(\Lambda)\right)\\
&\times \mathrm{Tr}\ \mathrm{tr}\left(\gamma^5 D^{-1}(x,y;M_i(\Lambda),U^{-1}\Phi U+\gamma^\mu U^{-1}(\partial_\mu U))\right)\\
&=\lim_{y\to x}\sum_i 2\mathcal{C}_i\left(m-M_i(\Lambda)\right)\mathrm{Tr}\ \mathrm{tr}\left(\gamma^5 U^{-1}(x)D^{-1}(x,y;M_i(\Lambda),\Phi)U(y)\right)\\
&=\lim_{y\to x}\sum_i 2\mathcal{C}_i\left(m-M_i(\Lambda)\right)\mathrm{Tr}\ \mathrm{tr}\left(\gamma^5 U(y)U^{-1}(x)D^{-1}(x,y;M_i(\Lambda),\Phi)\right)
\end{aligned}
\tag{59}
$$

which means that

$$
\begin{aligned}
&\mathcal{A}_\Lambda\left(x;U^{-1}\Phi U+\gamma^\mu U^{-1}(\partial_\mu U)\right)-\mathcal{A}_\Lambda(x;\Phi)\\
&=\lim_{y\to x}\mathrm{Tr}\ \mathrm{tr}\left(\gamma^5\left(U(y)U^{-1}(x)-1\right)\sum_i 2\mathcal{C}_i\left(m-M_i(\Lambda)\right)D^{-1}(x,y;M_i(\Lambda),\Phi)\right).
\end{aligned}
\tag{60}
$$

While combination $\sum_i \mathcal{C}_i D^{-1}(x,y;M_i(\Lambda),\Phi)$ has no divergence for $y\to x$ (with appropriately chosen coefficient $\mathcal{C}_i$), it is not necessarily true for $\sum_i \mathcal{C}_i M_i D^{-1}(x,y;M_i(\Lambda),\Phi)$. Due to that divergence, the limit in the expression above does not vanish, and it contains terms dependent on U.

Let us go back to the proof of Theorem 3.1. Expanding $D^{-1}(M)=(D_0(M)+\Phi)^{-1}$ into a series with regard to the powers of Φ

$$
\begin{aligned}
D^{-1}(M) &= D_0^{-1}(M)-D_0^{-1}(M)\Phi D_0^{-1}(M)\\
&\quad +D_0^{-1}(M)\Phi D_0^{-1}(M)\Phi D_0^{-1}(M)+\dots,
\end{aligned}
\tag{61}
$$

we obtain the expansion of $\mathcal{A}_{\mathrm{reg}}(x;\Lambda)$ into a series

$$
\mathcal{A}_\Lambda(x)=\mathcal{A}_\Lambda^{(0)}(x)+\mathcal{A}_\Lambda^{(1)}(x)+\mathcal{A}_\Lambda^{(2)}(x)+\dots
\tag{62}
$$

in which $\mathcal{A}_\Lambda^{(n)}$ is a homogeneous polynomial of n^{th} order in fields Φ.

5. Zeroth-order term

Using (A.5) and (A.6), we obtain

$$
\begin{aligned}
\mathcal{A}_\Lambda^{(0)}(x) &= \lim_{y\to x}\sum_i 2\mathcal{C}_i(m-M_i)\,\mathrm{Tr}\ \mathrm{tr}\left(\gamma^5 D_0^{-1}(x,y;M_i)\right)\\
&= N\int\frac{\mathrm{d}^4p}{(2\pi)^4}\sum_i 2\mathcal{C}_i(m-M_i)\,\mathrm{tr}\left(\gamma^5\frac{-\mathrm{i}\not{p}+M_i}{p^2+M_i^2}\right)\\
&= 0\,.
\end{aligned}
\tag{63}
$$

6. First-order term

$$
\begin{aligned}
&\mathcal{A}^{(1)}_{\Lambda}(x) = \\
&= \lim_{y\to x}(-1)\sum_i 2\mathcal{C}_i(m-M_i)\,\mathrm{Tr}\ \mathrm{tr}\left(\gamma^5\int \mathrm{d}^4 z D_0^{-1}(x,z;M_i)\Phi(z)D_0^{-1}(z,y;M_i)\right)\\
&= -\int\frac{\mathrm{d}^4 p}{(2\pi)^4}\int\frac{\mathrm{d}^4 q}{(2\pi)^4}\,\mathrm{e}^{\mathrm{i}px}\sum_i 2\mathcal{C}_i(m-M_i)\\
&\times \mathrm{Tr}\ \mathrm{tr}\left(\frac{(-\mathrm{i}\not{q}+M_i)\gamma^5(-\mathrm{i}(\not{q}+\not{p})+M_i)}{\left((q+p)^2+M_i^2\right)\left(q^2+M_i^2\right)}\Phi(p)\right)\,. \qquad (64)
\end{aligned}
$$

To calculate this expression, we will use identity (A.1) and then (A.5), (A.9), (A.10). We get

$$
\begin{aligned}
&\mathcal{A}^{(1)}_{\Lambda}(x) = -\int\frac{\mathrm{d}^4 p}{(2\pi)^4}\mathrm{e}^{\mathrm{i}px}\left(\frac{1}{(4\pi)^2}\sum_i 2\mathcal{C}_i(m-M_i)M_i^2\log\left(M_i^2\right)\mathrm{Tr}\,\mathrm{tr}\left(\gamma^5\Phi(p)\right)\right.\\
&+\frac{-\mathrm{i}p_\mu}{(4\pi)^2}\left(\sum_i 2\mathcal{C}_i(m-M_i)M_i\log(M_i)^2+\frac{1}{3}p^2+\mathcal{O}\left(\Lambda^{-1}\right)\right)\mathrm{Tr}\ \mathrm{tr}\left(\gamma^\mu\gamma^5\Phi(p)\right)\\
&\left.+\frac{p^2}{(4\pi)^2}\left(\frac{1}{2}\sum_i 2\mathcal{C}_i(m-M_i)\log\left(M_i^2\right)+\mathcal{O}\left(\Lambda^{-1}\right)\right)\mathrm{Tr}\ \mathrm{tr}\left(\gamma^5\Phi(p)\right)\right)\\
&=\frac{-1}{2\pi^2}\,\mathrm{Tr}(\lambda(x))\sum_i\mathcal{C}_i(m-M_i)M_i^2\log\left(M_i^2\right)\\
&+\frac{-1}{2\pi^2}\,\mathrm{Tr}\left(\partial_\mu C^\mu(x)\right)\sum_i\mathcal{C}_i(m-M_i)M_i\log\left(M_i^2\right)\\
&+\frac{1}{4\pi^2}\,\mathrm{Tr}\left(\partial^\mu\partial_\mu\lambda(x)\right)\sum_i\mathcal{C}_i(m-M_i)\log\left(M_i^2\right)\\
&+\frac{1}{12\pi^2}\,\mathrm{Tr}\left(\partial^\mu\partial_\mu\partial_\nu C^\nu(x)\right)+\mathcal{O}\left(\Lambda^{-1}\right)\,. \qquad (65)
\end{aligned}
$$

We see that some terms are divergent with Λ. For now, we will just keep them in mind, as similar divergent terms may arise in terms of higher order in Φ. The finite term will be denoted $\mathcal{A}^{(1)}$, and matches the formula given in (41).

7. Second-order term

$$\mathcal{A}_{\Lambda}^{(2)}(x) = \lim_{y\to x}\sum_i 2\mathcal{C}_i(m-M_i)\int \mathrm{d}^4z_1\int \mathrm{d}^4z_2$$
$$\times \,\mathrm{Tr}\ \mathrm{tr}\left(\gamma^5 D_0^{-1}(x,z_1;M_i)\Phi(z_1)D_0^{-1}(z_1,z_2;M_i)\Phi(z_2)D_0^{-1}(z_2,y;M_i)\right)$$
$$= \iint \frac{\mathrm{d}^4p\mathrm{d}^4k}{(2\pi)^8}\,\mathrm{e}^{\mathrm{i}(p+k)x}\int\frac{\mathrm{d}^4q}{(2\pi)^4}\sum_i \mathcal{C}_i 2(m-M_i)$$
$$\times \,\mathrm{Tr}\ \mathrm{tr}\left(\Phi(p)D_0^{-1}(q;M_i)\Phi(k)D_0^{-1}(q-k;M_i)\gamma^5 D_0^{-1}(q+p;M_i)\right)\ . \quad (66)$$

Using identity (A.1) and then (A.9)–(A.11), we get

$$\mathcal{A}_{\Lambda}^{(2)}(x) = \iint \frac{\mathrm{d}^4p\mathrm{d}^4k}{(2\pi)^8}\,\mathrm{e}^{\mathrm{i}(p+k)x}$$
$$\times\left(\frac{\mathrm{i}}{(4\pi)^2}\sum_i \mathcal{C}_i 2(m-M_i)\log\left(M_i^2\right)\right.$$
$$\times\left(-\frac{1}{2}\,\mathrm{Tr}\ \mathrm{tr}\left(\Phi(p)\not{p}\Phi(k)\gamma^5\right)+\frac{1}{4}\,\mathrm{Tr}\ \mathrm{tr}\left(\Phi(p)\gamma_\mu\Phi(k)\gamma^\mu(\not{p}+\not{k})\gamma^5\right)\right)$$
$$+\frac{-1}{(4\pi)^2}\left(\sum_i \mathcal{C}_i 2(m-M_i)M_i\log\left(M_i^2\right)\right)\mathrm{Tr}\ \mathrm{tr}\left(\Phi(p)\Phi(k)\gamma^5\right)$$
$$+\frac{1}{3(4\pi)^2}\,p^2\,\mathrm{Tr}\ \mathrm{tr}\left(\Phi(p)\Phi(k)\gamma^5\right)$$
$$+\frac{1}{3(4\pi)^2}\,\mathrm{Tr}\ \mathrm{tr}\left(\Phi(p)\Phi(k)(-2\not{k}-\not{p})(\not{p}+\not{k})\gamma^5\right)$$
$$+\frac{1}{3(4\pi)^2}\,\mathrm{Tr}\ \mathrm{tr}\left(\Phi(p)(\not{k}-\not{p})\Phi(k)(\not{p}+\not{k})\gamma^5\right)$$
$$\left.+\frac{-\mathrm{i}m}{2(4\pi)^2}\,\mathrm{Tr}\ \mathrm{tr}\left(\Phi(p)\Phi(k)(\not{p}+\not{k})\gamma^5\right)\right)+\mathcal{O}\left(\Lambda^{-1}\right)\ . \quad (67)$$

Calculating the traces, we obtain

$$\mathcal{A}_{\Lambda}^{(2)}(x) = \frac{-1}{\pi^2}\,\mathrm{Tr}\left(\kappa(x)\lambda(x)+B^{\mu\nu}(x)\tilde{B}_{\mu\nu}(x)\right)\sum_i \mathcal{C}_i(m-M_i)M_i\log\left(M_i^2\right)$$
$$+\mathcal{A}^{(2)}+\mathcal{O}\left(\Lambda^{-1}\right)\ , \quad (68)$$

where $\mathcal{A}^{(2)}$, the finite part, is given by Eq. (43).

Again, we can see a problematic term divergent when $\Lambda \to \infty$. It has the same kind of divergence as one of the terms that appeared in the linear order, and they will need to be considered together, leading to the condition (37). Among the many finite terms, there is one proportional to $\mathrm{Tr}(\epsilon^{\mu\nu\alpha\beta}\partial_\mu A_\nu(x)\partial_\alpha A_\beta(x))$, which leads to the standard, well-known axial anomaly in the presence of vector field.

8. Third-order term

$$\begin{aligned}
&\mathcal{A}^{(3)}_\Lambda(x) = \lim_{y\to x}(-1)\sum_i 2\mathcal{C}_i(m-M_i)\int \mathrm{d}^4z_1\int \mathrm{d}^4z_2\int \mathrm{d}^4z_3 \\
&\times \mathrm{Tr}\left(\gamma^5 D_0^{-1}(x,z_1;M_i)\Phi(z_1)D_0^{-1}(z_1,z_2;M_i)\Phi(z_2)\right. \\
&\left.\times D_0^{-1}(z_2,z_3;M_i)\Phi(z_3)D_0^{-1}(z_3,y;M_i)\right) \\
&= -\iiint \frac{\mathrm{d}^4p_1\mathrm{d}^4p_2\mathrm{d}^4p_3}{(2\pi)^{12}}\,\mathrm{e}^{\mathrm{i}(p_1+p_2+p_3)x} \\
&\times \int \frac{\mathrm{d}^4q}{(2\pi)^4}\sum_{i\neq 0}\mathcal{C}_i 2(m-M_i)\mathrm{tr}\left(\Phi(p_1)D_0^{-1}(q;M_i)\Phi(p_2)\right. \\
&\left.\times D_0^{-1}(q-p_2;M_i)\Phi(p_3)D_0^{-1}(q-p_2-p_3;M_i)\gamma^5 D_0^{-1}(q+p_1;M_i)\right)\,. \qquad (69)
\end{aligned}$$

Using identity (A.1) we get

$$\begin{aligned}
&D_0^{-1}(q-p_2-p_3;M_i)\gamma^5 D_0^{-1}(q+p_1;M_i) \\
&= \frac{1}{(q+p_1)^2+M_i^2}\left(1+D_0^{-1}(q-p_2-p_3;M_i)\mathrm{i}(\not{p}_1+\not{p}_2+\not{p}_3)\right)\gamma^5\,. \qquad (70)
\end{aligned}$$

The second term in this expression leads to the integrals which are convergent even without the sum $\sum_i \mathcal{C}_i 2(m-M_i)$. The leading term in those integrals is proportional to $1/M_i$, which makes them $\mathcal{O}(\Lambda^{-1})$ for $i\neq 0$. That means that only the first term from the expression above can lead to a term that is divergent in the limit $\Lambda\to\infty$. The second term can still give a finite contribution, thanks to the factor $(m-M_i)$, and simple dimensional analysis tells us that in the part of integral over q that contains this second term, we can omit all momenta p_i except for the factor $\mathrm{i}(\not{p}_1+\not{p}_2+\not{p}_2)$ — all terms coming from these omitted momenta would be at least of the order of $(m-M_i)/M_i^2$, which means they would vanish in the limit of $\Lambda\to\infty$

$$
\begin{aligned}
&\mathcal{A}_{\Lambda}^{(3)}(x) = -\iiint \frac{\mathrm{d}^4p_1\mathrm{d}^4p_2\mathrm{d}^4p_3}{(2\pi)^{12}}\,\mathrm{e}^{\mathrm{i}(p_1+p_2+p_3)x} \\
&\times\Bigg(\int \frac{\mathrm{d}^4q}{(2\pi)^4}\sum_i \mathcal{C}_i 2(m-M_i)\frac{1}{(q+p_1)^2+M_i^2} \\
&\times\mathrm{tr}\left(\Phi(p_1)D_0^{-1}(q;M_i)\Phi(p_2)D_0^{-1}(q-p_2;M_i)\Phi(p_3)\gamma^5\right) \\
&+\int \frac{\mathrm{d}^4q}{(2\pi)^4}\sum_i \mathcal{C}_i 2(m-M_i)\frac{\mathrm{i}(p_1+p_2+p_3)_\mu}{(q+p_1)^2+M_i^2} \\
&\times\mathrm{tr}\left(\Phi(p_1)D_0^{-1}(q;M_i)\Phi(p_2)D_0^{-1}(q-p_2;M_i)\right. \\
&\left.\times\Phi(p_3)D_0^{-1}(q-p_2-p_3;M_i)\gamma^\mu\gamma^5\right)\Bigg) \\
&= -\iiint \frac{\mathrm{d}^4p_1\mathrm{d}^4p_2\mathrm{d}^4p_3}{(2\pi)^{12}}\,\mathrm{e}^{\mathrm{i}(p_1+p_2+p_3)x}\int \frac{\mathrm{d}^4q}{(2\pi)^4}\sum_i \mathcal{C}_i 2(m-M_i) \\
&\times\Bigg(\frac{-q_\nu(q-p_2)_\mu}{\left((q+p_1)^2+M_i^2\right)\left(q^2+M_i^2\right)\left((q-p_2)^2+M_i^2\right)} \\
&\times\mathrm{tr}\left(\Phi(p_1)\gamma^\nu\Phi(p_2)\gamma^\mu\Phi(p_3)\gamma^5\right) \\
&+\frac{-\mathrm{i}q_\mu M_i}{\left((q+p_1)^2+M_i^2\right)\left(q^2+M_i^2\right)\left((q-p_2)^2+M_i^2\right)} \\
&\times\mathrm{tr}\left(\Phi(p_1)\gamma^\mu\Phi(p_2)\Phi(p_3)\gamma^5\right) \\
&+\frac{-\mathrm{i}(q-p_2)_\mu M_i}{\left((q+p_1)^2+M_i^2\right)\left(q^2+M_i^2\right)\left((q-p_2)^2+M_i^2\right)} \\
&\times\mathrm{tr}\left(\Phi(p_1)\Phi(p_2)\gamma^\mu\Phi(p_3)\gamma^5\right) \\
&+\frac{M_i^2}{\left((q+p_1)^2+M_i^2\right)\left(q^2+M_i^2\right)\left((q-p_2)^2+M_i^2\right)} \\
&\times\mathrm{tr}\left(\Phi(p_1)\Phi(p_2)\Phi(p_3)\gamma^5\right) \\
&+\mathrm{i}(p_1+p_2+p_3)_\alpha\left(\frac{-q_\mu q_\nu M_i}{\left(q^2+M_i^2\right)^4}+(\text{terms with } p_j)\right) \\
&\times\mathrm{tr}\left(\Phi(p_1)\left(\gamma^\mu\Phi(p_2)\Phi(p_3)\gamma^\nu+\gamma^\mu\Phi(p_2)\gamma^\nu\Phi(p_3)+\Phi(p_2)\gamma^\mu\Phi(p_3)\gamma^\nu\right)\gamma^\alpha\gamma^5\right) \\
&+\mathrm{i}(p_1+p_2+p_3)_\alpha\left(\frac{M_i^3}{\left(q^2+M_i^2\right)^4}+(\text{terms with } p_j)\right) \\
&\times\mathrm{tr}\left(\Phi(p_1)\Phi(p_2)\Phi(p_3)\gamma^\alpha\gamma^5\right)\Bigg).
\end{aligned}
\tag{71}
$$

Using identities (A.11), (A.13), (A.14), (A.15), (A.17), and (A.18), we get

$$\mathcal{A}_{\Lambda}^{(3)}(x) = -\iiint \frac{\mathrm{d}^4p_1\mathrm{d}^4p_2\mathrm{d}^4p_3}{(2\pi)^{12}}\,\mathrm{e}^{\mathrm{i}(p_1+p_2+p_3)x}\sum_{i\neq 0}\mathcal{C}_i 2(m-M_i)$$
$$\times\Bigg(\left(\frac{g_{\mu\nu}}{(4\pi)^2}\log\left(M_i^2\right)+\mathcal{O}\left(\Lambda^{-2}\right)\right)\mathrm{Tr}\left(\Phi(p_1)\gamma^\mu\Phi(p_2)\gamma^\nu\Phi(p_3)\gamma^5\right)$$
$$+\left(\frac{-\mathrm{i}}{6(4\pi)^2}\frac{(p_2-p_3)_\mu}{M_i}+\mathcal{O}\left(\Lambda^{-3}\right)\right)\mathrm{Tr}\left(\Phi(p_1)\gamma^\mu\Phi(p_2)\Phi(p_3)\gamma^5\right)$$
$$+\left(\frac{-\mathrm{i}}{6(4\pi)^2}\frac{(-2p_2-p_3)_\mu}{M_i}+\mathcal{O}\left(\Lambda^{-3}\right)\right)\mathrm{Tr}\left(\Phi(p_1)\Phi(p_2)\gamma^\mu\Phi(p_3)\gamma^5\right)$$
$$+\left(\frac{1}{2(4\pi)^2}+\mathcal{O}\left(\Lambda^{-2}\right)\right)\mathrm{Tr}\left(\Phi(p_1)\Phi(p_2)\Phi(p_3)\gamma^5\right)$$
$$+\mathrm{i}(p_1+p_2+p_3)_\alpha\left(\frac{-g_{\mu\nu}}{12(4\pi)^2M_i}+\mathcal{O}\left(\Lambda^{-3}\right)\right)$$
$$\times\,\mathrm{Tr}\left(\Phi(p_1)\left(\gamma^\mu\Phi(p_2)\Phi(p_3)\gamma^\nu+\gamma^\mu\Phi(p_2)\gamma^\nu\Phi(p_3)+\Phi(p_2)\gamma^\mu\Phi(p_3)\gamma^\nu\right)\gamma^\alpha\gamma^5\right)$$
$$+\mathrm{i}(p_1+p_2+p_3)_\alpha\left(\frac{1}{6(4\pi)^2M_i}+\mathcal{O}\left(\Lambda^{-3}\right)\right)$$
$$\times\,\mathrm{Tr}\left(\Phi(p_1)\Phi(p_2)\Phi(p_3)\gamma^\alpha\gamma^5\right)\Bigg)$$
$$= -\frac{1}{4(4\pi)^2}\sum_i \mathcal{C}_i 2(m-M_i)\log\left(M_i^2\right)\mathrm{Tr}\left(\Phi\gamma^\mu\Phi\gamma_\mu\Phi\gamma^5\right)$$
$$+\frac{-1}{3(4\pi)^2}\mathrm{Tr}\left(\Phi\left(\gamma^\mu\partial_\mu\Phi\Phi-\gamma^\mu\Phi\partial_\mu\Phi-2\partial_\mu\Phi\gamma^\mu\Phi-\Phi\gamma^\mu\partial_\mu\Phi\right)\gamma^5\right)$$
$$+\frac{1}{6(4\pi)^2}\partial_\mu\,\mathrm{Tr}\left(\Phi\left(2\Phi^2-\gamma^\nu\Phi^2\gamma_\nu-\gamma^\nu\Phi\gamma_\nu\Phi-\Phi\gamma^\nu\Phi\gamma_\nu\right)\gamma^\mu\gamma^5\right)$$
$$+\mathcal{O}\left(\Lambda^{-1}\right)\,, \tag{72}$$

where in the last expression, Φ means $\Phi(x)$ and ∂_μ means $\frac{\partial}{\partial x^\mu}$. After calculating the traces, we get

$$\mathcal{A}_{\Lambda}^{(3)}(x) = \left(\sum_i \mathcal{C}_i 2(m-M_i)\log\left(M_i^2\right)\right)$$
$$\times\frac{-1}{4\pi^2}\mathrm{Tr}\Big(\kappa^2\lambda-\mathrm{i}\kappa[A^\mu,C_\mu]-2\kappa B^{\mu\nu}\tilde{B}_{\mu\nu}+\mathrm{i}[A^\mu,A^\nu]\tilde{B}_{\mu\nu}+$$
$$-\mathrm{i}\tilde{B}_{\mu\nu}[C^\mu,C^\nu]-B^{\mu\nu}B_{\mu\nu}\lambda-2C^\mu C_\mu\lambda-\lambda^3\Big)+\mathcal{A}^{(3)}+\mathcal{O}\left(\Lambda^{-1}\right)\,, \tag{73}$$

where $\mathcal{A}^{(3)}$, the finite part, is given by Eq. (43), and the divergent part contributes to the condition (38).

9. Fourth-order term

$$\begin{aligned}
&\mathcal{A}^{(4)}_{\Lambda}(x) = \lim_{y\to x}\sum_i 2\mathcal{C}_i(m-M_i)\int \mathrm{d}^4z_1\int \mathrm{d}^4z_2\int \mathrm{d}^4z_3\int \mathrm{d}^4z_4 \\
&\times \mathrm{Tr}\ \mathrm{tr}\left(\gamma^5 D_0^{-1}(x,z_1;M_i)\Phi(z_1)D_0^{-1}(z_1,z_2;M_i)\Phi(z_2)\right.\\
&\left.\times D_0^{-1}(z_2,z_3;M_i)\Phi(z_3)D_0^{-1}(z_3,z_4;M_i)\Phi(z_4)D_0^{-1}(z_4,y;M_i)\right)\\
&= \iiiint \frac{\mathrm{d}^4p_1\mathrm{d}^4p_2\mathrm{d}^4p_3\mathrm{d}^4p_4}{(2\pi)^{16}}\,\mathrm{e}^{\mathrm{i}(p_1+p_2+p_3+p_4)x}\int\frac{\mathrm{d}^4q}{(2\pi)^4}\sum_i \mathcal{C}_i 2(m-M_i)\\
&\times \mathrm{Tr}\ \mathrm{tr}\left(D_0^{-1}(q+p_1+p_2+p_3+p_4;M_i)\Phi(p_1)D_0^{-1}(q+p_2+p_3+p_4;M_i)\right.\\
&\left.\times\Phi(p_2)D_0^{-1}(q+p_3+p_4;M_i)\Phi(p_3)D_0^{-1}(q+p_4;M_i)\Phi(p_4)D_0^{-1}(q;M_i)\gamma^5\right)\,.
\end{aligned} \tag{74}$$

Using identity (A.1), we get

$$\begin{aligned}
&D_0^{-1}(q;M_i)\gamma^5 D_0^{-1}(q+p_1+p_2+p_3+p_4;M_i)\\
&=\left(\frac{1}{(q+p_1+p_2+p_3+p_4)^2+M_i^2}+\frac{(-\mathrm{i}\not{q}+M_i)\mathrm{i}(\not{p}_1+\not{p}_2+\not{p}_3+\not{p}_4)}{\left(q^2+M_i^2\right)\left((q+p_1+p_2+p_3+p_4)^2+M_i^2\right)}\right)\gamma^5\,.
\end{aligned} \tag{75}$$

Using dimensional analysis, we can see that the second term will only produce contributions that vanish in the limit of $\Lambda\to\infty$. The first term may produce a finite contribution, but this contribution will be independent of p_i

$$\begin{aligned}
&\mathcal{A}^{(4)}_{\Lambda}(x) = \iiiint \frac{\mathrm{d}^4p_1\mathrm{d}^4p_2\mathrm{d}^4p_3\mathrm{d}^4p_4}{(2\pi)^{16}}\,\mathrm{e}^{\mathrm{i}(p_1+p_2+p_3+p_4)x}\\
&\times\int\frac{\mathrm{d}^4q}{(2\pi)^4}\sum_i \mathcal{C}_i 2(m-M_i)\frac{1}{q^2+M_i^2}\\
&\times \mathrm{Tr}\ \mathrm{tr}\left(\Phi(p_1)D_0^{-1}(q;M_i)\Phi(p_2)D_0^{-1}(q;M_i)\right.\\
&\left.\times\Phi(p_3)D_0^{-1}(q;M_i)\Phi(p_4)\gamma^5\right)+\mathcal{O}\left(\Lambda^{-1}\right)\\
&=\int\frac{\mathrm{d}^4q}{(2\pi)^4}\sum_i \mathcal{C}_i 2(m-M_i)\left(\frac{M_i^3}{\left(q^2+M_i^2\right)^4}\mathrm{Tr}\ \mathrm{tr}\left(\Phi^4\gamma^5\right)\right.\\
&\left.+\frac{-q_\mu q_\nu M_i}{\left(q^2+M_i^2\right)^4}\mathrm{Tr}\ \mathrm{tr}\left(\Phi\left(\gamma^\mu\Phi\gamma^\nu\Phi+\gamma^\mu\Phi^2\gamma^\nu+\Phi\gamma^\mu\Phi\gamma^\nu\right)\Phi\gamma^5\right)\right)+\mathcal{O}\left(\Lambda^{-1}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i\neq 0} \mathcal{C}_i 2(m-M_i)\Bigg(\frac{1}{6(4\pi)^2 M_i}\,\mathrm{Tr}\ \mathrm{tr}\left(\varPhi^4\gamma^5\right)\\
&+\frac{-g_{\mu\nu}}{12(4\pi)^2 M_i}\,\mathrm{Tr}\ \mathrm{tr}\left(\varPhi\left(\gamma^\mu\varPhi\gamma^\nu\varPhi+\gamma^\mu\varPhi^2\gamma^\nu+\varPhi\gamma^\mu\varPhi\gamma^\nu\right)\varPhi\gamma^5\right)\Bigg)+\mathcal{O}\left(\varLambda^{-1}\right)\\
&= \frac{1}{6(4\pi)^2}\,\mathrm{Tr}\ \mathrm{tr}\left(\varPhi\left(2\varPhi^2-\gamma^\mu\varPhi\gamma_\mu\varPhi-\gamma^\mu\varPhi^2\gamma_\mu-\varPhi\gamma^\mu\varPhi\gamma_\mu\right)\varPhi\gamma^5\right)+\mathcal{O}\left(\varLambda^{-1}\right)\\
&= \frac{1}{6(4\pi)^2}\,\mathrm{Tr}\Big(-32\kappa^3\lambda+32\mathrm{i}\kappa^2[A^\mu,C_\mu]-16A^\alpha A_\alpha(\kappa\lambda+\lambda\kappa)+32A^\alpha\kappa A_\alpha\lambda\\
&+48\mathrm{i}\lambda^2[A^\mu,C_\mu]+48C^\alpha C_\alpha(\kappa\lambda+\lambda\kappa)+96\kappa C^\mu\kappa C_\mu\\
&+64\kappa^2B^{\mu\nu}\tilde{B}_{\mu\nu}+32\kappa B^{\mu\nu}\kappa\tilde{B}_{\mu\nu}\\
&+24B^{\alpha\beta}B_{\alpha\beta}(\kappa\lambda+\lambda\kappa)+64\kappa B^{\mu\nu}\lambda B_{\mu\nu}\\
&-192\lambda^2B^{\mu\nu}\tilde{B}_{\mu\nu}-96\lambda B^{\mu\nu}\lambda\tilde{B}_{\mu\nu}\\
&-48\mathrm{i}\left(\tilde{B}_{\mu\nu}\kappa+\kappa\tilde{B}_{\mu\nu}\right)[A^\mu,A^\nu]+48\mathrm{i}(B_{\mu\nu}\lambda+\lambda B_{\mu\nu})[A^\mu,A^\nu]\\
&+16\mathrm{i}\left(\tilde{B}_{\mu\nu}\kappa+\kappa\tilde{B}_{\mu\nu}\right)[C^\mu,C^\nu]+128\mathrm{i}\tilde{B}_{\mu\nu}C^\mu\kappa C^\nu\\
&-16\mathrm{i}(B_{\mu\nu}\lambda+\lambda B_{\mu\nu})[C^\mu,C^\nu]-96\mathrm{i}B_{\mu\nu}C^\mu\lambda C^\nu\\
&+32(B_{\mu\nu}\kappa+\kappa B_{\mu\nu})[A^\mu,C^\nu]+32\left(\tilde{B}_{\mu\nu}\lambda+\lambda\tilde{B}_{\mu\nu}\right)[A^\mu,C^\nu]\\
&-64B_{\mu\nu}(A^\mu\kappa C^\nu+C^\nu\kappa A^\mu)\\
&-64\tilde{B}_{\mu\nu}(\lambda A^\mu C^\nu+C^\nu A^\mu\lambda)\\
&+64\tilde{B}_{\mu\nu}(A^\mu\lambda C^\nu+C^\nu\lambda A^\mu)\\
&+64\tilde{B}_{\mu\nu}(\lambda C^\nu A^\mu+A^\mu C^\nu\lambda)\\
&-256\mathrm{i}g_{\mu\nu}\kappa B^{\mu\alpha}\tilde{B}_{\alpha\beta}B^{\beta\nu}-512\mathrm{i}g_{\mu\nu}\lambda B^{\mu\alpha}B_{\alpha\beta}B^{\beta\nu}\\
&+32\mathrm{i}A^\alpha A_\alpha[A^\mu,C_\mu]\\
&+32A^\alpha A_\alpha B^{\mu\nu}\tilde{B}_{\mu\nu}-32A^\alpha B^{\mu\nu}A_\alpha\tilde{B}_{\mu\nu}\\
&+40\mathrm{i}B^{\alpha\beta}B_{\alpha\beta}[A^\mu,C_\mu]-16B^{\alpha\beta}B_{\alpha\beta}B^{\mu\nu}\tilde{B}_{\mu\nu}\\
&-96C^\alpha C_\alpha B^{\mu\nu}\tilde{B}_{\mu\nu}+32C^\alpha B^{\mu\nu}C_\alpha\tilde{B}_{\mu\nu}\\
&+192\mathrm{i}(A_\mu B_{\nu\alpha}C^\nu B^{\mu\alpha}-A_\mu B^{\mu\alpha}C^\nu B_{\nu\alpha})\\
&+64\mathrm{i}(A_\mu C^\nu B_{\nu\alpha}B^{\mu\alpha}-A_\mu B^{\mu\alpha}B_{\nu\alpha}C^\nu)\\
&-6\epsilon^{\mu\nu\alpha\beta}[A_\mu,A_\nu][A_\alpha,A_\beta]\\
&-16\mathrm{i}[A_\mu,A_\nu][A^\mu,C^\nu]\\
&-4\epsilon^{\mu\nu\alpha\beta}[A_\mu,A_\nu][C_\alpha,C_\beta]\\
&+8\epsilon^{\mu\nu\alpha\beta}[A_\mu,C_\nu][A_\alpha,C_\beta]
\end{aligned}
$$

$$+2\epsilon^{\mu\nu\alpha\beta}[C_\mu, C_\nu][C_\alpha, C_\beta]$$
$$-128 g_{\alpha\beta} B^{\mu\alpha}\tilde{B}^{\nu\beta}[A_\mu, A_\nu]$$
$$+64\mathrm{i} g_{\alpha\beta} B^{\mu\alpha}\tilde{B}^{\nu\beta}[C_\mu, C_\nu]$$
$$+64[B^{\alpha\mu}, B_{\beta\mu}]\left[B_{\alpha\nu}, \tilde{B}^{\beta\nu}\right]\Big) + \mathcal{O}\left(\Lambda^{-1}\right) , \tag{76}$$

where fields without an argument mean fields at a point x.

10. Summary

Higher-order integrals include at least six propagators $D_0^{-1}(p, M)$, so even with the factor $(m - M_i)$, they will not produce terms divergent or finite in the limit $\Lambda \to \infty$. The only divergent terms come from the terms up to the 3rd order in Φ, and the finite terms are up to the 4th order in Φ, which we calculated above. Writing all of them together, we have

$$\begin{aligned}\mathcal{A}_\Lambda(x) \;=\; & \left(\sum_i \mathcal{C}_i \left(m - M_i\right) M_i^2 \log\left(M_i^2\right)\right) \frac{-1}{2\pi^2}\,\mathrm{Tr}(\lambda) \\ & + \left(\sum_i \mathcal{C}_i (m - M_i) M_i \log\left(M_i^2\right)\right) \\ & \times \left(\frac{-1}{2\pi^2}\,\mathrm{Tr}(\partial_\mu C^\mu) + \frac{-1}{\pi^2}\,\mathrm{Tr}\left(\kappa\lambda + B_{\mu\nu}\tilde{B}_{\mu\nu}\right)\right) \\ & + \left(\sum_i \mathcal{C}_i 2(m - M_i) \log\left(M_i^2\right)\right) \left(\sum_I \frac{1}{4\pi^2}\,\mathrm{Tr}(\Box\lambda)\right. \\ & + \frac{-1}{4\pi^2}\,\mathrm{Tr}\Big(\kappa^2\lambda - \mathrm{i}\kappa[A^\mu, C_\mu] - 2\kappa B^{\mu\nu}\tilde{B}_{\mu\nu} + \mathrm{i}[A^\mu, A^\nu]\tilde{B}_{\mu\nu} \\ & \left.-\mathrm{i}\tilde{B}_{\mu\nu}[C^\mu, C^\nu] - 2B^{\mu\nu}B_{\mu\nu}\lambda - 2C^\mu C_\mu\lambda - \lambda^3\Big)\right) + \mathcal{O}\left(\Lambda^0\right) .\end{aligned} \tag{77}$$

For the divergent terms to vanish, we need then

$$\mathrm{tr}(\lambda) \;=\; 0\,, \tag{78}$$
$$\mathrm{tr}\left(\partial_\mu C^\mu + 2\kappa\lambda + 2B^{\mu\nu}\tilde{B}_{\mu\nu}\right) \;=\; 0\,, \tag{79}$$
$$\begin{aligned}\mathrm{tr}\Big(&\partial^\mu\partial_\mu\lambda - \kappa^2\lambda + \mathrm{i}\kappa[A^\mu, C_\mu] + 2\kappa B^{\mu\nu}\tilde{B}_{\mu\nu} - \mathrm{i}[A^\mu, A^\nu]\tilde{B}_{\mu\nu} \\ &+\mathrm{i}\tilde{B}_{\mu\nu}[C^\mu, C^\nu] + 2B^{\mu\nu}B_{\mu\nu}\lambda + 2C^\mu C_\mu\lambda + \lambda^3\Big) \;=\; 0\,.\end{aligned} \tag{80}$$

We require these conditions to be satisfied everywhere in the whole space. From the first condition, it follows then that $\mathrm{tr}(\partial^\mu\partial_\mu\lambda) = 0$ and this term can be omitted from the third condition, leaving us with

$$\begin{aligned}&\operatorname{tr}\Big(-\kappa^2\lambda + \mathrm{i}\kappa[A^\mu, C_\mu] + 2\kappa B^{\mu\nu}\tilde{B}_{\mu\nu} - \mathrm{i}[A^\mu, A^\nu]\tilde{B}_{\mu\nu}\\ &+\mathrm{i}\tilde{B}_{\mu\nu}[C^\mu, C^\nu] + 2B^{\mu\nu}B_{\mu\nu}\lambda + 2C^\mu C_\mu\lambda + \lambda^3\Big) = 0\,. \qquad (81)\end{aligned}$$

Optionally, we can also use the second condition to obtain the identity

$$\mathcal{A}^{(1)} = \frac{1}{12\pi^2}\mathrm{Tr}\left(\partial^\mu\partial_\mu\partial_\nu C^\nu\right) = \frac{1}{12\pi^2}\partial^\mu\partial_\mu\mathrm{Tr}\left(-2\kappa\lambda - 2B^{\rho\sigma}\tilde{B}_{\rho\sigma}\right)\,. \qquad (82)$$

This way, we can get rid of the term $\mathcal{A}^{(1)}$ completely, adding additional terms to $\mathcal{A}^{(2)}$ instead. Whether it is worth it, may be situational.

If these conditions are fulfilled, the regularized anomaly has a finite limit for $\Lambda \to \infty$, which contains terms from the first to the fourth order in fields

$$\mathcal{A}(x) = \lim_{\Lambda\to\infty}\mathcal{A}_\Lambda(x) = \mathcal{A}^{(1)}(x) + \mathcal{A}^{(2)}(x) + \mathcal{A}^{(3)}(x) + \mathcal{A}^{(4)}(x)\,. \qquad (83)$$

Appendix A

Formula library

In this appendix, we collect various formulas useful in the proof of Theorem 3.1. We start with the elementary identity

$$\frac{-\mathrm{i}\not{p}_2 + M_i}{p_2^2 + M_i^2}\gamma^5\frac{-\mathrm{i}\not{p}_1 + M_i}{p_1^2 + M_i^2} = \left(\frac{1}{p_1^2 + M_i^2} + \frac{(-\mathrm{i}\not{p}_2 + M_i)\mathrm{i}(\not{p}_1 - \not{p}_2)}{\left(p_2^2 + M_i^2\right)\left(p_1^2 + M_i^2\right)}\right)\gamma^5\,. \qquad (\mathrm{A}.1)$$

The following family of identities is very useful when we consider the so-called Feynman parameters

$$\int_0^\infty \mathrm{d}\rho\sum_i A_i\,\mathrm{e}^{-\rho B_i} = \sum_i\frac{A_i}{B_i}\,. \qquad (\mathrm{A}.2)$$

Assuming $\sum_i A_i = 0$,

$$\int_0^\infty \mathrm{d}\rho\,\rho^{-1}\sum_i A_i\,\mathrm{e}^{-\rho B_i} = -\sum_i A_i\log B_i\,. \qquad (\mathrm{A}.3)$$

Assuming $\sum_i A_i = 0$, $\sum_i A_iB_i = 0$,

$$\int_0^\infty \mathrm{d}\rho\,\rho^{-2}\sum_i A_i\,\mathrm{e}^{-\rho B_i} = \sum_i A_iB_i\log B_i\,. \qquad (\mathrm{A}.4)$$

Let us formulate the remaining identities as lemmas. Note that all integrands in the following integrals are in L^1.

Lemma A.1. *Assuming* $\sum_i A_i = 0$, $\sum_i A_i M_i^2 = 0$,

$$\int \frac{\mathrm{d}^4 q}{(2\pi)^4} \sum_i A_i \frac{1}{q^2 + M_i^2}, = \frac{1}{(4\pi)^2} \sum_i A_i M_i^2 \log\left(M_i^2\right) , \qquad (A.5)$$

$$\int \frac{\mathrm{d}^4 q}{(2\pi)^4} \sum_i A_i \frac{q^\mu}{q^2 + M_i^2} = 0 . \qquad (A.6)$$

Proof. The integrability of the integrand of (A.5) follows from:

$$\begin{aligned}\sum_i A_i \frac{1}{q^2 + M_i^2} &= \sum_i A_i \left(\frac{1}{q^2 + M_i^2} - \frac{1}{q^2 + m^2} + \frac{M_i^2 - m^2}{(q^2 + m^2)^2} \right) \\ &= \sum_i A_i \frac{\left(m^2 - M_i^2\right)^2}{\left(q^2 + M_i^2\right)\left(q^2 + m^2\right)^2} . \qquad (A.7)\end{aligned}$$

To obtain (A.5), we first introduce the so-called Feynman representation, then we evaluate the Gaussian integral, and finally we apply (A.4)

$$\begin{aligned}\int \frac{\mathrm{d}^4 q}{(2\pi)^4} \sum_i A_i \frac{1}{q^2 + M_i^2} &= \int_0^\infty \mathrm{d}\rho \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \sum_i A_i \, \mathrm{e}^{-\left(q^2 + M_i^2\right)\rho} \\ = \int_0^\infty \mathrm{d}\rho \frac{1}{(4\pi)^2} \sum_i \frac{A_i}{\rho^2} \mathrm{e}^{-M_i^2 \rho} &= \frac{1}{(4\pi)^2} \sum_i A_i M_i^2 \log\left(M_i^2\right) . \qquad (A.8)\end{aligned}$$

A proof of (A.6) is left to the reader. □

We also skip the proofs of the following identities:

Lemma A.2. *Assuming* $\sum_i A_i = 0$,

$$\begin{aligned}&\int \frac{\mathrm{d}^4 q}{(2\pi)^4} \sum_i A_i \frac{1}{\left((q+p)^2 + M_i^2\right)\left(q^2 + M_i^2\right)} \\ &= \frac{-1}{(4\pi)^2} \int_0^1 \mathrm{d}\beta_1 \int_0^1 \mathrm{d}\beta_2 \, \delta(1 - \beta_1 - \beta_2) \sum_i A_i \log\left(\beta_1 \beta_2 p^2 + M_i^2\right) , \qquad (A.9)\end{aligned}$$

$$\int \frac{\mathrm{d}^4 q}{(2\pi)^4} \sum_i A_i \frac{q_\mu}{\left((q+p)^2 + M_i^2\right)\left(q^2 + M_i^2\right)}$$

$$= \frac{p_\mu}{(4\pi)^2} \int_0^1 \mathrm{d}\beta_1 \int_0^1 \mathrm{d}\beta_2\, \delta(1-\beta_1-\beta_2) \sum_i A_i \beta_1 \log\left(\beta_1\beta_2 p^2 + M_i^2\right)\,, \tag{A.10}$$

$$\int \frac{\mathrm{d}^4 q}{(2\pi)^4} \sum_i A_i \frac{q_\nu (q-k)_\mu}{\left(q^2 + M_i^2\right)\left((q-k)^2 + M_i^2\right)\left((q+p)^2 + M_i^2\right)}$$

$$= \frac{1}{(4\pi)^2} \int_0^1 \mathrm{d}\beta_1 \int_0^1 \mathrm{d}\beta_2 \int_0^1 \mathrm{d}\beta_3\, \delta(1-\beta_1-\beta_2-\beta_3)$$

$$\times \left(\sum_i A_i \frac{(\beta_2 k - \beta_3 p)_\nu (\beta_3(-k-p) - \beta_1 k)_\mu}{\beta_1\beta_2 k^2 + \beta_1\beta_3 p^2 + \beta_2\beta_3 (k+p)^2 + M_i^2} \right.$$

$$\left. - \frac{g_{\mu\nu}}{2} \sum_i A_i \log\left(\beta_1\beta_2 k^2 + \beta_1\beta_3 p^2 + \beta_2\beta_3 (k+p)^2 + M_i^2\right) \right)\,, \tag{A.11}$$

$$\int \frac{\mathrm{d}^4 q}{(2\pi)^4} \sum_i A_i \frac{q_\mu q_\nu}{\left(q^2 + M_i^2\right)^3} = \frac{-g_{\mu\nu}}{4(4\pi)^2} \sum_i A_i \log\left(M_i^2\right)\,. \tag{A.12}$$

Lemma A.3.

$$\int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{1}{\left(q^2 + M_i^2\right)\left((q-k)^2 + M_i^2\right)\left((q+p)^2 + M_i^2\right)}$$

$$= \frac{1}{(4\pi)^2} \int_0^1 \mathrm{d}\beta_1 \int_0^1 \mathrm{d}\beta_2 \int_0^1 \mathrm{d}\beta_3\, \delta(1-\beta_1-\beta_2-\beta_3)$$

$$\times \frac{1}{\beta_1\beta_2 k^2 + \beta_1\beta_3 p^2 + \beta_2\beta_3 (k+p)^2 + M_i^2}\,, \tag{A.13}$$

$$\int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{q_\mu}{\left(q^2 + M_i^2\right)\left((q-k)^2 + M_i^2\right)\left((q+p)^2 + M_i^2\right)}$$

$$= \frac{1}{(4\pi)^2} \int_0^1 \mathrm{d}\beta_1 \int_0^1 \mathrm{d}\beta_2 \int_0^1 \mathrm{d}\beta_3\, \delta(1-\beta_1-\beta_2-\beta_3)$$

$$\times \frac{(\beta_2 k - \beta_3 p)_\mu}{\beta_1\beta_2 k^2 + \beta_1\beta_3 p^2 + \beta_2\beta_3 (k+p)^2 + M_i^2}\,, \tag{A.14}$$

$$\int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{(q-k)_\mu}{\left(q^2+M_i^2\right)\left((q-k)^2+M_i^2\right)\left((q+p)^2+M_i^2\right)}$$
$$= \frac{1}{(4\pi)^2} \int_0^1 \mathrm{d}\beta_1 \int_0^1 \mathrm{d}\beta_2 \int_0^1 \mathrm{d}\beta_3\, \delta(1-\beta_1-\beta_2-\beta_3)$$
$$\times \frac{(\beta_3(-k-p)-\beta_1 k)_\mu}{\beta_1\beta_2 k^2 + \beta_1\beta_3 p^2 + \beta_2\beta_3 (k+p)^2 + M_i^2}\,, \tag{A.15}$$

$$\int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{1}{\left(q^2+M_i^2\right)^3} = \frac{1}{2(4\pi)^2 M_i^2}\,, \tag{A.16}$$

$$\int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{1}{\left(q^2+M_i^2\right)^4} = \frac{1}{6(4\pi)^2 M_i^4}\,, \tag{A.17}$$

$$\int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{q_\mu q_\nu}{\left(q^2+M_i^2\right)^4} = \frac{g_{\mu\nu}}{12(4\pi)^2 M_i^2}\,. \tag{A.18}$$

Appendix B

Identity for Pauli–Villars regularization

A proof that (29) is a solution of (28):
Using (29), we have

$$\begin{aligned}\sum_i \mathcal{C}_i M_i^k &= \sum_{i=0}^{n} (-1)^i \binom{n}{i} (m + i\Lambda)^k \\ &= \sum_{p=0}^{k} \binom{k}{p} m^{k-p} \Lambda^p \sum_{i=0}^{n} (-1)^i \binom{n}{i} i^p\,.\end{aligned} \tag{B.1}$$

We have then

$$\begin{aligned}&\sum_{i=0}^{n} (-1)^i \binom{n}{i} i^p \\ &= \left(\sum_{i=0}^{n} (-1)^i \binom{n}{i} i^p x^i\right)\Bigg|_{x=1} \\ &= \left(\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^p \sum_{i=0}^{n} (-1)^i \binom{n}{i} x^i\right)\Bigg|_{x=1}\end{aligned}$$

$$= \left(\left(x \frac{\mathrm{d}}{\mathrm{d}x} \right)^p (1-x)^n \right) \Bigg|_{x=1} . \tag{B.2}$$

Since $x = 1$ is a n^{th} order zero of function $(1-x)^n$, then even after applying the operator $x\frac{\mathrm{d}}{\mathrm{d}x}$ less than n times, it will still be a zero of the resulting function; therefore, we get

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} i^p = 0 , \qquad \text{for} \quad p < n ; \tag{B.3}$$

$$\sum_i \mathcal{C}_i M_i^k = 0 , \qquad \text{for} \quad k < n . \tag{B.4}$$

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