Perturbations of Fefferman spaces over CR three-manifolds

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Motivation

Interaction between conformal geometry and Cauchy–Riemann geometry:

- `Fefferman (1976)`: Construction of a canonical Lorentzian conformal structure from a CR manifold
- `Lewandowski-Nurowski (1990)` (Robinson, Trautman, Penrose...): Description of algebraically special Einstein Lorentzian four-manifold in terms of CR geometry.

Aim

Exhibit the conformal structure of algebraically special Einstein Lorentzian four-fold as 'generalisations' of Fefferman conformal spaces.

This is a subtopic of (almost) Robinson geometry:
See `Nurowski-Trautman (2002)` and `Fino-Leistner-TC (2023)`

CR GEOMETRY

- A Cauchy-Riemann manifold consists of a triple $(\mathcal{M}, H, J)$ where
  - $\mathcal{M}$ is a smooth $(2m+1)$-manifold,
  - $H \subset T\mathcal{M}$ a rank-$2m$ distribution,
  - $J \in \Gamma(\mathrm{End}(H))$ s.t. $J^2 = -\mathrm{Id}$ with involutive eigenbundles, i.e. $\mathbf{C} \otimes H = H^{(1,0)} \oplus H^{(0,1)}$ with $[H^{(1,0)}, H^{(1,0)}] \subset H^{(1,0)}$
- Levi form $\mathcal{L} : H^{(1,0)} \times H^{(0,1)} \to \frac{\mathbf{C}\otimes T\mathcal{M}}{\mathbf{C}\otimes H} : (v, \overline{w}) \mapsto -2\mathrm{ipr}([v, \overline{w}])$ assumed to be non-degenerate

Model:

Hermitian $(\mathbb{V}, \langle\cdot,\cdot\rangle) \cong \mathbf{C}^{m+1,1} \longleftrightarrow (\mathbb{V}, \langle\cdot,\cdot\rangle_{\mathbf{R}}) \cong \mathbf{R}^{2m+2,2}$

$$\mathcal{C} = \{Z \in \mathbb{V}, \langle Z, \overline{Z}\rangle = 0\}$$

$\mathbb{P}_{\mathbf{R}}$: $\mathcal{C} \to (S^{2m+1} \times S^1)/\mathbf{Z}_2 \subset \mathbf{R}\mathbb{P}^{2m+3}$

$\mathbb{P}_{\mathbf{C}}$: $\mathcal{C} \to \mathbf{C}\mathbb{P}^{m+1} \supset S^{2m+1}$

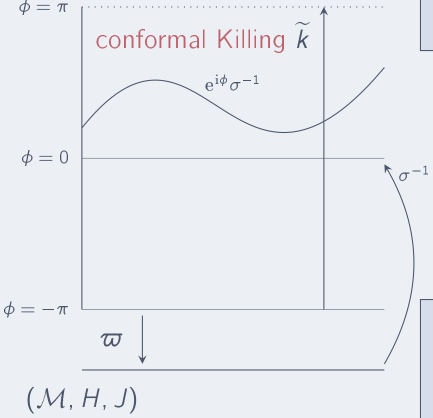
$(S^{2m+1} \times S^1)/\mathbf{Z}_2 \to \mathbf{C}\mathbb{P}^{m+1} \supset S^{2m+1}$

Fefferman space Fefferman (1976), Lee (1986), Čap-Gover (2008)

Density bundle $\mathcal{E}(-1,0) := \left(\wedge^{m+1}(\mathrm{Ann}\, H^{(0,1)})\right)^{\frac{1}{m+2}}$

$\mathcal{E}(w,w') := \mathcal{E}(-1,0)^{w} \otimes \overline{\mathcal{E}(-1,0)}^{w'}$

$\widetilde{\mathcal{M}} := \mathcal{E}^*(-1,0)/\mathbf{R}_{>0}$



Fefferman metric

$$\widetilde{g}_\theta := 4\varpi^*\theta \odot \left(\widetilde{\omega}^\theta - \tfrac{1}{m+2}\mathrm{P}^\theta\theta\right) + \varpi^* h^\theta$$

$\theta \mapsto \widehat{\theta} = \mathrm{e}^{\varphi}\theta \quad \rightsquigarrow \quad \widetilde{g}_{\widehat{\theta}} = \mathrm{e}^{\varphi}\widetilde{g}_\theta$

$\rightsquigarrow$ conformal structure $\widetilde{\mathbf{c}}$

Fefferman space $(\widetilde{\mathcal{M}}, \widetilde{\mathbf{c}}, \widetilde{k})$

canonical $\boldsymbol{\theta} \in \Gamma(T^*\mathcal{M} \otimes \mathcal{E}(1,1))$

$\Gamma(\mathcal{E}(1,0)) \ni \sigma \mapsto (\sigma\overline{\sigma})^{-1}\boldsymbol{\theta} = \theta$

$\theta \in \Gamma(\mathrm{Ann}(H))$, Levi form $h^\theta := \mathrm{d}\theta|_H$

Webster connection ∇^θ on $T\mathcal{M}$

Induced connection $\widetilde{\omega}^\theta$ on $\widetilde{\mathcal{M}}$

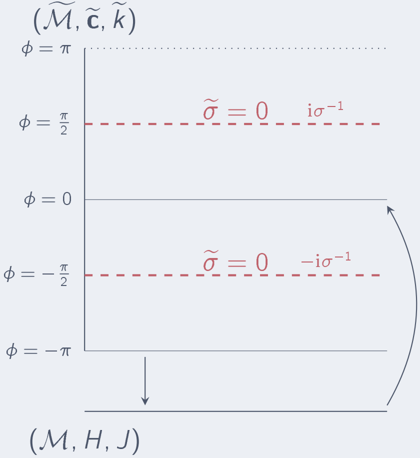
Webster–Schouten scalar P^θ

Characterisations: Sparling, Graham (1987), Čap-Gover (2008)

Almost Einstein scales Leitner (2005), Čap-Gover (2008)

Density bundle $\widetilde{\mathcal{E}}[1] := \left(\wedge^{2m+2} T^*\widetilde{\mathcal{M}}\right)^{-\frac{1}{2m+2}} \rightsquigarrow \widetilde{\mathcal{E}}[w] := (\widetilde{\mathcal{E}}[1])^w$

conformal metric $\widetilde{\mathbf{g}} \in \Gamma(\odot^2 T^*\widetilde{\mathcal{M}} \otimes \widetilde{\mathcal{E}}[2])$



σ^{-1}

Almost Einstein scale $\widetilde{\sigma} \in \Gamma(\widetilde{\mathcal{E}}[1])$

$$\left(\widetilde{\nabla}_a \widetilde{\nabla}_b \widetilde{\sigma} + \widetilde{\mathsf{P}}_{ab} \widetilde{\sigma}\right)_\circ = 0$$

$\uparrow$ $\widetilde{g} = \widetilde{\sigma}^{-2}\widetilde{\mathbf{g}}$

Einstein $\widetilde{g} = \sec^2\phi \cdot \widetilde{g}_\theta, \quad \phi \neq \pm\frac{\pi}{2}$

$\uparrow$

Fefferman metric $\widetilde{g}_\theta$

$\uparrow$ $\theta = (\sigma\overline{\sigma})^{-1}\boldsymbol{\theta}$

CR–Einstein scale $\sigma \in \Gamma(\mathcal{E}^*(1,0))$

$$\nabla_\alpha \nabla_\beta \sigma + \mathrm{i} A_{\alpha\beta} \sigma = 0 \qquad \nabla_{\bar{\alpha}} \sigma = 0$$

Lewandowski (1988):

No almost Einstein scales for non-conformally flat Fefferman **four**-spaces

Perturbations of Fefferman spaces

- Start with a Fefferman space $(\widetilde{\mathcal{M}}, \widetilde{\mathbf{c}}, \widetilde{k}) \to (\mathcal{M}, H, J)$
- Any 1-form $\widetilde{\xi}$ on $\widetilde{\mathcal{M}}$ with $\widetilde{\xi}(\widetilde{k}) = 0$ defines a perturbed Fefferman space $(\widetilde{\mathcal{M}}, \widetilde{\mathbf{c}}_{\widetilde{\xi}}, \widetilde{k})$ with the property that for any Fefferman metric $\widetilde{g}_\theta \in \widetilde{\mathbf{c}}$,

$$\widetilde{\mathbf{c}}_{\widetilde{\xi}} \ni \widetilde{g}_{\theta,\widetilde{\xi}} := \widetilde{g}_\theta + 4\theta \odot \widetilde{\xi}\,.$$

- Pick some $\sigma \in \Gamma(\mathcal{E}(1,0))$ and adapted coframe $(\theta, \theta^\alpha, \overline{\theta}^{\bar{\alpha}})$, and Fourier expand $\widetilde{\xi}$

$$\widetilde{\xi} = \left(\sum \xi_\alpha^{(k)} \mathrm{e}^{\mathrm{i}k\phi}\right)\theta^\alpha + c.c. + \left(\sum \xi_0^{(k)} \mathrm{e}^{\mathrm{i}k\phi}\right)\theta$$

$$\xi_\alpha^{(k)} \xrightarrow{\times \sigma^{\frac{k}{2}}\overline{\sigma}^{-\frac{k}{2}}} \boldsymbol{\xi}_\alpha^{(k)} \in \Gamma(\mathcal{E}_\alpha(\tfrac{k}{2}, -\tfrac{k}{2})) \qquad \xi_0^{(k)} \xrightarrow{\times \sigma^{\frac{k-2}{2}}\overline{\sigma}^{-\frac{k+2}{2}}} \boldsymbol{\xi}_0^{(k)} \in \Gamma(\mathcal{E}(\tfrac{k-2}{2}, -\tfrac{k+2}{2}))$$

- Collect non-zero densities as CR data $\left(\boldsymbol{\xi}_\alpha^{(i)}, [\nabla, \boldsymbol{\xi}_0^{(j)}]_\sim\right)_{i\in\mathcal{I}\subset\mathbf{Z}, j\in\mathcal{J}\subset\mathbf{Z}_{\geq 0}}$ where

$$(\nabla, \boldsymbol{\xi}_0^{(k)}) \sim (\widehat{\nabla}, \widehat{\boldsymbol{\xi}}_0^{(k)}) \iff \begin{cases} \widehat{\nabla}_\alpha = \nabla_\alpha + \Upsilon_\alpha \\ \widehat{\boldsymbol{\xi}}_0^{(k)} = \boldsymbol{\xi}_0^{(k)} - \mathrm{i}\boldsymbol{\xi}_\alpha^{(k)}\Upsilon^\alpha + \mathrm{i}\boldsymbol{\xi}_{\bar{\alpha}}^{(k)}\overline{\Upsilon}^{\bar{\alpha}} \end{cases}$$

Theorem I: Characterisation

Let $(\widetilde{\mathcal{M}}', \widetilde{\mathbf{c}}')$ be a Lorentzian conformal four-manifold admitting a vector field $\widetilde{k}'$ tangent to a twisting non-shearing congruence of null geodesics, i.e.

$$\widetilde{k}'_a \widetilde{k}'^a = 0 \,, \qquad \widetilde{\nabla}_{(a} \widetilde{k}'_{b)_\circ} = \widetilde{k}'_{(a} \widetilde{\alpha}_{b)} \,, \qquad \widetilde{k}'_{[a} \widetilde{\nabla}_b \widetilde{k}'_{c]} \neq 0 \,.$$

Suppose that the Weyl tensor and the Bach tensor satisfy

$$\widetilde{W}(\widetilde{k}', \widetilde{v}, \widetilde{k}', \cdot) = 0 \,, \qquad \widetilde{B}(\widetilde{k}', \widetilde{k}') = 0 \,, \qquad \widetilde{v} \in \Gamma(\langle \widetilde{k}' \rangle^\perp) \,,$$

respectively.
Then locally, $(\widetilde{\mathcal{M}}', \widetilde{\mathbf{c}}')$ is conformally isometric to a perturbed Fefferman space $(\widetilde{\mathcal{M}}, \widetilde{\mathbf{c}}_{\widetilde{\xi}}, \widetilde{k})$ with CR data

$$\left(\boldsymbol{\xi}_\alpha^{(-2)}, \boldsymbol{\xi}_\alpha^{(0)}, [\nabla, \boldsymbol{\xi}_0^{(0)}, \boldsymbol{\xi}_0^{(2)}], \boldsymbol{\xi}_0^{(4)} \right)$$

where $\boldsymbol{\xi}_0^{(0)} = \mathrm{i} \nabla_\alpha \boldsymbol{\xi}^\alpha_{(0)} - \mathrm{i} \nabla^\alpha \boldsymbol{\xi}_\alpha^{(0)} + 3 \boldsymbol{\xi}^\alpha_{(2)} \boldsymbol{\xi}_\alpha^{(-2)}$.

Theorem II: Pure radiation

Let $(\widetilde{\mathcal{M}}', \widetilde{\mathbf{c}}')$ be a Lorentzian conformal four-manifold admitting a vector field $\widetilde{k}'$ tangent to a twisting non-shearing congruence of null geodesics. Suppose it admits an almost pure radiation scale $\widetilde{\sigma} \in \Gamma(\widetilde{\mathcal{E}}[1])$, i.e.

$$\left(\widetilde{\nabla}_a \widetilde{\nabla}_b \widetilde{\sigma} + \widetilde{\mathrm{P}}_{ab} \widetilde{\sigma}\right)_\circ = \tfrac{1}{2} \widetilde{\Phi}_{ab} \widetilde{\sigma}\,, \qquad \widetilde{\Phi}_a{}^b \widetilde{\nabla}_b \widetilde{\sigma} + \tfrac{1}{2} \widetilde{\sigma} \widetilde{\nabla}_b \widetilde{\Phi}_a{}^b = 0\,.$$

where $\widetilde{\Phi}_{ab} = \widetilde{\Phi} \widetilde{k}_a \widetilde{k}_b$, $\widetilde{\Phi} \in \Gamma(\widetilde{\mathcal{E}}[-4])$. That is, away from $\widetilde{\sigma} = 0$, there exists a metric $\widetilde{g}'$ in $\widetilde{\mathbf{c}}'$ whose Ricci tensor satisfies

$$\widetilde{\mathrm{Ric}}_{ab} = \widetilde{\Phi} \widetilde{k}_a \widetilde{k}_b + \widetilde{\Lambda} \widetilde{g}_{ab}\,, \qquad \widetilde{\Lambda} \text{ constant.}$$

Then, $\widetilde{W}(\widetilde{k}', \widetilde{v}, \widetilde{k}', \cdot) = 0$, $\widetilde{B}(\widetilde{k}', \widetilde{k}') = 0$, $\widetilde{v} \in \Gamma(\langle \widetilde{k}' \rangle^\perp)$.
Thus, $(\widetilde{\mathcal{M}}', \widetilde{\mathbf{c}}')$ is conf. isometric to a perturbed Fefferman space $(\widetilde{\mathcal{M}}, \widetilde{\mathbf{c}}_{\widetilde{\xi}}, \widetilde{k}) \to (\mathcal{M}, H, J)$ with CR data $\left(\boldsymbol{\xi}_\alpha^{(-2)}, \boldsymbol{\xi}_\alpha^{(0)}, [\nabla, \boldsymbol{\xi}_0^{(0)}, \boldsymbol{\xi}_0^{(2)}], \boldsymbol{\xi}_0^{(4)}\right)$.
In addition, $(\mathcal{M}, H, J)$ admits a $\sigma \in \Gamma(\mathcal{E}(1,0))$ such that

$$\overset{\xi}{\nabla}_{\bar{\alpha}} \sigma + 2\mathrm{i} \boldsymbol{\xi}_{\bar{\alpha}}^{(2)} \overline{\sigma}^{-1} \sigma^2 = 0\,, \qquad \overset{\xi}{\nabla}_\alpha \overset{\xi}{\nabla}_\beta \sigma + \mathrm{i} A_{\alpha\beta} \sigma = 0\,,$$

$$\overset{\xi}{\nabla}_\alpha \boldsymbol{\xi}_0^{(4)} - \left(\sigma^{-1} \overset{\xi}{\nabla}_\alpha \sigma\right) \boldsymbol{\xi}_0^{(4)} = 0\,,$$

where $\overset{\xi}{\nabla}_\alpha := \nabla_\alpha + \boldsymbol{\xi}_\alpha^{(0)}$.

CR embeddability

Recast

$$\overset{\xi}{\nabla}_\alpha \overset{\xi}{\nabla}_\beta \sigma + \mathrm{i} A_{\alpha\beta}\sigma = 0 \quad (1) \quad \overset{\lambda_\alpha = \mathrm{i}\sigma^{-1}\overset{\xi}{\nabla}_\alpha \sigma}{\text{as}} \quad \nabla_\alpha \lambda_\beta - \mathrm{i}\lambda_\alpha\lambda_\beta - A_{\alpha\beta} = 0 \quad (2)$$

Theorem (Proof based on Hill-Lewandowski-Nurowski's Lemma (2008))

A CR three-manifold locally admits a CR function if and only if there exists a solution λ_α to (2).

Now recast

$$\overset{\xi}{\nabla}_\alpha \boldsymbol{\xi}_0^{(4)} - \left(\sigma^{-1}\overset{\xi}{\nabla}_\alpha \sigma\right)\boldsymbol{\xi}_0^{(4)} = 0 \quad \overset{\tau^{-3} := \overline{\sigma}^{-1}\boldsymbol{\xi}_0^{(-4)}}{\text{as}} \quad \overset{\xi}{\nabla}_{\bar{\alpha}}\tau = 0 \quad (3)$$

Theorem (Proof based on Jacobowitz' Theorem (1988))

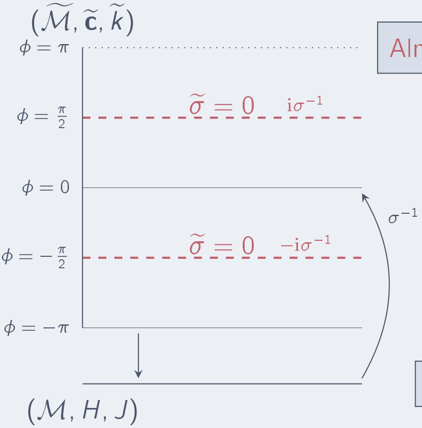
A CR three-manifold is locally realisable as a hypersurface in $\mathbf{C}^2$ if and only if there exist densities σ and τ of weight $(1,0)$ solving (1) and (3) respectively.

Pure radiation metrics

Corollary

A Lorentzian conformal four-manifold endowed with a twisting non-shearing congruence of null geodesics and admitting an almost pure radiation scale is locally fibered over an embeddable CR three-manifold.

Cf: `Lewandowski-Nurowski-Tafel (1990)`, `Hill-Lewandowski-Nurowski (2008)`



Almost pure radiation scale $\widetilde{\sigma} \in \Gamma(\widetilde{\mathcal{E}}[1])$

pure radiation metric

$$\widetilde{g} = \sec^2 \phi \cdot \widetilde{g}_{\theta, \widetilde{\xi}}, \qquad \phi \neq \pm \frac{\pi}{2}$$

perturbed Fefferman metric $\widetilde{g}_{\theta, \widetilde{\xi}}$

CR data + $\sigma \in \Gamma(\mathcal{E}(1,0))$ + d.e.

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