Generalisations of Fefferman's conformal structure and their Einstein metrics

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Motivation

Interaction between conformal geometry and Cauchy–Riemann geometry:

- Fefferman (1976): Construction of a canonical Lorentzian conformal structure from a CR manifold
- Lewandowski-Nurowski (1990) (Robinson, Trautman, Penrose...): Description of algebraically special Einstein Lorentzian four-fold in terms of CR geometry.

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Aim

Understand algebraically special Einstein Lorentzian four-fold as generalisations of Fefferman spaces.

Based on

- `TC (2022)`: Twisting non-shearing congruences of null geodesics, almost CR structures and Einstein metrics in even dimensions. *Ann. Mat. Pura Appl.* (4) **201** (2022), no. 2, 655–693.
- `TC`: In preparation

Caution

Most considerations will be local.

Almost CR geometry

- An almost CR manifold consists of a triple $(\mathcal{M}, H, J)$ where:
  - $(\mathcal{M}, H)$ is a smooth contact $(2m+1)$-fold, i.e. $T\mathcal{M} = H + [H, H]$;
  - $J \in \mathrm{End}(H)$ satisfies $J^2 = -\delta_H$, and its i-eigenbundle $H^{(1,0)}$ is partially integrable, i.e. $[H^{(1,0)}, H^{(1,0)}] \subset \mathbf{C} \otimes H$.

  Assume the Levi form $\mathbf{h} : H^{(1,0)} \times H^{(0,1)} \to (\mathbf{C} \otimes T\mathcal{M}/H)^*$ to be positive definite.

  We say that $(\mathcal{M}, H, J)$ is a CR manifold if $[H^{(1,0)}, H^{(1,0)}] \subset H^{(1,0)}$.

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  We say that $(\mathcal{M}, H, J)$ is a CR manifold if $[H^{(1,0)}, H^{(1,0)}] \subset H^{(1,0)}$.
- Complex density bundles $\mathcal{E}(w, w')$, $w, w' \in \mathbf{C}$, $w - w' \in \mathbf{Z}$
  Canonical bundle $\wedge^{m+1}\mathrm{Ann}(H^{(0,1)}) \cong \mathcal{E}(-m-2, 0)$
- A (real) CR scale $\tau \in \mathcal{E}(1,1)$ corresponds to a pseudo-Hermitian structure, i.e. a choice of contact form θ. For each choice of contact form, there is a unique compatible Webster connection ∇ with prescribed torsion. Under a change of contact forms:

$$\theta \to \widehat{\theta} = \mathrm{e}^{\varphi}\theta \quad \Longrightarrow \quad \nabla \to \widehat{\nabla} = \nabla + \Upsilon + \dots, \qquad (\Upsilon = \mathrm{d}\varphi).$$

- Abstract index notation: $\mathcal{E}^{\alpha} := H^{(1,0)}$, $\mathcal{E}^{\bar{\alpha}} := H^{(0,1)}$ with duals $\mathcal{E}_{\alpha}$, $\mathcal{E}_{\bar{\alpha}}$

Fefferman space

- Fefferman (1976): Original construction for real smooth hypersurfaces in complex space
- For abstract CR structures, we follow Lee (1986), Čap-Gover (2008); For almost CR structures: Leitner (2010):
- Define $\widetilde{\mathcal{M}}$ to be the S^1-bundle $\mathcal{E}(-1,0)/\mathbf{R}^*$. For each contact form θ with Levi form h and Webster-Schouten scalar P, we associate the (Lorentzian) Fefferman metric

$$\widetilde{g}_\theta = 4\theta \odot \left(\mathrm{d}\phi + \tfrac{\mathrm{i}}{2} \left(\sigma^{-1} \nabla \sigma - \overline{\sigma}^{-1} \nabla \overline{\sigma} \right) - \tfrac{1}{m+2} \mathrm{P} \theta \right) + h \,,$$

  for any choice of $\sigma \in \mathcal{E}(1,0)$ such that $\sigma\overline{\sigma}$ determines θ and a fibre coordinate ϕ.
- Under a change of contact forms,

$$\theta \mapsto \widehat{\theta} = \mathrm{e}^{\varphi} \theta \qquad \Longrightarrow \qquad \widetilde{g}_\theta \mapsto \widetilde{g}_{\widehat{\theta}} = \mathrm{e}^{\varphi} \widetilde{g}_\theta \,,$$

  from which we define the Fefferman conformal structure $\widetilde{\mathbf{c}}$.
- We refer to $(\widetilde{\mathcal{M}}, \widetilde{\mathbf{c}})$ the Fefferman space (for $(\mathcal{M}, H, J)$).

Characterisation

Theorem (TC)

A Lorentzian conformal $(2m+2)$-fold $(\widetilde{\mathcal{M}}, \widetilde{\mathbf{c}})$ is locally the Fefferman space of some almost CR manifold if and only if it admits a null conformal Killing field $\widetilde{k}$, i.e. $\pounds_{\widetilde{k}} \widetilde{\mathbf{g}} = 0$, and the following integrability conditions are satisfied for any $\widetilde{g} \in \widetilde{\mathbf{c}}$:

$$\frac{1}{(2m+2)^2}(\widetilde{\nabla}_a \widetilde{k}^a)^2 - \widetilde{\mathsf{P}}_{ab}\widetilde{k}^a\widetilde{k}^b - \frac{1}{2m+2}\widetilde{k}^a\widetilde{\nabla}_a\widetilde{\nabla}_b\widetilde{k}^b < 0\,, \qquad (\star)$$

$$\widetilde{k}^a \widetilde{W}_{abcd}\widetilde{k}^d = 0\,, \qquad \widetilde{k}^a \widetilde{Y}_{abc}\widetilde{k}^c = 0\,,$$

$$\widetilde{W}_{ab}{}^{cd}\widetilde{\boldsymbol{\tau}}_{cd} - 2\widetilde{k}^c\widetilde{Y}_{cab} - \frac{1}{2}\left(\widetilde{\boldsymbol{\tau}}_{c[a}\widetilde{k}^d\widetilde{W}_{b]d}{}^{ef}\widetilde{W}_{efg}{}^c\widetilde{k}^g - \widetilde{\boldsymbol{\kappa}}_{[a}\widetilde{k}^c\widetilde{W}_{b]c}{}^{de}\widetilde{Y}_{fde}\widetilde{k}^f\right) = 0\,,$$

where $\widetilde{\boldsymbol{\kappa}} = \widetilde{\mathbf{g}}(\widetilde{k}, \cdot)$, $\widetilde{\boldsymbol{\tau}} = \mathrm{d}^{\widetilde{\nabla}}\widetilde{\boldsymbol{\kappa}}$, $\widetilde{W}_{abcd}$ and $\widetilde{Y}_{abc}$ are the Weyl and Cotton tensors.

- Sparling, Graham (1987), Lewandowski (1991), Čap-Gover (2010): Involutive case characterised by $(\star)$ and $\widetilde{k}^a\widetilde{W}_{abcd} = \widetilde{k}^a\widetilde{Y}_{abc} = 0$.
- Henceforth, write $(\widetilde{\mathcal{M}}, \widetilde{\mathbf{c}}, \widetilde{k}) \longrightarrow (\mathcal{M}, H, J)$ for a Fefferman space.

Fefferman–Einstein metrics

Theorem (TC (2022), TC)

A Fefferman space $(\widetilde{\mathcal{M}}, \widetilde{\mathbf{c}}, \widetilde{k}) \longrightarrow (\mathcal{M}, H, J)$ *of dimension* $(2m+2)$ *admits an Einstein metric if and only if* $(\mathcal{M}, H, J)$ *is almost–CR–Einstein, i.e. it admits a contact form such that its Webster torsion* $A_{\alpha\beta}$ *and the Webster–Schouten tensor* $\mathsf{P}_{\alpha\bar{\beta}}$*, and the Nijenhuis torsion* $\mathsf{N}_{\alpha\beta\gamma}$ *satisfy*

$$A_{\alpha\beta} = 0\,, \qquad \nabla^{\gamma}\mathsf{N}_{\gamma(\alpha\beta)} = 0\,, \qquad \left(\mathsf{P}_{\alpha\bar{\beta}} - \tfrac{1}{m+2}\mathsf{N}_{\alpha\gamma\delta}\mathsf{N}_{\bar{\beta}}{}^{\gamma\delta}\right)_{\circ} = 0\,,$$

i.e. $\mathcal{M}$ *locally fibered over an almost Kähler–Einstein 2m-fold.*

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- Leitner (2007), Čap-Gover (2008): involutive case ($\mathsf{N}_{\alpha\beta\gamma} = 0$)
- Lewandowski (1988): Any Fefferman space of dimension four that admits an Einstein metric is conformally flat, and its underlying CR structure is flat.

However...

There are many non-conformally flat Einstein Lorentzian four-fold arising from CR three-folds...

Generalised Fefferman spaces

Prototypes: `Leitner (2010), Schmalz-Ganji (2019)`

- Let $(\widetilde{\mathcal{M}}, \widetilde{\mathbf{c}}, \widetilde{k}) \to (\mathcal{M}, H, J)$ be a Fefferman space. For any 1-form $\widetilde{\lambda}$ on $\mathcal{M}$ satisfying $\widetilde{\lambda}(\widetilde{k}) = 0$, we define a generalised Fefferman space $(\widetilde{\mathcal{M}}, \widetilde{\mathbf{c}}_{\widetilde{\lambda}}, \widetilde{k}) \to (\mathcal{M}, H, J)$ by the property that for any Fefferman metric $\widetilde{g}_\theta \in \mathbf{c}$,

$$\widetilde{\mathbf{c}}_{\widetilde{\lambda}} \ni \widetilde{g}_{\theta,\widetilde{\lambda}} := \widetilde{g}_\theta + 4\theta \odot \widetilde{\lambda}\,.$$

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$$\widetilde{\mathbf{c}}_{\widetilde{\lambda}} \ni \widetilde{g}_{\theta,\widetilde{\lambda}} := \widetilde{g}_\theta + 4\theta \odot \widetilde{\lambda}\,.$$

- Choose an adapted coframe $(\theta, \theta^\alpha, \overline{\theta}^{\bar{\alpha}})$ and Fourier expand:

$$\widetilde{\lambda} = \left(\sum \lambda_\alpha^{(k)} \mathrm{e}^{\mathrm{i}k\phi}\right)\theta^\alpha + c.c. + \left(\sum \lambda_0^{(k)} \mathrm{e}^{\mathrm{i}k\phi}\right)\theta\,,$$

where $\lambda_\alpha^{(k)} \in \mathcal{E}_\alpha(\frac{k}{2}, -\frac{k}{2})$, and $\lambda_0^{(k)} \in \mathcal{E}(\frac{k-2}{2}, -\frac{k+2}{2})$ transform as

$$\widehat{\lambda}_0^{(k)} = \lambda_0^{(k)} - \mathrm{i}\lambda_\alpha^{(k)}\Upsilon^\alpha + \mathrm{i}\lambda_{\bar{\alpha}}^{(k)}\overline{\theta}^{\bar{\alpha}}\,.$$

We refer to $\{\lambda_\alpha^{(k)}, [\lambda_0^{(k)}]\}_{k\in\mathbf{Z}}$ as the CR data for $(\widetilde{\mathcal{M}}, \widetilde{\mathbf{c}}, \widetilde{k})$.

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We refer to $\{\lambda_\alpha^{(k)}, [\lambda_0^{(k)}]\}_{k \in \mathbf{Z}}$ as the CR data for $(\widetilde{\mathcal{M}}, \widetilde{\mathbf{c}}, \widetilde{k})$.

- The vector field $\widetilde{k}$ generates a (twisting) non-shearing congruence of null geodesics, i.e.

$$\pounds_{\widetilde{k}} \widetilde{\mathbf{g}}_{\widetilde{\lambda}} = \widetilde{\mathbf{g}}_{\widetilde{\lambda}}(\widetilde{k}, \cdot) \odot \widetilde{\alpha} \qquad \text{for some 1-form } \widetilde{\alpha}.$$

EINSTEIN METRICS IN DIMENSIONS $2m + 2 > 4$

Theorem (TC (2022), TC)

Let $(\widetilde{\mathcal{M}}, \widetilde{\mathbf{c}}_{\widetilde{\lambda}}, \widetilde{k}) \to (\mathcal{M}, H, J)$ *be a generalised Fefferman space of dimension* $2m + 2 > 4$*, with CR data* $\{\lambda_\alpha^{(k)}, [\lambda_0^{(k)}]\}_{k \in \mathbf{Z}}$*. Suppose that the Weyl tensor of* $\widetilde{\mathbf{c}}_{\widetilde{\lambda}}$ *satisfies*

$$\widetilde{W}(\widetilde{k}, \widetilde{v}, \widetilde{k}, \cdot) = 0\,, \qquad \text{for all } \widetilde{v} \in \langle \widetilde{k} \rangle^\perp.$$

Then $\widetilde{\mathbf{c}}_{\widetilde{\lambda}}$ *contains an Einstein metric with constant* $\widetilde{\Lambda} = \frac{1}{2m}\widetilde{\mathrm{Sc}}$ *if and only if*

1. $(\mathcal{M}, H, J)$ *admits an almost-CR-Einstein structure given by some complex density* $\sigma \in \mathcal{E}(1,0)$*, with constant* $\Lambda = \frac{1}{m}\left(\mathrm{Sc} + \|\mathrm{N}\|_{\mathbf{h}}^2\right)$*,*
2. *all the CR data are zero except for*

$$\lambda_0^{(2k)} = \left(\tfrac{1}{2m+1}\widetilde{\Lambda} - \tfrac{1}{2m+2}\Lambda + \mathrm{i}M\right) c_{2k}\sigma^{k-1}\overline{\sigma}^{-k-1} \quad k = 1, \ldots, m+1\,,$$

where M *is a 'mass' parameter, and the* $c_{2k} = c_{2k}(k, m)$ *are constants.*

Einstein metrics in dimension four

Theorem (TC)

Let $(\widetilde{\mathcal{M}}, \widetilde{\mathbf{c}}_{\widetilde{\lambda}}, \widetilde{k}) \to (\mathcal{M}, H, J)$ *be a generalised Fefferman space of dimension four, with CR data* $\{\lambda_\alpha^{(k)}, [\lambda_0^{(k)}]\}_{k\in\mathbf{Z}}$. *Suppose* $\widetilde{\mathbf{c}}_{\widetilde{\lambda}}$ *contains an Einstein metric. Then* $\lambda_\alpha^{(0)}$, $\lambda_\alpha^{(-2)}$ *and* $\lambda_0^{(\pm 4)}$ *determine* $[\lambda_0^{(0)}]$ *and* $[\lambda_0^{(\pm 2)}]$, *and the remaining CR data are zero.*

- Main feature: $\lambda_\alpha^{(0)}$, $\lambda_\alpha^{(-2)}$ introduce non-linearity...

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Lemma (TC)

A CR three-fold admits a CR function if and only if it admits a solution $\lambda_\alpha \in \mathcal{E}_\alpha$ to

$$\nabla_\alpha \lambda_\beta - \mathrm{i} \lambda_\alpha \lambda_\beta - A_{\alpha\beta} = 0 \,.$$

This is CR invariant provided $\widehat{\lambda}_\alpha = \lambda_\alpha + \mathrm{i} \Upsilon_\alpha$.

- Kerr coordinates: Robinson-Trautman (1962), Kerr (1963)
  CR embeddability: Lewandowski-Nurowski-Tafel (1990), Hill-Lewandowski-Nurowski (2008)

Concluding remarks

- Any Einstein metric $\widetilde{g}_{Ein}$ on $(\widetilde{\mathcal{M}}, \widetilde{\mathbf{c}}_{\widetilde{\lambda}}, \widetilde{k}) \to (\mathcal{M}, H, J)$ can be written in a normal form

$$\widetilde{g}_{Ein} = \sec^2 \phi \cdot \widetilde{g}_{\theta, \widetilde{\lambda}} \,.$$

  where ϕ and θ are determined by some $\sigma \in \mathcal{E}(1, 0)$.
  Note that $\widetilde{g}_{Ein} \to \pm\infty$ as $\phi \to \pm\frac{\pi}{2}$. Interpret $\phi = \pm\frac{\pi}{2}$ as future/past infinities. We can then describe the asymptotics of $\widetilde{g}_{Ein}$ in terms of CR data.

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Note that $\widetilde{g}_{Ein} \to \pm\infty$ as $\phi \to \pm\frac{\pi}{2}$. Interpret $\phi = \pm\frac{\pi}{2}$ as future/past infinities. We can then describe the asymptotics of $\widetilde{g}_{Ein}$ in terms of CR data.
- Weaker conditions on the Ricci tensor are also possible.
- Prescription on the Weyl tensor restricts the CR data.

Claim TC

In dimension four, any Lorentzian conformal structure admitting a twisting non-shearing congruence of null geodesics locally looks like a generalised Fefferman space. For an Einstein metric, the Weyl tensor is algebraically special (Goldberg-Sachs (1962), Hill-Gover-Nurowski (2011)).
Similar considerations apply in higher even dimensions.

Example: The Kerr spacetime

- Kerr (1963): Ricci-flat Lorentzian manifold $\widetilde{\mathcal{M}} = \{u, \vartheta, \varphi, r\}$ describing a black hole with rotation parameter a and mass M:

$$\widetilde{g} = 4\theta \left(\mathrm{d}r + a\sin^2\vartheta \mathrm{d}\varphi + \left(\frac{Mr}{r^2+a^2\cos^2\vartheta} - \frac{1}{2}\right)\kappa\right) + 2(r^2 + a^2\cos^2\varphi)\theta^1\overline{\theta}^{\bar{1}},$$

$$\theta = \frac{1}{2}\left(\mathrm{d}t + a\sin^2\vartheta\mathrm{d}\varphi\right), \qquad \theta^1 = \mathrm{d}\vartheta + \mathrm{i}\sin\vartheta\mathrm{d}\varphi.$$

- Twisting non-shearing congruence of null geodesics generated by $k := \frac{\partial}{\partial r}$.
- The leaf space $\mathcal{M} = \{u, \vartheta, \varphi\}$ of k is endowed with a CR structure (H, J) where

$$H := \mathrm{Ann}(\theta), \qquad H^{(0,1)} := \mathrm{Ann}(\theta, \theta^1).$$

Easy exercise

Cast the Kerr metric into normal form and identify the CR data...

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