Lorentzian conformal manifolds from three-dimensional CR structures, and their Einstein metrics

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Pseudo-Riemannian Geometry and invariants in General Relativity
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The Kerr congruence

- Kerr spacetime (1963) $\mathcal{M} = \{u, \vartheta, \phi, r\}$ with metric

  $$g = 2\kappa\left(\mathrm{d}r + a\sin^2\vartheta\mathrm{d}\phi + \left(\frac{mr}{r^2+a^2\cos^2\vartheta} - \frac{1}{2}\right)\kappa\right) + 2(r^2 + a^2\cos^2\phi)\theta\bar{\theta}\,,$$

  $$\kappa = \mathrm{d}t + a\sin^2\vartheta\mathrm{d}\phi\,, \qquad \theta = \mathrm{d}\vartheta + \mathrm{i}\sin\vartheta\mathrm{d}\phi\,, \qquad a, m \in \mathbf{R}^*$$

- Twisting non-shearing congruence of null geodesics (NSCNG) $\mathcal{K}$ generated by null $k = \frac{\partial}{\partial r}$ where $\kappa = g(k, \cdot)$:

  $$\pounds_k g|_{K^\perp} \propto g|_{K^\perp}\,, \qquad K := \mathrm{span}(k)\,,$$

  $$\kappa \wedge \mathrm{d}\kappa \neq 0\,.$$

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- Robinson structure (N,K): involutive totally null complex 2-plane distribution

  $$N = \mathrm{Ann}(\kappa, \theta)\,, \qquad N \cap \overline{N} = \mathbf{C} \otimes K\,, \qquad [N, N] \subset N\,.$$

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- Contact Cauchy–Riemann (CR) structure $(\underline{H}, \underline{J})$ on the leaf space $\underline{\mathcal{M}} = \{u, \vartheta, \phi\}$ of $\mathcal{K}$:

$$\underline{H} := \mathrm{Ann}(\kappa)\,, \qquad \underline{H}^{(0,1)} := \mathrm{Ann}(\kappa, \theta)\,.$$

For the Kerr metric, $\underline{\mathcal{M}}$ can be realised as a real hypersurface in $\mathbf{C}^2$.

NSCNGs and Robinson structures

- Conformal Lorentzian 4-fold $(\mathcal{M}, \mathbf{c})$. For null $k \in T\mathcal{M}$, $g \in \mathbf{c}$ given as

$$g = 2\kappa\lambda + 2\theta\bar{\theta}\,, \qquad \kappa = g(k,\cdot)\,,$$

With $K = \mathrm{span}(k)$ and totally null complex $N = \mathrm{Ann}(\kappa, \theta)$,

K non-shearing geodesic $\iff$ $[K, N] \subset N$

$\iff$ $[N, N] \subset N$ Robinson structure.

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$$\begin{aligned} K \text{ non-shearing geodesic} \quad &\Longleftrightarrow \quad [K, N] \subset N \\ &\Longleftrightarrow \quad [N, N] \subset N \quad \text{Robinson structure}\,. \end{aligned}$$

- For the leaf space $\underline{\mathcal{M}}$:

$$\begin{array}{ccccc} (\mathcal{M}, \mathbf{c}) & \quad & K^{\perp}/K & \xrightarrow{\otimes \mathbf{C}} & N/^{\mathbf{C}}K \oplus \overline{N}/^{\mathbf{C}}K \\ \Big\downarrow k & & \Big\downarrow & & \Big\downarrow \\ \underline{\mathcal{M}} & & \underline{H} & \xrightarrow{\otimes \mathbf{C}} & \underline{H}^{(1,0)} \oplus \underline{H}^{(0,1)} \end{array} \qquad \text{CR structure}$$

- K twisting $\Longleftrightarrow$ $\underline{H}$ contact

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Problem

Reduce the vacuum Einstein field equations to CR data on the leaf space of a twisting NSCNG.

Timeline

1904-05 **Lorentz–Einstein–Poincaré**: special relativity
1907 **Poincaré**: real hypersurfaces in $\mathbf{C}^2$
1913-16 **Einstein–Grossmann–Hilbert**: general relativity
1932 **Cartan**: *géométrie pseudo-conforme* in dimension 3
1957 **Lewy**: non-solvable differential operator
1961 **Robinson theorem**: NSCNG and vacuum Maxwell equations
1962 **Goldberg–Sachs theorem**: NSCNG and Einstein equations
1963 **Kerr metric**: twisting NSCNG
1967 **Kerr–Penrose theorem** and **twistor theory**
1968 **Greenfield**: abstract CR manifolds — 1st use of the term 'CR'
1974 **Chern–Moser**: invariants of CR manifolds in any dimensions
1975 **Penrose**: hypersurface twistors and CR 5-folds
1976 **Tanaka**: CR manifolds; **Fefferman**'s conformal extension of CR structures; **Sommers**: NSCNG and CR 3-folds
1978 **Webster**: compatible CR connection
1983 **Hill–Penrose–Sparling**, **LeBrun**: non-realisable CR 5-folds
1984 **Mason**: hypersurface twistors
1985 **Tafel**: Lewy operator and non-analytic Robinson theorem
1986 **Robinson–Trautman**: CR structures in optical geometries
1990 **Lewandowski–Nurowski–Tafel**: Einstein equations and realisable CR 3-folds
2002 **Nurowski–Trautman**: Robinson manifolds as Lorentzian analogues of Hermitian manifolds
2021 **Fino–Leistner–TC**: Almost Robinson geometry

First CR function

- Kerr metric: the 1-form $\theta = \mathrm{d}\vartheta + \mathrm{i}\sin\vartheta \mathrm{d}\phi$ satisfies $\theta \wedge \mathrm{d}\theta = 0$, i.e.

$$\theta \wedge \mathrm{d}z = 0$$

  for some smooth $z : \underline{\mathcal{M}} \to \mathbf{C}$ s.t. $X(z) = 0$ for any $X \in \underline{H}^{(0,1)}$
- This Kerr coordinate z is referred to as a CR function.

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  Given a spacetime $(\mathcal{M}, g)$ equipped with NSCNG $\mathcal{K} \sim (N, K)$ and

$$\mathrm{Ric}(v, v) = 0 \qquad \text{for all } v \in N, \qquad\qquad \mathrm{Sc} = 0\,,$$

  then there exist coordinates $\{u, z, \bar{z}, r\}$ such that

$$g = 2\kappa\lambda + \frac{2}{r^2 + p^2}\theta\overline{\theta}\,, \qquad \lambda = \mathrm{d}r + W\mathrm{d}z + \overline{W}\mathrm{d}\bar{z} + H\kappa\,,$$
$$\kappa = \mathrm{d}u + f\mathrm{d}z + \overline{f}\mathrm{d}\bar{z} \qquad \theta = \mathrm{d}z\,,$$

  the r-dependence of all the functions is fully determined, and the form of the metric is subject to residual coordinate freedom.

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- `Goldberg-Sachs (1963), Hill-Gover-Nurowski (2011)`: Such spacetimes are algebraically special.

Einstein eqs and CR embeddability

- Lewandowski-Nurowski (1990):
  Lift $(\underline{\mathcal{M}}, \underline{J}, \underline{H})$ to $(\mathcal{M}, g, N, K)$ where $\mathcal{M} = \underline{\mathcal{M}} \times \mathbf{R}$ with metric

$$g = \mathrm{e}^{2\varphi}\left(4\underline{\theta}^0\lambda + 2\underline{\theta}^1\underline{\overline{\theta}}^{\bar{1}}\right),$$

$$\lambda = \mathrm{d}\phi + \lambda_1\underline{\theta}^1 + \lambda_{\bar{1}}\underline{\overline{\theta}}^{\bar{1}} + \lambda_0\underline{\theta}^0, \qquad \varphi, \lambda_1, \lambda_0 \in C^\infty(\mathcal{M})$$

  Vacuum field equations (+ cosmological constant and pure radiation)
  $\mathrm{Ric} = \Lambda g + \Phi\left(\underline{\theta}^0\right)^2$: ϕ-dependence entirely determined
  Metric $\mathrm{e}^{-2\varphi}g$ lives on a circle bundle and $\mathrm{e}^{2\varphi} = \mathrm{e}^{\underline{\varphi}}\sec^2(\phi + \underline{\psi})$

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Theorem (Lewandowski-Nurowski-Tafel (1990))

If a CR 3-fold admits a lift to a Ricci-flat metric then it is realisable as a real hypersurface in $\mathbf{C}^2$, i.e. it admits two CR functions z and w s.t. $\mathrm{d}z \wedge \mathrm{d}w \neq 0$.

- Related and further results: Mason (1984/1998), Hill-Lewandowski-Nurowski (2008), Schmalz-Ganji (2018)
- Applications — Type N vacuum metric with cosmological constant: Nurowski (2008), Zhang-Finley (2013)

Almost CR geometry

- Almost CR manifold $(\underline{\mathcal{M}}^{2m+1}, \underline{H}^{2m}, \underline{J})$: smooth $(2m+1)$-fold $\underline{\mathcal{M}}$, $\underline{H}^{2m} \subset T\underline{\mathcal{M}}$, bundle complex structure $\underline{J}$ on $\underline{H}$
- Assume contact and partially integrable, i.e. for any $\underline{\theta}^0 \in \mathrm{Ann}(\underline{H})$

$$\underline{\theta}^0 \wedge (\mathrm{d}\underline{\theta}^0)^m \neq 0\,, \qquad \mathrm{d}\underline{\theta}^0(\underline{v}, \underline{w}) = 0\,, \qquad \text{for all } \underline{v}, \underline{w} \in H^{(1,0)}\,.$$

- Levi form: weighted Hermitian form $\underline{\mathbf{h}}$ on $\underline{H}$:

$$\underline{h}(\underline{v}, \underline{w}) = -2\mathrm{i}\mathrm{d}\underline{\theta}^0(\underline{v}, \underline{w})\,, \qquad \underline{v} \in \underline{H}^{(1,0)}, \underline{w} \in \underline{H}^{(0,1)}\,.$$

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- Contact form $\underline{\theta}^0 \longrightarrow$ Canonical Webster–Tanaka connection $\underline{\nabla}$:

$$\underline{\theta}^0 \to \widehat{\underline{\theta}}^0 = \mathrm{e}^{\underline{\varphi}}\underline{\theta}^0 \quad \Longrightarrow \quad \underline{\nabla} \to \widehat{\underline{\nabla}} = \underline{\nabla} + \underline{\Upsilon} + \ldots\,, \qquad (\underline{\Upsilon} = \mathrm{d}\underline{\varphi})\,.$$

- CR invariants:
  - Nijenhuis tensor $\underline{\mathrm{N}}$ $(m > 1)$: Involutivity of $\underline{H}^{(1,0)}$
  - Chern–Moser $(m > 1)$ and Cartan $(m = 1)$ tensors: CR flatness
- Pseudo-Hermitian invariants (depend on contact form):
  - Pseudo-Hermitian Webster torsion $\underline{A}$: transverse CR symmetry
  - Schouten–Webster tensor $\underline{\mathrm{P}}$

Almost Robinson geometry

Definition (Nurowski-Trautman (2002), Fino-Leistner-TC (2021))

An almost Robinson manifold consists of a quadruple $(\mathcal{M}, g, N, K)$ where

- $(\mathcal{M}, g)$ is a smooth Lorentzian manifold of dimension $2m+2$,
- N is a totally null complex $(m+1)$-plane distribution,
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Fino-Leistner-TC (2021): Intrinsic torsion of (N, K)

- Structure group $\mathbf{R}_{>0} \cdot \mathbf{U}(m) \ltimes \mathbf{R}^{2m}$ stabilises $\kappa \in \mathrm{Ann}(K^\perp)$ and "Hermitian" 3-form $\rho := 3\kappa \wedge \omega$
- Induced geometries on the leaf space $\underline{\mathcal{M}}$ of congruence tangent to K
- Three important (conformally invariant) classes:

geodesic	nearly Robinson	Robinson
$[K, K^\perp] \subset K^\perp$	$[K, N] \subset N$	$[N, N] \subset N$
$\underline{H}^{2m} \subset T\underline{\mathcal{M}}$	$(\underline{H}, \underline{J})$ almost CR	$(\underline{H}, \underline{J})$ CR

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$\underline{H}^{2m} \subset T\underline{\mathcal{M}}$	$(\underline{H}, \underline{J})$ almost CR	$(\underline{H}, \underline{J})$ CR

- ...and a 4th one:

twist-induced almost Robinson $\iff$ $\kappa \wedge \mathrm{d}\kappa \propto \rho$

Fefferman conformal structure

- Fefferman (1976), Lee (1986), Sparling, Graham (1987), Čap-Gover (2010):
  Associate to a CR manifold a Lorentzian conformal structure $\mathbf{c}$:

$$\begin{array}{ccccccc} (\mathcal{M}^{2m+2} := C/\mathbf{R}^*, \mathbf{c}) & & \mathbf{c} & \ni & g & \rightsquigarrow & \widehat{g} = \mathrm{e}^{\underline{\varphi}} g \\ \big\downarrow k & & & & \big\uparrow & & \big\uparrow \\ (\underline{\mathcal{M}}^{2m+1}, \underline{H}, \underline{J}) & & \mathrm{Ann}(\underline{H}) & \ni & \underline{\theta}^0 & \rightsquigarrow & \widehat{\underline{\theta}}^0 = \mathrm{e}^{\underline{\varphi}} \underline{\theta}^0 \end{array}$$

  where
  - $C := \wedge^{m+1} \mathrm{Ann}(\underline{T}^{(0,1)} \underline{\mathcal{M}})$,
  - $g = \frac{4}{m+2} \underline{\theta}^0 \odot \left(\mathrm{d}\phi + \mathrm{i} \underline{\Gamma}_\alpha{}^\alpha - \frac{1}{2} \mathrm{i} \underline{h}^{\alpha\bar{\beta}} \mathrm{d} \underline{h}_{\alpha\bar{\beta}} - \underline{\mathsf{P}} \underline{\theta}^0 \right) + \underline{h}$
  - $k = \frac{\partial}{\partial \phi}$ null conformal Killing field $\longrightarrow$ twisting NSCNG
- Robinson structure determined by $\underline{\theta}^0 \wedge \mathrm{d}\underline{\theta}^0$

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- Leitner (2007), Čap-Gover (2008):
  $g \in \mathbf{c}$ Einstein $\Longrightarrow (\underline{\mathcal{M}}, \underline{H}, \underline{J}, \underline{\theta}^0)$ CR–Einstein $\longrightarrow$ Kähler–Einstein
  Lewandowski (1988):
  Any Fefferman–Einstein 4-fold must be conformally flat.

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- Leitner (2010), TC (unpublished): Partially integrable case

Twisting NSCNGs in higher dimensions

- Index notation: abstractly $\underline{\mathcal{E}}^{\alpha} := \underline{H}^{(1,0)}$, concretely $\alpha = 1, \ldots m$, etc.

Theorem (TC (2021))

Let $(\mathcal{M}, \mathbf{c})$ be a Lorentzian conformal manifold of dimension $2m+2>4$ with null line distribution K tangent to a twisting NSCNG $\mathcal{K}$. Denote by $\underline{\mathcal{M}}$ the local leaf space of $\mathcal{K}$ and by W the Weyl tensor of $\mathbf{c}$.

1. *If $W(k,v,k,v)=0$ for any $k \in K, v \in K^{\perp}$, then the twist of $\mathcal{K}$ induces a nearly Robinson structure (N,K), and $\underline{\mathcal{M}}$ inherits a p.i. contact almost CR structure $(\underline{H}, \underline{J})$ with positive definite Levi form.*

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2. *If in addition $W(k, v, k, \cdot) = 0$ for any $k \in K$, $v \in K^{\perp}$, any Einstein metric in $\mathbf{c}$ determines a contact form $\underline{\theta}^0$ for $\underline{H}$ such that $(\underline{H}, \underline{J}, \underline{\theta}^0)$ is an almost CR–Einstein structure, i.e.*

$$\underline{A}_{\alpha\beta} = 0\,, \qquad \underline{\nabla}^{\gamma}\underline{\mathsf{N}}_{\gamma(\alpha\beta)} = 0\,, \qquad \left(\underline{\mathsf{P}}_{\alpha\bar{\beta}} - \tfrac{1}{m+2}\underline{\mathsf{N}}_{\alpha\gamma\delta}\underline{\mathsf{N}}_{\bar{\beta}}{}^{\gamma\delta}\right)_{\circ} = 0\,,$$

i.e. $\underline{\mathcal{M}}$ locally fibered over an almost Kähler–Einstein $2m$-fold.

Twisting NSCNGs in higher dimensions

- Index notation: abstractly $\underline{\mathcal{E}}^{\alpha} := \underline{H}^{(1,0)}$, concretely $\alpha = 1, \ldots m$, etc.

Theorem (TC (2021))

Let $(\mathcal{M}, \mathbf{c})$ be a Lorentzian conformal manifold of dimension $2m + 2 > 4$ with null line distribution K tangent to a twisting NSCNG $\mathcal{K}$. Denote by $\underline{\mathcal{M}}$ the local leaf space of $\mathcal{K}$ and by W the Weyl tensor of $\mathbf{c}$.

1. *If $W(k, v, k, v) = 0$ for any $k \in K, v \in K^{\perp}$, then the twist of $\mathcal{K}$ induces a nearly Robinson structure (N, K), and $\underline{\mathcal{M}}$ inherits a p.i. contact almost CR structure $(\underline{H}, \underline{J})$ with positive definite Levi form.*
2. *If in addition $W(k, v, k, \cdot) = 0$ for any $k \in K, v \in K^{\perp}$, any Einstein metric in $\mathbf{c}$ determines a contact form $\underline{\theta}^0$ for $\underline{H}$ such that $(\underline{H}, \underline{J}, \underline{\theta}^0)$ is an almost CR–Einstein structure, i.e.*

$$\underline{A}_{\alpha\beta} = 0\,, \qquad \underline{\nabla}^{\gamma}\underline{\mathsf{N}}_{\gamma(\alpha\beta)} = 0\,, \qquad \left(\underline{\mathsf{P}}_{\alpha\bar{\beta}} - \tfrac{1}{m+2}\underline{\mathsf{N}}_{\alpha\gamma\delta}\underline{\mathsf{N}}_{\bar{\beta}}{}^{\gamma\delta}\right)_{\circ} = 0\,,$$

i.e. $\underline{\mathcal{M}}$ locally fibered over an almost Kähler–Einstein 2m-fold.

- 3-parameter family of Einstein metrics:
  (massive) Fefferman–Einstein and (massless) Taub–NUT-type metrics

Twisting NSCNGs in dimension four

- Index notation: abstractly $\underline{\mathcal{E}}^{\alpha} := \underline{H}^{(1,0)}$, concretely $\alpha = 1$!

Theorem (TC)

Let $(\mathcal{M}, g)$ be a Lorentzian 4-fold with a twisting NSCNG $\mathcal{K} \sim (N, K)$.

1. *Suppose $\mathrm{Ric}(v, v) = 0$ for all $v \in N$. Then g is determined by*
   - *a pseudo-Hermitian structure $(\underline{H}, \underline{J}, \underline{\theta}^0)$ on the leaf space $\underline{\mathcal{M}}$ of $\mathcal{K}$,*
   - *a solution $\underline{\lambda}_{\alpha} \in (\underline{H}^{(1,0)})^*$ to*

$$\underline{\nabla}_{\alpha}\underline{\lambda}_{\beta} - \mathrm{i}\underline{\lambda}_{\alpha}\underline{\lambda}_{\beta} - \underline{A}_{\alpha\beta} = 0\,.$$

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   $$\underline{\nabla}_\alpha \underline{\lambda}_\beta - \mathrm{i} \underline{\lambda}_\alpha \underline{\lambda}_\beta - \underline{A}_{\alpha\beta} = 0 \, .$$

2. *Suppose that g satisfies the vacuum Einstein field equations with cosmological constant Λ and possibly pure radiation. Then g is uniquely determined by $\underline{\theta}^0$ and $\underline{\lambda}_\alpha$ as in 1. and a real density $\underline{c}$ satisfying*

   $$\underline{\nabla}_\alpha (\underline{b} - \mathrm{i}\underline{c}) = 3\mathrm{i} \underline{\lambda}_\alpha (\underline{b} - \mathrm{i}\underline{c}) \, ,$$

   where $\underline{b} := -\frac{8}{3}\Lambda + 8\underline{\mathrm{P}} - 6\underline{\lambda}_\alpha \underline{\lambda}^\alpha + 6\mathrm{i} \, (\underline{\nabla}_\alpha \underline{\lambda}^\alpha - \underline{\nabla}^\alpha \underline{\lambda}_\alpha)$.

Some properties

- Agrees with Mason and Hill–Lewandowski–Nurowski–Tafel
- Formulation now in terms of pseudo-Hermitian tensorial quantities
- Form of the metric:

$$g = \sec^2\phi\left(4\underline{\theta}^0\left(\mathrm{d}\phi + \left(1+\tfrac{1}{2}\mathrm{e}^{-2\mathrm{i}\phi}\right)\underline{\lambda}_\alpha\underline{\theta}^\alpha + c.c. + \lambda_0\underline{\theta}^0\right) + \underline{h}\right)\,,$$

  where $\mathrm{d}\underline{\theta}^0 = \mathrm{i}\underline{h}_{\alpha\bar{\beta}}\underline{\theta}^\alpha \wedge \overline{\underline{\theta}}^{\bar{\beta}}$.
- For vacuum, possibly with pure radiation,

$$\lambda_0 = \underline{a}_0 + \underline{a}_1\cos^2\phi + \underline{a}_2\cos\phi\sin\phi + \underline{b}\cos^4\phi + \underline{c}\cos^3\phi\sin\phi\,,$$

  where $\underline{a}_0$, $\underline{a}_1$, $\underline{a}_2$, $\underline{b}$ and $\underline{c}$ pseudo-Hermitian quantities.

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Problem: Conformal/CR invariance not transparent...

Need an analogue of the Fefferman conformal structure:

$$\begin{array}{ccccccc}
(\mathcal{M}, \mathbf{c}) & \quad & \mathbf{c} & \ni & g & \rightsquigarrow & \widehat{g} = \mathrm{e}^{\underline{\varphi}} g \\
\downarrow k & & & & \uparrow & & \uparrow \\
(\underline{\mathcal{M}}, \underline{H}, \underline{J}) & & \mathrm{Ann}(\underline{H}) & \ni & \underline{\theta}^0 & \rightsquigarrow & \widehat{\underline{\theta}}^0 = \mathrm{e}^{\underline{\varphi}}\underline{\theta}^0
\end{array}$$

Alg. special conformal structures

Theorem (TC)

Let $(\underline{\mathcal{M}}, \underline{H}, \underline{J})$ be a contact CR 3-fold and $\mathcal{M} \to \underline{\mathcal{M}}$ be a (trivial) circle bundle with fiber coordinate ϕ. Let $\underline{\lambda}_0^{(4)} \in \underline{\mathcal{E}}(-1,-1)$, $\underline{\lambda}_\alpha^{(-2)} \in \underline{\mathcal{E}}_\alpha$ and $[\underline{\lambda}_\alpha^{(0)}] \in \underline{\mathcal{E}}_\alpha\{\frac{\mathrm{i}}{2}\}$, i.e. $\widehat{\underline{\lambda}}_\alpha^{(0)} = \underline{\lambda}_\alpha^{(0)} + \frac{\mathrm{i}}{2}\underline{\Upsilon}_\alpha$ under a change of contact forms. Choosing a contact form $\underline{\theta}^0$ with Levi form $\underline{h}$, we define a Lorentzian metric on $\mathcal{M}$ by

$$g = 4\underline{\theta}^0 \left(\mathrm{d}\phi + \lambda_\alpha \underline{\theta}^\alpha + \lambda_{\bar{\alpha}} \overline{\underline{\theta}}^{\bar{\alpha}} + \lambda_0 \underline{\theta}^0\right) + \underline{h}\,, \qquad \mathrm{d}\underline{\theta}^0 = \mathrm{i}\underline{h}_{\alpha\bar{\beta}}\underline{\theta}^\alpha \wedge \overline{\underline{\theta}}^{\bar{\beta}}$$

where $\lambda_\alpha = \underline{\lambda}_\alpha^{(-2)} \mathrm{e}^{-2\mathrm{i}\phi} + \underline{\lambda}_\alpha^{(0)}$ and

$$\begin{aligned}
\lambda_0 &= \underline{\lambda}_0^{(-4)} \mathrm{e}^{-4\mathrm{i}\phi} + \underline{\lambda}_0^{(-2)} \mathrm{e}^{-2\mathrm{i}\phi} + \underline{\lambda}_0^{(0)} + \underline{\lambda}_0^{(-2)} \mathrm{e}^{-2\mathrm{i}\phi} + \underline{\lambda}_0^{(4)} \mathrm{e}^{4\mathrm{i}\phi}\,,\\
\underline{\lambda}_0^{(0)} &= \mathrm{i}\underline{\nabla}_\gamma \underline{\lambda}_{(0)}^\gamma - \mathrm{i}\underline{\nabla}^\gamma \underline{\lambda}_\gamma^{(0)} + 3\underline{\lambda}_\gamma^{(-2)} \underline{\lambda}_{(2)}^\gamma + \underline{\mathsf{P}}\,,\\
\underline{\lambda}_0^{(2)} &= \tfrac{\mathrm{i}}{2}\underline{\nabla}_\gamma \underline{\lambda}_{(2)}^\gamma + \underline{\lambda}_\gamma^{(0)} \underline{\lambda}_{(2)}^\gamma + 2\underline{\lambda}_0^{(4)}\,.
\end{aligned}$$

Any other contact form $\widehat{\underline{\theta}}^0 = \mathrm{e}^{\underline{\varphi}}\underline{\theta}^0$ yields the conformal related metric $\widehat{g} = \mathrm{e}^{\underline{\varphi}} g$. Thus, $\mathcal{M}$ acquires a conformal structure **c**, *which is in fact algebraically special, and the fibration is a NSCNG.*

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- Idea: Conditions on the Fourier expansion coefficients of the Weyl tensor components.

Vacuum Einstein field equations

Theorem (TC)

Let $(\mathcal{M}, \mathbf{c}) \xrightarrow{\varpi} (\underline{\mathcal{M}}, \underline{H}, \underline{J})$ be the conformal structure of the previous Theorem. The following statements are equivalent:

1. $\mathbf{c}$ *(locally) contains a metric g that satisfies the vacuum Einstein field equations with cosmological constant Λ and possibly pure radiation;*
2. *There exists a CR scale $\underline{\sigma} \in \underline{\mathcal{E}}_{\mathbb{R}}(1,1)$ (i.e. a contact form) such that the following CR-invariant equations hold:*

$$\underline{\nabla}_\alpha \underline{\sigma} + \mathrm{i}\left(2\underline{\lambda}_\alpha^{(0)} - 4\underline{\lambda}_\alpha^{(-2)}\right)\underline{\sigma} = 0\,, \tag{1}$$

$$\underline{\nabla}_\alpha\left(\underline{\lambda}_\beta^{(0)} - \underline{\lambda}_\beta^{(-2)}\right) - 2\mathrm{i}\left(\underline{\lambda}_\alpha^{(0)} - \underline{\lambda}_\alpha^{(-2)}\right)\left(\underline{\lambda}_\beta^{(0)} - \underline{\lambda}_\beta^{(-2)}\right) - \tfrac{1}{2}\underline{A}_{\alpha\beta} = 0\,, \tag{2}$$

$$\begin{aligned}\underline{\lambda}_0^{(4)} + \underline{\lambda}_0^{(-4)} &= \mathrm{i}\left(\underline{\nabla}_\alpha \underline{\lambda}^\alpha_{(0)} - \underline{\nabla}^\alpha \underline{\lambda}_\alpha^{(0)}\right) - 2\underline{\lambda}_\alpha^{(0)}\underline{\lambda}^\alpha_{(0)} + \underline{\mathsf{P}} - \tfrac{1}{3}\Lambda\underline{\sigma}^{-1} \\ &\quad - \tfrac{1}{4}\mathrm{i}\left(\underline{\nabla}_\alpha \underline{\lambda}^\alpha_{(2)} - \underline{\nabla}^\alpha \underline{\lambda}_\alpha^{(-2)}\right) + \tfrac{1}{2}\underline{\lambda}_\alpha^{(-2)}\underline{\lambda}^\alpha_{(0)} + \tfrac{1}{2}\underline{\lambda}_\alpha^{(0)}\underline{\lambda}^\alpha_{(2)} + 3\underline{\lambda}_\alpha^{(-2)}\underline{\lambda}^\alpha_{(2)}\,,\end{aligned}$$

$$\underline{\nabla}_\alpha \underline{\lambda}_0^{(4)} - 2\mathrm{i}\left(\underline{\lambda}_\alpha^{(0)} + \underline{\lambda}_\alpha^{(-2)}\right)\underline{\lambda}_0^{(4)} = 0\,.$$

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$$\underline{\nabla}_\alpha\underline{\lambda}_0^{(4)} - 2\mathrm{i}\left(\underline{\lambda}_\alpha^{(0)} + \underline{\lambda}_\alpha^{(-2)}\right)\underline{\lambda}_0^{(4)} = 0\,.$$

- For an Einstein metric (no pure radiation), additional equation required.
- Locally, any algebraically special Einstein spacetime (by the Goldberg-Sachs theorem) arises in this way.

Relation to CR functions

- If $\underline{\nabla}_\alpha \underline{\sigma} = 0$ and $\underline{\sigma} \neq 0$ then (1) implies $\underline{\lambda}_\alpha := \underline{\lambda}_\alpha^{(0)} = 2\underline{\lambda}_\alpha^{(-2)}$ and (2) becomes

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Lemma (TC)

A contact CR 3-fold admits a CR function if and only if it admits a Webster–Weyl structure, i.e. a solution $[\underline{\lambda}_\alpha] \in \underline{\mathcal{E}}_\alpha\{\mathrm{i}\}$ to the CR-invariant equation $(\star)$. (Here, $\widehat{\underline{\lambda}}_\alpha = \underline{\lambda}_\alpha + \mathrm{i}\underline{\Upsilon}_\alpha$ under a change of contact forms.)

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- Two types of solutions:
  1. Trivial solutions:

$$\underline{\lambda}_\alpha = -\mathrm{i}\underline{\Upsilon}_\alpha = -\mathrm{i}\underline{\nabla}_\alpha \underline{\varphi}\,, \qquad \text{for some smooth function } \underline{\varphi}$$

     i.e. there exists a transverse CR symmetry
  2. Non-trivial solutions: there exists a family of adapted coframes such that the $(0,1)$-component of the Webster–Tanaka connection 1-form satisfies

$$\underline{\Gamma}_{\bar{\alpha}} = -\mathrm{i}\underline{\lambda}_{\bar{\alpha}}\,.$$

Concluding remarks

- ✓ Conformal and CR invariant properties of algebraically special Einstein metrics (with possibly pure radiation).
- … Work on a Fefferman-type circle bundle (Cf `Schmalz-Ganji (2018)`) $\longrightarrow$ reinterpretation of the Fourier coefficients $\underline{\lambda}_{\alpha}^{(k)}$, $\underline{\lambda}_{0}^{(k)}$, gauged connection 1-form, etc.
- … More muscular tractorial approach...
- … Relation to asymptotic and global properties of spacetime...
- … Higher dimensions: non-shearing property of null geodesic congruences too strong. But many other 'classes' of almost Robinson manifolds to be investigated...

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Thank you for your attention!

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