Almost Robinson geometry

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The Robinson congruence (1960ies)

- Minkowski space $\mathbb{M} = \{u, z, \bar{z}, r\}$ with null $k = \frac{\partial}{\partial r}$:

$$\eta = 2\kappa \mathrm{d}r + 2(r^2+1)\theta\bar{\theta}$$
$$\kappa = \eta(k,\cdot) = \mathrm{d}u - \mathrm{i}\bar{z}\mathrm{d}z + \mathrm{i}z\mathrm{d}\bar{z}\,, \qquad \theta = \mathrm{d}z\,.$$

- Twisting non-shearing congruence of null geodesics (NSCNG) $\mathcal{K}$ generated by k:

$$\pounds_k \eta|_{K^\perp} \propto \eta|_{K^\perp}\,, \qquad K := \mathrm{span}(k)\,,$$
$$\kappa \wedge \mathrm{d}\kappa \neq 0\,.$$

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- Robinson structure (N,K): involutive totally null complex 2-plane distribution

$$N = \mathrm{Ann}(\kappa, \theta)\,, \qquad N \cap \overline{N} = \mathbf{C} \otimes K\,, \qquad [N, N] \subset N\,.$$

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- Contact Cauchy–Riemann (CR) structure $(\underline{H}, \underline{J})$ on the leaf space $\underline{\mathcal{M}} = \{u, z, \bar{z}\}$ of $\mathcal{K}$:

$$\underline{H} := \mathrm{Ann}(\kappa)\,, \qquad \underline{H}^{(0,1)} := \mathrm{Ann}(\kappa,\theta)\,.$$

Hyperquadric $\underline{\mathcal{M}} = \{(z,w)\in\mathbf{C}^2 : \Im(w) = |z|^2\}$

NSCNGs and Robinson structures

- Conformal Lorentzian 4-fold $(\mathcal{M}, \mathbf{c})$. For null $k \in T\mathcal{M}$, $g \in \mathbf{c}$ given as

$$g = 2\kappa\lambda + 2\theta\bar{\theta}\,, \qquad \kappa = g(k, \cdot)\,,$$

With $K = \mathrm{span}(k)$ and totally null complex $N = \mathrm{Ann}(\kappa, \theta)$,

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$$\iff [N, N] \subset N \quad \text{Robinson structure}\,.$$

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- For the leaf space $\underline{\mathcal{M}}$:
$$\begin{array}{ccccc} (\mathcal{M}, \mathbf{c}) & & K^{\perp}/K & \xrightarrow{\otimes \mathbf{C}} & N/^{\mathbf{C}}K \oplus \overline{N}/^{\mathbf{C}}K \\ \downarrow k & & \downarrow & & \downarrow \\ \underline{\mathcal{M}} & & \underline{H} & \xrightarrow{\otimes \mathbf{C}} & \underline{H}^{(1,0)} \oplus \underline{H}^{(0,1)} \end{array} \qquad \text{CR structure}$$
- K twisting $\iff$ $\underline{H}$ contact

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Problem

Reduce the vacuum Einstein field equations to CR data on the leaf space of a twisting NSCNG.

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2002 **Nurowski–Trautman**: Robinson manifolds as Lorentzian analogues of Hermitian manifolds
2021 **Fino–Leistner–TC**: Almost Robinson geometry

The Kerr congruence

- Kerr metric (1963): Petrov type D vacuum spacetime
  $\mathcal{M} = \{u, \vartheta, \phi, r\}$ with parameters a and m:

  $$g = 2\kappa\left(\mathrm{d}r + a\sin^2\vartheta\mathrm{d}\phi + \left(\frac{mr}{r^2+a^2\cos^2\vartheta} - \frac{1}{2}\right)\kappa\right) + 2(r^2 + a^2\cos^2\phi)\theta\overline{\theta}\,,$$

  $$\kappa = \mathrm{d}t + a\sin^2\vartheta\mathrm{d}\phi\,, \qquad \theta = \mathrm{d}\vartheta + \mathrm{i}\sin\vartheta\mathrm{d}\phi\,.$$

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- Note $\theta = \mathrm{d}\vartheta + \mathrm{i} \sin \vartheta \mathrm{d}\phi$ satisfies $\theta \wedge \mathrm{d}\theta = 0$, i.e.

$$\theta \wedge \mathrm{d}z = 0$$

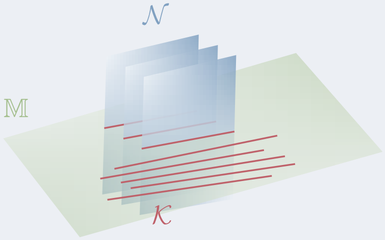
for some smooth $z : \underline{\mathcal{M}} \to \mathbf{C}$ s.t. $X(z) = 0$ for any $X \in \underline{H}^{(0,1)}$
- z is a CR function (also known as a Kerr coordinate among relativists)
- In fact, two CR functions $\Longrightarrow (\underline{\mathcal{M}}, \underline{H}, \underline{J})$ realisable

Kerr surfaces in Twistor space

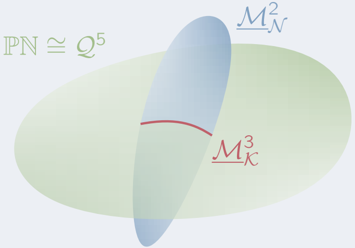
Kerr theorem (Penrose (1967))

Any analytic NSCNG in Minkowski space $\mathbb{M}$ locally gives rise to a complex (Kerr) surface in twistor space $\mathbb{PT}$. Conversely, any such NSCNG arises in this way.

- Twistor space $\mathbb{PT} \cong \mathbf{C}\mathbb{P}^3$: space of α-planes in ${}^{\mathbf{C}}\mathbb{M}$
- CR 5-hypersphere $\mathbb{PN} \subset \mathbb{PT}$: space of null geodesics
- NSCNG $\mathcal{K} = \mathbb{M} \cap \mathcal{N}$ where $\mathcal{N}$ is a foliation by α-planes



${}^{\mathbf{C}}\mathbb{M}$



$\mathbb{PT} \cong \mathbf{C}\mathbb{P}^3$

Reduced Einstein equations as CR data

- Kerr (1963), Debney-Kerr-Schild (1969):
  Given a spacetime $(\mathcal{M}, g)$ equipped with NSCNG $\mathcal{K} \sim (N, K)$ and

  $$\mathrm{Ric}(v, v) = 0 \qquad \text{for all } v \in N, \qquad\qquad \mathrm{Sc} = 0\,, \qquad\qquad (\dagger)$$

  then there exist coordinates $\{u, z, \bar{z}, r\}$ such that

  $$g = 2\kappa\lambda + \frac{2}{r^2 + p^2}\theta\overline{\theta}\,, \qquad \lambda = \mathrm{d}r + W\mathrm{d}z + \overline{W}\mathrm{d}\bar{z} + H\kappa\,,$$
  $$\kappa = \mathrm{d}u + f\mathrm{d}z + \overline{f}\mathrm{d}\bar{z} \qquad \theta = \mathrm{d}z\,,$$

  the r-dependence is fully determined, and the form of the metric is subject to residual coordinate freedom.

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Theorem (Mason (1984,1998))

Let $(\underline{\mathcal{M}}, \underline{H}, \underline{J})$ be a contact CR 3-fold. Then, any choice of

- *a weighted $(1, 0)$-form $\underline{\boldsymbol{\theta}}$ such that $\underline{\boldsymbol{\theta}} \wedge \mathrm{d}\underline{\boldsymbol{\theta}} = 0$, and*
- *a complex density ψ^0_2,*

determines a metric on a circle bundle associated to $\wedge^2\mathrm{Ann}(\underline{H}^{(0,1)})$ that satisfies the reduced Einstein equations $(\dagger)$.

Einstein eqs and CR embeddability

- Lewandowski-Nurowski (1990):
  Lift $(\underline{\mathcal{M}}, \underline{J}, \underline{H})$ to $(\mathcal{M}, g, N, K)$ where $\mathcal{M} = \underline{\mathcal{M}} \times \mathbf{R}$ with metric

$$g = \Omega^2 \left(4\underline{\theta}^0 \lambda + 2\underline{\theta}^1 \underline{\overline{\theta}}^{\bar{1}}\right) ,$$

$$\lambda = \mathrm{d}\phi + \lambda_1 \underline{\theta}^1 + \lambda_{\bar{1}} \underline{\overline{\theta}}^{\bar{1}} + \lambda_0 \underline{\theta}^0 , \qquad \Omega, \lambda_1, \lambda_0 \in C^\infty(\mathcal{M})$$

  Field equations:
  1. $\mathrm{Ric}(k, k) = 0$ for all $k \in K$: $\Omega^2 = \mathrm{e}^{\underline{\varphi}} \sec^2(\phi + \underline{\psi})$ for $\underline{\varphi}, \underline{\psi} \in C^\infty(\underline{\mathcal{M}})$
  2. $\mathrm{Ric}(v, v) = 0$ for all $v \in N$: ϕ-dependence is integrated out in λ_1
  3. Vacuum (+ pure radiation): ϕ-dependence is integrated out in λ_1, λ_0

  Reduction in terms of $\underline{\varphi}$, $\underline{\psi}$, structure functions and derivatives.

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Theorem (Lewandowski-Nurowski-Tafel (1990))

If a CR 3-fold admits a lift to a Ricci-flat metric then it is realisable as a real hypersurface in $\mathbf{C}^2$.

- Generalisations:
  Hill-Lewandowski-Nurowski (2008), Schmalz-Ganji (2018)
- Applications — Type N vacuum metric with cosmological constant:
  Nurowski (2008), Zhang-Finley (2013)

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- Almost CR manifold $(\underline{\mathcal{M}}^{2m+1}, \underline{H}^{2m}, \underline{J})$: smooth $(2m+1)$-fold $\underline{\mathcal{M}}$, $\underline{H}^{2m} \subset T\underline{\mathcal{M}}$, bundle complex structure $\underline{J}$ on $\underline{H}$
- Assume contact and partially integrable, i.e. for any $\underline{\theta}^0 \in \mathrm{Ann}(\underline{H})$

$$\underline{\theta}^0 \wedge (\mathrm{d}\underline{\theta}^0)^m \neq 0 \,, \qquad \mathrm{d}\underline{\theta}^0(\underline{v}, \underline{w}) = 0 \,, \qquad \text{for all } \underline{v}, \underline{w} \in H^{(1,0)} \,.$$

- Levi form: weighted Hermitian form $\underline{\mathbf{h}}$ on $\underline{H}$:

$$\underline{h}(\underline{v}, \underline{w}) = -2\mathrm{i}\mathrm{d}\underline{\theta}^0(\underline{v}, \underline{w}) \,, \qquad \underline{v} \in \underline{H}^{(1,0)}, \underline{w} \in \underline{H}^{(0,1)} \,.$$

  Assume the signature of $\underline{\mathbf{h}}$ to be positive definite.

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  Assume the signature of $\underline{\mathbf{h}}$ to be positive definite.
- Contact form $\underline{\theta}^0 \longrightarrow$ Canonical Webster–Tanaka connection $\underline{\nabla}$:

$$\underline{\theta}^0 \to \underline{\widehat{\theta}}^0 = \mathrm{e}^{\underline{\varphi}}\underline{\theta}^0 \quad \Longrightarrow \quad \underline{\nabla} \to \underline{\widehat{\nabla}} = \underline{\nabla} + \underline{\Upsilon} + \dots\,, \qquad (\underline{\Upsilon} = \mathrm{d}\underline{\varphi})\,.$$

- CR invariants:
  - Nijenhuis tensor $\underline{\mathrm{N}}$ $(m > 1)$: Involutivity of $\underline{H}^{(1,0)}$
  - Chern–Moser $(m > 1)$ and Cartan $(m = 1)$ tensors: CR flatness
- Pseudo-Hermitian invariants (depend on contact form):
  - Pseudo-Hermitian Webster torsion $\underline{A}$: transverse CR symmetry
  - Schouten–Webster tensor $\underline{\mathrm{P}}$

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Definition (Nurowski-Trautman (2002), Fino-Leistner-TC (2021))

An almost Robinson manifold consists of a quadruple $(\mathcal{M}, g, N, K)$ where

- $(\mathcal{M}, g)$ is a smooth Lorentzian manifold of dimension $2m+2$,
- N is a totally null complex $(m+1)$-plane distribution,
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Fino-Leistner-TC (2021): Intrinsic torsion of (N, K)

- Structure group $\mathbf{R}_{>0} \cdot \mathbf{U}(m) \ltimes \mathbf{R}^{2m}$ stabilises
  $\kappa \in \mathrm{Ann}(K^{\perp})$ and "Hermitian" 3-form $\rho := 3\kappa \wedge \omega$
- Induced geometries on the leaf space $\underline{\mathcal{M}}$ of congruence tangent to K
- Three important (conformally invariant) classes:

geodesic	nearly Robinson	Robinson
$[K, K^{\perp}] \subset K^{\perp}$	$[K, N] \subset N$	$[N, N] \subset N$
$\underline{H}^{2m} \subset T\underline{\mathcal{M}}$	$(\underline{H}, \underline{J})$ almost CR	$(\underline{H}, \underline{J})$ CR

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An almost Robinson manifold consists of a quadruple $(\mathcal{M}, g, N, K)$ where

- $(\mathcal{M}, g)$ is a smooth Lorentzian manifold of dimension $2m+2$,
- N is a totally null complex $(m+1)$-plane distribution,
- K is the null line distribution given by $\mathbf{C} \otimes K = N \cap \overline{N}$.

Fino-Leistner-TC (2021): Intrinsic torsion of (N, K)

- Structure group $\mathbf{R}_{>0} \cdot \mathbf{U}(m) \ltimes \mathbf{R}^{2m}$ stabilises $\kappa \in \mathrm{Ann}(K^\perp)$ and "Hermitian" 3-form $\rho := 3\kappa \wedge \omega$
- Induced geometries on the leaf space $\underline{\mathcal{M}}$ of congruence tangent to K
- Three important (conformally invariant) classes:

geodesic	nearly Robinson	Robinson
$[K, K^\perp] \subset K^\perp$	$[K, N] \subset N$	$[N, N] \subset N$
$\underline{H}^{2m} \subset T\underline{\mathcal{M}}$	$(\underline{H}, \underline{J})$ almost CR	$(\underline{H}, \underline{J})$ CR

- ...and a 4th one:

$$\text{twist-induced almost Robinson} \qquad \Longleftrightarrow \qquad \kappa \wedge d\kappa \propto \rho$$

Fefferman conformal structure

- Fefferman (1976), Lee (1986), Sparling, Graham (1987), Čap-Gover (2010):
  Associate to a CR manifold a Lorentzian conformal structure $\mathbf{c}$:

$$\begin{array}{ccc} (\mathcal{M}^{2m+2} := C/\mathbf{R}^*, \mathbf{c}) & \quad & \mathbf{c} \ni g \rightsquigarrow \widehat{g} = \mathrm{e}^{\underline{\varphi}} g \\ \big\downarrow k & & \uparrow \qquad\quad \uparrow \\ (\underline{\mathcal{M}}^{2m+1}, \underline{H}, \underline{J}) & & \mathrm{Ann}(\underline{H}) \ni \underline{\theta}^0 \rightsquigarrow \widehat{\underline{\theta}}^0 = \mathrm{e}^{\underline{\varphi}} \underline{\theta}^0 \end{array}$$

  where
  - $C := \wedge^{m+1} \mathrm{Ann}(\underline{T}^{(0,1)} \underline{\mathcal{M}})$,
  - $g = 4\underline{\theta}^0 \odot \left(\mathrm{d}\phi + \frac{1}{m+2} \left(\mathrm{i}\underline{\Gamma}_\alpha{}^\alpha - \underline{\mathrm{P}}\underline{\theta}^0 \right) \right) + \underline{h}$
  - $k = \frac{\partial}{\partial \phi}$ null conformal Killing field $\longrightarrow$ twisting NSCNG
- Robinson structure determined by $\underline{\theta}^0 \wedge \mathrm{d}\underline{\theta}^0$

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- Leitner (2007), Čap-Gover (2008):
  $g \in \mathbf{c}$ Einstein $\Longrightarrow (\underline{\mathcal{M}}, \underline{H}, \underline{J}, \underline{\theta}^0)$ CR–Einstein $\longrightarrow$ Kähler–Einstein
  Lewandowski (1988):
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- Leitner (2010), TC (unpublished): Partially integrable case

Kerr congruence in higher dimensions

Theorem (Mason-TC (2010))

Let $(\mathcal{M}, g)$ be a Lorentzian manifold of dimension $2m+2$ equipped with a closed conformal Killing–Yano 2-form Φ, i.e.

$$\nabla_v \Phi = -\frac{1}{2m+1} g(v, \cdot) \wedge \mathrm{d}^* \Phi \,, \qquad \text{for all } v \in T\mathcal{M}.$$

Suppose Φ is generic. Then $(\mathcal{M}, g)$ admits two congruences of null geodesics, each associated to 2^{m-1} Robinson structures. Each (N, K) satisfies $\Phi(v, w) = 0$, for all $v, w \in N$.

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- Congruences are shearing when $m > 1$.
- Examples:
  - Kerr–NUT–(A)dS Chen-Lü-Pope (2006), Frolov-Kubizňák (2007)
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- Kerr theorem in even dimensions Hughston-Mason (1988)
- Description of the Kerr surface in twistor space TC (2017)

Twisting NSCNGs in higher dimensions

Theorem (TC (2021))

Let $(\mathcal{M}, \mathbf{c})$ *be a Lorentzian conformal manifold of dimension* $2m + 2 > 4$ *with null line distribution* K *tangent to a twisting NSCNG* $\mathcal{K}$. *Denote by* $\underline{\mathcal{M}}$ *the local space of* $\mathcal{K}$ *and by* W *the Weyl tensor of* $\mathbf{c}$.

1. *If* $W(k, v, k, v) = 0$ *for any* $k \in K, v \in K^{\perp}$, *then the twist of* $\mathcal{K}$ *induces a nearly Robinson structure* (N, K), *and* $\underline{\mathcal{M}}$ *inherits a p.i. contact almost CR structure* $(\underline{H}, \underline{J})$.

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2. *If in addition $W(k,v,k,\cdot)=0$ for any $k \in K, v \in K^{\perp}$, any Einstein metric in $\mathbf{c}$ determines a contact form $\underline{\theta}^0$ for $\underline{H}$ such that $(\underline{H}, \underline{J}, \underline{\theta}^0)$ is an almost CR–Einstein structure, i.e.*

$$\underline{A}_{\alpha\beta}=0\,, \quad \underline{\nabla}^{\gamma}\underline{N}_{\gamma(\alpha\beta)}=0\,, \quad \left(\underline{P}_{\alpha\bar{\beta}}-\frac{1}{m+2}\underline{N}_{\alpha\gamma\delta}\underline{N}_{\bar{\beta}}{}^{\gamma\delta}\right)_{\circ}=0\,,$$

i.e. $\underline{\mathcal{M}}$ locally fibered over an almost Kähler–Einstein $2m$-fold.

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- 3-parameter family of Einstein metrics: (massive) Fefferman–Einstein and (massless) Taub–NUT-type metrics

Twisting NSCNGs in dimension four

Theorem (TC)

Let $(\mathcal{M}, g)$ be a Lorentzian 4-fold with a twisting NSCNG $\mathcal{K} \sim (N, K)$.

1. *Suppose $\mathrm{Ric}(v, v) = 0$ for all $v \in N$. Then g is determined by*
   - *a pseudo-Hermitian structure $(\underline{H}, \underline{J}, \underline{\theta}^0)$ on the leaf space $\underline{\mathcal{M}}$ of $\mathcal{K}$,*
   - *a solution $\underline{\lambda}_\alpha \in (\underline{H}^{(1,0)})^*$ to CR Einstein–Weyl-type equation on $\underline{\mathcal{M}}$*

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2. *Suppose that g satisfies the vacuum Einstein field equations with cosmological constant Λ and possibly pure radiation. Then g is uniquely determined by $\underline{\theta}^0$ and $\underline{\lambda}_\alpha$ as in 1. and a real density $\underline{c}$ satisfying*

   $$\underline{\nabla}_\alpha (\underline{b} - \mathrm{i}\underline{c}) = 3\mathrm{i} \underline{\lambda}_\alpha (\underline{b} - \mathrm{i}\underline{c}) \, ,$$

   where $\underline{b} := -\frac{8}{3}\Lambda + 8\underline{P} - 6\underline{\lambda}_\alpha \underline{\lambda}^\alpha + 6\mathrm{i}\left(\underline{\nabla}_\alpha \underline{\lambda}^\alpha - \underline{\nabla}^\alpha \underline{\lambda}_\alpha\right)$.

Some properties

- Agrees with Mason and Hill–Lewandowski–Nurowski–Tafel
- Formulation now purely in terms of pseudo-Hermitian quantities
- Form of the metric:

$$g = \sec^2\phi \left(4\underline{\theta}^0 \left(\mathrm{d}\phi + \left(1 + \frac{1}{2}\mathrm{e}^{-2\mathrm{i}\phi}\right) \underline{\lambda}_1 \underline{\theta}^1 + c.c. + \lambda_0 \underline{\theta}^0 \right) + 2\underline{\theta}^1 \underline{\overline{\theta}}^{\bar{1}} \right) .$$

- For vacuum possibly with pure radiation

$$\lambda_0 = \underline{a}_0 + \underline{a}_1 \cos^2\phi + \underline{a}_2 \cos\phi \sin\phi + \underline{b} \cos^4\phi + \underline{c} \cos^3\phi \sin\phi ,$$

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Lemma (TC)

A CR 3-fold admits a CR function if and only if it admits either a transverse CR symmetry or a solution $\underline{\lambda}_\alpha$ to

$$\underline{\nabla}_\alpha\underline{\lambda}_\beta - \mathrm{i}\underline{\lambda}_\alpha\underline{\lambda}_\beta - \underline{A}_{\alpha\beta} = 0\,. \qquad (\star)$$

Eq. $(\star)$ is CR-invariant: $\underline{\lambda}_\alpha \to \underline{\lambda}_\alpha + \mathrm{i}\underline{\Upsilon}_\alpha$ whenever $\underline{\theta} \to \mathrm{e}^{\underline{\varphi}}\underline{\theta}$.

Concluding remarks

- Reduction of the Einstein field equations in terms of pseudo-Hermitian data
- Better integration of the CR-conformal correspondence...
- More muscular tractorial approach...
- How does the CR data fit into the asymptotic properties of spacetime?
- Higher dimensions: many 'classes' of almost Robinson manifolds to be investigated...

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Thank you for your attention!

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