Lectures on almost Robinson geometry

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The aim of these notes is to give a relatively self-contained account of almost Robinson geometry based on the recent articles [FLTC20; FLTC21] by Fino, Leistner and myself. Referred to as a Lorentzian analogue of almost Hermitian geometry by Nurowski and Trautman [NT02], almost Robinson geometry provides a link between Lorentzian conformal geometry and Cauchy–Riemann geometry.

Acknowledgments These notes arose from a series of lectures that I gave at the Artic University of Norway, Tromsø, (UiT) in October 2021 as guest TFS professor. I am grateful to Dennis The for his invitation and hospitality. I also wish to thank him, Boris Kruglikov and Andrea Santi, for providing a pleasant environment and for their valuable feedback during the mini-course.

The preparation of these notes was carried out with the support of the fellowship "Conformal and CR methods in general relativity", 2020/37/K/ST1/02788, funded as part of the POLS NCN competition research projects financed from the Norwegian Financial Mechanism for 2014-2021.

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Norway grants

Motivation

Let us introduce the geometric structures at play in these lectures by means of the following example credited to Ivor Robinson in [NT02]. We choose coordinates $(u, v, z, \bar{z})$, where u and v are real, and z complex, so that the Minkowski metric takes the form

$$g = 2(\mathrm{d}u + \mathrm{i}z\mathrm{d}\bar{z} - \mathrm{i}\bar{z}\mathrm{d}z)\mathrm{d}v + 2(v^2 + 1)\mathrm{d}z\mathrm{d}\bar{z}\,.$$

Let us define the vector field and 1-forms

$$k = \frac{\partial}{\partial v}\,, \qquad \kappa = g(k, \cdot) = \mathrm{d}u + \mathrm{i}z\mathrm{d}\bar{z} - \mathrm{i}\bar{z}\mathrm{d}z\,, \qquad \mu = \mathrm{d}z\,, \qquad \overline{\mu} = \mathrm{d}\bar{z}\,.$$

Then k is *null*, i.e. $g(k, k) = 0$ and the *congruence* $\mathcal{K}$ of curves generator by k, i.e. the aggregate of the integral curves of k, satisfies the following properties:

1. the curves of $\mathcal{K}$ are *geodesics*, i.e.

$$\pounds_k \kappa = 0\,, \qquad \text{or equivalently,} \qquad \nabla_k k = 0\,;$$

2. $\mathcal{K}$ is *non-shearing*, i.e.

$$\pounds_k g = \epsilon g + 2\,\kappa \odot \alpha\,, \qquad \text{for some function } \epsilon \text{ and 1-form } \alpha.$$

3. $\mathcal{K}$ is *twisting*, i.e.

$$\kappa \wedge \mathrm{d}\kappa \neq 0\,.$$

We note that (2) implies (1), i.e. one can talk of a non-shearing congruence of null curves only if they are geodesics. In addition, these definitions are all *conformally invariant*, i.e. invariant under rescalings of g, and depend on the *distribution*

$$K := \mathrm{span}(k)\,,$$

not on k alone — with the understanding that 'geodesic' means $\nabla_k k \propto k$.

An involutive complex distribution The presence of *complex* coordinates z and $\bar{z}$ is not fortuitous: the distribution K can equivalently be expressed in terms of the rank-2 complex distribution

$$N := \mathrm{Ann}(\kappa, \mu)\,,$$

so that $N \cap \overline{N} := \mathbf{C} \otimes K$. In addition, one may readily check that N is *involutive*, i.e. $[N, N] \subset N$, a property that turns out to be *equivalent* to $\mathcal{K}$ being geodesic and non-shearing. The pair (N, K) will later be referred to as a *Robinson structure*.

A CR structure on the leaf space Denoting the leaf space of $\mathcal{K}$ by $\underline{\mathcal{M}}$, we note that

- by virtue of (1), the rank-2 quotient bundle $K^\perp/K$ descends to a rank-2 distribution $\underline{H}$ on $\underline{\mathcal{M}}$,
- by virtue of (2), the complex rank-1 quotient $N/\mathbf{C} \otimes K$ descends to a complex rank-1 distribution $\underline{H}^{(0,1)} \subset \mathbf{C} \otimes \underline{H}$ on $\underline{\mathcal{M}}$, which may be taken as the defining property of a so-called *Cauchy–Riemann (CR) structure*, and
- by virtue of (3), this CR structure is *contact*.

Why study such congruences? Not only do non-shearing congruences of null geodesics encode the geometry of special families of light rays, they are also one of the central notions of the Golden Age of mathematical relativity, and form the backbone of some of the most celebrated results in the field:

- The *Robinson theorem* [Rob61] in the context of the *vaccum Maxwell equations*;
- The *Goldberg–Sachs theorem* [GS62; GS09] in the context of the *vacuum Einstein field equations*;
- The *Kerr theorem*, one of the pillars of Penrose's *Twistor theory* [Pen67].

The road leading from theses congruences to CR geometry was however rather long and tortuous. The *spinorial approach* to general relativity, as promoted by Witten [Wit59] and especially Penrose [Pen60], certainly contributed to the shift from the 'real' world to the 'complex' one. These ideas were later developed in Penrose's *twistor theory* [Pen67]. But it was not until the early 1980ies that the terminology 'Cauchy–Riemann' made its appearance in connection with the geometry of light rays. Building on the legacy of Robinson, Trautman and Penrose, this CR aspect of general relativity was notably explored in the work of Tafel, Mason, Lewandowski, Nurowski, and Sparling, to name but a few, and has ramifications into analytical aspects of CR geometry [Taf85; LN90; HLN08] and the theory of spinors [HM88; BT89; KT92].

Almost Robinson geometry In higher dimensions, there are different ways of generalising these ideas to higher-dimensional Lorentzian geometry:

- In terms of a null (real) line distribution, which leads to the so-called *optical geometry* in the terminology of [FLTC20], or the *null alignment formalism* of [OPP13];
- In terms of a totally null complex distribution of maximal rank, now referred to *almost Robinson geometry* [NT02; FLTC21], and provides a Lorentzian counterpart of almost Hermitian geometry.

The recent articles [FLTC20; FLTC21] give a comprehensive account of optical geometry and almost Robinson geometry in the language of G-structures, and provides a conceptual starting point to the topic. These notes reflect this perspective.

Notation The notation adopted is fairly standard and follows closely the one given in the aforementioned articles. For instance, the complexification $\mathbf{C} \otimes_{\mathbf{R}} \mathbb{W}$ of a real vector space $\mathbb{W}$, say, will be denoted ${}^{\mathbf{C}}\mathbb{W}$.

1 Algebraic preliminaries

In this section, we introduce our main protagonists at the level of linear algebra.

1.1 Null structures

Let $\mathbb{W}$ be a $2m$-dimensional complex vector space equipped with a non-degenerate symmetric bilinear form g.

Definition 1.1. Let $\mathbb{N}$ be a vector subspace of $\mathbb{W}$, and denote by $\mathbb{N}^\perp$ its orthogonal complement with respect to g. We say that $\mathbb{N}$ is *totally null* if $\mathbb{N} \subset \mathbb{N}^\perp$, and *maximal totally null (MTN)* if $\mathbb{N} = \mathbb{N}^\perp$, i.e. $\dim \mathbb{N} = m$, in which case, we shall also refer to $\mathbb{N}$ as a *null structure*.

One can always choose a (not uniquely defined) complementary subspace $\mathbb{P}$ of $\mathbb{N}$ in $\mathbb{W}$ such that $\mathbb{W} = \mathbb{P} \oplus \mathbb{N}$. In particular, $\mathbb{P}$ is also MTN and isomorphic to $\mathbb{N}^*$

via g. We can then describe the isotropy group of $\mathbb{N} \cong (\mathbf{C}^m)^*$ as the Lie subgroup $\mathbf{GL}(m, \mathbf{C}) \ltimes \wedge^2(\mathbf{C}^m)^*$ of $\mathbf{SO}(2m, \mathbf{C})$, i.e. it consists of elements of the form

$$\begin{pmatrix} A & 0 \\ B & A \end{pmatrix}, \qquad \text{where } A \in \mathbf{GL}(m, \mathbf{C}),\, B \in \wedge^2(\mathbf{C}^m)^*.$$

Remark 1.2. Maximal totally null vector spaces are intrinsically connected to the notion of *pure spinors* introduced by [Car67], and later elaborated on notably by Budinich, Trautman and Kopczyńsky [BT88; BT89; KT92; Kop97].

1.2 Hermitian structures

Let $\mathbb{V}$ be an n-dimensional vector space.

Definition 1.3. A *complex structure* on $\mathbb{V}$ is an endomorphism J on $\mathbb{V}$ that satisfies $J \circ J = -\mathrm{Id}$, where Id is the identity on $\mathbb{V}$.

It easily follows that $n = 2m$. In fact, one may equivalent define a complex structure as a splitting of the complexification of $\mathbb{V}$

$$^{\mathbf{C}}\mathbb{V} = \mathbb{V}^{(1,0)} \oplus \mathbb{V}^{(0,1)},$$

where $\mathbb{V}^{(1,0)}$ and $\mathbb{V}^{(0,1)}$ are each of dimension m, and correspond to the $+\mathrm{i}$ and $-\mathrm{i}$ eigenspaces of J, i.e.

$$\mathbb{V}^{(1,0)} = \{v - \mathrm{i}J(v) : v \in \mathbb{V}\}, \qquad \mathbb{V}^{(0,1)} = \{v + \mathrm{i}J(v) : v \in \mathbb{V}\}.$$

Dually, we have a splitting

$$^{\mathbf{C}}\mathbb{V}^* = \wedge^{(1,0)}\mathbb{V}^* \oplus \wedge^{(0,1)}\mathbb{V}^*.$$

The isotropy group of $(\mathbb{V}, J)$ is isomorphic to $\mathbf{GL}(m, \mathbf{C}) \subset \mathbf{GL}(2m, \mathbf{R})$.

Definition 1.4. A *Hermitian structure* on $\mathbb{V}$ consists of a pair (g, J) where g is a positive-definite symmetric bilinear form and J a complex structure on $\mathbb{V}$ compatible with g in the sense that J is orthogonal, i.e. $g(J(v), w) = -g(v, J(w))$ for any $v, w \in \mathbb{V}$. We refer to $\omega := g \circ J$, i.e. $\omega(v, w) := g(J(v), w)$ for any $v, w \in \mathbb{V}$, as the *Hermitian* 2*-form* of $(\mathbb{V}, g, J)$.

Lemma 1.5. *The eigenspaces $\mathbb{V}^{(1,0)}$ and $\mathbb{V}^{(0,1)}$ of a Hermitian structure are maximal totally null subspaces of $(^{\mathbf{C}}\mathbb{V}, {}^{\mathbf{C}}g)$, where $^{\mathbf{C}}g$ is the complex linear extension of g. Conversely, any MTN on $(\mathbb{V}, g)$ defines a compatible complex structure J.*

Proof. To show the implication, it suffices to take arbitrary elements $v, w \in \mathbb{V}$ so that $v - \mathrm{i}J(v), w - \mathrm{i}J(w) \in \mathbb{V}^{(1,0)}$. Using the linearity of g and the defining properties of J, we then find $g(v - \mathrm{i}J(v), w - \mathrm{i}J(w))$=0, i.e. $\mathbb{V}^{(1,0)}$ is maximal totally null.

For the converse implication, we note that we can always write $^{\mathbf{C}}\mathbb{V} = \overline{\mathbb{N}} \oplus \mathbb{N}$, with $\overline{\mathbb{N}} = \mathbb{N}^*$, which is simply the splitting defining a complex structure compatible with g. □

The isotropy group of $(\mathbb{V}, g, J)$ is well-known to be the unitary group $\mathbf{U}(m) := \mathbf{SO}(2m) \cap \mathbf{GL}(m, \mathbf{C})$. From the perspective of MTNs, this is also the intersection of $\mathbf{SO}(2m, \mathbf{R})$ and the isotropy groups of $\mathbb{V}^{(1,0)}$ and $\mathbb{V}^{(0,1)}$ in $\mathbf{SO}(2m, \mathbf{C})$.

See references [Che95; Sal89] for more information on hermitian geometry.

1.3 Optical structures

Let $\mathbb{V}$ be an $(n+2)$-dimensional real vector space equipped with a symmetric bilinear form g of signature $(n+1,1)$, i.e. $(+,\ldots,+,-)$. A non-zero vector v in $\mathbb{V}$ is said to be

- *spacelike* if $g(v,v) > 0$,
- *timelike* if $g(v,v) < 0$,
- *lightlike* or *null* if $g(v,v) = 0$.

Definition 1.6. An optical structure on $(\mathbb{V}, g)$ is a null one-dimensional vector subspace $\mathbb{K}$ of $\mathbb{V}$.

Remark 1.7.

1. Clearly, $\mathbb{K} \subset \mathbb{K}^\perp$. The n-dimensional quotient $\mathbb{H}_\mathbb{K} := \mathbb{K}^\perp/\mathbb{K}$ is referred to as the *screen space* of $\mathbb{K}$. It is equipped with a positive-definite symmetric bilinear form h induced from g, i.e.

$$h(v+\mathbb{K}, w+\mathbb{K}) = g(v,w)\,, \qquad \text{for any } v,w \in \mathbb{K}^\perp.$$

   It also inherits an orientation from $\mathbb{V}$.

2. It is often convenient to split $\mathbb{V}$ as

$$\mathbb{V} = \mathbb{L} \oplus \mathbb{H}_{\mathbb{K},\mathbb{L}} \oplus \mathbb{K}\,, \qquad \mathbb{H}_{\mathbb{K},\mathbb{L}} := \mathbb{K}^\perp \cap \mathbb{L}^\perp\,. \tag{1.1}$$

   for some choice of null one-dimensional vector subspace of $\mathbb{V}$ dual to $\mathbb{K}$, i.e. $g(k,\ell) \neq 0$ for any non-zero vectors $k \in \mathbb{K}$ and $\ell \in \mathbb{L}$.

3. The isotropy group of $(\mathbb{V}, g, \mathbb{K})$ together with the orientation and time orientation is $\mathbf{Sim}(n) := \mathbf{R}_{>0} \cdot \mathbf{SO}(n) \ltimes (\mathbf{R}^n)^* \subset \mathbf{SO}^0(n+1,1)$. Here, $\mathbf{SO}^0(2m+1,1)$ is the identity component of the special orthogonal group. Any element of $\mathbf{Sim}(n)$ takes the form

$$\begin{pmatrix} e^{-\varphi} & 0 & 0 \\ -e^{\varphi}\psi\phi^\perp & \psi & 0 \\ -\frac{e^{\varphi}}{2}\phi\phi^\perp & \phi & e^{\varphi} \end{pmatrix}\,, \qquad \text{where } \varphi \in \mathbf{R},\, \phi \in (\mathbf{R}^n)^*,\, \psi \in \mathbf{SO}(n).$$

4. Clearly, $\mathbb{K}$, $\mathbb{K}^\perp$ and $\mathbb{H}_\mathbb{K}$ are $\mathbf{Sim}(n)$-modules, but $\mathbb{L}$ and $\mathbb{H}_{\mathbb{K},\mathbb{L}}$ are $\mathbf{CO}(n)$-modules, where $\mathbf{CO}(n) := \mathbf{R}_{>0} \cdot \mathbf{SO}(n)$ is the reductive part of $\mathbf{Sim}(n)$. Nevertheless, as vector spaces, $\mathbb{H}_{\mathbb{K},\mathbb{L}} \cong \mathbb{H}_\mathbb{K}$.

1.4 Robinson structures

Let $\mathbb{V}$ be a $(2m+2)$-dimensional vector space equipped with a symmetric bilinear form g of signature $(2m+1,1)$.

Definition 1.8. A *Robinson structure* on $(\mathbb{V}, g)$ consists of a pair $(\mathbb{N}, \mathbb{K})$ where $\mathbb{N}$ is an MTN subspace of $\mathbb{V}$ and $\mathbb{K} := \mathbb{V} \cap \mathbb{N}$ is a null one-dimensional vector subspace, i.e. ${}^{\mathbf{C}}\mathbb{K} = \mathbb{N} \cap \overline{\mathbb{N}}$.

An alternative definition is given below:

Definition 1.9. A *Robinson structure* on $(\mathbb{V}, g)$ consists of a triple $(\mathbb{K}, \mathbb{H}_{\mathbb{K}}, J)$ where $\mathbb{K}$ is an optical structure with screen space $\mathbb{H}_{\mathbb{K}} := \mathbb{K}^{\perp}/\mathbb{K}$ and J a complex structure on $\mathbb{H}_{\mathbb{K}}$ compatible with the screen space bilinear form h.

Lemma 1.10. *Definitions 1.8 and 1.9 are equivalent.*

Proof. Assume Definition 1.8. Then, clearly $\mathbb{N}$ determines an optical structure $\mathbb{K}$. But then, we have a splitting

$$^{\mathbf{C}}\mathbb{H}_{\mathbb{K}} = \left(\overline{\mathbb{N}}/^{\mathbf{C}}\mathbb{K}\right) \oplus \left(\mathbb{N}/^{\mathbf{C}}\mathbb{K}\right) ,$$

where $\overline{\mathbb{N}}/^{\mathbf{C}}\mathbb{K}$ and $\mathbb{N}/^{\mathbf{C}}\mathbb{K}$ are MTNs of $^{\mathbf{C}}\mathbb{H}_{\mathbb{K}}$ with respect to h, i.e. $\mathbb{H}_{\mathbb{K}}$ admits a complex structure comptaible with h, and we recover Definition 1.9.

Conversely, let us start with Definition 1.9. Then a complex structure J on $\mathbb{H}_{\mathbb{K}}$ compatible with h is equivalent to a splitting of $^{\mathbf{C}}\mathbb{H}_{\mathbb{K}}$ into two MTNs $\mathbb{H}_{\mathbb{K}}^{(1,0)}$ and $\mathbb{H}_{\mathbb{K}}^{(0,1)}$. Choose a splitting

$$\mathbb{V} = \mathbb{L} \oplus \mathbb{H}_{\mathbb{K},\mathbb{L}} \oplus \mathbb{K} , \qquad \qquad \mathbb{H}_{\mathbb{K},\mathbb{L}} := \mathbb{K}^{\perp} \cap \mathbb{L}^{\perp} .$$

for some $\mathbb{L}$. In particular, $^{\mathbf{C}}\mathbb{H}_{\mathbb{K},\mathbb{L}} = \mathbb{H}_{\mathbb{K},\mathbb{L}}^{(1,0)} \oplus \mathbb{H}_{\mathbb{K},\mathbb{L}}^{(0,1)}$ where $\mathbb{H}_{\mathbb{K},\mathbb{L}}^{(0,1)}$ is linearly isomorphic to $\mathbb{H}_{\mathbb{K}}^{(0,1)}$. Then $\mathbb{N} := \mathbb{H}_{\mathbb{K},\mathbb{L}}^{(0,1)} + \mathbb{K}$ is a MTN of $(\mathbb{V}, g)$. The argument being analogous for $\overline{\mathbb{N}}$, one eventually recovers Definition 1.8. □

Under the assumption that the orientation and time orientation are also preserved, the isotropy group of $(\mathbb{V}, g, \mathbb{N}, \mathbb{K})$ is $\mathbf{R}_{>0} \cdot \mathbf{U}(m) \ltimes (\mathbf{R}^{2m})^* \subset \mathbf{Sim}(2m) \subset \mathbf{SO}^0(2m+1,1)$ as can be seen from Definition 1.9. From the perspective of Definition 1.8 One can also describe this group as the intersection of $\mathbf{SO}^0(2m+1,1)$ with the respective isotropy groups of $\mathbb{N}$ and $\overline{\mathbb{N}}$ in $\mathbf{SO}(2m+2, \mathbf{C})$.

Lemma 1.11. *In dimension four, an optical structure is equivalent to a Robinson structure.*

Proof. This follows from the fact that $\mathbb{H}_{\mathbb{K}}$ is two-dimensional, so the positive-definite bilinear form h is equivalent to a complex structure J. Indeed, if ε is the volume form on $\mathbb{H}_{\mathbb{K}}$ corresponding to h, we have that $J = h^{-1} \circ \varepsilon$. Another way to see this is in terms of the isotropy groups of the respective structures, which, in dimension four, coincides since $\mathbf{SO}(2) \cong \mathbf{U}(1)$. □

2 Geometric structures on manifolds

2.1 Intrinsic torsion

Let $(\mathcal{M}, g)$ be a pseudo-Riemannian manifold of signature (p, q), and $\mathcal{F}^G(\mathcal{M})$ the bundle of frames with structure group $G \cong \mathbf{SO}^0(p, q)$, the identity component of the special orthogonal group. Any geometric structure s on $(\mathcal{M}, g)$ compatible with g is equivalent to a reduction of G to $H \subset G$, the isotropy group of s at any point — here s

could be, for instance, a tensor field or a vector distribution. To derive the differential geometrical properties of s, one may always have recourse to a connection compatible with it, on the reduced frame bundle $\mathcal{F}^H(\mathcal{M})$ or its associated bundles $\mathcal{F}^H(\mathcal{M}) \times_H \mathbb{A}$ where $\mathbb{A}$ is an H-module. Invariants associated to such a connection include its curvature and torsion. For instance, the curvature of the Levi-Civita connection, the unique torsion-free connection preserving g, is the obstruction to $(\mathcal{M}, g)$ being locally isometric to pseudo-Euclidean space.

For a proper subgroup H of G, the existence of a torsion-free connection compatible with s is in general not guaranteed. The obstruction to the existence of such a connection is referred to as the *intrinsic torsion* of s — see e.g. [Sal89]. Viewing a connection on $\mathcal{F}^H(\mathcal{M})$ as a H-equivariant horizontal distribution, we may identify the intrinsic torsion $\mathring{T}$ of s as a H-equivariant function with values in $\mathbb{V}^* \otimes \mathfrak{g}/\mathfrak{h}$, where $\mathfrak{g} \cong \mathfrak{so}(p,q)$ and $\mathfrak{h}$ are the respective Lie algebras of G and H, and $\mathbb{V} \cong \mathbf{R}^{p,q}$ denotes the tangent space at a point. In particular, one can describe $\mathring{T}$ in terms of the H-submodules of $\mathbb{V}^* \otimes \mathfrak{g}/\mathfrak{h}$.

One can be a little more concrete in the description of the intrinsic torsion. Since the Levi-Civita connection ∇ is the *unique* torsion-free connection taking values in $\mathfrak{g}$, any torsion-free connection taking values in $\mathfrak{h} \subset \mathfrak{g}$ must coincide with ∇. Thus, taking s to be a tensor field, the intrinsic torsion of s can be identified with $\nabla s \pmod{\alpha \otimes s}$. Thus, the intrinsic torsion of s vanishes if and only if the holonomy of ∇ is contained in $\mathfrak{h}$.

We shall now apply these general remarks to the geometric structures described in Section 1 with the exception of null structures — these are considered in [TC16; TC17a]. They are nevertheless relevant to almost Robinson geometry.

2.2 Almost Hermitian geometry

Recall that an *almost Hermitian manifold* is defined to be a triple $(\mathcal{M}, g, J)$ where $(\mathcal{M}, g)$ is a Riemannian manifold of dimension $2m$ and J an almost complex structure compatible with g. With reference to the previous section, the geometric structure s at play here can be taken as either J, or its associated almost Hermitian 2-form $\omega := g \circ J$, or the eigenbundles $T^{(1,0)}$ and $T^{(0,1)}$ of J. The structure group is $H \cong \mathbf{U}(m)$ and we have a canonical splitting of the Lie algebra $\mathfrak{g} \cong \mathfrak{so}(2m) = \mathfrak{u}(m) \oplus \mathfrak{u}(m)^{\perp}$. The H-module $\mathbb{G}$ of intrinsic torsions is then isomorphic to $(\mathbf{R}^{2m})^* \otimes \mathfrak{u}(m)^{\perp}$. We shall denote the real spans of the complex $\mathbf{U}(m)$-modules by enclosing them around $[\![\cdot]\!]$ or $[\cdot]$ — see [Sal89] for this notation. Thus, writing $\mathbb{V} \cong \mathbf{R}^{2m}$ for the tangent space at a point,

$$\mathfrak{u}(m) \cong [\wedge^{(1,1)}\mathbb{V}^*] , \qquad \mathfrak{u}(m)^{\perp} \cong [\![\wedge^{(2,0)}\mathbb{V}^*]\!] .$$

The *Gray–Hervella classification of almost Hermitian manifolds* [GH80] is essentially based on the decomposition of $\mathbb{G}$ into four irreducible $\mathbf{U}(m)$-submodules: $\mathbb{G} = \mathbb{G}_0 \oplus \mathbb{G}_1 \oplus \mathbb{G}_2 \oplus \mathbb{G}_3$ where

$$\begin{aligned} \mathbb{G}_0 &\cong \mathbb{V}^* , & \mathbb{G}_1 &\cong [\![\wedge^{(3,0)}\mathbb{V}^*]\!] , \\ \mathbb{G}_2 &\cong [\![\boxplus\mathbb{V}^*]\!] , & \mathbb{G}_3 &\cong [\![\wedge^{(0,1)}\mathbb{V}^* \otimes_\circ \wedge^{(2,0)}\mathbb{V}^*]\!] , \end{aligned}$$

where, by definition, $\yng(2,1)\mathbb{V}^*$ is the complex vector subspace of $\wedge^{(1,0)}\mathbb{V}^* \otimes \wedge^{(2,0)}\mathbb{V}^*$ whose elements α satisfy

$$\alpha(u,v,w) + \alpha(v,w,u) + \alpha(w,u,v) = 0\,, \qquad \text{for any } (1,0)\text{-vectors } u,v,w.$$

Using the Levi-Civita connection ∇, the intrinsic torsion of $(\mathcal{M},g,J)$ can be expressed in two different ways:

- as the tensor field $\nabla\omega$, which clearly takes values in $\mathbb{G} = \mathbb{V}^* \otimes \mathfrak{u}(m)^\perp$ at every point,
- by choosing an arbitrary unitary frame (e_α), and viewing

  $$\Gamma_{\alpha\beta\gamma} := g(\nabla_{e_\alpha} e_\beta, e_\gamma)\,, \qquad\qquad \Gamma_{\bar{\alpha}\beta\gamma} := g(\nabla_{\bar{e}_{\bar{\alpha}}} e_\beta, e_\gamma)\,,$$

  as the (complex) components of a tensor with values in $\mathbb{G}$ at any point. The indices on Γ can also be understood abstractly.

Example 2.1. 1. For a *Hermitian manifold*, $T^{(1,0)}$ and $T^{(0,1)}$ are involutive, i.e. the Nijenhuis tensor of J vanishes, and $\nabla\omega \in \mathbb{G}_0 \oplus \mathbb{G}_3$ at any point.

2. For an almost Kähler manifold, $\mathrm{d}\omega = 0$ i.e. $\nabla\omega \in \mathbb{G}_2$ at any point.

3. For a Kähler manifold, the intrinsic torsion vanishes, i.e. the holonomy of ∇ is contained in $\mathbf{U}(m)$, i.e. $\nabla\omega = 0$.

2.3 Optical geometry

Definition 2.2. An *optical geometry* consists of a triple $(\mathcal{M},g,K)$ where $(\mathcal{M},g)$ is an (oriented and time-oriented) Lorentzian manifold of dimension $n+2$ and K a real null line distribution. We shall refer to K as an *optical structure*.

Note that the screen bundle $H_K = K^\perp/K$ of K inherits a bundle metric h from g and an orientation.

2.3.1 Intrinsic torsion

An optical structure, viewed as a G-structure, has structure group $P \cong \mathbf{Sim}(n)$. Its intrinsic torsion is a section of the vector bundle associated with the P-module $\mathbb{G} = \mathbb{V}^* \otimes \mathfrak{g}/\mathfrak{p}$ where $\mathbb{V} \cong \mathbf{R}^{n+1,1}$. Unlike the Hermitian case, there is no canonical subspace complementary to $\mathfrak{p}$ in $\mathfrak{g}$, which means that in general $\mathbb{G}$ does not split into a direct sum of P-submodules. Instead, one has filtrations of P-submodules whose associated graded modules splits into irreducibles. In the following, the reductive part $\mathbf{R}_{>0} \cdot \mathbf{SO}(n)$ of P will be denoted by P_0.

Before we proceed, for each $w \in \mathbf{R}$, we introduce the 1-dimensional representation $\mathbf{R}(w)$ of P on $\mathbf{R}$, given by

$$\mathbf{Sim}(n) \ni (\mathrm{e}^\varphi, \psi, \phi) \cdot r := \mathrm{e}^{w\varphi} r\,,$$

We can then identify $\mathbb{K}$ with $\mathbf{R}(-1)$ and $\mathbb{L}$ with $\mathbf{R}(1)$. We shall refer to any section of the line bundle $\mathcal{E}(w)$ associated to $\mathbf{R}(w)$ as a *density of boost weight w*. Concretely, a density σ of boost weight w transforms as

$$\sigma \mapsto \mathrm{e}^{w\varphi}\sigma\,,$$

under a *boost transformation* of an adapted frame (ℓ, e_i, k), i.e.

$$k \mapsto \mathrm{e}^{\varphi}k\,, \qquad e_i \mapsto e_i\,, \qquad \ell \mapsto \mathrm{e}^{-\varphi}\ell\,, \tag{2.1}$$

for some smooth function φ.

To make the analysis more tractable, we introduce a splitting so that using

$$\mathfrak{g} \cong \wedge^2\mathbb{V}^* = \underbrace{\left(\mathbb{H}^*_{\mathbb{K},\mathbb{L}} \otimes \mathbf{R}(1)\right)}_{\cong \mathfrak{g}/\mathfrak{p}} \oplus \underbrace{\left(\wedge^2\mathbb{H}^*_{\mathbb{K},\mathbb{L}} \oplus \mathbf{R}\right) \oplus \left(\mathbb{H}^*_{\mathbb{K},\mathbb{L}} \otimes \mathbf{R}(-1)\right)}_{\mathfrak{p}}\,,$$

we find

$$\mathbb{G} \cong \left(\mathbf{R}(1) \oplus \mathbb{H}^*_{\mathbb{K},\mathbb{L}} \oplus \mathbf{R}(-1)\right) \otimes \left(\mathbb{H}^*_{\mathbb{K},\mathbb{L}} \otimes \mathbf{R}(1)\right)$$

which, upon distributing and decomposing into irreducible P_0-modules, gives

$$\mathbb{G} \cong \left\{ \begin{array}{ccccc} & & \mathbb{H}^*_{\mathbb{K},\mathbb{L}} \otimes \mathbf{R}(2) & & \\ & & \oplus & & \\ \mathbf{R}(1) & \oplus & \wedge^2\mathbb{H}^*_{\mathbb{K},\mathbb{L}} \otimes \mathbf{R}(1) & \oplus & \odot^2_\circ\mathbb{H}^*_{\mathbb{K},\mathbb{L}} \otimes \mathbf{R}(1) \\ & & \oplus & & \\ & & \mathbb{H}^*_{\mathbb{K},\mathbb{L}} & & \end{array} \right. \tag{2.2}$$

To make contact with differential geometry, we choose a frame (ℓ, e_i, k) adapted to the optical structure. We set $\kappa = g(k,\cdot)$, which is a section of $\mathrm{Ann}(K^\perp)$, and consider $\nabla\kappa$. Define

$$\begin{aligned} \nabla\kappa(k, e_j) &=: \gamma_j\,, \\ \nabla\kappa(e_i, e_j) &=: \frac{\epsilon}{n}h_{ij} + \tau_{ij} + \sigma_{ij}\,, \\ \nabla\kappa(\ell, e_i) &=: E_i\,, \end{aligned} \tag{2.3}$$

where $h_{ij} =: h(e_i + K, e_j + K)$ are the screen bundle metric components, $\tau_{ij} = \tau_{[ij]}$ and $\sigma_{ij} = \sigma_{(ij)_\circ}$.

Note that under a boost transformation (2.1), these quantities transform as

$$\epsilon \mapsto \mathrm{e}^{\varphi}\epsilon\,, \qquad \begin{aligned} \gamma_j &\mapsto \mathrm{e}^{2\varphi}\gamma_j\,, \\ \tau_{ij} &\mapsto \mathrm{e}^{\varphi}\tau_{ij}\,, \\ E_i &\mapsto E_i\,, \end{aligned} \qquad \sigma_{ij} \mapsto \mathrm{e}^{\varphi}\sigma_{ij}\,,$$

which shows that their boost weights do indeed agree with the weighted vector bundles of (2.2). Hence:

Lemma 2.3. *The quantities defined in* (2.3) *are the components of sections of the bundles arising from the irreducible P_0-submodules of $\mathbb{G}$ given in* (2.2).

Clearly, any sum of these P_0-submodules is also a P_0-submodule. We are however interested in bundles arising from P-modules. To this end, it is enough to determine how γ_i, ϵ, τ_{ij}, σ_{ij} and E_i transform under a *null rotation along* k, i.e.

$$k \mapsto k\,, \qquad e_i \mapsto e_i + \phi_i k\,, \qquad \ell \mapsto \ell - \phi^i e_i - \frac{1}{2}\phi^i \phi_i k\,,$$

for some $\phi^i \in \mathbf{R}^n$. This is essentially the action of the nilpotent part P_+ of P on the frame. We find

$$\begin{aligned} & & \gamma_j &\mapsto \gamma_j\,, & & \\ \epsilon &\mapsto \epsilon + \gamma_i \phi^i\,, & \tau_{ij} &\mapsto \tau_{ij} + \phi_{[i}\gamma_{j]}\,, & \sigma_{ij} &\mapsto \sigma_{ij} + \phi_{(i}\gamma_{j)_\circ}\,, \\ & & E_j &\mapsto E_j - \sigma_{ij}\phi^j + \tau_{ij}\phi^j - \frac{\epsilon}{n}\phi_j - \frac{1}{2}\phi^i\phi_i\gamma_j\,. & & \end{aligned}$$

From these transformations, we immediately single out those P_0-submodules subspaces of $\mathbb{G}$ that are also P-submodules. For instance, a P_0-submodule whose elements are characterised by $\gamma_i = \tau_{ij} = 0$ gives rise to a P-submodule. More generally, we have

Proposition 2.4. *The following P_0-modules are also P-modules:*

- $\mathbb{V}^* \otimes \mathfrak{g}$,
- $\mathbf{R}(1) \oplus \left(\wedge^2 \mathbb{H}^*_{\mathbb{K},\mathbb{L}} \otimes \mathbf{R}(1)\right) \oplus \left(\odot^2_\circ \mathbb{H}^*_{\mathbb{K},\mathbb{L}} \otimes \mathbf{R}(1)\right) \oplus \mathbb{H}^*_{\mathbb{K},\mathbb{L}} \oplus (\mathbb{V}^* \otimes \mathfrak{p})$,
- $\left(\wedge^2 \mathbb{H}^*_{\mathbb{K},\mathbb{L}} \otimes \mathbf{R}(1)\right) \oplus \left(\odot^2_\circ \mathbb{H}^*_{\mathbb{K},\mathbb{L}} \otimes \mathbf{R}(1)\right) \oplus \mathbb{H}^*_{\mathbb{K},\mathbb{L}} \oplus (\mathbb{V}^* \otimes \mathfrak{p})$,
- $\mathbf{R}(1) \oplus \left(\odot^2_\circ \mathbb{H}^*_{\mathbb{K},\mathbb{L}} \otimes \mathbf{R}(1)\right) \oplus \mathbb{H}^*_{\mathbb{K},\mathbb{L}} \oplus (\mathbb{V}^* \otimes \mathfrak{p})$,
- $\mathbf{R}(1) \oplus \left(\wedge^2 \mathbb{H}^*_{\mathbb{K},\mathbb{L}} \otimes \mathbf{R}(1)\right) \oplus \mathbb{H}^*_{\mathbb{K},\mathbb{L}} \oplus (\mathbb{V}^* \otimes \mathfrak{p})$,
- $\mathbb{V}^* \otimes \mathfrak{p}$.

In addition, any P-submodule of $\mathbb{G}$ arises as an intersection of a subset of these modulo $\mathbb{V}^* \otimes \mathfrak{p}$. In particular, there are precisely 10 distinct P-submodules of $\mathbb{G}$.

2.3.2 Congruences of null curves

Let us now give a geometric interpretation of the intrinsic torsion of an optical structure. For this purpose, we shall pay attention to the *congruence of null curves* tangent to K, that is, the aggregate $\mathcal{K}$ of the integral curves of any non-vanishing section of K.

We start with the weakest possible condition on the intrinsic torsion of K.

Lemma 2.5. *The following statements are equivalent:*

1. *The congruence $\mathcal{K}$ is* geodesic, *i.e. every curve in $\mathcal{K}$ is a geodesic.*

2. $\gamma_i = 0$.

3. $K^\perp$ *is preserved along the curves of* $\mathcal{K}$*, i.e. for any generator* k *of* $\mathcal{K}$*,* $\pounds_k \kappa(v) = 0$*, for any* $v \in \Gamma(K^\perp)$.

Suppose now that $(\mathcal{M}, g, K)$ is an optical geometry with congruence of null geodesics $\mathcal{K}$. From the discussion of Section 2.3.1, we can distinguish three P-invariant pieces of the intrinsic torsion, given in our earlier notation by ϵ, τ_{ij} and σ_{ij}. The invariants are well-known in general relativity, and can be expressed as follows. Fix a choice of generator k of $\mathcal{K}$ and set $\kappa = g(k, \cdot)$. Then

$$\begin{aligned} \epsilon\kappa &= \kappa \mathrm{div} k - \nabla_k \kappa \,, & \\ \tau(v + K, w + K) &= \mathrm{d}\kappa(v, w) \,, & \text{for all } v, w \in \Gamma(K^\perp), \\ \sigma(v + K, w + K) &= \frac{1}{2} \pounds_k g(v, w) - \frac{\epsilon}{2n} g(v, w) \,, & \text{for all } v, w \in \Gamma(K^\perp). \end{aligned}$$

We refer to ϵ, τ and σ as the *expansion*, *twist* and *shear* of k respectively, and by extension of $\mathcal{K}$, where we view them as sections of $\mathcal{E}(1)$, $\wedge^2 H_K^* \otimes \mathcal{E}(1)$ and $\odot_\circ^2 H_K^* \otimes \mathcal{E}(1)$ respectively.

2.3.3 Geometric structures on the leaf space

Probably the most interesting features of a null congruence $\mathcal{K}$ are the geometric structures it gives rise to on its *local leaf space*, which we shall denote by $\underline{\mathcal{M}}$.

It is clear from Lemma 2.5 that if $\mathcal{K}$ is geodesic, the screen bundle H_K descends to a rank-n distribution $\underline{H}$ on $\underline{\mathcal{M}}$. In fact, since the exterior derivative is functorial, $\underline{H}$ is also equipped with a skew-symmetric bilinear form induced from the twist of $\mathcal{K}$. Note that the rank of the twist, being defined to be the largest integer r such that

$$\kappa \wedge (\mathrm{d}\kappa)^r \neq 0 \,,$$

provides another invariant of the congruence.

There are two extreme cases of interest in that respect (and these are the only cases in dimension four, i.e. $n = 2$):

- $\mathcal{K}$ is non-twisting, i.e. the twist has rank zero, $K^\perp$ is integrable, and so is $\underline{H}$. We then have a local foliation of $\underline{\mathcal{M}}$ by null hypersurfaces tangent to $K^\perp$, and a local foliation of $\mathcal{M}$ by hypersurfaces tangent to $\underline{H}$.

- $\mathcal{K}$ is maximally twisting, i.e. the twist has maximal rank. When $n = 2m$, this means that the distribution $\underline{H}$ on $\underline{\mathcal{M}}$ is *contact*.

How about the *shear* of $\mathcal{K}$? From its defining property, we obtain the following result:

Lemma 2.6. *For a non-shearing congruence of null geodesics, the conformal class* $[h]$ *of the screen bundle metric* h *is preserved along* $\mathcal{K}$*. In particular,* $[h]$ *descends to a conformal class* $[\underline{h}]$ *of bundle metrics on* $(\underline{\mathcal{M}}, \underline{H})$*. If in addition,* $\mathcal{K}$ *is non-expanding, then* h *descends to a bundle metric* $\underline{h}$*.*

Remark 2.7. We shall often use the acronym NSCNG for 'non-shearing congruence of null geodesics'.

Example 2.8. A null conformal Killing field k, i.e. $\pounds_k g \propto g$, generates a non-shearing congruence of null geodesics. It is Killing, i.e. $\pounds_k g = 0$, if the congruence is non-expanding.

As a corollary of Lemma 2.6, we have

Corollary 2.9. *Assume $n = 2m$. Let $\mathcal{K}$ be a maximally twisting non-shearing congruence of null geodesics. Then the leaf space of $\mathcal{K}$ inherits a positive-definite sub-conformal contact structure, and a sub-Riemannian contact structure when $\mathcal{K}$ is non-expanding.*

Spacetimes admitting a non-twisting non-shearing congruence of null geodesics $\mathcal{K}$ have been extensively studied in arbitrary dimensions, and come in two flavours:

- *Robinson–Trautman spacetimes* if $\mathcal{K}$ is *expanding*: among these are Schwarzschild metrics, which are well-known to solve the vacuum Einstein field equations.
- *Kundt spacetimes* if $\mathcal{K}$ is *non-expanding*, which include pp-waves, Brinkman waves, and metrics for which the holonomy of ∇ is contained in $\mathbf{Sim}(n)$, i.e. K is parallel — in other words, optical geometries with vanishing intrinsic torsion.

2.3.4 Conformal invariance

The property of null geodesics is well-known to be *conformally invariant*, i.e. invariant under conformal changes of metrics $g \mapsto \mathrm{e}^{2f} g$ for some smooth function f. Similarly, given a congruence of null geodesics, it is immediate from their definitions that *the twist and shear of $\mathcal{K}$ are conformally invariant*.

On the other hand, the expansion of $\mathcal{K}$ is not a conformal invariant. Nevertheless, this lack of invariance has a certain advantage: it singles out a subclass of metrics $[g]_{n.e.}$ conformally related to g for which $\mathcal{K}$ is non-expanding. Any two metrics in $[g]_{n.e.}$ differ by a factor constant along K. In particular, Kundt spacetimes and Robinson–Trautman spacetimes are conformally related.

2.3.5 Lifts

Following Robinson and Trautman, rather than considering the leaf space of a null geodesics congruence, one may start with a $(n+1)$-dimensional smooth manifold $\underline{\mathcal{M}}$ equipped with a rank-n distribution $\underline{H}$, and extends $\underline{\mathcal{M}}$ to the trivial line bundle $\mathcal{M} := \underline{\mathcal{M}} \times \mathbf{R} \xrightarrow{\varpi} \underline{\mathcal{M}}$. Choose a coframe $(\underline{\theta}^0, \underline{\theta}^i)$ on $\underline{\mathcal{M}}$ where $\underline{\theta}^0$ annihilates $\underline{H}$. Define

- K to be the line distribution on $\mathcal{M}$ tangent to the $\mathbf{R}$-factor, and
- g to be the metric given by

$$g = \mathrm{e}^{2f} \left(2\kappa\lambda + h_{ij}\theta^i\theta^j \right) \tag{2.4}$$

  where

– f is a smooth function,
– $\kappa := 2\varpi^* \underline{\theta}^0$ and $\theta^i := \varpi^* \underline{\theta}^i$,
– h_{ij} is a non-degenerate symmetric matrix depending smoothly on $\mathcal{M}$,
– λ a non-vanishing 1-form on $\mathcal{M}$ such that $\lambda(k) \neq 0$ for any $k \in \Gamma(K)$.

Then, it is straightforward to check $(\mathcal{M}, g, K)$ is an optical geometry with congruence of null geodesics. Conversely, the metric of any optical geometry $(\mathcal{M}, g, K)$ with congruence of null geodesics $\mathcal{K}$ locally takes the form (2.4).

Remark 2.10. In the particular case where $h_{ij} := \varpi^* \underline{h}_{ij}$ for some bundle metric $\underline{h}_{ij}$ on $\underline{H}$, then $\mathcal{K}$ is also non-shearing, and non-expanding with respect to the metric $e^{-2f} g$.

Remark 2.11. Note that if $\widehat{g}$ is a metric related to g via

$$\widehat{g} = e^{2f} g + 2\kappa\alpha\,, \tag{2.5}$$

for some 1-form α and function f and $\kappa \in \Gamma(\mathrm{Ann}(K^\perp))$, we obtain another optical geometry $(\mathcal{M}, \widehat{g}, K)$. Under the change (2.5), the congruence $\mathcal{K}$ remains geodesic, and its twist and shear remain unchanged. The equivalence class of metrics related by (2.5) is discussed in e.g. [RT85] and is referred to as *generalised optical geometry* in [FLTC20].

2.4 Almost Robinson geometry

Definition 2.12. An *almost Robinson geometry* consists of a quadruple $(\mathcal{M}, g, N, K)$ where $(\mathcal{M}, g)$ is an (oriented and time-oriented) Lorentzian manifold of dimension $2m+2$, N is a totally null complex $(m+1)$-plane distribution and $K = T\mathcal{M} \cap N$ is a real null line distribution, i.e. ${}^{\mathbf{C}}K = N \cap \overline{N}$. The pair (N, K) is referred to as an *almost Robinson structure*.

Equivalent, $(\mathcal{M}, g, N, K)$ can be viewed as

- an optical geometry $(\mathcal{M}, g, K)$ of dimension $2m+2$ whose screen bundle $H_K := K^\perp / K$ is equipped with a bundle complex structure J compatible with the induced screen bundle metric h;
- a G-structure on a smooth manifold $\mathcal{M}$ of dimension $2m+2$ where the structure group is $Q \cong \mathbf{R}_{>0} \cdot \mathbf{U}(m) \ltimes (\mathbf{R}^{2m})^* \subset \mathbf{Sim}(2m) \subset \mathbf{SO}^0(2m+1,1) \cong P$. As before, the reductive part of Q will be denoted by Q_0.

Three of the invariant tensor fields associated to an almost Robinson structure (N, K) are

- any 1-form κ (defined up to scale) which annihilates $K^\perp$;
- any 3-form $\rho := 3\,\kappa \wedge \omega$, where κ annihilates $K^\perp$, and ω is a 2-form representing the Hermitian form on the screen bundle H_K — note that ω is defined only up to the addition of a term of the form $\kappa \wedge \alpha$ for some 1-form α;
- any section of $\wedge^{m+1}\mathrm{Ann}(N)$.

Remark 2.13. Any of these objects can be derived from a *pure spinor field*, defined up to scale, and its *charge conjugate*. We recall [Car67] that the defining property of a pure spinor field is that it annihilates a field of MTNs.

2.4.1 Intrinsic torsion

Just as in the case of an optical geometry, we can consider the intrinsic torsion of $(\mathcal{M}, g, N, K)$ as a tensor taking values in $\mathbb{G} = \mathbb{V} \otimes \mathfrak{g}/\mathfrak{q}$ where $\mathbb{V} \cong \mathbf{R}^{2m+1,1}$. Again, there is no canonical complementary Q-submodule of $\mathfrak{q}$ in $\mathfrak{g}$. Choosing an arbitrary splitting, we arrive at the following direct sum decomposition of $\mathbb{G}$ into Q_0-submodules:

$$
\mathbb{G} \cong \left\{ \begin{array}{ccccc}
\mathbb{H}^*_{\mathbb{K},\mathbb{L}} \otimes \mathbf{R}(2) & & & & \\
\oplus & & & & \\
\bigotimes^2 \mathbb{H}^*_{\mathbb{K},\mathbb{L}} \otimes \mathbf{R}(1) & \oplus & [\![\wedge^{(2,0)} \mathbb{H}^*_{\mathbb{K},\mathbb{L}}]\!] \otimes \mathbf{R}(1) & & \\
\oplus & & \oplus & & \\
\mathbb{H}^*_{\mathbb{K},\mathbb{L}} & \oplus & \mathbb{H}^*_{\mathbb{K},\mathbb{L}} \otimes [\![\wedge^{(2,0)} \mathbb{H}^*_{\mathbb{K},\mathbb{L}}]\!] & & \\
& & \oplus & & \\
& & [\![\wedge^{(2,0)} \mathbb{H}^*_{\mathbb{K},\mathbb{L}}]\!] \otimes \mathbf{R}(-1) & &
\end{array} \right. \tag{2.6}
$$

These Q_0-submodules further decompose into irreducibles. Computing the action of the nilpotent part $P_+ \subset Q$ on these allows use to determine all the Q-submodules of $\mathbb{G}$ just as we did for optical structures. We omit the rather lenghty details that are given in [FLTC21], and we will content ourselves with the following remarks:

- Having fixed a null 1-form κ and a 3-form $\rho = 3\kappa \wedge \omega$ stabilised by Q at any point, the intrinsic torsion of (N, K) at any point is determined by $\nabla \rho \pmod{\alpha \otimes \rho}$. In fact, the $\nabla \kappa$ part of $\nabla \rho$ will give us information on the modules of (2.6) marked in blue, while the $\nabla \omega$ part on the modules marked in red.

- The Q_0-submodules of $\mathbb{G}$ highlighted in blue in (2.6) are none other than the P_0-submodules considered in Section 2.3.1 in the treatment of optical structures. In particular, we have the following respective decompositions of the 'twist' and 'shear' into further irreducibles:
$$
\begin{aligned}
\wedge^2 \mathbb{H}^*_{\mathbb{K},\mathbb{L}} \otimes \mathbf{R}(1) &\cong \left([\![\wedge^{(2,0)} \mathbb{H}^*_{\mathbb{K},\mathbb{L}} \otimes \mathbf{R}(1)]\!] \right) \oplus \left([\wedge^{(1,1)}_\circ \mathbb{H}^*_{\mathbb{K},\mathbb{L}}] \otimes \mathbf{R}(1) \right) \\
&\qquad \oplus \left(\mathrm{span}(\omega) \otimes \mathbf{R}(1) \right) , \\
\odot^2_\circ \mathbb{H}^*_{\mathbb{K},\mathbb{L}} \otimes \mathbf{R}(1) &\cong \left([\![\odot^{(2,0)} \mathbb{H}^*_{\mathbb{K},\mathbb{L}}]\!] \otimes \mathbf{R}(1) \right) \oplus \left([\odot^{(1,1)}_\circ \mathbb{H}^*_{\mathbb{K},\mathbb{L}}] \otimes \mathbf{R}(1) \right) .
\end{aligned}
$$

- The module $\mathbb{H}^*_{\mathbb{K},\mathbb{L}} \otimes [\![\wedge^{(2,0)} \mathbb{H}^*_{\mathbb{K},\mathbb{L}}]\!]$ can be identified with the module of intrinsic torsions of an almost Hermitian structure considered in Section 2.2, and as such, splits into four irreducible Q_0-submodules.

- Some of the Q_0-submodules of $\mathbb{G}$ are isotypic, i.e. several copies of the same modules. For instance, the module $[\![\wedge^{(2,0)} \mathbb{H}^*_{\mathbb{K},\mathbb{L}} \otimes \mathbf{R}(1)]\!]$ occurs twice: as the $(2,0)$-part of the twist, and as the covariant derivative of ω along k. It may thus happens that the intrinsic torsion satisfies some algebraic relation among its components in these isotypic modules — see for instance Proposition 2.16 below.

2.4.2 (Almost) CR structures and (nearly) Robinson manifolds

Since an almost Robinson structure is in particular an optical structure, the leaf space of its associated congruence of null curves is of relevance. Before we delve into the matter, it is necessary to recall some important facts regarding Cauchy–Riemann geometry — see [Jac90] for a more substantial introduction.

Let $\underline{\mathcal{M}}$ be a smooth manifold of dimension $2m+1$. An *almost Cauchy–Riemann (CR) structure* on $\underline{\mathcal{M}}$ consists of a pair $(\underline{H}, \underline{J})$ where $\underline{H}$ is a rank-$2m$ distribution and $\underline{J}$ is a bundle complex structure on $\underline{H}$, i.e. ${}^{\mathbf{C}}\underline{H} = \underline{H}^{(1,0)} \oplus \underline{H}^{(0,1)}$ where $\underline{H}^{(1,0)}$ and $\underline{H}^{(0,1)}$ are the $+\mathrm{i}$- and $-\mathrm{i}$-eigenbundles of $\underline{J}$. We say that $(\underline{H}, \underline{J})$ is

- *involutive* or *integrable* if $\underline{H}^{(1,0)}$ is involutive, i.e. $[\underline{H}^{(1,0)}, \underline{H}^{(1,0)}] \subset \underline{H}^{(1,0)}$,
- *contact* if $\underline{H}$ is contact,
- *partially integrable* if the Levi form of $\underline{H}$ is of type $(1,1)$, i.e. for any 1-form $\underline{\theta}^0$ annihilating $\underline{H}$, $\mathrm{d}\underline{\theta}^0(\underline{v}, \underline{w}) = 0$ for any sections $\underline{v}$, $\underline{w}$ of $\underline{H}^{(1,0)}$.

An involutive almost CR structure is simply referred to as a *CR structure*.

Remark 2.14. We shall not here distinguish between 'involutive' and 'integrable', although the reader should bear in mind that the distinction is important with regards to analytical issues.

With these considerations, we may return to our main story:

Definition 2.15. An almost Robinson manifold $(\mathcal{M}, g, N, K)$ is called a *nearly Robinson manifold* whenever $[K, N] \subset N$.

Equivalently, in the light of Definition 1.9, a nearly Robinson manifold is an almost Robinson manifold $(\mathcal{M}, g, K, H_K, J)$ such that the bundle complex structure J is preserved along the congruence $\mathcal{K}$ tangent to K. To be precise, the screen bundle H_K of $\mathcal{K}$, together with its bundle Hermitian structure J, descends to a rank-$2m$ distribution $\underline{H}$ equipped with a bundle complex structure $\underline{J}$ on $\underline{\mathcal{M}}$. It follows that the leaf space $\underline{\mathcal{M}}$ of $\mathcal{K}$ is endowed with an almost CR structure. The Levi form is determined by the twist of $\mathcal{K}$ — essentially $\mathrm{d}\kappa$ 'restricted' to H_K.

Before we proceed we shall make the following definition. Let $(\mathcal{M}, g, N, K)$ be an almost Robinson manifold with congruence of null geodesics $\mathcal{K}$. Fix a generator k of $\mathcal{K}$ and a null vector field ℓ dual to k, i.e. $g(k, \ell) = 1$. Define a skew-symmetric bilinear form on H_K by

(2.7)
$$\zeta(v+K, w+K) := \nabla_k \rho(\ell, v, w) - \nabla_k \kappa(\ell) \rho(\ell, v, w)\,, \quad \text{for any sections } v, w \text{ of } K^{\perp}.$$

One can easily check that ζ takes values in $[\![\wedge^{(2,0)}\mathbb{H}_{\mathbb{K}}^*]\!] \otimes \mathbf{R}(1)$. It is indeed not dependent on the choice of ℓ since under a null transformation, we have

$$\zeta_{\alpha\beta} \mapsto \zeta_{\alpha\beta} - 2\mathrm{i}\gamma_\alpha \phi_\beta + 2\mathrm{i}\gamma_\beta \phi_\alpha\,,$$

and by assumption, $\mathcal{K}$ is geodesic, i.e. $\gamma_\alpha = 0$. That ζ has boost weight 1 is immediate.

A straightforward computation will yield the following characterisation of a nearly Robinson manifold in terms of its associated congruence of null curves.

Proposition 2.16. *An almost Robinson manifold $(\mathcal{M}, g, N, K)$ with congruence of null curves $\mathcal{K}$ is a nearly Robinson manifold if and only if*

- *$\mathcal{K}$ is a congruence of null geodesics, and*
- *the shear σ and twist τ satisfy*

$$\sigma(v+K, w+K) = 0\,, \qquad \text{for all } v, w \in \Gamma(N), \tag{2.8}$$

$$2\mathrm{i}\tau(v+K, w+K) - \zeta(v+K, w+K) = 0\,, \qquad \text{for all } v, w \in \Gamma(N), \tag{2.9}$$

where ζ is the skew-symmetric bilinear form defined by (2.7)*.*

Definition 2.17. We call $(\mathcal{M}, g, N, K)$ a *Robinson manifold* whenever $[N, N] \subset N$.

Clearly, a Robinson manifold is a nearly Robinson manifold so they share the same geometric properties. In addition, the leaf space $(\underline{\mathcal{M}}, \underline{H}, \underline{J})$ of a Robinson manifold is a *CR manifold*, i.e. the eigenbundles of $\underline{J}$ are involutive.

In dimension four, a nearly Robinson structure is necessarily involutive. In addition, condition (2.8) is equivalent to $\mathcal{K}$ being non-shearing, and condition (2.9) is vacuous. We thus conclude:

Theorem 2.18. *In dimension four, an optical geometry with non-shearing congruence of null geodesics is equivalent to a Robinson geometry.*

2.4.3 Lifts

There is a nearly Robinson analogue of the lift construction given in Section 2.3.5: Let $(\underline{\mathcal{M}}, \underline{H}, \underline{J})$ be an almost CR manifold of dimension $2m+1$, and consider the trivial extension $\mathcal{M} = \underline{\mathcal{M}} \times \mathbf{R} \xrightarrow{\varpi} \underline{\mathcal{M}}$.

Fix an adapted coframe $(\underline{\theta}^0, \underline{\theta}^\alpha, \overline{\underline{\theta}}^{\bar{\alpha}})$ for $(\underline{M}, \underline{H}, \underline{J})$, and define

- K to be the line distribution on $\mathcal{M}$ tangent to the $\mathbf{R}$-factor, and
- g to be the metric given by

$$g = \mathrm{e}^{2f}\left(2\kappa\lambda + 2h_{\alpha\bar{\beta}}\theta^\alpha\overline{\theta}^{\bar{\beta}}\right)$$

  where

  - f is a smooth function,
  - $\kappa := 2\varpi^*\underline{\theta}^0$ and $\theta^\alpha := \varpi^*\underline{\theta}^\alpha$,
  - $h_{\alpha\bar{\beta}}$ is a non-degenerate Hermitian matrix depending smoothly on $\mathcal{M}$,
  - λ a non-vanishing 1-form on $\mathcal{M}$ such that $\lambda(k) \neq 0$ for any $k \in \Gamma(K)$.

Then $(\mathcal{M}, g, N, K)$ is a nearly Robinson manifold where

$$N := \mathrm{Ann}(\varpi^*\underline{\theta}^0, \varpi^*\underline{\theta}^\alpha)\,, \qquad K := \mathrm{Ann}(\varpi^*\underline{\theta}^0)^\perp = N \cap \overline{N}\,.$$

Conversely, locally, any nearly Robinson manifold arises in this way.

Following on from Remark 2.11, if $\widehat{g}$ is a metric related to g by (2.5), the resulting geometry $(\mathcal{M}, \widehat{g}, N, K)$ is also nearly Robinson. The respective intrinsic torsions of (g, N, K) and $(\widehat{g}, N, K)$ share many (but not all) properties — see the notion of *generalised almost Robinson geometry* given in [FLTC21].

Remark 2.19. In dimension four, if $\mathcal{K}$ is twisting and non-shearing, $h_{\alpha\bar{\beta}}$ can be identified with the Levi form of $(\underline{H}, \underline{J})$ up to some factor. However, this is not the case in higher dimensions in general. There are however notable exceptions that we briefly discuss below.

Example 2.20 (Fefferman space)**.** Suppose that $(\underline{\mathcal{M}}, \underline{H}, \underline{J})$ be a contact CR manifold, then one can choose an adapted frame $(\underline{\theta}^0, \underline{\theta}^\alpha, \overline{\underline{\theta}}^{\bar{\alpha}})$ for $(\underline{\mathcal{M}}, \underline{H}, \underline{J})$ such that

$$\mathrm{d}\underline{\theta}^0 = \mathrm{i}\underline{h}_{\alpha\bar{\beta}}\underline{\theta}^\alpha \wedge \overline{\underline{\theta}}^{\bar{\beta}} , \qquad \mathrm{d}\underline{\theta}^\alpha = \underline{\theta}^\beta \wedge \underline{\Gamma}_\beta{}^\alpha + \underline{A}^\alpha{}_{\bar{\beta}}\underline{\theta}^0 \wedge \underline{\theta}^{\bar{\beta}} ,$$

and similarly for $\mathrm{d}\overline{\underline{\theta}}^{\bar{\alpha}}$. Here, $\underline{\Gamma}_\beta{}^\alpha$ is the connection 1-form of the *Webster–Tanaka connection*, the unique connection that preserves the contact form $\underline{\theta}^0$ and its Levi form $\underline{h}_{\alpha\bar{\beta}}$, and whose *pseudo-Hermitian* torsion tensor satisfies $\underline{A}_{[\alpha\beta]} = 0$. The vanishing of $\underline{A}_{\alpha\beta}$ is equivalent to the Reeb vector field of $\underline{\theta}^0$ being a *transverse CR symmetry*.

In the integrable case, Fefferman [Fef76] constructed a conformal structure $(\mathcal{M}, \mathbf{c})$ on a canonical circle bundle over $(\underline{\mathcal{M}}, \underline{H}, \underline{J})$. Here, $(\mathcal{M}, \mathbf{c})$ admits a null conformal Killing vector field k whose integrable curves generates the fibration. In particular, the foliation tangent to k is a non-shearing congruence of null geodesics, whose twist induces a Robinson structure, which descends to the CR manifold $(\underline{\mathcal{M}}, \underline{H}, \underline{J})$. For each metric g in $\mathbf{c}$ for which k is Killing, (i.e. $\mathcal{K}$ is non-expanding), we can write

$$g = 4\underline{\theta}^0 \odot \lambda + 2\underline{h}_{\alpha\bar{\beta}}\underline{\theta}^\alpha \odot \overline{\underline{\theta}}^{\bar{\beta}} ,$$

where λ can be viewed as a connection 1-form on the circle bundle. Explicitly,

$$\lambda = \mathrm{d}\phi + \frac{1}{m+2}\left(\mathrm{i}\underline{\Gamma}_\alpha{}^\alpha - \frac{\mathrm{i}}{2}\underline{h}^{\alpha\bar{\beta}}\mathrm{d}\underline{h}_{\alpha\bar{\beta}} - \underline{\mathrm{P}}\underline{\theta}^0\right) ,$$

where ϕ is a coordinate on S^1, and $\underline{\mathrm{P}}$ is the *Webster–Schouten scalar* — see [Lee86] for details. One can check that any change of contact form induces a conformal change of metrics in $\mathbf{c}$.

The Fefferman construction was generalised to the non-integrable partially integrable contact CR manifold in [Lei10]. In dimension four, Lewandowski [Lew88] showed that for the Fefferman conformal class to admit any Einstein metric (even locally) it must be conformally flat – in fact, its underlying CR structure must be flat (i.e. locally CR-equivalent to the CR 3-sphere. In higher dimensions, there are however non-conformally flat Fefferman–Einstein metrics [Lei07].

The Fefferman conformal structure is of course very special, and non-conformal flat Einstein Lorentzian metrics with NSCNGs (i.e. Robinson structures) can be found in plentiful in dimension four [Ste+03]. One such solution is the so-called Taub–NUT metric [Tau51; NTU63], which can be seen as a radial extension of the Hopf fibration $S^3 \to S^2$. These also admit higher-dimensional analogues [BB85; AC02; Ale+21]. In fact:

Theorem 2.21 ([TC21]). *Let $(\mathcal{M}, g, K)$ be an Einstein optical geometry of dimension $2m+2>4$, with twisting non-shearing congruence of null geodesics $\mathcal{K}$. Suppose that the Weyl tensor satisfies*

$$W(k, v, k, \cdot) = 0\,, \qquad\qquad \text{for all } k \in \Gamma(K),\, v \in \Gamma(K^\perp).$$

Then the twist of $\mathcal{K}$ determines a nearly Robinson manifold, and the local leaf space of $\mathcal{K}$ is an almost CR–Einstein manifold $(\underline{\mathcal{M}}, \underline{H}, \underline{J}, \underline{\theta}^0)$ *and g takes the form*

$$g = \sec^2\phi \left(4\underline{\theta}^0 \odot \lambda + 2\underline{h}_{\alpha\bar{\beta}}\underline{\theta}^\alpha \odot \overline{\underline{\theta}}^{\bar{\beta}}\right)\,,$$

where ϕ is a fiber coordinate on $\mathcal{M} \to \underline{\mathcal{M}}$, $\lambda = \mathrm{d}\phi + \lambda_0\underline{\theta}^0$ and $\lambda_0 = \lambda_0(\phi)$ depends on three parameters.

Remark 2.22.
- The *almost CR–Einstein* condition is a CR analogue of the Einstein condition. An almost CR–Einstein manifold admits a transverse CR symmetry, and the local quotient space is an *almost Kähler–Einstein manifold*.
- The precise form of $\lambda_0(\phi)$ is given in the aforementioned reference: two of the parameters can be identified with the Ricci scalar of g and the Webster–Ricci scalar of the almost CR–Einstein structure
- For suitable parameters, the metrics of Theorem 2.21 are locally isometric to Fefferman–Einstein metrics or Taub–NUT-(A)dS metrics.
- One feature that both the Fefferman conformal structure and the Taub–NUT metrics share is that their nearly Robinson structure is induced by the twist of their associated non-shearing congruence of null geodesics, i.e. the twist gives rise to a Hermitian form on the screen bundle of the congruence.

Theorem 2.21 tells us that the Einstein condition for almost Robinson structures with a twisting non-shearing congruence of null geodesics is too strong in dimensions greater than four. How about non-shearing examples in higher dimensions?

Example 2.23 (The Kerr–NUT–(A)dS metric). A special case of the *Plebański–Demiański metric* [PD76] is the *Kerr–NUT–(A)dS metric*, an Einstein metric which was generalised to higher dimensions in [CLP06]. In dimension $2m+2$, in coordinates (r, x_α, t, ψ_i), where $\alpha, i = 1, \dots m$, this metric takes the form

$$g = \frac{U}{X}\mathrm{d}r^2 - \frac{X}{U}\left(\mathrm{d}t + \sum_{k=1}^{m} A^{(k)}\mathrm{d}\psi_k\right)^2 + \sum_{\alpha=1}^{m}\left(\frac{U_\alpha}{X_\alpha}\mathrm{d}x_\alpha^2 + \frac{X_\alpha}{U_\alpha}\left(\mathrm{d}t + \sum_{k=1}^{m} A_\alpha^{(k)}\mathrm{d}\psi_k\right)^2\right)\,,$$

where

$$U = \prod_{\beta=1}^{m}(r^2 + x_\beta^2)\,, \qquad X = \sum_{k=0}^{m+1}(-1)^k c_k r^{2k} + Mr\,, \qquad A^{(k)} = \sum_{\nu_1 < \dots < \nu_k} x_{\nu_1}^2 \dots x_{\nu_k}^2\,,$$

and, for $\alpha = 1, \ldots, m$,

$$U_\alpha = (r^2 + x_\alpha^2) \prod_{\alpha \neq \beta} (x_\beta^2 - x_\alpha^2) \,, \qquad X_\alpha = \sum_{k=0}^{m+1} c_k x_\alpha^{2k} + L_\alpha r \,,$$
$$A_\alpha^{(k)} = \sum_{\substack{\nu_1 < \ldots < \nu_k \\ \nu_i \neq \mu}} x_{\nu_1}^2 \ldots x_{\nu_k}^2 - r^2 \sum_{\substack{\nu_1 < \ldots < \nu_{k-1} \\ \nu_i \neq \mu}} x_{\nu_1}^2 \ldots x_{\nu_{k-1}}^2 \,.$$

Here, M, L_α, $\alpha = 1, \ldots m$ and c_α are constants related to the mass, NUT parameters, the cosmological constant and rotation parameters of the black hole.

Defining

$$\begin{aligned}
\kappa &= \sqrt{\frac{U}{2X}} \left(\mathrm{d}r + \frac{X}{U} \left(\mathrm{d}t + \sum_{k=1}^{m} A^{(k)} \mathrm{d}\psi_k \right) \right) \,, \\
\lambda &= \sqrt{\frac{U}{2X}} \left(\mathrm{d}r - \frac{X}{U} \left(\mathrm{d}t + \sum_{k=1}^{m} A^{(k)} \mathrm{d}\psi_k \right) \right) \,, \\
\theta^\alpha &= \sqrt{\frac{U_\alpha}{2X_\alpha}} \left(\mathrm{d}x_\alpha + \mathrm{i} \frac{X_\alpha}{U_\alpha} \left(\mathrm{d}t + \sum_{i=1}^{m} A_\alpha^{(k)} \mathrm{d}\psi_k \right) \right) \,, \quad (\alpha = 1, \ldots, m) \,,
\end{aligned}$$

one can check that each of the sets of null 1-forms (κ, θ^α) and (λ, θ^α) define two involutive almost Robinson structure [MT10].

The congruences of null curves associated associated to each of these Robinson structures are geodesic, expanding, maximally twisting and shearing, when $m > 1$ [PPO07]. — and non-shearing when $m = 1$ as is well-known.

In fact, each of these congruences admits 2^{m-1} Robinson structures [MT10]! In particular, the leaf space of each congruence is endowed with 2^{m-1} CR structures for a given contact structure. The existence of these Robinson structures was shown to arise from a closed *conformal Killing–Yano* 2*-form*.

3 Key theorems of mathematical relativity

In this final section, we explain some of the main theorems of general relativity that led to a more conceptual understanding of non-shearing congruences of null geodesics and eventually almost Robinson geometry.

These results are more easily understood in the analytic category, where one can get a more pictorial idea of the geometry at play. Let us recall that if $(\mathcal{M}, g)$ is an four-dimensional analytic Lorentzian manifold, one may locally, complexify $(\mathcal{M}, g)$ to a *complex Riemannian manifold* $(\widetilde{\mathcal{M}}, \widetilde{g})$, where $\widetilde{\mathcal{M}}$ is a complex manifold of real dimension eight and $\widetilde{g}$ is a holomorphic complex metric on $\widetilde{\mathcal{M}}$. Thus, we now view $\mathcal{M}$ as a real slice of $\widetilde{\mathcal{M}}$ and the restriction of $\widetilde{g}$ is simply g.

Consider now an analytic Robinson structure (N, K) on $(\mathcal{M}, g)$, which we extend to a holomorphic almost null structure, also denoted N, on $(\widetilde{\mathcal{M}}, \widetilde{g})$. Similarly, we extend $(\overline{N}, K)$ to a holomorphic almost null structure $\widetilde{N}$, on $(\widetilde{\mathcal{M}}, \widetilde{g})$. Assume that N

and $\widetilde{N}$ are involutive, so that by the Frobenius theorem, these are tangent to foliations $\mathcal{N}$ and $\widetilde{\mathcal{N}}$ on $\widetilde{\mathcal{M}}$, and one can show that these leaves are *totally geodesic*, that is, if $\widetilde{\nabla}$ is the holomorphic Levi-Civita connection of $\widetilde{g}$, then

$$\widetilde{\nabla}_X Y \in \Gamma(N)\,, \qquad \text{for any } X, Y \in \Gamma(N)\,. \tag{3.1}$$

In particular, these two foliations intersect in a foliation $\widetilde{\mathcal{K}}$ by complex null geodesics in $\widetilde{\mathcal{M}}$. On restriction to $(\mathcal{M}, g)$, the foliation $\widetilde{\mathcal{K}}$ restricts to a non-shearing congruence of null geodesics $\mathcal{K}$.

3.1 The Robinson theorem

Definition 3.1. A 2-form F on a four-dimensional Lorentzian manifold $(\mathcal{M}, g)$ is said to be *null* or *algebraically special* if it satisfies

$$F \wedge F = 0\,, \qquad F \wedge \star F = 0\,,$$

where $\star$ is the Hodge duality operator.

A null 2-form enjoys the following properties:

- F is null if and only if it admits a null vector field k such that

$$k \lrcorner F = 0\,, \qquad \kappa \wedge F = 0\,,$$

  where $\kappa = g(k, \cdot)$. We refer to k as a *principal null direction (PND)* of F.
- F is null if and only if $F = \phi + \overline{\phi}$ for some totally null 2-form ϕ. In particular, ϕ is a simple SD or ASD 2-form.

Theorem 3.2 (Robinson (1961))**.** *Any null solution to the vacuum Maxwell field equations,*

$$\mathrm{d}F = 0\,, \qquad \mathrm{d} \star F = 0\,, \tag{3.2}$$

locally gives rise to a NSCNG. Conversely, any analytic *solution of the vacuum Maxwell field equation locally arises in this way.*

Proof. Note that $F = \phi + \overline{\phi}$ satisfying (3.2) is equivalent to $\mathrm{d}\phi = 0$. But $\phi = \kappa \wedge \alpha$, where $\mathrm{span}(\kappa, \alpha)$ annihilates an almost Robinson structure (N, K). Since ϕ is closed, N is involutive, and so K is tangent to an NSCNG.

For the converse, we work in the analytic category as described at the beginning of Section 3, i.e. locally, complexify $(\mathcal{M}, g)$ to a complex Riemannian manifold $(\widetilde{\mathcal{M}}, \widetilde{g})$. Identify an NSCNG with a Robinson structure (N, K), which we assume to be self-dual for definiteness. Extend N to a holomorphic null structure tangent to a foliation $\mathcal{N}$ in $(\widetilde{\mathcal{M}}, \widetilde{g})$. Denote by $\varpi : \widetilde{\mathcal{M}} \to \underline{\widetilde{\mathcal{M}}}_{\widetilde{\mathcal{N}}}$ the local submersion onto the local leaf space of $\mathcal{N}$. Let $\underline{\phi}$ be any 2-form on $\underline{\widetilde{\mathcal{M}}}_{\widetilde{\mathcal{N}}}$. Then $\underline{\phi}$ is simple and closed, and so is its pull-back $\phi := \varpi^* \underline{\phi}$. Clearly, ϕ annihilates N, and is thus self-dual and totally null. Play the same game with $(\overline{N}, K)$ to obtain a closed anti-self-dual 2-form $\widetilde{\phi}$. On restriction to $\mathcal{M}$, we have $\widetilde{\phi} = \overline{\phi}$, and the real 2-form $F = \phi + \overline{\phi}$ satisfies the vacuum Maxwell field equation. □

Remark 3.3.
- In [Taf85], Tafel showed that the Robinson theorem may not hold if the assumption of analyticity is dropped.
- The theorem generalises to irreducible spinor fields of higher valence, e.g. Rarita–Schwinger equations, linear graviton, etc. [Som76].
- A generalisation to even dimensions is given in [HM88] and to odd dimensions in [TC17a].

3.2 The Goldberg–Sachs theorem

Let $(\mathcal{M}, g)$ be a Lorentzian manifold. Recall that the Weyl tensor W is the totally tracefree part of the Riemann tensor, and is conformally invariant.

Definition 3.4. The span of a null vector k at a point of $\mathcal{M}$ is said to be a *principal null direction (PND)* of the Weyl tensor W if it satisfies

$$W(k, v, k, v) = 0\,, \qquad \text{for any vector } v \in \operatorname{span}(k)^{\perp}.$$

We say that the span of k is a *repeated PND* if it satisfies

$$W(k, v, k, \cdot) = 0\,, \qquad \text{for any vector } v \in \operatorname{span}(k)^{\perp}.$$

Remark 3.5. The existence of a repeated PND is equivalent to W being *algebraically special*, i.e. W is of *Petrov* type II or more degenerate.

The relation with non-shearing congruences of null geodesics is given by the next well-known result, which essentially follows from the work of Sachs (see e.g. [Sac62]).

Proposition 3.6. *Any generator of a NSCNG is a PND of the Weyl tensor.*

This condition is rather weak: in fact, Cartan [Car22] had already remarked that at any point, there exist at most four PNDs of the Weyl tensor. Further degeneracy of the Weyl tensor however turns out to have a powerful geometric interpretation in the context of Einstein metrics:

Theorem 3.7 (Goldberg–Sachs [GS62])**.** *Let $(\mathcal{M}, g)$ be an Einstein manifold. Then the Weyl tensor is algebraically special if and only if it admits a non-shearing congruence of null geodesics.*

Remark 3.8.
- The Einsten condition can be weakened to a conformally invariant condition on the Cotton tensor. There also exist versions for other metric signatures — see [GHN11] and references therein.
- Closely related to the Goldberg–Sachs theorem is the question of the *embeddability* of the CR structure underlying the non-shearing congruence of null geodesics [LNT90; HLN08].

3.3 The Kerr theorem

In 1963, Kerr discovered a spacetime $(\mathcal{M}, g)$ describing a rotating black hole [Ker63]. The metric is Ricci-flat and of Petrov type D, so, by the Goldberg–Sachs theorem, it admits a pair of NSCNGs. Let k be a generator of one of them. Then the Kerr metric can be cast as an exact first-order perturbation of the Minkowski metric η

$$g = \eta + 2H\kappa^2 ,$$

where $\kappa = g(k, \cdot)$ and H is a function. One can check that the congruence is geodesic and non-shearing for g if and only if it is for η. For metrics of the above form, we can thus reduce the problem of finding a NSCNG for $(\mathcal{M}, g)$ to one on Minkowski space $\mathbb{M}$. Kerr found that any analytic NSCNG on $\mathbb{M}$ can be obtained from a *single holomorphic function of three complex variables*. It was however Penrose in [Pen67] who provided a geometric proof of the Kerr theorem in the garb of *twistor geometry*. We sketch the idea below — see also [PR86].

- We work in the complexification ${}^{\mathbb{C}}\mathbb{M}$ of $\mathbb{M}$ — the Minkowski metric complexifies to a non-degenerate symmetric bilinear form.
- By conformal invariance, we view ${}^{\mathbb{C}}\mathbb{M}$ as a dense open subset of a smooth complex quadric $\mathcal{Q}$ of dimension four.
- Locally, a NSCNG $\mathcal{K}$ then arises as a complex foliation $\mathcal{N}$ in $\mathcal{Q}$. The key point, here, is that the totally geodesic null leaves of $\mathcal{N}$ (see (3.1)) are *two-dimensional linear subspaces of* $\mathcal{Q}$.
- Now define the *twistor space* $\mathbb{PT}$ to be the space of all (self-dual) linear subspaces of $\mathcal{Q}$. Twistor space can be shown to be three-dimensional complex projective space $\mathbf{C}\mathbb{P}^3$.
- The leaf space of $\mathcal{N}$ can therefore be identified with a hypersurface $\underline{\mathcal{M}}_{\mathcal{N}}$ in $\mathbb{PT}$: locally, this is precisely given by single holomorphic function of three complex variables as Kerr had found!

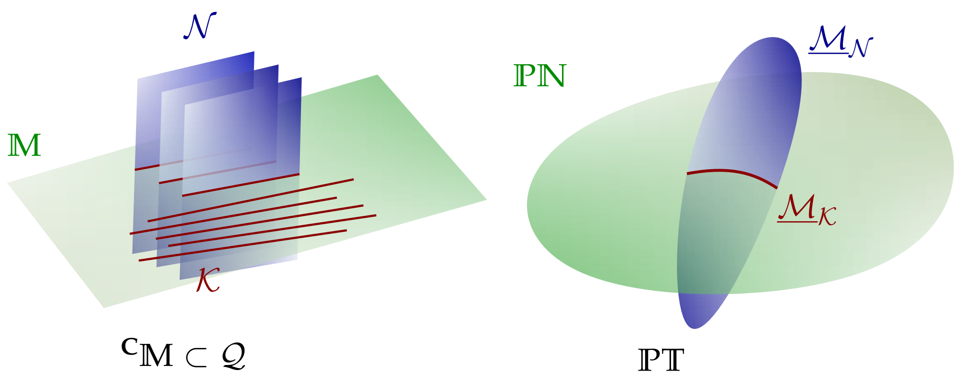

Figure 1: The Kerr theorem in terms of the twistor correspondence

So much for the complex picture of the Kerr theorem. But we can go a step further: the space of null lines (i.e. null geodesics) in the compactification of $\mathbb{M}$ turns out to be a five-dimensional CR hypersurface $\mathbb{PN}$ in $\mathbb{PT}$. The leaf space $\underline{\mathcal{M}}_{\mathcal{K}}$ of the NSCNG can then be identified with the three-dimensional CR submanifold arising from the intersection of $\mathbb{PN}$ with $\underline{\mathcal{M}}_{\mathcal{N}}$! See Figure 1. The details can be found in the aforementioned references.

Remark 3.9. There are two examples illustrating the Kerr theorem. One is the so-called *Robinson congruence*, the other is the *Kerr congruence*, which shows up in the Kerr metric. The former arises from the locus of a linear homogeneous polynomial in $\mathbb{PT}$, the latter from the locus of a quadratic homogeneous polynomial. These are in turn connected with the *twistor equation* and the *conformal Killing-Yano equation* on 2-forms respectively.

Remark 3.10. A generalisation of the Kerr theorem to even dimensions is given in [HM88], and to odd dimensions in [TC17b], which also contains an explicit generalisation of the Robinson and Kerr congruences.

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