Basic boson model

$$\hat{H} = \varepsilon \hat{b}^+ \hat{b} \qquad [\hat{b}, \hat{b}^+] = 1$$

Trace partition function

$[\varepsilon] = [J]$

$[\hat{b}] = [1]$

$[\hbar] = [J]$

$$Z = \mathrm{Tr}\, e^{-\beta \hat{H}} = \sum_{n=0}^{\infty} \langle n| e^{-\beta \varepsilon \hat{b}^+ \hat{b}} |n\rangle =$$

$$= \sum_{n=0}^{\infty} e^{-\beta \varepsilon n} = \frac{1}{1 - e^{-\beta \varepsilon}}$$

$\hat{b}^+ \hat{b} |n\rangle = n |n\rangle$

$[\beta] = [\frac{1}{k_B T}] = [\frac{1}{J}]$

Partition function Path integral

$$Z = \int \mathcal{D}[b^* b]\, e^{-\frac{1}{\hbar} \int_0^{\hbar\beta} d\tau\, b^*(\tau) (\hbar \partial_\tau + \varepsilon) b(\tau)}$$

$[t] = [J \cdot s]$

$[\tau] = [s]$

$[\beta \hbar] = [\frac{1}{J} \cdot J \cdot s] = [s]$

$\tau = it$, boundary condition

$$b(\tau + \beta\hbar) = b(\tau)$$

Matsubara-Fourier transformation

$$b(\tau) = \frac{1}{\sqrt{\beta\hbar}} \sum_{n=-\infty}^{\infty} b_n e^{-i\omega_n \tau} \qquad \omega_n = \frac{2\pi n}{\beta\hbar}$$

with

$$\delta_{nm} = \frac{1}{\beta\hbar} \int_0^{\beta\hbar} d\tau\, e^{i(\omega_n - \omega_m)\tau} \qquad [\omega_n] = [\frac{1}{s}]$$

$$\int_0^{\beta\hbar} d\tau\, b^*(\tau)(\hbar\partial_\tau + \varepsilon) b(\tau) = \frac{1}{\beta\hbar} \sum_{nm} b_n^* b_m \int_0^{\beta\hbar} d\tau (-i\omega_m \hbar + \varepsilon) e^{i(\omega_n - \omega_m)\tau}$$

$$= \sum_n (-i\hbar\omega_n + \varepsilon) b_n^* b_n$$

$$Z = \int_{-\infty}^{\infty} \prod_{n=-\infty}^{\infty} \frac{d b_n^* d b_n}{2\pi i (\hbar\beta)} e^{-\frac{1}{\hbar} \sum_n (-i\hbar\omega_n + \varepsilon) b_n^* b_n}$$

$[b_n] = [\sqrt{s}]$

$$b_n = \frac{1}{\sqrt{\beta\hbar}} \int_0^{\beta\hbar} d\tau\, b(\tau) e^{i\omega_n \tau}$$

Gaussian integral over complex variables

$$\int_{-\infty}^{\infty} \frac{dz^* dz}{2\pi i} e^{-\alpha z^* z} = \frac{1}{\alpha}$$

$$Z = \prod_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d b_n^* d b_n}{2\pi i} \frac{e^{-\frac{1}{\hbar}(-i\hbar\omega_n+\varepsilon) b_n^* b_n}}{\beta\hbar} =$$

$$= \prod_{n=-\infty}^{\infty} \frac{1}{\beta(-i\omega_n\hbar+\varepsilon)} = e^{-\sum_n \ln\left(\beta(-i\hbar\omega_n+\varepsilon)\right)}$$

To compute the sum we need to add a regularization and make a reference to $\varepsilon = 0$ system

$$-\sum_n \ln\left(\beta(-i\hbar\omega_n+\varepsilon)\right) \longrightarrow -\sum_n \ln\left(\frac{-i\hbar\omega_n+\varepsilon}{-i\hbar\omega_n}\right) =$$

$$= \sum_n \int_0^{\varepsilon} \frac{dx}{i\omega_n\hbar - x}$$

$$\left\{ \int \frac{dx}{ax+b} = \frac{1}{a}\ln|ax+b| \right\}$$

$$\int_0^{\varepsilon} \frac{dx}{i\omega_n\hbar - x} = -\ln|-x+i\hbar\omega_n|\Big|_0^{\varepsilon} =$$

$$= -\ln\left|\frac{-\varepsilon+i\hbar\omega_n}{i\hbar\omega_n}\right|$$

$$\sum_n \frac{1}{i\hbar\omega_n - x} \xrightarrow{\eta\to 0^+} \sum_n \frac{e^{\eta\omega_n}}{i\hbar\omega_n - x} = -\frac{\beta}{2\pi i}\oint \frac{dz}{e^{\beta z}-1}\frac{e^{\eta z}}{z-x} =$$

$$\underset{\eta\to 0}{=} -\frac{\beta}{e^{\beta x}-1}$$

$$\left[\int \frac{dx}{ae^{mx}+b} = \frac{1}{bm}\ln\left|\frac{e^{mx}}{ae^{mx}+b}\right| \right]$$

Hence

$$\int_0^{\varepsilon} dx \sum_n \frac{1}{i\omega_n\hbar - x} = -\int_0^{\varepsilon} dx \frac{\beta}{e^{\beta x}-1} =$$

$$= -\beta\left(-\frac{1}{\beta}\right)\ln\frac{e^{\beta x}}{e^{\beta x}-1}\Big|_0^{\varepsilon} = \ln\frac{1}{1-e^{-\beta\varepsilon}}$$

Stand

$$\boxed{Z = e^{\ln \frac{1}{1-e^{-\beta\varepsilon}}} = \frac{1}{1-e^{-\beta\varepsilon}}}$$

$\square$

Green's function

Heisenberg picture

- real time $\qquad \hat{b}(t) = e^{i\frac{\hat{H}t}{\hbar}} \hat{b}\, e^{-i\frac{\hat{H}t}{\hbar}}$
- imaginary time $\qquad \hat{b}(\tau) = e^{\frac{\hat{H}\tau}{\hbar}} \hat{b}\, e^{-\frac{\hat{H}\tau}{\hbar}}$

  $it = \tau$

  $i\hbar\partial_t = -\hbar\partial_\tau$
- one particle Green's function

$$G(t) = -i\langle T_t \hat{b}(t)\hat{b}^+(0)\rangle = -i\,\Theta(t)\langle\hat{b}(t)\hat{b}^+(0)\rangle - i\,\Theta(-t)\langle\hat{b}^+(0)\hat{b}(t)\rangle$$

$[G(t)] = [1]$

$$G(\tau) = -\langle T_\tau \hat{b}(\tau)\hat{b}^+(0)\rangle = -\Theta(\tau)\langle\hat{b}(\tau)\hat{b}^+(0)\rangle - \Theta(-\tau)\langle\hat{b}^+(0)\hat{b}(\tau)\rangle$$

Equation of motion

$$i\hbar\,\partial_t \hat{b}(t) = -[\hat{H}, \hat{b}(t)]$$

$$\hbar\,\partial_\tau \hat{b}(\tau) = [\hat{H}, \hat{b}(\tau)]$$

$$[\hat{H}, \hat{b}] = -\varepsilon\hat{b}$$

$\frac{d}{dx}\Theta(x) = \delta(x)$

$$i\hbar\partial_t G(t) = \hbar\delta(t) + \varepsilon G(t) \qquad [\delta(t)] = [1/s]$$

$$-\hbar\partial_\tau G(\tau) = \hbar\delta(\tau) + \varepsilon G(\tau) \qquad [\delta(\tau)] = [1/s]$$

Fourier transforms

$$G(t) = \frac{1}{2\pi} \int d\omega\, e^{-i\omega t}\, G(\omega)$$

$$G(\tau) = \frac{1}{\beta\hbar} \sum_n e^{-i\omega_n \tau}\, G_n$$

with

$$\delta(t) = \frac{1}{2\pi} \int d\omega\, e^{-i\omega t}$$

$$\delta(\tau) = \frac{1}{\beta\hbar} \sum_n e^{-i\omega_n \tau}$$

Hence

$$G(\omega) = \frac{\hbar}{\hbar\omega - \varepsilon} \qquad [G(\omega)] = [s]$$

$$G_n = G(\omega_n) = \frac{\hbar}{i\omega_n - \varepsilon} \qquad [G_n] = [s]$$

Path integral Green's function

$$G(\tau) = -\theta(\tau) \frac{1}{\hbar\beta} \sum_n e^{-i\omega_n \tau} \langle b_n b_n^* \rangle -$$

$$-\theta(-\tau) \frac{1}{\hbar\beta} \sum_n e^{-i\omega_n \tau} \langle b_n^* b_n \rangle =$$

$$= -\frac{1}{\beta\hbar} \sum_n e^{-i\omega_n \tau} \langle b_n^* b_n \rangle$$

where $\langle b_n b_m^* \rangle = \delta_{nm} \langle b_n b_m^* \rangle$ in equilibrium.

Note

$$\int \frac{dz^* dz}{2\pi i} z^* z\, e^{-\alpha z^* z} =$$

$$= -\frac{d}{d\alpha} \int \frac{dz^* dz}{2\pi i} e^{-\alpha z^* z} = -\frac{d}{d\alpha} \frac{1}{\alpha} = \frac{1}{\alpha^2}$$

$$\langle b_m^* b_m \rangle = \frac{\int_{-\infty}^{\infty} \prod_{n=-\infty}^{\infty} \frac{db_n^* db_n}{2\pi i} \frac{1}{\beta\hbar} b_m^* b_m e^{-\frac{1}{\hbar}\sum_n (-i\hbar\omega_n + \varepsilon) b_n^* b_n}}{\int_{-\infty}^{\infty} \prod_{n=-\infty}^{\infty} \frac{db_n^* db_n}{2\pi i} \frac{1}{\beta\hbar} e^{-\frac{1}{\hbar}\sum_n (-i\hbar\omega_n + \varepsilon) b_n^* b_n}} =$$

$$\frac{1}{Z} \mathrm{Tr}(\hat{b}_m^+ \hat{b}_m e^{-\beta \hat{H}})$$

$$= \frac{\prod_{n\neq m} \frac{1}{\beta(-i\hbar\omega_n+\varepsilon)} \frac{1}{\hbar\beta} \frac{1}{\left(\frac{1}{\hbar}(-i\hbar\omega_m+\varepsilon)\right)^2}}{\prod_n \frac{1}{\beta(-i\hbar\omega_n+\varepsilon)}} = \boxed{\frac{\hbar}{-i\hbar\omega_m+\varepsilon} = -G_m}$$

Different formulation

$$Z = e^{-\sum_n \ln(\beta(-i\hbar\omega_n+\varepsilon))} = e^{-\sum_n \ln(-\hbar\beta G_n^{-1})}$$

Symbolically writing

$$Z = \int D(b^* b)\, e^{-\int_0^{\beta\hbar} d\tau \int_0^{\beta\hbar} d\tau'\, b^*(\tau)(-G^{-1}(\tau-\tau'))b(\tau')} =$$

$$= \frac{1}{\mathrm{Det}(-G^{-1}(\tau-\tau'))} = e^{-\mathrm{Tr}\ln(-G^{-1}(\tau-\tau'))}$$

Where the Green's operator is

$$-G^{-1}(\tau-\tau') = \frac{1}{\hbar}\delta(\tau-\tau')(\hbar\partial_\tau+\varepsilon) \qquad [G^{-1}(\tau)] = [1/s^2]$$

since $\quad \delta(\tau-\tau'') = \int d\tau'\, G^{-1}(\tau-\tau')\, G(\tau'-\tau'')$

we derive

$$-\delta(\tau-\tau'') = \int d\tau' \frac{1}{\hbar}\delta(\tau-\tau')(\hbar\partial_{\tau'}+\varepsilon)\, G(\tau'-\tau'') =$$

$$= \frac{1}{\hbar}(\hbar\partial_\tau+\varepsilon)\, G(\tau-\tau'') = \frac{1}{\hbar}(\hbar\partial_\tau+\varepsilon)\frac{1}{\beta\hbar}\sum_n e^{-i\omega_n(\tau-\tau'')} G_n =$$

$$= \frac{1}{\hbar\beta}\sum_n \frac{1}{\hbar}(-i\hbar\omega_n+\varepsilon)\, G_n\, e^{-i\omega_n(\tau-\tau'')}$$

Hence

$$\boxed{G_n = \frac{\hbar}{i\hbar\omega_n-\varepsilon}}$$