Cosmology Course
problems for the written exam

1. "Derive" the Friedmann equation within the Newtonian gravity and discuss its drawbacks.

2. Show that for a gas of non-relativistic particles of mass m, the energy density (ρ), pressure (p) and the number density (n) are related by $\rho = nm + \frac{3}{2}p$.

3. Show that for a gas of relativistic particles, the energy density (ρ) and pressure (p) are related by $\rho = 3p$.

4. The fundamental equations for cosmology are

   - The Friedmann equation
   $$\dot{a}^2 + k = \frac{8\pi G}{3}\rho a^2$$
   - The acceleration equation:
   $$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = -8\pi G p$$
   - The energy-momentum conservation (the first law of thermodynamics):
   $$\dot{p}a^3 = \frac{d}{dt}\left[a^3(\rho + p)\right]$$

   Show that only two of them are independent.

5. Calculate the non-vanishing components of the curvature tensor $R_{\mu\nu}$ for a static and isotropic metric (as for the Schwarzschild solution).

6. Knowing the non-vanishing components of the curvature tensor $R_{\mu\nu}$:
   $$\begin{aligned} R_{tt} &= -\frac{B''}{2A} + \frac{1}{4}\frac{B'}{A}\left(\frac{B'}{B} + \frac{A'}{A}\right) - \frac{1}{r}\frac{B'}{A} \\ R_{rr} &= \frac{B''}{2B} - \frac{1}{4}\frac{B'}{B}\left(\frac{B'}{B} + \frac{A'}{A}\right) - \frac{1}{r}\frac{A'}{A} \\ R_{\theta\theta} &= -1 + \frac{r}{2A}\left(\frac{B'}{B} - \frac{A'}{A}\right) + \frac{1}{A} \\ R_{\varphi\varphi} &= \sin^2\theta R_{\theta\theta} \\ R_{\mu\nu} &= 0 \quad \text{for} \quad \nu \neq \mu \end{aligned}$$
   for
   $$d\tau^2 = B(r)dt^2 - A(r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$
   find the Schwarzschild solution and determine the Schwarzschild radius.

7. Show that if the energy density is dominated by positive cosmological constant and Universe is spatially flat ($k = 0$) then the expansion is exponential.

8. Assuming a single component Universe, find the condition which must be satisfied by the equation of state in order to ensure positive acceleration of the Universe.

9. Find the present Universe age in terms of $\Omega_{\rm rad}$ and H_0 assuming radiation domination and zero curvature ($k = 0$).

10. Find the present Universe age in terms of $\Omega_{\rm m}$ and H_0 assuming matter domination and zero curvature ($k=0$).

11. Find the present Universe age in terms of Ω_Λ and H_0 assuming that it is flat ($k=0$) and it contains both matter and cosmological constant.
Hint:
$$\int_0^1 \frac{dx}{(ax^{-1}+bx^2)^{1/2}}\Bigg|_{a+b=1} = \frac{1}{3b^{1/2}}\ln\left[\frac{(b+b^{1/2})^2}{b(1-b)}\right]$$

12. Assuming $\Lambda>0$ and $k=1$ construct the static Universe containing cosmological constant Λ and non-relativistic matter, i.e. determine Λ_E and a_E, such that $\dot a(t)=\ddot a(t)=0$ and discuss its stability. Describe the evolution if
   - $\Lambda<\Lambda_E$
   - $\Lambda>\Lambda_E$

13. For a material point in a static isotropic gravitational field
   - Derive parametric equations of motion using
$$\begin{array}{llll}
\Gamma^t_{tr}=\Gamma^t_{rt}=\frac{1}{2}B'B^{-1} & & & \\
\Gamma^r_{tt}=\frac{1}{2}B'A^{-1} & \Gamma^r_{rr}=\frac{1}{2}A'A^{-1} & \Gamma^r_{\theta\theta}=-rA^{-1} & \Gamma^r_{\varphi\varphi}=-rA^{-1}\sin^2\theta \\
\Gamma^\theta_{r\theta}=\Gamma^\theta_{\theta r}=-r^{-1} & \Gamma^\theta_{\varphi\varphi}=-\sin\theta\cos\theta & & \\
\Gamma^\varphi_{r\varphi}=\Gamma^\varphi_{\varphi r}=r^{-1} & \Gamma^\varphi_{\varphi\theta}=\cot\theta & &
\end{array}$$
   - Determine constants of motion.
   - Derive the equation for $r=r(\varphi)$.

14. Starting from the quadrature solution for the trajectory of a material point in a static isotropic gravitational field
$$\varphi-\varphi_0=\pm\int_{r(\varphi_0)}^{r(\varphi)}\frac{A^{1/2}(r')\,dr'}{r'^2\left[J^{-2}B^{-1}(r')-EJ^{-2}-r'^{-2}\right]^{1/2}}$$
where E is the energy of the particle while
$$B(r)=\frac{1}{A(r)}=1-\frac{r_s}{r}\qquad\text{for}\qquad r_s=2GM$$
show that the deflection angle
$$\Delta\varphi=2[\varphi(r_0)-\varphi_\infty]-\pi$$
for light traveling in the vicinity (r_0 is the closest distance to the center) of a massive spherical object of mass M is given by
$$\Delta\varphi=2[\varphi(r_0)-\varphi_\infty]-\pi=\frac{4MG}{r_0}(1+\cdots)$$

15. Assuming small angles derive the lens equation for gravitational lensing.

16. Determine region for the flat, open and closed Universes in figure 1.

17. Determine region for the accelerating and decelerating Universes in figure 1.

18. Determine the boundary (i.e. find the appropriate condition for $\Omega^0_\Lambda=\Omega^0_\Lambda(\Omega^0_m)$) between regions "expands forever" and "recollapses eventually" in figure 1.

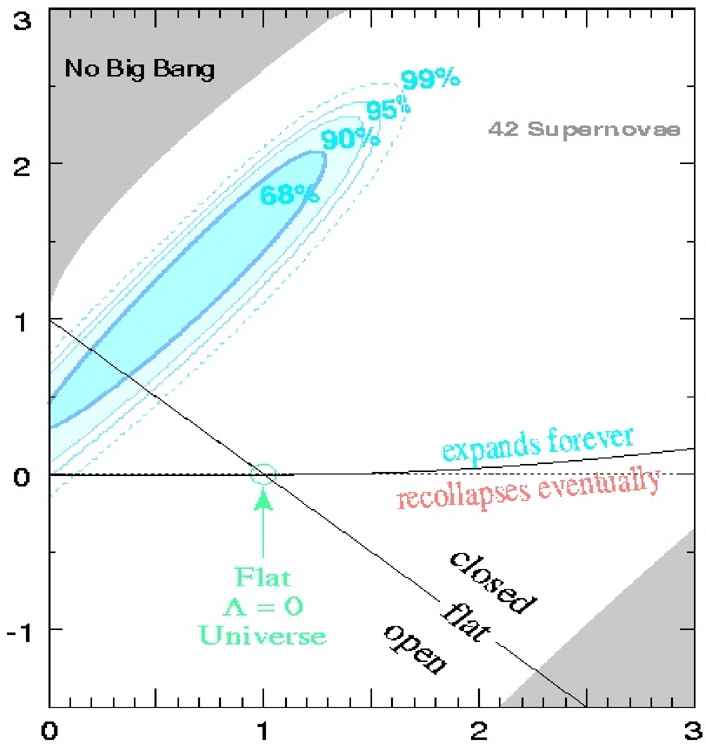

Figure 1: Confidence region for Ω_m vs. Ω_Λ plane, from SCP.

19. Using results obtained for the static Einstein Universe ($k = 1$ and $\Lambda > 0$):

$$a_E = \frac{3}{2}b, \qquad \Lambda_E = \left(\frac{2}{3b}\right)^2, \qquad \text{for} \qquad b \equiv \frac{1}{3} 8\pi G \rho_0 a_0^3$$

where ρ_0 corresponds to the scale factor a_0, determine the boundary of the region "No Big Bang" in the figure 1.

20. Assuming flat Universe ($k = 0$) find distance to the particle horizon for a "matter" that satisfies the general equation of state $p = w\rho$.

21. For $k = 0$ show that if $w < -\frac{1}{3}$ then the distance to the particle horizon is ∞.

22. Find the distance to the horizon for matter dominated Universe with $k \neq 0$.
Hint:

$$\int \frac{dy}{(y^2 - y_0^2)^{1/2}} = \cosh^{-1} \frac{y}{y_0}$$

23. Determine the fraction of the matter-dominated Universe that is visible as a function of time for $k = 1$.

24. For $k = 1$ find the fraction of the matter-dominated Universe that is visible at the moment of maximal expansion and at the "Big Crunch".

25. Assume that the size of presently observed Universe is $\simeq 8 \cdot 10^3$ Mpc, how large was it one millisecond after the Big Bang. Hint: assume radiation domination during first millisecond after the Big Bang.

26. Show that in the radiation dominated Universe the deceleration parameter q_0 equals $\Omega^0_{\rm rad}$.

27. In terms of Ω^0_r and Ω^0_m, determine the red-shift for which radiation and matter contributions to the energy density are equal.

28. Assuming that the cross-section for graviton $\leftrightarrow$ (elementary particles) interaction is

$$\sigma_{\text{grav}} \sim \frac{T^2}{M_{Pl}^4}$$

find relation between graviton background temperature and the temperature of CMB. Estimate present graviton contribution to the energy density. Hint: graviton is massless.

29. Assuming that there exist an extra (comparing to the Standard Model) single relativistic species which decouples form the SM at $T_f > 246$ GeV, finds its contribution to the energy density at the BBN.

30. Assuming that the BBN constraints an extra contribution to the energy density ρ^{NP} such that

$$\left.\frac{\rho^{NP}}{\rho^{SM}}\right|_{BBN} < 0.07 \quad \text{at} \quad 95\%\text{CL}$$

find the upper limit for the number of extra degrees of freedom that is implied by the constraint.

31. Derive a limit for the number of extra light, stable neutrinos which is implied by the BBN constraint

$$\left.\frac{\rho^{NP}}{\rho^{SM}}\right|_{BBN} < 0.07 \quad \text{at} \quad 95\%\text{CL}$$

Assume that those neutrinos decouple at the same temperature as the standard neutrinos.

32. Assume monopoles are produced at $T \simeq 2.5 \cdot 10^{15}$ GeV with $\Omega_{\text{in}}^{\text{mon}} = 10^{-10}$. For the flat Universe dominated by radiation find the temperature at which $\rho_{\text{rad}} = \rho_{\text{mon}}$.