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Fundamental Interactions – the Standard Model

♠ Gauge symmetry: $SU(3)_C \times SU(2)_L \times U(1)_Y$

$$\mathcal{L} \supset -\frac{1}{4} \underbrace{F_a^{\mu\nu} F_{a\,\mu\nu}}_{SU(3)_C} - \frac{1}{4} \underbrace{W_i^{\mu\nu} W_{i\,\mu\nu}}_{SU(2)_L} - \frac{1}{4} \underbrace{B^{\mu\nu} B_{\mu\nu}}_{U(1)_Y}$$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$G_a^\mu|_{a=1,\ldots 8} \qquad\qquad W_\mu^\pm, Z_\mu, A_\mu$$

- $F_a^{\mu\nu} \equiv \partial^\mu G_a^\nu - \partial^\nu G_a^\mu - g_{QCD} f^{abc} G_b^\mu G_c^\nu$
- $W_i^{\mu\nu} \equiv \partial^\mu W_i^\nu - \partial^\nu W_i^\mu - g_{weak} \varepsilon^{ijk} W_\mu^j W_\nu^k$
- $B^{\mu\nu} \equiv \partial^\mu B^\nu - \partial^\nu B^\mu$

♠ The Higgs sector:

- The minimal choice $H = \begin{pmatrix} G^+ \\ (h + iG^0)/\sqrt{2} \end{pmatrix}$ necessary for $SU(2)_L \times U(1)_Y \to U(1)_{EM}$.

$$\mathcal{L} \supset (D_\mu H)^\dagger D^\mu H - V(H)$$

for $D_\mu \equiv \partial_\mu + igW_\mu^i T^i + ig' \frac{1}{2} YB_\mu, \quad V(H) = \mu^2 |H|^2 + \lambda |H|^4$ and $Y_H = \frac{1}{2}$

- If $\mu^2 < 0$ then $\langle 0 || H|^2 | 0 \rangle = -\frac{1}{2} \frac{\mu^2}{\lambda} \equiv \frac{v^2}{2}$ (spontaneous symmetry breaking, the origin of mass)
- Boson masses: $m_h = \sqrt{2\lambda} v$, $m_{W^\pm} = \frac{1}{2} gv$ and $m_Z = m_W / c_W$, for $c_W \equiv \cos\theta_W = g/(g^2 + g'^{\,2})^{1/2}$

♠ Fermions

fermion	T	T_3	$\frac{1}{2}Y$	Q
$\nu_{i\,L}$	$\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$	0
$l_{i\,L}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	-1
$u_{i\,L}$	$\frac{1}{2}$	$+\frac{1}{2}$	$\frac{1}{6}$	$\frac{2}{3}$
$d_{i\,L}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{6}$	$-\frac{1}{3}$
$l_{i\,R}$	0	0	-1	-1
$u_{i\,R}$	0	0	$\frac{2}{3}$	$\frac{2}{3}$
$d_{i\,R}$	0	0	$-\frac{1}{3}$	$-\frac{1}{3}$
$\nu_{i\,R}$	0	0	0	0

$i = 1, \ldots, N_f = 3$, $\psi_{L,R} \equiv \frac{1}{2}(1 \mp \gamma_5)\psi$ (parity violation), $Q = T_3 + \frac{1}{2}Y$
Neutrino masses:

- Dirac mass: $f_{ij}\ \bar{L}_{i\,L}\nu_{j\,R}\ \tilde{H}$ + H.c. for $\tilde{H} \equiv i\tau_2 H^\star$
- Majorana mass: $\frac{1}{2}M_{ij}\ \overline{\nu_{i\,R}{}^C}\nu_{j\,R}$ + H.c.

Gauge transformations: $\psi(x) \to \exp\left\{-igT^i\theta_i(x) - ig'\frac{1}{2}Y\beta(x)\right\}\psi(x)$
Gauge interactions:

$$\mathcal{L} \supset \sum_{\psi} \bar{\psi} i\gamma^\mu D_\mu \psi \qquad \text{for} \qquad D_\mu \equiv \partial_\mu + igW_\mu^i T^i + ig'\frac{1}{2}YB_\mu$$

Yukawa interactions:

$$\mathcal{L} \supset -\sum_{i,j=1}^{3} \left(\tilde{\Gamma}_{ij}\bar{u}_{i\,R}\tilde{H}^\dagger Q_{j\,L} + \Gamma_{ij}\bar{d}_{i\,R}H^\dagger Q_{j\,L} + \text{H.c.}\right)$$

$$\Downarrow$$

$$\text{if } \langle H \rangle \neq 0 \text{ then } m_q \neq 0$$

$$\mathcal{L}_{\text{q mass}} = -\sum_{i,j=1}^{3} \left(\bar{u}_{i\,R}\mathcal{M}_{ij}^u u_{j\,L} + \bar{d}_{i\,R}\mathcal{M}_{ij}^d d_{j\,L} + \text{H.c.}\right)$$

for

$$\mathcal{M}_{ij}^u = \frac{v}{\sqrt{2}}\tilde{\Gamma}_{ij} \qquad \mathcal{M}_{ij}^d = \frac{v}{\sqrt{2}}\Gamma_{ij} \qquad \Rightarrow \qquad \text{no FCNC for one Higgs boson doublet}$$

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}_{L,R} = U_{L,R} \begin{pmatrix} u \\ c \\ t \end{pmatrix}_{L,R} \qquad \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}_{L,R} = D_{L,R} \begin{pmatrix} d \\ s \\ b \end{pmatrix}_{L,R}$$

$$U_R^\dagger \mathcal{M}^u U_L = \mathrm{diag}(m_u, m_c, m_t) \qquad D_R^\dagger \mathcal{M}^d D_L = \mathrm{diag}(m_d, m_s, m_b)$$

$$\Downarrow$$

$\tilde{\Gamma}, \Gamma$ diagonal $(g_f = \sqrt{2}\frac{m_f}{v})$ $\Rightarrow$ no FCNC

- charged currents: $\sum \bar{u}_{i\,L} \gamma^\mu d_{i\,L} = (\bar{u}, \bar{c}, \bar{t})_L \underbrace{U_L^\dagger D_L}_{U_{CKM}} \gamma^\mu \begin{pmatrix} d \\ s \\ b \end{pmatrix}_L$
- neutral currents: $\sum \bar{u}_{i\,L} \gamma^\mu u_{i\,L}$, $\sum \bar{d}_{i\,L} \gamma^\mu d_{i\,L}$ remain unchanged upon $U_{L,R}$, $D_{L,R}$ transformations

U_{CKM}:

- unitary complex $N \times N$ matrix, $q_{i\,L} \to e^{i\alpha_i} q_{i\,L} \;\Rightarrow\; \frac{1}{2}(N-1)(N-2)$ phases in U_{CKM}
- $N \geq 3 \qquad \Rightarrow \qquad$ CP violation in charged currents

♠ Masses in the SM: $\quad m_V \propto g v \qquad m_h \propto \lambda^{1/2} v \qquad m_f \propto g_f v$

Leptons:

$$m_{\nu_e} \lesssim 3 \text{ eV} \qquad m_{\nu_\mu} \lesssim 0.2 \text{ MeV} \qquad m_{\nu_\tau} \lesssim 18 \text{ MeV}$$
$$m_e = 0.5 \text{ MeV} \qquad m_\mu = 105.5 \text{ MeV} \qquad m_\tau = 1.78 \text{ GeV}$$

Quarks:

$$m_u \simeq 2 \text{ MeV} \qquad m_c \simeq 1.2 \text{ GeV} \qquad m_t \simeq 174 \text{ GeV}$$
$$m_d = 5 \text{ MeV} \qquad m_s = 0.1 \text{ GeV} \qquad m_b = 4.3 \text{ GeV}$$

Bosons:

$$m_{W^\pm} = 80.4 \text{ GeV} \qquad m_Z = 91.2 \text{ GeV} \qquad m_\gamma = 0 \qquad m_h = 125.3 \text{ GeV}$$

$$\Downarrow$$

Fine tuning:

$$\frac{m_{\nu_e}}{m_t} \lesssim 1.72 \cdot 10^{-11} \qquad \Rightarrow \qquad \frac{g_{\nu_e}}{g_t} \lesssim 1.72 \cdot 10^{-11}$$

Introduction to the Standard Model: Experimental constraints

- Perfect agreement with the existing data
- The scalar sector not fully tested
  - Higgs-boson representation:

$$\rho \equiv \frac{m_W^2}{m_Z^2 \cos^2\theta_W}, \qquad \text{SM} \qquad \Rightarrow \qquad \rho = 1 + \mathcal{O}(\alpha)$$

    for general Higgs multiplets: $\rho = \frac{\sum_i [T_i(T_i+1) - T_{i\,3}^2] v_i^2}{\sum_i 2T_{i\,3}^2 v_i^2}$

    data: $\rho = 1.0002 \begin{cases} +0.0024 \\ -0.0009 \end{cases} \qquad \Rightarrow \qquad T = \frac{1}{2}$ (doublets are favored)

  - $m_h = 125.3$ GeV
  - Higgs-boson interactions: no direct tests of quartic Higgs interactions (potential)

Outstanding problems of the SM

♠ Gauge-Higgs sector:

- Why is there only one Higgs boson?
  - The Higgs field was introduced just to make the model renormalizable (unitary)
  - There exist many fermions and vector bosons, so why only one scalar? Why, for instance, not a dedicated scalar for each fermion?
- The strong CP problem:
  - symmetries of the SM allow for

$$\mathrm{Tr}\left(F_{\mu\nu}\tilde{F}^{\mu\nu}\right) \equiv \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}\mathrm{Tr}\left(F_{\mu\nu}F_{\alpha\beta}\right) \xrightarrow{P} -\mathrm{Tr}\left(F_{\mu\nu}\tilde{F}^{\mu\nu}\right)$$

  - odd under CP

$$\mathcal{L}_\theta = \theta\frac{g_s^2}{32\pi^2}F^{a\,\mu\nu}\tilde{F}^a_{\mu\nu} \quad \Rightarrow \quad \text{neutron-EDM} \quad D_n \simeq 2.7\cdot 10^{-16}\theta \text{ e cm}$$

$$\Downarrow$$

$$\text{data: } D_n \lesssim 1.1\cdot 10^{-25} \text{ e cm} \quad \Rightarrow \quad \theta \lesssim 3\cdot 10^{-10}$$

The strong CP problem: why is θ so small?

♠ The flavor sector:

- parity violation:

$$W^{+\,\mu}\ \bar{u}_i \gamma_\mu (1 - \gamma_5) d_j \quad \xrightarrow{P} \quad W^{+\,\mu}\ \bar{u}_i \gamma_\mu (1 + \gamma_5) d_j$$

  Maximal parity violation, why?

- Charge quantization, why $q_u = \frac{2}{3}$, $q_d = -\frac{1}{3}$ and $q_l = -1$?
- Number of generations, why $N = 3$?
- Why is the top quark so heavy ($m_t \simeq 174$ GeV while $m_b \simeq 4.3$ GeV) ?

$$m_t \simeq v = \langle 0|H|0\rangle \simeq 246 \text{ GeV}$$

$$\Downarrow$$

top quark is very different (possibly sensitive to the mechanism of gauge symmetry breaking)

- Mixing angles and fermion masses:

$$\mathcal{L} \supset -\sum_{i,j=1}^{3} \left(\tilde{\Gamma}_{ij} \bar{u}_{i\,R} \tilde{H}^{\dagger} Q_{j\,L} + \Gamma_{ij} \bar{d}_{i\,R} H^{\dagger} Q_{j\,L} + \text{H.c.} \right)$$

$$\Downarrow$$

$$\mathcal{L}_{\text{q mass}} = -\sum_{i,j=1}^{3} \left(\bar{u}_{i\,R} \mathcal{M}^{u}_{ij} u_{j\,L} + \bar{d}_{i\,R} \mathcal{M}^{d}_{ij} d_{j\,L} + \text{H.c.} \right) \quad \text{for} \quad \mathcal{M}^{u}_{ij} = \frac{v}{\sqrt{2}} \tilde{\Gamma}_{ij}, \ \mathcal{M}^{d}_{ij} = \frac{v}{\sqrt{2}} \Gamma_{ij}$$

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}_{L,R} = U_{L,R} \begin{pmatrix} u \\ c \\ t \end{pmatrix}_{L,R} \qquad \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}_{L,R} = D_{L,R} \begin{pmatrix} d \\ s \\ b \end{pmatrix}_{L,R}$$

$$U_R^{\dagger} \mathcal{M}^u U_L = \text{diag}(m_u, m_c, m_t) \qquad D_R^{\dagger} \mathcal{M}^d D_L = \text{diag}(m_d, m_s, m_b)$$

$$\Downarrow$$

$$\sum \bar{u}_{i\,L} \gamma^{\mu} d_{i\,L} = (\bar{u}, \bar{c}, \bar{t})_L \underbrace{U_L^{\dagger} D_L}_{U_{CKM}} \gamma^{\mu} \begin{pmatrix} d \\ s \\ b \end{pmatrix}_L$$

It is natural to expect that $U_{CKM} = U_{CKM}(m_q/m_q')$.

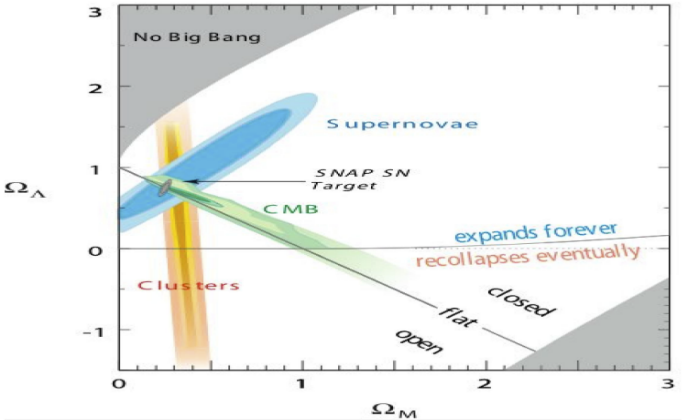
♠ Parameters of the SM:

$$\begin{array}{cccccc} m_e & m_\mu & m_\tau & m_u & m_c & m_t \\ m_{\nu_e} & m_{\nu_\mu} & m_{\nu_\tau} & m_d & m_s & m_b \end{array}$$

$$\underbrace{g \quad , \quad g'}_{(\alpha_{QED}, \sin\theta_W)} , \underbrace{g_s}_{(\alpha_{QCD})} , \underbrace{m_h, \ \lambda}_{(\mu, \lambda)} , \underbrace{U_{CKM}}_{\theta_{1,2,3}, \delta_{CP}}$$

21 parameters !

♠ Cosmology:



$$\Omega_i \equiv \frac{\rho_i}{\rho_c} \qquad \text{for} \qquad \rho_c = \frac{3H_0^2}{8\pi G_N}$$

data $\Rightarrow$ $\Omega_\Lambda = \frac{\Lambda}{3H_0^2} \simeq 70\%$, $\Omega_{DM} \simeq 27\%$ and $\Omega_B \simeq 3\%$

- SM has no candidate for dark matter
- $\Omega_\Lambda = \frac{\rho_\Lambda}{\rho_c} \simeq 0.7 \quad \Rightarrow \quad \rho_\Lambda \simeq 10^{-120} M_{Pl}^4 = (10^{-3}\ \text{eV})^4$ while typical scale of the SM is $\mathcal{O}(100\ \text{GeV})$! Fine tuning again!
- Inflation: period of fast expansion of the very early Universe, $a(t) \propto \exp\left(\sqrt{\frac{\Lambda}{3}}\, t\right)$
  Again the SM has no means to explain the inflation (no inflaton in the SM). For a typical inflaton $m_\phi \sim 10^{13}$ GeV and $\lambda \sim 10^{-13}$, so the SM Higgs boson is not an inflaton (assuming standard interactions with gravity).
- Baryogenesis and SM CP violation $\eta \equiv \frac{n_b - n_{\bar{b}}}{n_\gamma} \simeq \frac{n_b}{n_\gamma} \simeq 6 \cdot 10^{-10}$
  The Sakharov's necessary conditions for baryogenesis:
    - B number violation
    - C and CP violation
    - Departure from thermal equilibrium

  SM:
    - B number violation: **OK**
    - C and CP violation: too weak CP violation $\propto \Im Q$, for $Q \equiv U_{ud} U_{cb} U^\star_{ub} U^\star_{cd}$ (re-phasing invariant)
    - Departure from thermal equilibrium: first-order electro-weak phase transition requires $m_h \lesssim 72$ GeV

  Conclusion: the SM doesn't explain the baryogenesis
- Why is gravity so weak? Or, why $M_{Pl} = 10^{19}$ GeV $\gg v = 246$ GeV?

The interaction rates Γ_i

♠ Definition of the cross-section:
The S-matrix element $w_{i\to f}$ gives the probability for the transition to occur:

$$P_{i\to f} = |w_{i\to f}|^2 = |\langle f|i\rangle|^2$$

The translational invariance allows to write the matrix element as

$$w_{i\to f} = \delta_{if} + i(2\pi)^4\delta^4(p_f - p_i)T_{i\to f}$$

The above formula defines the transition matrix T.
Let's consider the following scattering process

$$a + b \to c_1 + c_2 + \cdots + c_n$$

We assume that b is at rest, and the velocity of a is $v = |\vec{p}_a|/E_a$. The number of particles b per target volume is (that defines the normalization of plane waves $\propto (2E)^{-1/2}e^{ipx}$): $2E_b = 2m_b$ as b is at rest. The incident flux is the velocity $(\vec{p}_a/E_a)$ of a times their number density $2E_a$, so $2|\vec{p}_a|$.

If the reaction volume is V and the reaction takes place during the time T, then *the cross-section σ is defined* such that the transition probability per unit time and unit volume equals the target density $\times$ the incident flux $\times$ the cross-section σ, that is, $2m_b \times 2|\vec{p}_a| \times \sigma$. On the other hand it is equal to $|w_{i\to f}|^2/(VT)$. Hence summing over all available momenta for the final state we get

$$\begin{aligned}\sigma(a+b \quad &\rightarrow \quad c_1 + c_2 + \cdots + c_n) = \\ &= \frac{1}{4m_b|\vec{p}_a|} \int \prod_{j=1}^{n} \frac{d^3 p_j}{2E_j(2\pi)^3} (2\pi)^4 \delta^4(p_a + p_b - p_1 - \cdots - p_n) |\tilde{T}|^2\end{aligned}$$

where for unpolarized initial state we have

$$|\tilde{T}|^2 = \frac{1}{S} \frac{1}{(2s_a+1)(2s_b+1)} \sum_{\text{spins}} |T_{i\to f}|^2$$

The spins of initial states are denoted by s_a and s_b. The symmetry factor S appears because in quantum mechanics we can't distinguish between two final states which differ only by an exchange of identical particles, in general, if there are k groups of n_i $(i = 1, 2, \ldots, k)$ identical particles in the final state, one has $S = n_1!\, n_2! \ldots n_k!$.

In order to have the cross-section in a Lorentz invariant form one has to replace

$$m_b|\vec{p}_a| \rightarrow \left[(p_a \cdot p_b)^2 - m_a^2 m_b^2\right]^{1/2}$$

For decays

$$a \to c_1 + c_2 + \cdots + c_n$$

we get instead of the cross-section the decay width

$$\begin{aligned}\Gamma(a \quad \to \quad & c_1 + c_2 + \cdots + c_n) = \\ & = \frac{1}{4m_a} \int \prod_{j=1}^{n} \frac{d^3 p_j}{2E_j (2\pi)^3} (2\pi)^4 \delta^4 (p_a - p_1 - \cdots - p_n) |\tilde{T}|^2\end{aligned}$$

for

$$|\tilde{T}|^2 = \frac{1}{S} \frac{1}{(2s_a + 1)} \sum_{\text{spins}} |T_{i \to f}|^2$$

Summing over all final states we get the total width

$$\Gamma_{\text{tot}} = \sum_{\text{final states f}} \Gamma(a \to f)$$

Then the life time is given by

$$\tau = \frac{1}{\Gamma_{\text{tot}}}$$

while the branching ratio reads

$$BR(a \to f) = \frac{\Gamma(a \to f)}{\Gamma_{\text{tot}}(a)}$$

♠ Strong and Electroweak Transitions:
Estimates of cross-sections:

- $$\sigma_{\text{em}}(e^+e^- \to \mu^+\mu^-) \sim \left(\frac{e^2}{4\pi}\right)^2 \frac{1}{s} \qquad \text{for} \qquad s \equiv (p_{e^+} + p_{e^-})^2 \gg m_e^2$$

  where $\frac{e^2}{4\pi} \equiv \alpha_{\text{QED}} \simeq \frac{1}{128}$, for $\sqrt{s} \simeq 100$ GeV.
- $$\sigma_{\text{strong}}(q\bar{q} \to q\bar{q}) \sim \left(\frac{g_{\text{QCD}}^2}{4\pi}\right)^2 \frac{1}{s} \qquad \text{for} \qquad s \gg m_q^2$$

  where $\frac{g_{\text{QCD}}^2}{4\pi} \equiv \alpha_{\text{QCD}} \simeq 10^{-1}$,
- $$\sigma_{\text{weak}}(\nu_e + e^+ \to \nu_\mu + \mu^+) \sim \left(\frac{g_{\text{weak}}^2}{4\pi}\right)^2 \frac{s}{(s - m_W^2)^2}$$

  where $\frac{g_{\text{weak}}^2}{4\pi} = \frac{e^2}{4\pi \sin^2\theta_W} = \frac{\alpha_{\text{QED}}}{0.23}$

♠ The Interaction Rate:

If interactions between species are fast enough they could be in local/kinetic equilibrium (state of maximal entropy). The reaction rate responsible for establishing equilibrium can be characterized by the *collision time*:

$$t_c \equiv 1/(n\sigma v)$$

where σ is the cross-section, n is the number density of target particles and v is the relative velocity. Note that $\sigma = 1/(n\lambda)$, where λ is a mean free path, so $n\sigma v = v/\lambda$ is roughly a number of collisions per time, while its inverse is a time per collision. For estimates we will be using an equilibrium number density. In order to maintain the equilibrium this time must be much shorter than the Universe age $t_H \sim H^{-1}$:

$$t_c \ll t_H \tag{1}$$

Then the local equilibrium is reached before the expansion becomes relevant.

High-energy example

Let's consider $T \gtrsim 500$ GeV, then the cross-section for strong and electroweak interactions could be estimated applying just dimensional analysis for typical energy-momentum $p \sim T$ (masses are irrelevant at that energy range)

$$\sigma \sim \alpha^2/T^2$$

where α is the fine structure constant $\alpha \simeq 10^{-1} - 10^{-2}$.

Taking into account that the equilibrium number density of relativistic species behaves (see next section for details) as $n \sim a^{-3} \sim T^3$ we obtain

$$t_c \sim \frac{1}{\alpha^2 T}$$

If the universe is dominated by a single relativistic species then we have (see next section for details)

$$t_H \sim \frac{1}{H} \sim \frac{1}{(\rho_{\rm rad}/M_{Pl}^2)^{1/2}} \sim \frac{M_{Pl}}{T^2},$$

where we have introduced the Planck mass defined as $M_{Pl} \equiv G^{-1/2}$. Hence we can see that the collision (reaction) time t_c decreases slower (when T increases) than the Hubble time t_H, so if T is too large then (1) can not be satisfied. Note that since $\rho_{\rm rad} \sim T^4$ during the radiation dominated epoch we have $H \sim T^2/M_{Pl}$ (see next section for details). Therefore at temperatures $T \sim \alpha^2 M_{Pl} \simeq 10^{15} - 10^{17}$ GeV, we obtain $t_c \simeq t_H$. So for $T \lesssim 10^{15} - 10^{17}$ GeV but above few hundred GeV (where $\sigma \sim \frac{\alpha^2}{T^2}$) the inequality (1) is satisfied and the Universe made of quarks, leptons, gauge bosons and Higgses remains in equilibrium. Above 10^{17} GeV the interaction that we know are too slow too keep the universe in equilibrium.

Low-energy example

For $\sqrt{s} \ll 100$ GeV, the masses of gauge bosons $W^{\pm}$ ($m_W \simeq 80.4$ GeV) and Z ($m_Z \simeq 91.2$ GeV) become relevant and the cross-section for e.g. $\sigma(\nu_e + e^+ \to \nu_\mu + \mu^+)$ scales as $\alpha_{\rm weak}^2 T^2/m_W^4$, so

$$t_c \sim \frac{1}{\alpha_{\rm weak}^2} \left(\frac{m_W}{T}\right)^4 \frac{1}{T}$$

Again assuming the universe is dominated by a single relativistic species we find that in order to have $t_c \ll t_H$ one needs

$$T \gg 3.5 \text{ MeV}$$

For lower temperatures the weak interaction becomes too slow to maintain the equilibrium, as a consequence, e.g. neutrinos decouple at $T \simeq 1$ MeV (more on that later).

Assumptions

- The Universe is a dilute and weakly interacting gas.
- If rates of interactions between constituents of the Universe are large enough, then we assume the Universe is in *local/kinetic equilibrium* (so the state of maximal entropy, see Mukhanov for detailed discussion).

Then the number density n_i, the energy density ρ_i, and the pressure for particles with g_i internal degrees of freedom (massless gauge boson has g=2, massive gauge boson has g=3, massless fermion has $g = 1$, massive fermion has $g = 2$, same for anti-fermions) are given by the following integrals of the expected number density of particles in states with energy E_i (phase space distribution or occupancy functions) $f_i(\vec{p}, T)$:

$$n_i(T) = g_i \int f_i(\vec{p}, T) \frac{d^3p}{(2\pi)^3} \tag{2}$$

$$\rho_i(T) = g_i \int E_i(\vec{p}) f_i(\vec{p}, T) \frac{d^3p}{(2\pi)^3} \quad \text{for} \quad E_i(\vec{p}) = (|\vec{p}|^2 + m_i^2)^{1/2} \tag{3}$$

$$p_i(T) = g_i \int \frac{|\vec{p}|^2}{3E_i(\vec{p})} f_i(\vec{p}, T) \frac{d^3p}{(2\pi)^3} \tag{4}$$

(See tutorials for the derivation of (4).)

The phase space distribution (the expected number of particles in an energy state) is given by the Fermi-Dirac (for fermions, + sign below) or Bose-Einstein (for bosons, − sign below) distributions

$$f_i(\vec{p}, T) = \frac{1}{e^{[E_i(\vec{p})-\mu_i]/T} \pm 1}$$

where μ_i is the chemical potential of the species, for our unit choice $k_B = 1$. It will be usually assumed that μ_i can be neglected in the early Universe. Performing the angular integrations and changing variables from $|\vec{p}|$ to $E = (|\vec{p}|^2+m^2)^{1/2}$, so $|\vec{p}|d|\vec{p}|= EdE$, so that $d^3p \to 4\pi(E^2 - m^2)^{1/2}EdE$ and we obtain

$$n(T) = \frac{g}{2\pi^2}\int_m^\infty \frac{(E^2-m^2)^{1/2}}{\exp[E-\mu]/T \pm 1}EdE$$

$$\rho(T) = \frac{g}{2\pi^2}\int_m^\infty \frac{(E^2-m^2)^{1/2}}{\exp[E-\mu]/T \pm 1}E^2dE$$

$$p(T) = \frac{g}{6\pi^2}\int_m^\infty \frac{(E^2-m^2)^{3/2}}{\exp[E-\mu]/T \pm 1}dE$$

In the relativistic limit ($T \gg m$) with $\mu = 0$ we get (see class)

$$n(T) = \begin{cases} \frac{\zeta(3)}{\pi^2} g T^3 & \text{bosons} \\ \frac{3}{4}\frac{\zeta(3)}{\pi^2} g T^3 & \text{fermions} \end{cases} \quad \rho(T) = \begin{cases} \frac{\pi^2}{30} g T^4 & \text{bosons} \\ \frac{7}{8}\frac{\pi^2}{30} g T^4 & \text{fermions} \end{cases} \quad p(T) = \frac{\rho(T)}{3} \tag{5}$$

where $\zeta(3) = 1.202\ldots$ is the Riemann zeta function of 3.

In the non-relativistic limit ($T \ll m$) there is no difference between fermions and bosons, result for $\mu = 0$ reads (see class)

$$n(T) = g\left(\frac{mT}{2\pi}\right)^{3/2} \exp(-m/T), \qquad \rho(T) = m\, n(T), \qquad p(T) = n(T)T \ll \rho(T) \tag{6}$$

For relativistic species the average energy per particle reads

$$\langle E \rangle \equiv \frac{\rho}{n} = \begin{cases} \frac{\pi^4}{30\zeta(3)} T \simeq 2.701\ T & \text{for bosons} \\ \frac{7\pi^4}{180\zeta(3)} T \simeq 3.151\ T & \text{for fermions} \end{cases} \tag{7}$$

For the rhs of Friedmann equations we need the total contribution to the energy density and the pressure, that is

$$\rho_{\text{tot}} = T^4 \sum_i \left(\frac{T_i}{T}\right)^4 \frac{g_i}{2\pi^2} \int_{x_i}^{\infty} \frac{(y^2 - x_i^2)^{1/2} y^2 dy}{\exp(y) \pm 1} \tag{8}$$

$$p_{\text{tot}} = T^4 \sum_i \left(\frac{T_i}{T}\right)^4 \frac{g_i}{2\pi^2} \int_{x_i}^{\infty} \frac{(y^2 - x_i^2)^{3/2} y^2 dy}{\exp(y) \pm 1} \tag{9}$$

where $x_i \equiv m_i/T$ and $y = E/T$, and it has been taken into account that some species may have decoupled (maintaining an equilibrium distribution) so that they may have different "temperatures" T_i.

Note that at a given temperature the ratio of the energy density for non-relativistic species to the relativistic one reads

$$\frac{\rho_{\text{nrel}}}{\rho_{\text{rel}}} \propto \left(\frac{m}{T}\right)^{5/2} e^{-m/T}$$

For the species to be non-relativistic one needs $m \gg T$ so the $e^{-m/T}$ is a strong suppression factor, therefore we will neglect contributions from non-relativistic species while calculating total energy density. In that case we get

$$\rho_{\text{tot}}(T) = \frac{\pi^2}{30} g_\star T^4 \qquad \text{and} \qquad p(T) = \frac{\rho(T)}{3} = \frac{\pi^2}{90} g_\star T^4 \tag{10}$$

where $g_\star$ counts only massless/relativistic ($m_i \ll T$) degrees of freedom:

$$g_\star = \sum_{\text{bosons}} g_i \left(\frac{T_i}{T}\right)^4 + \frac{7}{8} \sum_{\text{fermions}} g_i \left(\frac{T_i}{T}\right)^4 \tag{11}$$

Note that $g_\star = g_\star(T)$ is a function of temperature. An exact form of $g_\star(T)$ could be easily (see tutorials) obtained from (8) and (9). For $T \gg 100$ MeV $g_\star = 106\frac{3}{4}$, for $T \ll 1$ MeV $g_\star = 3.36$, while for $100 \text{ MeV} \gtrsim T \gtrsim 1$ MeV one gets $g_\star = 10\frac{3}{4}$ (see tutorials).

particle	flavour	spin	colour	particle + anti-particle	total
quarks(u, d, c, s, t, b)	6	2	3	2	72
charged leptons (e, μ, τ)	3	2	1	2	12
neutrinos (ν_e, ν_μ, ν_τ)	3	1	1	2	6
gluons (g)	1	2	8	1	16
photon (γ)	1	2	1	1	2
charged massive gauge bosons ($W^\pm$)	1	3	1	2	6
neutral massive gauge bosons (Z)	1	3	1	1	3
Higgs boson (H)	1	1	1	1	1

Table 1: Standard Model internal degrees of freedom, 118 total.

Note that a single flavour neutrino is contributing only 1 dof and anti-neutrino another 1. This is because in the SM there are only left-handed neutrinos (1 dof) and right-handed anti-neutrinos.

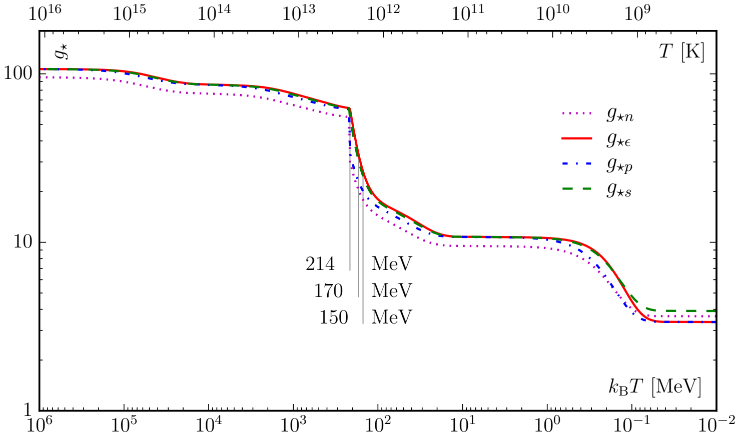


Figure 1: The evolution of the number density ($g_{\star n}$), energy density ($g_{\star\epsilon}$), pressure ($g_{\star p}$), and entropy density ($g_{\star s}$) as functions of temperature.

(plot from L. Husdal, "On Effective Degrees of Freedom in the Early Universe", Galaxies **4**, no. 4, 78 (2016), doi:10.3390/galaxies4040078, arXiv:1609.04979)

During the radiation dominated epoch ($t \lesssim 4 \times 10^{10}$ s, see class for this number), $\rho_{\rm tot} = \rho_{\rm rad}$ hence, for $k = 0$, inserting (10) into the Friedmann equation one gets the very important formula for the physics of early Universe:

$$H = \left[\frac{8\pi G}{3}\rho_{\rm tot}(T)\right]^{1/2} = \left[\frac{8\pi G}{3}\frac{\pi^2}{30}g_\star T^4\right]^{1/2} = 1.66\frac{g_\star^{1/2}T^2}{M_{Pl}}$$

For the radiation dominated Universe we have obtained earlier the following time dependence of the scale factor

$$a(t) \propto t^{1/2}$$

So, for the radiation domination one has

$$H \equiv \frac{\dot{a}}{a} = \frac{1}{2t}$$

Hence the following time – temperature relation could be obtained

$$t = 0.30\frac{M_{Pl}}{g_\star^{1/2}T^2} = \left(\frac{5.2}{g_\star}\right)^{1/2}\left(\frac{1\ \text{MeV}}{T}\right)^2 \text{s} \sim \left(\frac{1\ \text{MeV}}{T}\right)^2 \text{s},$$

where in the last step $g_\star \sim 5.2$ was adopted, note that for $100\ \text{MeV} \gtrsim T \gtrsim 1$ MeV one gets $g_\star = 10\frac{3}{4}$, while for $T \ll 1$ MeV $g_\star = 3.36$. The above is a useful formula to memorize as $T \simeq 1$ MeV is a very important temperature in the evolution of the early Universe.

Distribution functions in expanding Universe

The momentum of freely moving particles redshifts with the expansion of the universe as follows (see class):

$$\vec{p}(t_1) = \frac{a(t_2)}{a(t_1)}\,\vec{p}(t_2)$$

while the physical coordinates (position vectors) scale as

$$\vec{x}(t_1) = \frac{a(t_1)}{a(t_2)}\,\vec{x}(t_2)$$

Massless particles

We will show that relativistic non-interacting particles that decoupled from the thermal bath preserve equilibrium distribution during the expansion of the universe.

At moment t_1 a phase space element $d^3p_1 d^3x_1$ contains

$$dn = \frac{g}{(2\pi)^3} f(\vec{p}_1) d^3p_1 d^3x_1$$

particles with distribution (note that $E_1 = |\vec{p}_1| \equiv p_1$ for relativistic particles) at the time t_1

$$f(\vec{p}_1) = \frac{1}{e^{(p_1 - \mu_1)/T_1} \pm 1} \qquad (12)$$

At time t_2 these same dn particles are in a phase space element $d^3p_2 d^3x_2$. We will find out how are the distributions at t_2 and t_1 related. For $f(\vec{p}_2)$ we have

$$f(\vec{p}_2) = \frac{(2\pi)^3}{g} \frac{dn}{d^3p_2 d^3x_2} \qquad (13)$$

Since the phase space volumes scale as

$$d^3p_1 = \left(\frac{a(t_2)}{a(t_1)}\right)^3 d^3p_2 \qquad \text{and} \qquad d^3x_1 = \left(\frac{a(t_1)}{a(t_2)}\right)^3 d^3x_2$$

therefore

$$\begin{aligned} dn &= \frac{g}{(2\pi)^3} \frac{d^3p_1 d^3x_1}{e^{(p_1 - \mu_1)/T_1} \pm 1} \\ &= \frac{g}{(2\pi)^3} \frac{\left(\frac{a(t_2)}{a(t_1)}\right)^3 d^3p_2 \left(\frac{a(t_1)}{a(t_2)}\right)^3 d^3x_2}{e^{(\frac{a(t_2)}{a(t_1)} p_2 - \mu_1)/T_1} \pm 1} = \frac{g}{(2\pi)^3} \frac{d^3p_2 d^3x_2}{e^{(p_2 - \mu_2)/T_2} \pm 1}, \end{aligned}$$

The latter line determines the temperature and chemical potential at time t_2

$$T_2 = \frac{a(t_1)}{a(t_2)} T_1 \qquad \text{and} \qquad \mu_2 = \frac{a(t_1)}{a(t_2)} \mu_1$$

So, the distribution retains its thermal character (although particles have decoupled) at red-shifted temperature and chemical potentials

$$T(t) \propto \frac{1}{a(t)} \qquad \text{and} \qquad \mu(t) \propto \frac{1}{a(t)}$$

Massive, non-relativistic particles

Now we assume that particles are decoupling from the thermal bath while being non-relativistic.

The phase space distribution (the expected number of particles in an energy state) is given by the Fermi-Dirac (for fermions, **+** sign below) or Bose-Einstein (for bosons, $-$ sign below) distributions

$$f(\vec{p}, T) = \frac{1}{e^{[E(\vec{p})-\mu]/T} \pm 1}$$

Assuming ± 1 in the equilibrium distribution could be neglected we obtain in the non-relativistic regime

$$f(\vec{p}_1) = e^{-\frac{m-\mu_1}{T_1}} \; e^{-\frac{\vec{p}_1^2}{2mT_1}}$$

From (13) one finds

$$f(\vec{p}_2) = e^{-\frac{m-\mu_1}{T_1}} \; e^{-\frac{a^2(t_2)\;\vec{p}_2^2}{2m\;a(t_1)^2\,T_1}}$$

The above could be rewritten as follows

$$f(\vec{p}_2) = e^{-\frac{m-\mu_2}{T_2}} \; e^{-\frac{\vec{p}_2^2}{2mT_2}}$$

where

$$T_2 = T_1 \left(\frac{a(t_1)}{a(t_2)} \right)^2 \qquad \text{and} \qquad \frac{m-\mu_2}{T_2} = \frac{m-\mu_1}{T_1}$$

So, the distribution function still has the same form of equilibrium distribution (although particles have decoupled) however the temperature evolves as

$$T(t) \propto \frac{1}{a^2(t)}$$

Entropy

Let's define the entropy through its differential

$$TdS(V,T) \equiv d[\rho(T)V] + p(T)dV = Vd\rho + (\rho+p)dV = V\frac{d\rho}{dT}dT + (\rho+p)dV \quad (14)$$

In general we have

$$dS(V,T) = \frac{\partial S(V,T)}{\partial T}dT + \frac{\partial S(V,T)}{\partial V}dV$$

So, we get from (14)

$$\frac{\partial S(V,T)}{\partial T} = \frac{V}{T}\frac{d\rho(T)}{dT} \qquad \text{and} \qquad \frac{\partial S(V,T)}{\partial V} = \frac{1}{T}[\rho(T)+p(T)]$$

The integrability condition tells us that

$$\frac{\partial^2 S(V,T)}{\partial T \partial V} = \frac{\partial^2 S(V,T)}{\partial V \partial T} \quad \Rightarrow \quad \frac{\partial}{\partial T}\left[\frac{1}{T}[\rho(T)+p(T)]\right] = \frac{\partial}{\partial V}\left[\frac{V}{T}\frac{d\rho(T)}{dT}\right]$$

$$\Downarrow$$

$$\frac{dp(T)}{dT} = \frac{1}{T}\left[\rho(T)+p(T)\right] \quad \Rightarrow \quad dp(T) = \frac{\rho(T)+p(T)}{T}dT \quad (15)$$

Rewriting dS from (14) as

$$dS = \frac{1}{T} d\left[(\rho + p)V\right] - dp\frac{V}{T}$$

and inserting (15) we get

$$dS = \frac{1}{T} d\left[V(\rho + p)\right] \underbrace{- \frac{dT}{T^2}}_{d\left(\frac{1}{T}\right)} V\left[\rho(T) + p(T)\right] = d\left\{\frac{V}{T}[\rho(T) + p(T)] + \text{const.}\right\}$$

So the entropy, up to an integration constant is given by

$$S(V, T) = \frac{V}{T}[\rho(T) + p(T)]$$

Recall now the "first law of thermodynamics" (equivalently $T^{\mu\nu}{}_{;\nu} = 0$)

$$a^3 \frac{dp(T)}{dt} = \frac{d}{dt}\left\{a^3[\rho(T) + p(T)]\right\}$$

Combining with (15) we get

$$a^3 \underbrace{\frac{1}{T}\frac{dT}{dt}}_{-T\frac{d}{dt}\left(\frac{1}{T}\right)} [\rho(T) + p(T)] = \frac{d}{dt}\left\{a^3[\rho(T) + p(T)]\right\}$$

Hence

$$\frac{d}{dt}\left\{\frac{a^3}{T}[\rho(T) + p(T)]\right\} = 0$$

Therefore, identifying volume with a^3 we can conclude that the entropy of the volume V is conserved. It proves useful to define the entropy density

$$s(T) \equiv \frac{S(T)}{V} = \frac{\rho(T) + p(T)}{T}$$

Since relativistic particles dominate both $\rho(T)$ and $p(T)$, the same happens for the entropy density. Using (5) one gets:

$$s = \frac{2\pi^2}{45} g_{\star\, S} T^3$$

where

$$g_{\star\, S} = \sum_{\text{bosons}} g_i \left(\frac{T_i}{T}\right)^3 + \frac{7}{8} \sum_{\text{fermions}} g_i \left(\frac{T_i}{T}\right)^3 \tag{16}$$

Since $n_\gamma \propto T^3$:

$$n_\gamma = \frac{2\zeta(3)}{\pi^2} T^3$$

therefore one can derive the following relation

$$s = \frac{\pi^4}{45\zeta(3)} g_{\star\, S} n_\gamma \simeq 1.8 g_{\star\, S} n_\gamma$$

Note that the entropy conservation implies that $g_{\star\, S} T^3 a^3 = \text{const.}$, therefore in the early Universe ($a \sim 0$) the temperature was maximal (roughly $T \propto a^{-1}$), consequently all species can be treated as highly relativistic.

Let's now illustrate the possibility of some species having different temperatures by the decoupling of neutrinos at about $T \sim 1$ MeV. For weak interactions we had

$$\sigma_{\text{weak}}(e^+ + e^- \to \nu_i + \bar{\nu}_i) \sim \left(\frac{g_{\text{weak}}^2}{4\pi}\right)^2 \frac{s}{(s - m_Z^2)^2} \overset{s \ll m_Z^2}{\simeq} \left(\frac{g_{\text{weak}}^2}{4\pi}\right)^2 \frac{s}{m_Z^4}$$

So, since $\langle E \rangle \sim 3T$ therefore at $T \ll m_Z$ we get

$$\sigma_{\text{weak}}(e^+ + e^- \to \nu_i + \bar{\nu}_i) \simeq \left(\frac{g_{\text{weak}}^2}{4\pi}\right)^2 \frac{T^2}{m_Z^4}$$

Since the interaction rate $\Gamma_{\text{int}} \equiv t_c^{-1} = n\sigma v$ therefore we get for $n \sim T^3$ and $v \simeq 1$

$$\Gamma_{\text{int}} \simeq \frac{\alpha_{\text{weak}}^2 T^5}{m_Z^4} \simeq G_F^2 T^5$$

where $G_F = 1.1664 \times 10^{-5}$ GeV^{-2} is the Fermi constant ($G_F/\sqrt{2} \equiv g_{weak}^2/(8m_W^2)$). Let's compare the interaction rate with the expansion rate $H \sim g_\star^{1/2} T^2/M_{Pl}$

$$\frac{t_H}{t_c} = \frac{\Gamma_{\text{int}}}{H} \simeq \frac{G_F^2 T^5}{g_\star^{1/2} T^2/M_{Pl}} \simeq \frac{G_F^2 T^5}{T^2/M_{Pl}} \simeq \left(\frac{T}{0.7\ \text{MeV}}\right)^3$$

So, at $T \lesssim 1$ MeV the interactions are too slow to provide an equilibrium between leptons and neutrinos. Neutrinos decouple ("the freeze-out") from the SM and evolve separately, so the possibility for neutrinos to have different temperature appears. Their energy (temperature) is being redshifted the same way as for photons

$$T_\nu = T_{\text{dec}} \frac{a_{\text{dec}}}{a} \propto \frac{1}{a}$$

Let's investigate consequences of entropy conservation <u>for the thermal bath</u>, i.e. photons and $e^\pm$:

$$g_{\star\, S}(aT)^3 = \text{const.} \qquad \Rightarrow \qquad T \sim (g_{\star\, S})^{-1/3} \frac{1}{a}$$

As long as $(g_{\star\, S})$ does not change the thermal bath temperature changes only as a consequence of the expansion, i.e. $T \propto a^{-1}$, the same way the neutrino temperature evolves. However around the same temperature neutrinos decouple, electrons become non-relativistic $m_e \simeq 0.5$ MeV so that the number of relativistic degrees of freedom (rdf) $g_{\star\, S}$ drops. $e^\pm$ annihilate $e^+ e^- \to \gamma\gamma$, while the inverse process is being suppressed as the averaged energy decreases roughly below $2m_e$.

Therefore:

- for $T \gtrsim 2m_e \simeq 1$ MeV:

$$g_{\star\, S} = \sum_{\text{bosons}} g_i \left(\frac{T_i}{T}\right)^3 + \frac{7}{8} \sum_{\text{fermions}} g_i \left(\frac{T_i}{T}\right)^3 = 2 + \frac{7}{8} \times 4 = \frac{11}{2}$$

- for $T \ll 2m_e$:

$$g_{\star\, S} = 2$$

From continuity of the entropy we get the following condition

$$\left[g_{\star\, S}(aT)^3\right]_{\text{before}} = \left[g_{\star\, S}(aT)^3\right]_{\text{after}}$$

which implies

$$\frac{11}{2}(aT)^3_{\text{before}} = 2(aT)^3_{\text{after}} \qquad \Rightarrow \qquad T_{\text{before}} = \left(\frac{4}{11}\right)^{1/3} T_{\text{after}}$$

For the temperature "before", the neutrinos even though they decoupled a bit earlier, have the same temperature as photons, however at $T \sim 2m_e$ photons are heated up by $e^+e^- \to \gamma\gamma$ as the entropy is transferred (since it is a continuous function of T) from e^+e^- to photons. The already decoupled neutrinos do not benefit from that reheating, since they do not interact with the thermal bath (photons and electrons) any more (in other words the entropy of neutrinos is conserved separately after the decoupling). Consequently there is a difference in temperatures for neutrinos and photons after e^+e^- freeze-out:

$$T_\nu = \left(\frac{4}{11}\right)^{1/3} T_\gamma$$

Strictly speaking photon's temperature does not jump at $T = 2m_e$, but rather starts to decrease slower already at temperatures slightly above $T = 2m_e$ (in reality the freeze-out process is smooth and starts already before $T = 2m_e$). So, for CMB photons of temperature $T_\gamma = 2.73$ K, there should be also *the cosmic neutrino background* of temperature $T_\nu = 1.95$ K.

Let's now determine the present energy density, number density and entropy density for CMB photons and neutrinos assuming $T_0 = 2.75$ K.

	γ	ν
$g_\star = \sum_b g_i \left(\frac{T_i}{T}\right)^4 + \frac{7}{8}\sum_f g_i \left(\frac{T_i}{T}\right)^4$	2	$\frac{7}{8} \cdot 2 \cdot 3 \cdot \left(\frac{4}{11}\right)^{4/3} = 1.36$
$g_{\star\,S} = \sum_b g_i \left(\frac{T_i}{T}\right)^3 + \frac{7}{8}\sum_f g_i \left(\frac{T_i}{T}\right)^3$	2	$\frac{7}{8} \cdot 2 \cdot 3 \cdot \frac{4}{11} = 1.91$
$\rho = \frac{\pi^2}{30} g_\star T^4$	$4.64 \cdot 10^{-34}$ g cm^{-3}	$3.16 \cdot 10^{-34}$ g cm^{-3}
$n = \frac{2\zeta(3)}{\pi^2} T^3$	410 cm^{-3}	149 cm^{-3}
$s = \frac{2\pi^2}{45} g_{\star\,S} T^3$	1478 cm^{-3}	1412 cm^{-3}
$\Omega h^2 = \rho \frac{8\pi G}{3H_0^2}$	$2.47 \cdot 10^{-5}$	$1.68 \cdot 10^{-5}$

Table 2: Present Universe parameters for massless neutrinos.

I used the following conversion factors:
1 K = 4.3668 cm^{-1} = $8.6170 \cdot 10^{-14}$ GeV = $1.5361 \cdot 10^{-37}$ g,
1 Mpc = $1.5637 \cdot 10^{38}$ GeV^{-1}, G = $6.7065 \cdot 10^{-39}$ GeV^{-2} and
$H_0 = h\, 2.1317 \cdot 10^{-42}$ GeV.

There exists also a possibility for another kind of radiation present as a relic of the early Universe, this is the graviton, the massless quantum fluctuation of the gravitational field. The reaction responsible for maintaining the equilibrium would be e.g. $\bar{\psi}\psi \leftrightarrow hh$, where h is the graviton and ψ is a massless fermion. Gravitons $h_{\mu\nu}$ interact with ordinary matter through the standard Lagrangian $\propto 1/M_{Pl} \times h_{\mu\nu} T^{\mu\nu}$, where $T^{\mu\nu}$ is the energy momentum tensor, therefore the reaction width (the inverse of the reaction rate) is

$$\Gamma_{\text{grav}} = n\sigma v \sim T^3 \frac{T^2}{M_{Pl}^4} \sim \frac{T^5}{M_{Pl}^4}$$

Since at the early Universe $H \sim g_\star^{1/2} T^2/M_{Pl}$ therefore we get ($g_\star^{1/2} \sim 10$ for the SM at $T \gtrsim 100$ GeV)

$$\frac{t_H}{t_c} = \frac{\Gamma_{\text{grav}}}{H} \sim \frac{1}{10}\left(\frac{T}{M_{Pl}}\right)^3$$

So gravitons freeze-out roughly at the Planck temperature $T \sim 2M_{Pl} \sim 10^{19}$ GeV. Using the continuity of entropy at the moment of graviton freeze-out and all the SM thresholds we get the relation between graviton temperature and the CMB photon temperature at the present moment (see class for the discussion):

$$T_{\rm grav} = \left(\frac{g_{\star S}^{\rm now}}{g_{\star S}^{\rm Planck}}\right)^{1/3} \cdot T_0 \simeq 1 \text{ K}$$

where we have approximated $g_{\star S}^{\rm Planck}$ by its SM value for $T \gtrsim 100$ GeV, i.e. ~ 100. Their contribution to the present energy density is $\rho_{\rm grav} \sim T^4 \sim 0.018\rho_\gamma$.