Brief "Course" of General Relativity and the Friedmann-Lemaître-Robertson-Walker (FLRW) Metric

Content

- The Principle of Equivalence and the Principle of General Covariance
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- The Friedmann-Lemaître-Robertson-Walker Metric and the Friedmann Equations

The Principle of Equivalence and the Principle of General Covariance

♠ The Principle of Equivalence

The equations of motion for a system of material points moving with non-relativistic velocities under the influence of forces $\vec{F}(\vec{x}_n - \vec{x}_m)$ and an external homogeneous and static gravitational field $\vec{g}$ reads

$$m_n \frac{d^2 \vec{x}_n}{dt^2} = m_n \vec{g} + \sum_m \vec{F}(\vec{x}_n - \vec{x}_m)$$

Perform the following non-Galilean space-time coordinate transformation

$$\vec{x}' = \vec{x} - \frac{1}{2}\vec{g}t^2 \qquad t' = t \tag{1}$$

Then $\vec{g}$ will be canceled by an inertial "force" so that the equation of motion in the new reference frame become

$$m_n \frac{d^2 \vec{x}'_n}{dt^2} = \sum_m \vec{F}(\vec{x}'_n - \vec{x}'_m)$$

Remarks:

- The observer $\mathcal{O}$ who uses the coordinates $t, \vec{x}$ and his *freely falling* colleague $\mathcal{O}'$ with coordinates $t', \vec{x}'$ are going to detect the same laws of mechanics but $\mathcal{O}'$ will conclude that there is no gravitational interactions while $\mathcal{O}$ will say that there is one.
- The gravitational field was homogeneous and static. Had $\vec{g}$ depended on $\vec{x}$ or t, we would not have been able to eliminate it through (1).

The equivalence principle (strong):

At every space-time point in an arbitrary gravitational field it is possible to choose a "locally inertial coordinate system" such that, within a sufficiently small region around the point in question, the laws of nature take the same form as in unaccelerated coordinate system, consistent with the special relativity **and** in the absence of gravity.

Comments:

- "locally inertial coordinate system" means that the gravitational field in the vicinity of the point in question could be considered as static and homogeneous.

♠ Equation of Motion

Consider a particle moving freely under the influence of purely gravitational forces. From the Principle of Equivalence (PE) we conclude that there is a freely falling system of coordinates ξ^α such that the equations of motion (EoM) read

$$\frac{d^2\xi^\alpha}{d\tau^2} = 0 \quad \text{for} \quad d\tau^2 = \eta_{\alpha\beta} d\xi^\alpha d\xi^\beta \quad \text{with} \quad \eta_{\alpha\beta} = \mathrm{diag}(1, -1, -1, -1)\,,$$

where τ is so called proper time. Note that the special relativity EoM are: $f^\alpha = m\frac{d^2\xi^\alpha}{d\tau^2}$. In any other coordinate system x^μ ($\xi^\alpha = \xi^\alpha(x^\mu)$) the EoM would look as

$$0 = \frac{d}{d\tau}\left(\frac{\partial\xi^\alpha}{\partial x^\mu}\frac{dx^\mu}{d\tau}\right) = \frac{\partial\xi^\alpha}{\partial x^\mu}\frac{d^2x^\mu}{d\tau^2} + \frac{\partial^2\xi^\alpha}{\partial x^\mu\partial x^\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\,.$$

Adopting $\frac{\partial\xi^\alpha}{\partial x^\mu}\frac{\partial x^\lambda}{\partial\xi^\alpha} = \delta^\lambda_\mu$ the EoM reads:

$$0 = \frac{d^2x^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \qquad \text{for} \qquad \Gamma^\lambda_{\mu\nu} \equiv \frac{\partial x^\lambda}{\partial\xi^\alpha}\frac{\partial^2\xi^\alpha}{\partial x^\mu\partial x^\nu}\,,$$

where $\Gamma^\lambda_{\mu\nu}$ is the *affine connection*. The proper time could also be expressed in the new frame:

$$d\tau^2 = \eta_{\alpha\beta}\frac{\partial\xi^\alpha}{\partial x^\mu}dx^\mu\frac{\partial\xi^\beta}{\partial x^\nu}dx^\nu = g_{\mu\nu}dx^\mu dx^\nu\,,$$

where $g_{\mu\nu} \equiv \eta_{\alpha\beta}\frac{\partial\xi^\alpha}{\partial x^\mu}\frac{\partial\xi^\beta}{\partial x^\nu}$ is the *metric tensor*.

♠ Metric v.s. Connection

From the definition of the metric one can derive the relation between $g_{\mu\nu}$ and $\Gamma^{\lambda}_{\mu\nu}$:

$$\Gamma^{\sigma}_{\lambda\mu} = \frac{1}{2} g^{\nu\sigma} \left\{ \frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} + \frac{\partial g_{\lambda\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\lambda}}{\partial x^{\nu}} \right\}$$

where $g^{\nu\sigma}$ *is defined* through

$$g_{\kappa\nu} g^{\nu\sigma} = \delta^{\sigma}_{\kappa}$$

♠ The Newtonian Limit

Consider a particle moving *slowly in a weak and static* gravitational field. Then the general EoM

$$0 = \frac{d^2 x^{\lambda}}{d\tau^2} + \Gamma^{\lambda}_{\mu\nu} \frac{dx^{\mu}}{d\tau} \frac{dx^{\nu}}{d\tau}$$

will be simplified by neglecting $d\vec{x}/d\tau$ with respect to $dt/d\tau$

$$0 = \frac{d^2 x^{\lambda}}{d\tau^2} + \Gamma^{\lambda}_{00} \left(\frac{dt}{d\tau} \right)^2$$

Expanding to the first order (i.e. for a weak field) in $h_{\mu\nu}$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad \text{with} \qquad |h_{\mu\nu}| \ll 1$$

and utilizing time-independence of the metric together with $\Gamma^{\lambda}_{00} = -\frac{1}{2}\eta^{\alpha\lambda}\frac{\partial h_{00}}{\partial x^{\alpha}}$ $\left(\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\alpha\lambda}\left\{\frac{\partial g_{\nu\alpha}}{\partial x^{\mu}} + \frac{\partial g_{\mu\alpha}}{\partial x^{\nu}} - \frac{\partial g_{\nu\mu}}{\partial x^{\alpha}}\right\}\right)$, one gets

$$\frac{d^2x^{\mu}}{d\tau^2} = \frac{1}{2}\eta^{\mu\nu}\frac{\partial h_{00}}{\partial x^{\nu}}\left(\frac{dt}{d\tau}\right)^2$$

and

$$\frac{d^2\vec{x}}{dt^2} = -\frac{1}{2}\nabla h_{00}$$

to be compared with the Newtonian result

$$\frac{d^2\vec{x}}{dt^2} = -\nabla\varphi$$

Finally, we get $g_{00} = 1 + 2\varphi$.

Tensors

Contravariant vector V^{μ}, *by definition*, transforms under a coordinate transformation $x^{\mu} \to x'^{\,\mu}(x)$ as

$$V'^{\,\mu} = V^{\nu} \frac{\partial x'^{\,\mu}}{\partial x^{\nu}}$$

Covariant vector U_{μ}, *by definition*, transforms as

$$U'_{\mu} = \frac{\partial x^{\nu}}{\partial x'^{\,\mu}} U_{\nu}$$

♠ Covariant Derivative

The connection

$$\Gamma^{\lambda}_{\mu\nu} \equiv \frac{\partial x^{\lambda}}{\partial \xi^{\alpha}} \frac{\partial^2 \xi^{\alpha}}{\partial x^{\mu} \partial x^{\nu}}$$

is not a tensor

$$\Gamma'^{\lambda}_{\mu\nu} \equiv \frac{\partial x'^{\lambda}}{\partial \xi^{\alpha}} \frac{\partial^2 \xi^{\alpha}}{\partial x'^{\mu} \partial x'^{\nu}} = \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x^{\tau}}{\partial x'^{\mu}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} \Gamma^{\rho}_{\tau\sigma} + \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial^2 x^{\rho}}{\partial x'^{\mu} \partial x'^{\nu}} \tag{2}$$

A derivative of a tensor, in general, does not yield another tensor. Consider for instance a vector

$$V'^{\mu} = V^{\nu} \frac{\partial x'^{\mu}}{\partial x^{\nu}}$$

$$\frac{\partial V'^{\mu}}{\partial x'^{\lambda}} = \frac{\partial x^{\rho}}{\partial x'^{\lambda}} \frac{\partial V^{\nu}}{\partial x^{\rho}} \frac{\partial x'^{\mu}}{\partial x^{\nu}} + \frac{\partial^2 x'^{\mu}}{\partial x^{\nu} \partial x^{\rho}} \frac{\partial x^{\rho}}{\partial x'^{\lambda}} V^{\nu} \tag{3}$$

Combining (2) and (3) we can define the covariant derivative of a contravariant vector which is a tensor:

$$V^{\mu}{}_{;\lambda} \equiv \frac{\partial V^{\mu}}{\partial x^{\lambda}} + \Gamma^{\mu}_{\lambda\kappa} V^{\kappa} \qquad \text{with} \qquad V'^{\mu}{}_{;\lambda} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} \frac{\partial x^{\rho}}{\partial x'^{\lambda}} V^{\nu}{}_{;\rho}$$

Similarly for the covariant derivative of a covariant vector

$$U_{\mu;\nu} \equiv \frac{\partial U_{\mu}}{\partial x^{\nu}} - \Gamma^{\lambda}_{\mu\nu} U_{\lambda} \qquad \text{with} \qquad U'_{\mu;\nu} = \frac{\partial x^{\rho}}{\partial x'^{\mu}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} U_{\rho;\sigma}$$

♠ Gradient, Curl, and Divergence

Properties of covariant derivatives:

For a scalar S

$$S_{;\mu} = \frac{\partial S}{\partial x^\mu}$$

$$U_{\mu;\nu} - U_{\nu;\mu} = \frac{\partial U_\mu}{\partial x^\nu} - \frac{\partial U_\nu}{\partial x^\mu}$$

$$V^\mu{}_{;\mu} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\mu} \{\sqrt{g} V^\mu\} \qquad \text{and} \qquad \int d^4x \sqrt{g} V^\mu{}_{;\mu} = 0 \qquad (\text{if } V^\mu \text{ vanishes at infinity})$$

where $g \equiv -\det(g_{\mu\nu})$.

$$T^{\mu\nu}{}_{;\mu} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\mu} \{\sqrt{g} T^{\mu\nu}\} + \Gamma^\nu_{\mu\lambda} T^{\mu\lambda}$$

For $A^{\mu\nu} = -A^{\nu\mu}$ one gets:

- $$A^{\mu\nu}{}_{;\mu} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\mu} \{\sqrt{g} A^{\mu\nu}\} \tag{4}$$

- $$A_{\mu\nu;\lambda} + A_{\nu\lambda;\mu} + A_{\lambda\mu;\nu} = \frac{\partial A_{\mu\nu}}{\partial x^\lambda} + \frac{\partial A_{\nu\lambda}}{\partial x^\mu} + \frac{\partial A_{\lambda\mu}}{\partial x^\nu} \tag{5}$$

♠ The Principle of General Covariance

A physical equation holds in a general gravitational field if

- The equation holds in the absence of gravitation; i.e., it agrees with laws of special relativity when $g_{\mu\nu} = \eta_{\mu\nu}$ and $\Gamma^{\alpha}_{\mu\nu} = 0$.
- The equation is generally covariant; i.e. it preserves its form under a general coordinate transformation $x \to x'$.

$$\Downarrow$$

It is useful to adopt quantities which have well defined transformation properties, i.e. tensors

♠ Electrodynamics

The Maxwell equations:

$$\frac{\partial}{\partial x^\alpha} F^{\alpha\beta} = J^\beta \qquad \text{and} \qquad \frac{\partial}{\partial x_\alpha} F_{\beta\gamma} + \frac{\partial}{\partial x_\beta} F_{\gamma\alpha} + \frac{\partial}{\partial x_\gamma} F_{\alpha\beta} = 0, \qquad (6)$$

where

$$J^\mu(x) \equiv \sum_n q_n \delta^3(\vec{x} - \vec{x}_n(t)) \frac{dx_n^\mu(t)}{dt}$$

is the four-current, while $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$.

To make (6) covariant we replace ordinary derivatives by covariant derivatives and the Minkowski metric $\eta_{\mu\nu}$ by a general metric $g_{\mu\nu}$ (note raising and lowering indices):

$$F^{\mu\nu}{}_{;\mu} = J^\nu \qquad \text{and} \qquad F_{\mu\nu;\lambda} + F_{\nu\lambda;\mu} + F_{\lambda\mu;\nu} = 0$$

Using identities (4) and (5) we get

$$\frac{\partial}{\partial x^\alpha} \sqrt{g} F^{\alpha\beta} = \sqrt{g} J^\beta \qquad \text{and} \qquad \frac{\partial}{\partial x_\lambda} F_{\mu\nu} + \frac{\partial}{\partial x_\nu} F_{\lambda\mu} + \frac{\partial}{\partial x_\mu} F_{\nu\lambda} = 0$$

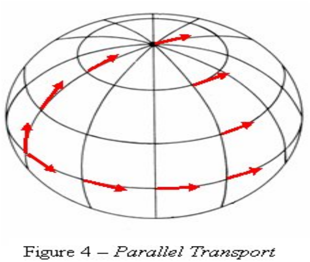

Figure 1: *The parallel transport on the surface of a sphere.*

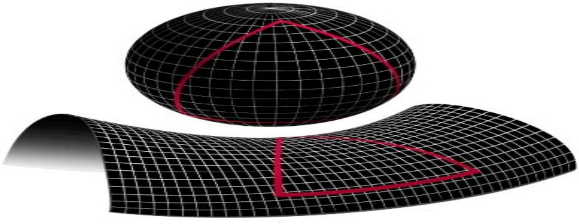

Figure 2: *Positive and negative curvatures: the sum of angles in a triangle.*

♠ The Riemann-Christoffel curvature tensor

The simplest tensor made out of $g_{\mu\nu}$ and its first and second derivatives:

$$R^{\lambda}{}_{\mu\nu\kappa} \equiv \frac{\partial \Gamma^{\lambda}_{\mu\nu}}{\partial x^{\kappa}} - \frac{\partial \Gamma^{\lambda}_{\mu\kappa}}{\partial x^{\nu}} + \Gamma^{\eta}_{\mu\nu}\Gamma^{\lambda}_{\kappa\eta} - \Gamma^{\eta}_{\mu\kappa}\Gamma^{\lambda}_{\nu\eta}$$

♠ Commutation of covariant derivatives

$$V_{\mu;\nu;\kappa} - V_{\mu;\kappa;\nu} = -V_{\sigma} R^{\sigma}{}_{\mu\nu\kappa}$$

Similar formulas for other tensors:

$$T^{\lambda}{}_{\mu;\nu;\kappa} - T^{\lambda}{}_{\mu;\kappa;\nu} = T^{\sigma}{}_{\mu} R^{\lambda}{}_{\sigma\nu\kappa} - T^{\lambda}{}_{\sigma} R^{\sigma}{}_{\mu\nu\kappa}$$

Conclusion: covariant derivatives of tensors commute if the metric is equivalent (to be defined) to $\eta_{\mu\nu}$.

♠ Properties

$$R_{\lambda\mu\nu\kappa} = \frac{1}{2}\left[\frac{\partial^2 g_{\lambda\nu}}{\partial x^{\kappa}\partial x^{\mu}} - \frac{\partial^2 g_{\mu\nu}}{\partial x^{\kappa}\partial x^{\lambda}} - \frac{\partial^2 g_{\lambda\kappa}}{\partial x^{\nu}\partial x^{\mu}} + \frac{\partial^2 g_{\mu\kappa}}{\partial x^{\nu}\partial x^{\lambda}}\right] + g_{\eta\sigma}\left[\Gamma^{\eta}_{\nu\lambda}\Gamma^{\sigma}_{\mu\kappa} - \Gamma^{\eta}_{\kappa\lambda}\Gamma^{\sigma}_{\mu\nu}\right]$$

for $R_{\lambda\mu\nu\kappa} \equiv g_{\lambda\sigma} R^{\sigma}{}_{\mu\nu\kappa}$

- Symmetry: $R_{\lambda\mu\nu\kappa} = R_{\nu\kappa\lambda\mu}$
- Antisymmetry: $R_{\lambda\mu\nu\kappa} = -R_{\mu\lambda\nu\kappa} = -R_{\lambda\mu\kappa\nu} = +R_{\mu\lambda\kappa\nu}$
- Cyclicity: $R_{\lambda\mu\nu\kappa} + R_{\lambda\nu\kappa\mu} + R_{\lambda\kappa\mu\nu} = 0$
- The Bianchi identities: $R_{\lambda\mu\nu\kappa;\eta} + R_{\lambda\mu\kappa\eta;\nu} + R_{\lambda\mu\eta\nu;\kappa} = 0$
  Contracting λ and ν one gets (using $g_{\mu\nu;\lambda} = 0$, see class)

$$G^{\mu\nu}{}_{;\mu} = 0 \qquad \text{for} \qquad G^{\mu\nu} \equiv R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R$$

  - $R_{\mu\kappa} \equiv g^{\lambda\nu} R_{\lambda\mu\nu\kappa}$ is the Ricci tensor ($R_{\mu\kappa} = R_{\kappa\mu}$)
  - $R \equiv g^{\lambda\nu} g^{\mu\kappa} R_{\lambda\mu\nu\kappa}$ is the Ricci scalar

Def. A metric is equivalent to the Minkowski metric if there is a set of Minkowskian coordinates $\xi^\alpha(x)$ that *everywhere* satisfy the conditions

$$\eta^{\alpha\beta} = g^{\mu\nu}(x) \frac{\partial \xi^\alpha(x)}{\partial x^\mu} \frac{\partial \xi^\beta(x)}{\partial x^\nu}$$

Theorem:
The necessary and sufficient conditions for a metric $g_{\mu\nu}(x)$ to be equivalent to the Minkowski metric $\eta_{\mu\nu}$ are:

- $R^{\lambda}{}_{\mu\nu\kappa} = 0$
- At some point X, the matrix $g_{\mu\nu}(X)$ has three negative and one positive eigenvalues.

A perfect fluid is defined as having at each point a velocity $\vec{v}$, such that an observer moving with this velocity sees the fluid (energy-momentum tensor $T_{\mu\nu}$) around him/her as isotropic.

Suppose that we are in a frame of reference in which the fluid is at rest at some particular position and time. Then at this point the isotropy implies

$$\tilde{T}^{ij} = p\delta^{ij}, \qquad \tilde{T}^{i0} = \tilde{T}^{0i} = 0, \qquad \tilde{T}^{00} = \rho$$

where p and ρ are the pressure and energy density, respectively. After a Lorentz transformation to the lab frame we get the general form of the energy-momentum tensor for the perfect fluid

$$T^{\alpha\beta} = -p\eta^{\alpha\beta} + (p + \rho)U^{\alpha}U^{\beta}$$

where U^{α} is the velocity four-vector of a fluid point

$$U^0 = \frac{dt}{d\tau} = \gamma, \qquad \vec{U} = \frac{d\vec{x}}{d\tau} = \gamma\vec{v} \quad \text{for} \quad \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

Note that $U_\alpha U^\alpha = 1$.

The "energy-momentum conservation" implies (if there is no external forces)

$$0 = \partial_\beta T^{\beta\alpha} = \frac{\partial p}{\partial x_\alpha} + \frac{\partial}{\partial x^\beta}\left[(\rho + p)U^\alpha U^\beta\right]$$

The particle number conservation

$$0 = \frac{\partial N^\alpha}{\partial x^\alpha} = \frac{\partial}{\partial x^\alpha}(nU^\alpha)$$

where n is the particle number density. Consider a fluid composed of point particles

$$I_M = -\sum_n m_n \int_{-\infty}^{\infty} dp \left[g_{\mu\nu}(x_n(p))\frac{dx_n{}^\mu(p)}{dp}\frac{dx_n{}^\nu(p)}{dp}\right]^{1/2}$$

The energy momentum tensor is defined by

$$\delta I = -\frac{1}{2}\int d^4x\, g^{1/2} T^{\mu\nu}\delta g_{\mu\nu}$$

So for the action I_M we get

$$T^{\alpha\beta}(x) = \sum_n \frac{p_n(t)^\alpha p_n(t)^\beta}{E_n(t)} \delta^3(\vec{x} - \vec{x}_n(t))$$

- For non relativistic gas $\rho \simeq nm + \frac{3}{2}p$
- For highly relativistic gas $\rho \simeq 3p$

In the presence of gravity the energy-momentum tensor reads

$$T^{\alpha\beta} = -pg^{\alpha\beta} + (p + \rho)U^\alpha U^\beta$$

The energy-momentum covariant conservation implies:

$$0 = T^{\alpha\beta}{}_{;\beta} = -\frac{\partial p}{\partial x_\beta} g^{\alpha\beta} + g^{-1/2} \frac{\partial}{\partial x^\beta} \left[g^{1/2}(\rho + p)U^\alpha U^\beta\right] + \Gamma^\alpha_{\beta\lambda}(p + \rho)U^\beta U^\lambda$$

The energy-momentum tensor for a system described by the action $I_M = \int d^4x\, g^{1/2}\mathcal{L}_M$:

$$\delta I_M = -\frac{1}{2}\int d^4x\, g^{1/2}T^{\mu\nu}\delta g_{\mu\nu}$$

for the variation of the metric $g_{\mu\nu}(x) \to g_{\mu\nu}(x) + \delta g_{\mu\nu}(x)$ such that $\delta g_{\mu\nu}(x) \to 0$ for $|x^\lambda| \to \infty$. For instance for electrodynamics:

$$\mathcal{L}_M - \frac{1}{4}g^{\mu\nu}g^{\lambda\rho}F_{\mu\lambda}F_{\nu\rho}$$

one finds that $T^{\lambda\kappa} = \frac{1}{4}g^{\lambda\kappa}F_{\mu\nu}F^{\mu\nu} - F_\mu{}^\lambda F^{\mu\kappa}$.

The Einstein's Field Equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -8\pi G T_{\mu\nu}$$

are derived from the Einstein-Hilbert action I_{E-H} with a matter action I_M:

$$I = \underbrace{\frac{1}{16\pi G}\int d^4x\, g^{1/2}(x)R(x)}_{I_{E-H}} \quad + \quad I_M$$

Contracting μ and ν one gets: $R = 8\pi G T^{\lambda}{}_{\lambda}$, hence

$$R_{\mu\nu} = -8\pi G \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^{\lambda}{}_{\lambda} \right)$$

♠ Gravitational Radiation

The weak field approximation: $g_{\mu\nu} \equiv \eta_{\mu\nu} + h_{\mu\nu}$ for $|h_{\mu\nu}| \ll 1$. First we calculate the connection expanding in powers of $h_{\mu\nu}$:

$$\Gamma^{\sigma}_{\lambda\mu} = \frac{1}{2} g^{\nu\sigma} \left\{ \frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} + \frac{\partial g_{\lambda\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\lambda}}{\partial x^{\nu}} \right\}$$

Then the Ricci tensor:

$$R_{\mu\kappa} = \frac{\partial \Gamma^{\nu}_{\mu\nu}}{\partial x^{\kappa}} - \frac{\partial \Gamma^{\nu}_{\mu\kappa}}{\partial x^{\nu}} + \Gamma^{\eta}_{\mu\nu} \Gamma^{\nu}_{\kappa\eta} - \Gamma^{\eta}_{\mu\kappa} \Gamma^{\nu}_{\nu\eta}$$

Finally we get

$$R_{\mu\nu} = \frac{1}{2}\left(\partial_\alpha \partial^\alpha \, h_{\mu\nu} - \partial_\lambda \partial_\mu \, h^\lambda{}_\nu - \partial_\lambda \partial_\nu \, h^\lambda{}_\mu + \partial_\mu \partial_\nu h^\lambda{}_\lambda\right) + \mathcal{O}(h^2)$$

For a given action

$$I = \int d^4x \, \mathcal{L}(\phi^i(x), \partial_\mu \phi^i(x))$$

EoM are obtained through the Principle of Least Action:

$$-\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \phi^i_{,\mu}}\right) + \frac{\partial \mathcal{L}}{\partial \phi^i} = 0$$

The free EoM for the linearized gravity reads:

$$\partial_\alpha \partial^\alpha h_{\mu\nu} - \partial_\lambda \partial_\mu h^\lambda{}_\nu - \partial_\lambda \partial_\nu h^\lambda{}_\mu + \partial_\mu \partial_\nu h^\lambda{}_\lambda = 0$$

For coordinate transformations of the form

$$x^\mu \to x'^{\,\mu} = x^\mu + \varepsilon^\mu(x)$$

the corresponding transformations of $h_{\mu\nu}$ ($g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$) are as follows (it is assumed that $\frac{\partial \varepsilon_\mu}{\partial x^\nu} \sim h_{\mu\nu}$):

$$g'^{\,\mu\nu} = \frac{\partial x'^{\,\mu}}{\partial x^\lambda}\frac{\partial x'^{\,\nu}}{\partial x^\rho} g^{\lambda\rho} \qquad \Rightarrow \qquad h_{\mu\nu}(x) \to h'_{\mu\nu}(x) = h_{\mu\nu}(x) - \partial_\mu \varepsilon_\nu - \partial_\nu \varepsilon_\mu$$

Note that

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + \mathcal{O}(h^2) \qquad \text{since} \qquad g^{\mu\nu} g_{\nu\lambda} = \delta^\mu_\lambda\,.$$

It is convenient to adopt the harmonic coordinate gauge conditions

$$\Gamma^\lambda \equiv g^{\mu\nu}\Gamma^\lambda_{\mu\nu} = 0 \qquad \Longleftrightarrow \qquad \partial_\kappa (g^{1/2} g^{\lambda\kappa}) = 0$$

Up to the first order in h: $\partial_\mu h^\mu{}_\nu = \frac{1}{2}\partial_\nu h^\mu{}_\mu$. Then the equations of motion in vacuum ($R_{\mu\nu} = 0$) simplify

$$\Box h_{\mu\nu} = 0 \qquad \text{(together with the gauge condition)}$$

Plane wave solutions:

$$h_{\mu\nu}(x) = e_{\mu\nu} e^{ik_\lambda x^\lambda} + e^\star_{\mu\nu} e^{-ik_\lambda x^\lambda}$$

with $k_\mu k^\mu = 0$ and $k_\mu e^\mu{}_\nu = \frac{1}{2} k_\nu e^\mu{}_\mu$ and $e_{\mu\nu} = e_{\nu\mu}$.

Using the residual gauge freedom $x^\mu \to x^\mu + \varepsilon^\mu(x)$ with

$$\varepsilon^\mu(x) = i\varepsilon^\mu e^{ik_\lambda x^\lambda} - i\varepsilon^{\mu\,\star} e^{-ik_\lambda x^\lambda}$$

$(e_{\mu\nu} \to e'_{\mu\nu} = e_{\mu\nu} + k_\mu \varepsilon_\nu + k_\nu \varepsilon_\mu)$ one concludes that there are only two independent degrees of freedom (as it should be), e.g. e_{11} and e_{12} $(e_{22} = -e_{11})$, then the solution (gravitational plane waves) is

$$h_{\mu\nu}(x) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & e_{11} & e_{12} & 0 \\ 0 & e_{12} & -e_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{ik_\lambda x^\lambda} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & e_{11}^\star & e_{12}^\star & 0 \\ 0 & e_{12}^\star & -e_{11}^\star & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{-ik_\lambda x^\lambda}$$

for $k^\mu = (k, 0, 0, k)$.

♠ The Action Principle

The action for charged massive particles in the electromagnetic field and gravitational background

$$
\begin{aligned}
I_M = \quad & -\sum_n m_n \int_{-\infty}^{\infty} dp \left[g_{\mu\nu}(x_n(p)) \frac{d{x_n}^{\mu}(p)}{dp} \frac{d{x_n}^{\nu}(p)}{dp} \right]^{1/2} \\
& -\frac{1}{4} \int d^4x \, g^{1/2}(x) F_{\mu\nu}(x) F^{\mu\nu}(x) + \sum_n e_n \int_{-\infty}^{\infty} dp \frac{d{x_n}^{\mu}}{dp} A_{\mu}(x_n(p))
\end{aligned}
$$

The gravitational action

$$
I_{GR} = \frac{1}{16\pi G} \int d^4x \, g^{1/2}(x) \, R(x)
$$

The Friedmann-Lemaître-Robertson-Walker Metric and the Friedmann Equations

The Friedmann-Lemaître-Robertson-Walker (FLRW) metric describes a homogeneous, isotropic expanding/contracting/static universe.

In cosmology we often assume that the spacetime is homogeneous (in space, not in time). If the space-time is homogeneous and isotropic, then *it is possible to choose coordinates* such that the length element reads:

$$d\tau^2 \equiv g_{\mu\nu} dx^\mu dx^\nu = dt^2 - a^2(t) \left\{ \frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2 \right\}$$

where $a(t)$ is called the scale factor and $k = \pm 1, 0$.
The coordinates (r, θ, φ) of the (FLRW) metric are called comoving coordinates. This reflects the fact that this coordinate system follows the expansion of space, so that the space coordinates of objects which have zero peculiar velocity remain the same. **The homogeneity of the universe fixes a special frame of reference, the cosmic rest frame. In other words this means that there exists a coordinate system in which t = const. hypersurfaces are homogeneous. The time coordinate of this system is called the cosmic time.**

♠ The two sphere first

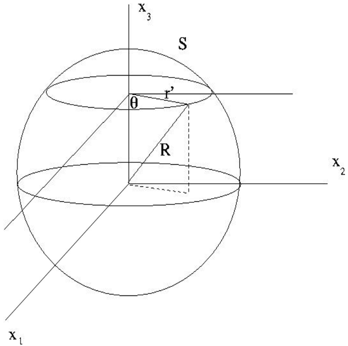

Figure 3: *The two sphere.*

$$x_1^2 + x_2^2 + x_3^2 = R^2 \qquad d\vec{l}^{\,2} = dx_1^2 + dx_2^2 + dx_3^2$$

Eliminate the fictitious coordinate x_3:

$$x_1^2 + x_2^2 + x_3^2 = R^2 \quad \Rightarrow \quad x_1 dx_1 + x_2 dx_2 + x_3 dx_3 = 0$$

$$\Downarrow$$

$$dx_3 = -\frac{x_1 dx_1 + x_2 dx_2}{x_3} = -\frac{x_1 dx_1 + x_2 dx_2}{(R^2 - x_1^2 - x_2^2)^{1/2}}$$

$$\Downarrow$$

$$d\vec{l}^{\,2} = dx_1^2 + dx_2^2 + \frac{(x_1 dx_1 + x_2 dx_2)^2}{R^2 - x_1^2 - x_2^2}$$

Introduce new coordinates (r', θ): $x_1 = r' \cos\theta, \qquad x_2 = r' \sin\theta$.
Then

$$dx_1^2 + dx_2^2 = dr'^{\,2} + r'^{\,2} d\theta^2 \qquad x_1 dx_1 + x_2 dx_2 = r' dr' \qquad (\Leftarrow x_1^2 + x_2^2 = r'^{\,2})$$

In terms of (r', θ) we get

$$d\vec{l}^{\,2} = \frac{R^2 dr'^{\,2}}{R^2 - r'^{\,2}} + r'^{\,2} d\theta^2$$

Define $r \equiv \frac{r'}{R}$, then

$$d\vec{l}^{\,2} = R^2 \left[\frac{dr^2}{1 - r^2} + r^2 d\theta^2 \right]$$

Note similarity between that and the FLRW metric for $k = 1$.
Another convenient coordinate system is (θ, φ):

$$x_1 = R \sin\theta \cos\varphi \qquad x_2 = R \sin\theta \sin\varphi \qquad x_3 = R \cos\theta$$

Then

$$d\vec{l}^{\,2} = R^2(d\theta^2 + \sin^2\theta d\varphi^2) \qquad \Rightarrow \qquad g_{ij} = R^2 \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}$$

- Cosmological Principle $\longrightarrow R = R(t)$ (homogeneity)
- As the sphere expands or contracts, the coordinates (r, θ) remain unchanged (the comoving coordinates)
- For the negative curvature: $R \to iR$ and $x_3 \to ix_3$

$$d\vec{l}^{\,2} = R^2\left[\frac{dr^2}{1+r^2} + r^2 d\theta^2\right]$$

♠ The two sphere $\rightarrow$ the three sphere

$$d\vec{l}^{\,2} = dx_1^2 + dx_2^2 + dx_3^2 + \frac{(x_1 dx_1 + x_2 dx_2 + x_3 dx_3)^2}{R^2 - x_1^2 - x_2^2 - x_3^2}$$

$$x_1 = r'\sin\theta\cos\varphi \qquad x_2 = r'\sin\theta\sin\varphi \qquad x_3 = r'\cos\theta$$

After rescaling r' ($r \equiv \frac{r'}{R}$) one gets

$$d\vec{l}^2 = R^2 \left[\frac{dr^2}{1-r^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2 \right]$$

In the spherical 4-d coordinates (see class perhaps)

$$\begin{aligned} x_1 &= R \sin\chi \sin\theta \cos\varphi \\ x_2 &= R \sin\chi \sin\theta \sin\varphi \\ x_3 &= R \sin\chi \cos\theta \\ x_4 &= R \cos\chi \end{aligned}$$

$$d\vec{l}^2 = R^2 [d\chi^2 + \sin^2\chi (d\theta^2 + \sin^2\theta d\varphi^2)] \longrightarrow g_{ij} = R^2 \text{ diag}(1, \sin^2\chi, \sin^2\chi \sin^2\theta)$$

Introducing time we get the FLRW metric

$$d\tau^2 \equiv g_{\mu\nu} dx^\mu dx^\nu = dt^2 - a^2(t) \left\{ \frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2 \right\}$$

where $R(t)$ is the scale factor denoted by $a(t)$ ($R(t) \to a(t)$) and $k = \pm 1, 0$.
Note that r is dimensionless.

Comments:

- The spatial coordinates r, θ, φ form a comoving system in the sense that typical galaxies have constant spatial coordinates r, θ, φ.
- Since $\Gamma^{\mu}_{tt} = 0$ for the FLRW metric, it is easy to show (see class) that the trajectories $\vec{x}$ = const. are geodesics. Thus the statement that a galaxy has constant r, θ, φ is perfectly consistent with the the supposition that galaxies are in free fall.

♠ Distance in general relativity

Along an arbitrary space-like path P, the proper (physical) distance is given as the line integral

$$D_P = \int_P d\tau = \int_P (-g_{\mu\nu} dx^\mu dx^\nu)^{1/2}$$

$$= \int_P \left[-dt^2 + a^2(t) \left\{ \frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2 \right\} \right]^{1/2}$$

Let's calculate the distance between a galaxy placed in the origin $(0, 0, 0)$ and another one at comoving coordinates (r, θ, φ) along the curve t, θ, φ = const. so $dt = d\theta = d\varphi = 0$

$$\int_0^r a(t) \frac{dr'}{(1 - kr'^2)^{1/2}} = a(t) \begin{cases} \sin^{-1} r & k = +1 \\ r & k = 0 \\ \sinh^{-1} r & k = -1 \end{cases}$$

♠ The most useful formula in cosmology

We will show that

$$\frac{\lambda_{\rm emit}}{a(t_{\rm emit})} = \frac{\lambda_{\rm obs}}{a(t_{\rm obs})}$$

Emission at $t = t_1$ and $r = r_1$, detection at $t = t_0$ and $r = 0$. The massless wave travels along a geodesic, $d\tau^2 = 0$, so

$$d\tau^2 = dt^2 - a(t)^2 \frac{dr^2}{1 - kr^2} = 0$$

Integrating from the emission to the detection and taking into account that $dr < 0$ one finds

$$\int_{t_1}^{t_0} \frac{dt'}{a(t')} = \int_0^{r_1} \frac{dr}{\sqrt{1 - kr^2}} = f(r_1) \tag{7}$$

where the rhs is fixed (independent of time) as the comoving coordinate of the source remains unchanged.

Consider two subsequent emissions at $t = t_1$ and $t = t_1 + \delta t_1$ (corresponding to two successive wave-crests), which were detected at $t = t_0$ and $t = t_0 + \delta t_0$. Then the rhs of (7) does not change, so we get

$$\int_{t_1}^{t_0} \frac{dt'}{a(t')} = \int_{t_1+\delta t_1}^{t_0+\delta t_0} \frac{dt'}{a(t')} = \int_{t_1}^{t_0} \frac{dt'}{a(t')} - \int_{t_1}^{t_1+\delta t_1} \frac{dt'}{a(t')} + \int_{t_0}^{t_0+\delta t_0} \frac{dt'}{a(t')}$$

$$\Downarrow$$

$$\int_{t_1}^{t_1+\delta t_1} \frac{dt'}{a(t')} = \int_{t_0}^{t_0+\delta t_0} \frac{dt'}{a(t')}$$

Assume that $\delta t_i = \lambda_i$ ($c = 1$), $i = 0, 1$, are small enough such that $a(t) \simeq$ const. in the integrand, so

$$\frac{\delta t_1}{a(t_1)} = \frac{\delta t_0}{a(t_0)}$$

$$\Downarrow$$

$$\text{emission} \quad \rightarrow \quad \frac{\lambda(t_1)}{a(t_1)} = \frac{\lambda(t_0)}{a(t_0)} \quad \leftarrow \quad \text{detection}$$

♠ The cosmic time -– redshift relation

The redshift z

$$1 + z \equiv \frac{\lambda_{\rm obs}}{\lambda_{\rm emit}}$$

Then from the $\lambda \leftrightarrow a$ relation for $t_{\rm obs} = t_0$ and $t_{\rm emit} = t$ we find

$$1 + z = \frac{a(t_0)}{a(t)}$$

For a given geometry the above is the relation between the redshift observed now and the time of light emission.

♠ The Friedmann Equations

We will solve the Einstein's equations

$$R_{\mu\nu} = -8\pi G\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T^{\lambda}{}_{\lambda}\right)$$

for

$$T^{\alpha\beta} = -pg^{\alpha\beta} + (p+\rho)U^{\alpha}U^{\beta}$$

for $U^t = 1$ and $U^i = 0$ (this is a consequence of the cosmological principle).
Using the FLRW metric

$$d\tau^2 \equiv g_{\mu\nu}dx^{\mu}dx^{\nu} = dt^2 - a^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2d\theta^2 + r^2\sin^2\theta d\varphi^2\right\}$$

The metric $g_{\mu\nu}$:

$$g_{tt} = 1, \qquad g_{it} = 0, \qquad g_{ij} = -a^2(t)\tilde{g}_{ij}(x)$$

with $\tilde{g}_{rr} = (1-kr^2)^{-1}$, $\tilde{g}_{\theta\theta} = r^2$, $\tilde{g}_{\varphi\varphi} = r^2\sin^2\theta$ and $\tilde{g}_{ij} = 0$ for $i \neq j$.
The inverse metric $g^{\mu\nu}$ ($g^{\lambda\rho}g_{\rho\kappa} \equiv \delta^{\lambda}_{\kappa}$):

$$g^{tt} = 1, \qquad g^{it} = 0, \qquad g^{ij} = -a^{-2}(t)\tilde{g}^{ij}(x)$$

with $\tilde{g}^{rr} = (1-kr^2)$, $\tilde{g}^{\theta\theta} = r^{-2}$, $\tilde{g}^{\varphi\varphi} = r^{-2}\sin^{-2}\theta$ and $\tilde{g}^{ij} = 0$ for $i \neq j$.

Then we calculate the affine connection from the metric

$$\Gamma^{\sigma}_{\lambda\mu} = \frac{1}{2} g^{\nu\sigma} \left\{ \frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} + \frac{\partial g_{\lambda\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\lambda}}{\partial x^{\nu}} \right\}$$

Non-zero entries (see class) are

$$\begin{aligned}
\Gamma^{t}_{ij} &= a\dot{a}\tilde{g}_{ij} \\
\Gamma^{i}_{tj} &= \frac{\dot{a}}{a}\delta^{i}_{j} \\
\Gamma^{i}_{jk} &= \frac{1}{2}\tilde{g}^{il} \left\{ \frac{\partial \tilde{g}_{kl}}{\partial x^{j}} + \frac{\partial \tilde{g}_{jl}}{\partial x^{k}} - \frac{\partial \tilde{g}_{jk}}{\partial x^{l}} \right\} = \tilde{\Gamma}^{i}_{jk}
\end{aligned}$$

Then the Ricci tensor

$$R_{\mu\kappa} = \frac{\partial \Gamma^{\nu}_{\mu\nu}}{\partial x^{\kappa}} - \frac{\partial \Gamma^{\nu}_{\mu\kappa}}{\partial x^{\nu}} + \Gamma^{\eta}_{\mu\nu}\Gamma^{\nu}_{\kappa\eta} - \Gamma^{\eta}_{\mu\kappa}\Gamma^{\nu}_{\nu\eta}$$

The non-vanishing elements (see class) are:

$$R_{tt} = 3\frac{\ddot{a}}{a}, \qquad R_{ij} = \tilde{R}_{ij} - \tilde{g}_{ij}(a\ddot{a} + 2\dot{a}^{2})$$

It is easy to show that $\tilde{R}_{ij} = -2k\tilde{g}_{ij}$ (see class), hence

$$R_{ij} = -\tilde{g}_{ij}(a\ddot{a} + 2\dot{a}^2 + 2k)$$

We also need the components of the rhs of the Einstein equations:

$$S_{\mu\nu} \equiv T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T^{\lambda}{}_{\lambda} = (\rho + p)U_\mu U_\nu - \frac{1}{2}(\rho - p)g_{\mu\nu}$$

for $T_{\mu\nu} = -pg_{\mu\nu} + (p + \rho)U_\mu U_\nu$ ($T^{\alpha}{}_{\alpha} = \rho - 3p$, as $g^{\mu\nu}g_{\nu\alpha} = \delta^\mu_\alpha$ and $U^\mu U_\mu = 1$). So for $U^t = 1$ and $U^i = 0$ we have

$$\begin{aligned} R_{tt} &= 3\tfrac{\ddot{a}}{a} & S_{tt} &= \tfrac{1}{2}(\rho + 3p), & S_{it} &= 0 \\ R_{ij} &= -\tilde{g}_{ij}(a\ddot{a} + 2\dot{a}^2 + 2k) & S_{ij} &= \tfrac{1}{2}(\rho - p)a^2\tilde{g}_{ij} \end{aligned}$$

Substituting into the Einstein's equations

$$R_{\mu\nu} = -8\pi G\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T^{\lambda}{}_{\lambda}\right) = -8\pi G S_{\mu\nu}$$

One gets the celebrated Friedmann's equations.

$$R_{tt} = 3\tfrac{\ddot{a}}{a} \qquad S_{tt} = \tfrac{1}{2}(\rho + 3p), \qquad S_{it} = 0$$
$$R_{ij} = -\tilde{g}_{ij}(a\ddot{a} + 2\dot{a}^2 + 2k) \quad S_{ij} = \tfrac{1}{2}(\rho - p)a^2\tilde{g}_{ij}$$

Substituting into the Einstein's equations

$$R_{\mu\nu} = -8\pi G\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T^{\lambda}{}_{\lambda}\right) = -8\pi G S_{\mu\nu}$$

One gets the Friedmann's equations

- $(0,0)$ component:
$$3\ddot{a} = -4\pi G(\rho + 3p)a \tag{8}$$
- (i,i) component:
$$a\ddot{a} + 2\dot{a}^2 + 2k = 4\pi G(\rho - p)a^2 \tag{9}$$

Eliminating $\ddot{a}$ one gets the Friedmann equation which determines the evolution of the Hubble parameter $H(t)$

$$\dot{a}^2 + k = \frac{8\pi G}{3}\rho a^2 \quad \Rightarrow \quad H^2(t) = \frac{8\pi G}{3}\rho - \frac{k}{a^2(t)} \quad \text{for} \quad H(t) \equiv \frac{\dot{a}}{a} \tag{10}$$

Using (10) one can eliminate ρ from the second equation to obtain the acceleration equation

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = -8\pi G p$$

To investigate consequences of the energy-momentum conservation let's recall the following identity

$$T^{\mu\nu}{}_{;\mu} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^\mu}\left\{\sqrt{g}T^{\mu\nu}\right\} + \Gamma^\nu_{\mu\lambda}T^{\mu\lambda}$$

For the energy-momentum tensor $T^{\mu\nu} = -pg^{\mu\nu} + (p+\rho)U^\mu U^\nu$ one gets for the "energy-momentum conservation" ($T^{\mu\nu}{}_{;\mu} = 0$):

$$0 = T^{\alpha\beta}{}_{;\beta} = -\frac{\partial p}{\partial x^\beta}g^{\alpha\beta} + g^{-1/2}\frac{\partial}{\partial x^\beta}\left[g^{1/2}(\rho+p)U^\alpha U^\beta\right] + \Gamma^\alpha_{\beta\lambda}(p+\rho)U^\beta U^\lambda \quad (11)$$

Are the "energy-momentum conservation" (11) and the Friedmann equations independent? Hint: the Bianchi identities.
It is easy to find (see class) that the time component of $T^{\mu\nu}{}_{;\nu} = 0$ implies

$$\dot{p}a^3 = \frac{d}{dt}\left[a^3(\rho+p)\right]$$

Then

$$\dot{\rho}a^3 = \dot{a}\frac{d}{da}\left(a^3\rho\right) + \frac{d}{dt}\left(a^3 p\right) = \dot{a}\frac{d}{da}\left(a^3\rho\right) + 3a^2\dot{a}p + a^3\dot{p} \quad \Rightarrow \quad \frac{d}{da}\left(\rho a^3\right) = -3pa^2 \tag{12}$$

The above equation could be rewritten in a more familiar way

$$d(\rho a^3) = -pd(a^3) \tag{13}$$

that comprise the first law of thermodynamics and has a simple interpretation: the rate of change of the total energy in a volume element of size $V = a^3$ is equal minus the pressure times the change of volume, $-pdV$, which is the work responsible for the energy change. Note however, that in the case of cosmology that kind of reasoning is hardly applicable, since a change of energy $d(\rho a^3)$ is *not* equivalent to work done against a piston as such does not exist. Therefore in cosmology, although we can calculate change of energy using (13) but we can not say where is the energy coming from or going to. We must conclude that the energy of the fluid is not conserved.

If $p = p(\rho)$ (the equation of state) is known then using (12) one can determine $\rho = \rho(a)$. For instance:

- If $p \ll \rho$ ("dust") then

$$\frac{d}{da}\left(a^3\rho\right) = 0 \qquad \Rightarrow \qquad \rho \propto a^{-3}$$

  Then the total energy contained in a volume $V(t) \propto a^3(t)$ scales as

$$E(t) \propto a^3(t) \cdot \rho(t) \propto a^3(t) \cdot a^{-3}(t) = \mathrm{const.}$$

  So, for the dust its energy is conserved.

- For ultra-relativistic fluid $p = \frac{1}{3}\rho$, then

$$\frac{d}{da}\left(\rho a^3\right) = \frac{d\rho}{da}a^3 + \rho 3a^2 = -3pa^2 = -3\frac{1}{3}\rho a^2$$

$$\Downarrow$$

$$\frac{d\rho}{\rho} = -4\frac{da}{a} \qquad \Rightarrow \qquad \rho \propto a^{-4}$$

Then the total energy contained in a volume $V(t) \propto a^3(t)$ scales as

$$E(t) \propto a^3(t) \cdot \rho(t) \propto a^3(t) \cdot a^{-4}(t) \propto a^{-1}(t)$$

So, for the radiation its energy is not conserved.

The fundamental equations are:

- The Friedmann equation

$$\dot{a}^2 + k = \frac{8\pi G}{3}\rho a^2 \qquad \Rightarrow \qquad H^2(t) = \frac{8\pi G}{3}\rho - \frac{k}{a^2(t)} \qquad \text{for} \qquad H(t) \equiv \frac{\dot{a}}{a}$$

- The acceleration equation:

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = -8\pi G p$$

- The "energy-momentum conservation" (the first law of thermodynamics):

$$\dot{p}a^3 = \frac{d}{dt}\left[a^3(\rho + p)\right] \qquad \Rightarrow \qquad \frac{d}{da}\left(\rho a^3\right) = -3pa^2$$

However, only two of the above three equations are independent!

The Schwarzschild Solution

We are looking for a solution of the Einstein equations which are static and isotropic. So the metric does not depend on t but only on $|\vec{x}|$ while $d\tau^2$ may contain $\vec{x} \cdot d\vec{x}$:

$$d\tau^2 = F(r)dt^2 - 2E(r)dt\ \vec{x} \cdot d\vec{x} - D(r)(\vec{x} \cdot d\vec{x})^2 - C(r)d\vec{x}^2$$

for $r \equiv |\vec{x}|$. In the spherical coordinates

$$x^1 = r \sin\theta \cos\varphi \qquad x^2 = r \sin\theta \sin\varphi \qquad x^3 = r \cos\theta$$

we get

$$d\tau^2 = F(r)dt^2 - 2rE(r)dt\,dr - r^2 D(r)dr^2 - C(r)\underbrace{[dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2]}_{d\vec{x}^2}$$

Define new time: $t' \equiv t + \phi(r)$, so that

$$dt = dt' - \frac{d\phi}{dr}dr \qquad \text{and} \qquad dt^2 = dt'^2 - 2\frac{d\phi}{dr}dt'dr + \left(\frac{d\phi}{dr}\right)^2 dr^2$$

Then

$$\begin{aligned} d\tau^2 &= F(r)dt'^2 - 2\left[\frac{d\phi}{dr}F(r) + rE(r)\right] dt'\,dr + \\ &+ \left[\left(\frac{d\phi}{dr}\right)^2 F(r) + 2rE(r)\frac{d\phi}{dr} - r^2 D(r)\right] dr^2 + \\ &- C(r)[dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2] \end{aligned}$$

Choose $\phi(r)$ such that: $\frac{d\phi}{dr}F(r) + rE(r) = 0$, so $\frac{d\phi}{dr} = -r\frac{E(r)}{F(r)}$ then

$$\left(\frac{d\phi}{dr}\right)^2 F(r) + 2rE(r)\frac{d\phi}{dr} - r^2 D(r) = -r^2\left[D(r) + \frac{E^2(r)}{F(r)}\right] \equiv -G(r)$$

Finally we can redefine the radius: $r'^2 \equiv C(r)r^2$, hence

$$dr^2 = \frac{4r^2 C(r)}{[2rC(r) + r^2 C'(r)]^2} dr'^2 \qquad \text{for} \qquad C'(r) \equiv \frac{dC(r)}{dr}$$

Then we obtain the *standard form* of the length element

$$d\tau^2 = B(r')dt'^2 - A(r')dr'^2 - r'^2(d\theta^2 + \sin^2\theta d\varphi^2])$$

for

$$B(r') \equiv F(r) \qquad \text{and} \qquad A(r') \equiv \left[1 + \frac{G(r)}{C(r)}\right]\left[1 + \frac{r}{2C(r)}C'(r)\right]^{-2}$$

We drop primes from now on, so the metric and its inverse read:

$$g_{tt} = B(r), \qquad g_{rr} = -A(r), \qquad g_{\theta\theta} = -r^2, \qquad g_{\varphi\varphi} = -r^2 \sin^2\theta$$

$$g^{tt} = \frac{1}{B(r)}, \qquad g^{rr} = -\frac{1}{A^{(}r)}, \qquad g^{\theta\theta} = -\frac{1}{r^2}, \qquad g^{\varphi\varphi} = -\frac{1}{r^2 \sin^2\theta}$$

Then we calculate the affine connection from the metric

$$\Gamma^{\sigma}_{\lambda\mu} = \frac{1}{2} g^{\nu\sigma} \left\{ \frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} + \frac{\partial g_{\lambda\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\lambda}}{\partial x^{\nu}} \right\}$$

Non-zero entries (see class) are:

$$\begin{array}{llll}
\Gamma^{t}_{tr} = \Gamma^{t}_{rt} = \frac{1}{2} B' B^{-1} & & & \\
\Gamma^{r}_{tt} = \frac{1}{2} B' A^{-1} & \Gamma^{r}_{rr} = \frac{1}{2} A' A^{-1} & \Gamma^{r}_{\theta\theta} = -rA^{-1} & \Gamma^{r}_{\varphi\varphi} = -rA^{-1} \sin^2\theta \\
\Gamma^{\theta}_{r\theta} = \Gamma^{\theta}_{\theta r} = -r^{-1} & \Gamma^{\theta}_{\varphi\varphi} = -\sin\theta\cos\theta & & \\
\Gamma^{\varphi}_{r\varphi} = \Gamma^{\varphi}_{\varphi r} = r^{-1} & \Gamma^{\varphi}_{\varphi\theta} = \cot\theta & &
\end{array}$$

Then the Ricci tensor

$$R_{\mu\kappa} = \frac{\partial \Gamma^{\nu}_{\mu\nu}}{\partial x^{\kappa}} - \frac{\partial \Gamma^{\nu}_{\mu\kappa}}{\partial x^{\nu}} + \Gamma^{\eta}_{\mu\nu}\Gamma^{\nu}_{\kappa\eta} - \Gamma^{\eta}_{\mu\kappa}\Gamma^{\nu}_{\nu\eta}$$

The non-vanishing elements (see class) are:

$$\begin{aligned}
R_{tt} &= -\frac{B''}{2A} + \frac{1}{4}\frac{B'}{A}\left(\frac{B'}{B} + \frac{A'}{A}\right) - \frac{1}{r}\frac{B'}{A} \\
R_{rr} &= \frac{B''}{2B} - \frac{1}{4}\frac{B'}{B}\left(\frac{B'}{B} + \frac{A'}{A}\right) - \frac{1}{r}\frac{A'}{A} \\
R_{\theta\theta} &= -1 + \frac{r}{2A}\left(\frac{B'}{B} - \frac{A'}{A}\right) + \frac{1}{A} \\
R_{\varphi\varphi} &= \sin^2\theta R_{\theta\theta} \\
R_{\mu\nu} &= 0 \quad \text{for} \quad \nu \neq \mu
\end{aligned}$$

Now we are ready to look for solutions of the Einstein’s equations in the empty space

$$R_{\mu\nu} = 0$$

It is sufficient to require $R_{rr} = R_{\theta\theta} = R_{tt} = 0$. Note also that

$$\frac{R_{rr}}{A} + \frac{R_{tt}}{B} = -\frac{1}{r}\frac{1}{A}\left(\frac{B'}{B} + \frac{A'}{A}\right)$$

Therefore $R_{\mu\nu} = 0$ implies that $\frac{B'}{B} + \frac{A'}{A} = 0$, so

$$A \cdot B = \text{const.}$$

The constant is determined by the boundary conditions: $g_{\mu\nu} \xrightarrow[r\to\infty]{} \eta_{\mu\nu}$. Since

$$d\tau^2 = B(r)dt^2 - A(r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

that implies

$$B(r) \xrightarrow[r\to\infty]{} +1, \qquad A(r) \xrightarrow[r\to\infty]{} +1, \qquad \Rightarrow \qquad A(r) = \frac{1}{B(r)}$$

Now it is sufficient to impose

$$R_{rr} = 0 \qquad \text{and} \qquad R_{\theta\theta} = 0$$

Eliminate $A(r)$ through $A = B^{-1}$ ($A' = -\frac{B'}{B^2}$), then

$$R_{\theta\theta} = -1 + \frac{r}{2A}\left(\frac{B'}{B} - \frac{A'}{A}\right) + \frac{1}{A} = -1 + rB' + B$$

$$R_{rr} = \frac{B''}{2B} - \frac{1}{4}\frac{B'}{B}\left(\frac{B'}{B} + \frac{A'}{A}\right) - \frac{1}{r}\frac{A'}{A} = \frac{B''}{2B} + \frac{1}{r}\frac{B'}{B} = \frac{R'_{\theta\theta}}{2rB}$$

Therefore it is sufficient to require

$$R_{\theta\theta} = 0$$

$$\Downarrow$$

$$rB' + B = 1 \qquad \Rightarrow \qquad \frac{d}{dr}(rB) = 1 \qquad \Rightarrow \qquad B(r) = 1 + \frac{\text{const.}}{r}$$

Again the boundary behavior determines the constant, since at large r the $(0,0)$ component of the metric should be related to the Newton's gravitational potential:

$$g_{tt} \underset{r\to\infty}{\sim} 1 + 2\phi = 1 - 2\frac{GM}{r}$$

Therefore the constant = $-2GM$, hence

$$B(r) = \frac{1}{A(r)} = 1 - \frac{r_s}{r} \qquad \text{for} \qquad r_s = 2GM$$

So, finally the solution for a space-time outside of a static massive body of mass M:

$$d\tau^2 = \left(1 - \frac{r_s}{r}\right) dt^2 - \frac{1}{1 - r_s/r} dr^2 - r^2(d\theta^2 + \sin^2\theta \, d\varphi^2)$$

where $r_s = 2GM$ is the Schwarzschild radius.
Comments:

- A test particle which orbits around a central mass on an elliptical orbit will undergo "perihelion motion", which means a rotation of the long axis of the ellipse with respect to distant stars. (Measured e.g. for Mercury is one the earliest triumphs of GR.)
- A passing light-ray which travels at the closest distance b from the central body will be deflected by an angle $\Delta\theta = 4GM/b$. (Measured for a starlight near the obscured Sun during the eclipse.)

Comments:

- Look at a photon ($d\tau^2 = 0$), traveling radially in the Schwarzschild metric, then $cdt = \frac{dr}{1 - r_s/r}$, so that the time to leave from $r = r_s$ to an outside point becomes infinite. Thus, if an object is so dense that its radius is inside the Schwarzschild radius, the object does not emit any light - it is a *black hole*.
- In deriving the Schwarzschild metric, it was assumed that the metric was in the vacuum, spherically symmetric and static. In fact, the static assumption is stronger than required, as Birkhoff's theorem states that any spherically symmetric vacuum solution of Einstein's field equations is stationary; then one obtains the Schwarzschild solution. Birkhoff's theorem has the consequence that any pulsating star which remains spherically symmetric cannot generate gravitational waves (as the region exterior to the star must remain static).