Homework problems #1

1. The energy momentum tensor is defined by

$$\delta I = -\frac{1}{2}\int d^4x g^{1/2} T^{\mu\nu}\delta g_{\mu\nu}$$

The action for a fluid composed of point particles reads

$$I_M = -\sum_n m_n \int_{-\infty}^{\infty} dp \left[g_{\mu\nu}(x_n(p)) \frac{d{x_n}^{\mu}(p)}{dp}\frac{d{x_n}^{\nu}(p)}{dp}\right]^{1/2},$$

where p is a parameter that parametrizes a trajectory ${x_n}^{\mu}(p)$.

- Show that the energy-momentum tensor is of the form

$$\begin{aligned} T^{\alpha\beta}(x) &= g(x)^{-1/2}\sum_n m_n \frac{dx_n^{\alpha}(t)}{dt}\frac{dx_n^{\beta}(t)}{dt}\left(\frac{d\tau_n}{dt}\right)^{-1}\delta^3(\vec{x}-\vec{x}_n(t)) \\ &= g(x)^{-1/2}\sum_n \frac{p_n(t)^{\alpha}p_n(t)^{\beta}}{E_n(t)}\delta^3(\vec{x}-\vec{x}_n(t)) \end{aligned}$$

- Derive the equation of state for the fluid, i.e. the relation between ρ and p for non-relativistic and ultra-relativistic cases.

2. Show that (tt), (ij) components of the Einstein equations and the energy-momentum conservation equation (${T^{\mu\nu}}_{;\nu} = 0$) for the FRW metric are not independent. Explain why does it happen.

3. Derive the energy-momentum tensor for pure electrodynamics.

4. Show that

$${G^{\mu\nu}}_{;\nu} = 0 \quad \text{for} \quad G^{\mu\nu} \equiv R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R$$

5. The harmonic coordinate gauge conditions are defined by

$$\Gamma^{\lambda} \equiv g^{\mu\nu}\Gamma^{\lambda}_{\mu\nu} = 0\,,$$

where $\Gamma^{\lambda}_{\mu\nu}$ are connection components. Show that the condition is equivalent to

$$\partial_{\kappa}(g^{1/2}g^{\lambda\kappa}) = 0$$

and up to the first order in the graviton field h it is equivalent to

$$\partial_{\mu}{h^{\mu}}_{\nu} = \frac{1}{2}\partial_{\nu}{h^{\mu}}_{\mu}\,.$$

Literatura

[1] S. Weinberg, "Graviatation and Cosmology", John Wiley & Sons.