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A Goldberg–Sachs theorem in dimension three

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Abstract

We prove a Goldberg–Sachs theorem in dimension three. To be precise, given a three-dimensional Lorentzian manifold satisfying the topological massive gravity equations, we provide necessary and sufficient conditions on the trace-free Ricci tensor for the existence of a null line distribution whose orthogonal complement is integrable and totally geodetic. This includes, in particular, Kundt spacetimes that are solutions of the topological massive gravity equations.

Keywords: three-dimensional pseudo-Riemannian geometry, Goldberg–Sachs theorem, congruences of geodesics, algebraically special spacetimes, topological massive gravity

1. Introduction

The classical Goldberg–Sachs theorem [GS09] states that *a four-dimensional Ricci-flat Lorentzian manifold* $(\mathcal{M}, \boldsymbol{g})$ *admits a shear-free congruence of null geodesics if and only if its Weyl tensor is algebraically special*. Here, the property of being algebraically special is based on the Petrov classification of the Weyl tensor [Pet00]. Since its original publication in 1962 the theorem has been generalized along two main directions.

- First, it was shown to hold for *Einstein* Lorentzian manifolds, i.e. $R_{ab} = \Lambda\, g_{ab}$ for some constant Λ, and more general energy–momentum tensors, such as $R_{ab} = \Lambda\, g_{ab} + \Phi\, k_a k_b$, for some function Φ and 1-form k_a with $k_a k_b g^{ab} = 0$. Even weaker conditions involving the Cotton tensor have been formulated [KT62, RS63] for which the theorem holds too, thereby highlighting its *conformal invariance*.
- Second, the theorem admits versions in any metric signatures [AG97, GHN10, Nur93, Nur96, NT02, PB83], providing, among others, an interesting result in four-dimensional

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Riemannian geometry stating that *a four-dimensional Einstein Riemannian manifold locally admits a Hermitian structure if and only if its Weyl tensor is algebraically special*. The key to the understanding of this generalization is the fact that in four dimensions, a shear-free congruence of null geodesics and a Hermitian structure are both equivalent to an *integrable totally null complex 2-plane distribution*. The distinction between them is made by different reality structures. Thus, the Goldberg–Sachs theorem relates the existence of integrable null 2-plane distributions to the algebraic speciality of the Weyl tensor.

The recent interest in solutions of Einstein equations in higher dimensions has generated much research into the generalization of the Petrov classification of the Weyl tensor and the Goldberg–Sachs theorem to higher dimensions. One approach to the problem, advocated by [HM88, NT02], is to consider an *(almost) null structure*, i.e. a totally null complex m-plane distribution, on a $2m$-dimensional (pseudo-)Riemannian manifold. Their importance in higher-dimensional black holes was highlighted in [MT10]. Motivated by the conformal invariance and the underlying complex geometry of the theorem in dimension four, one of the authors (AT-C) proved a Goldberg–Sachs theorem in dimension five in [TC11] and higher in [TC12a]. Particularly relevant here is the version of the theorem in dimension $2m + 1$, in which an almost null structure is also defined to be a totally null complex m-plane distribution $\mathcal{N}$, say. The difference now is that $\mathcal{N}$ has an orthogonal complement $\mathcal{N}^{\perp}$ of rank $m + 1$, and the crucial point, here, is that the theorem of [TC12a] gives sufficient, conformally invariant, conditions on the Weyl tensor and the Cotton tensor for the integrability of *both* $\mathcal{N}$ *and* $\mathcal{N}^{\perp}$, i.e. $[\Gamma(\mathcal{N}), \Gamma(\mathcal{N})] \subset \Gamma(\mathcal{N})$ and $[\Gamma(\mathcal{N}^{\perp}), \Gamma(\mathcal{N}^{\perp})] \subset \Gamma(\mathcal{N}^{\perp})$. More refined algebraic classifications of the curvature tensors depending on the concept of almost null structure can be found in [TC12b, TC13, TC14] inspired by [Jef95]. An alternative approach to the classification of the Weyl tensor of higher-dimensional Lorentzian manifolds is given in [CMPP04], and a Goldberg–Sachs theorem in this setting has been given in [DR09, OPPR12, OPP13].

The aim of this paper is to consider yet another generalization, by formulating the theorem in *three dimensions*. A priori, one would expect such a putative Goldberg–Sachs theorem to follow the same lines as in four and higher dimensions. However, there are a number of features of (pseudo-)Riemannian geometry specific to dimension three that prevent such a straightforward generalization.

- First, there is no Weyl tensor.
- Then, Einstein metrics are necessarily of constant curvature.
- Finally, an almost null structure (i.e. a totally null (complex) line distribution) $\mathcal{N}$ is always integrable, and the (conformally invariant) condition that its orthogonal complement $\mathcal{N}^{\perp}$ be integrable too does not impose any constraint on the curvature. Related to this is the fact that congruences of null geodesics are necessarily shear-free.

To remedy these shortcomings, one is led to seek stronger conditions that must depend on the *metric* rather than the conformal structure of our manifold. To this end, we shall exploit the following special features of three-dimensional (pseudo-)Riemannian geometry.

- From an algebraic point of view, the tracefree Ricci tensor Φ_{ab} in dimension three behaves in the same way as the (anti-)self-dual Weyl tensor in dimension four, since they both belong to a five-dimensional irreducible (complex) representation of $SL(2, \mathbb{C})$ or any of its real forms This leads to a notion of *algebraically special* tracefree Ricci tensors in dimension three analogous to the one on the (anti-)self-dual Weyl tensor in dimension four, and a notion of *multiple principal null structure*, i.e. a preferred (complex) null line distribution.

- The Cotton tensor A_{abc} can be Hodge-dualized to yield a tracefree symmetric tensor $(*A)_{ab}$ of valence 2, and must then belong to the same (complex) representation as Φ_{ab}.
- These properties allow us to weaken the Einstein equations to the equations governing *topological massive gravity*, [DJT82], which relate Φ_{ab} and $(*A)_{ab}$ as

$$\Phi_{ab} = \frac{1}{m}(*A)_{ab}, \qquad\qquad R = 6\,\Lambda = \text{constant},$$

where Λ is the cosmological constant and m is a 'mass' parameter. In analogy to the Einstein equations, these equations can also be written as

$$R_{ab} - \frac{1}{2}g_{ab}R + \Lambda\, g_{ab} - \frac{1}{m}(*A)_{ab} = 0.$$

- There is a natural *non-conformally invariant* condition that a (multiple principal) null structure $\mathcal{N}$ can satisfy, namely that not only $\mathcal{N}$ and $\mathcal{N}^{\perp}$ be integrable, but that $\mathcal{N}^{\perp}$ be *totally geodetic*, i.e. $\boldsymbol{g}(\nabla_{\boldsymbol{X}}\boldsymbol{Y}, \boldsymbol{Z}) = 0$ for all $\boldsymbol{X}, \boldsymbol{Y} \in \Gamma(\mathcal{N}^{\perp})$, $\boldsymbol{Z} \in \Gamma(\mathcal{N})$.

Reality conditions imposed on the top of these features will also yield various geometric interpretations of a null structure in terms of congruences of *real* curves. With these considerations in mind, we shall prove ultimately the following theorem.

Theorem 1.1. *Let $(\mathcal{M}, \boldsymbol{g})$ be a three-dimensional Lorentzian manifold satisfying the topological massive gravity equations. Then the tracefree Ricci tensor is algebraically special if and only if $(\mathcal{M}, \boldsymbol{g})$ admits a divergence-free congruence of null geodesics (i.e. it is a Kundt spacetime) or a shear-free congruence of timelike geodesics.*

In fact, it is shown in [CPS10b] that a Kundt spacetime that is also a solution of the topological massive gravity equations must be algebraically special, while the converse is left as an open problem. Theorem 1.1 thus gives an answer to this question.

The strategy adapted in this paper is to use a Newman–Penrose formalism and the Petrov classification of the tracefree Ricci tensor in dimension three. While these have already been used in [MW13], we develop these tools from scratches, and our conventions will certainly differ. Theorem 1.1 will in fact follow from more general theorems that we shall prove in the course of the article. A number of solutions of the topological massive gravity equations have been discovered in recent years, see [CPS10b, CPS10a] and references therein. In a subsequent paper, we shall give further explicit algebraically special solutions of the topological massive gravity equations.

The structure of the paper is as follows. In section 2, we set up the scene with a short introduction of the Newman–Penrose formalism, and we review the background on the general geometric properties of null structures on (pseudo-)Riemannian manifolds in dimension three. Particularly relevant here are the notions of co-integrable and co-geodetic null structures of definition 2.3, the latter being central to the Goldberg–Sachs theorem.

This is taken further in section 3 where we examine the consequences of the reality conditions on a null structure, which may be understood as a congruence of curves that are either null or timelike. Propositions 3.5 and 3.11 in particular give real interpretations of co-integrable and co-geodetic null structures.

Section 4 focuses on the algebraic classification of the tracefree Ricci tensor. We introduce the definition of algebraically special Ricci tensors in definition 4.3 based on the notion of principal null structure of definitions 4.1 and 4.2. This leads to definitions of the complex Petrov types in section 4.1, and their real signature-dependent analogues in sections 4.2.1 and 4.2.2.

Curvature conditions for the existence of co-geodetic and parallel null structures are given in proposition 5.1 of section 5.

The main results of this paper are contained in section 6. We initially give general results for a metric of any signature. We first give in proposition 6.1 obstructions for a multiple principal null structure to be co-geodetic in terms of the Cotton tensor and the derivatives of the Ricci scalar. We then show in theorems 6.3, 6.4, 6.5 and 6.6 how the various algebraically special Petrov types guarantee the existence of a co-geodetic null structure. The converse, that a co-geodetic null structure implies algebraic speciality, is given in theorem 6.7. The application to topological massive gravity in theorem 6.10 then follows naturally. The section is wrapped up by giving real versions 6.11 and 6.12 of the Goldberg–Sachs theorem.

We end the paper with three appendices. Appendix A contains a spinor calculus in three dimensions, which we then apply in appendix B to derive a Newman–Penrose formalism adapted to a null structure. Finally, in appendix C we have given alternative, manifestly invariant, proofs of the main theorems of section 6 in the language of spinors.

2. Geometric considerations

Throughout this section, we consider an oriented three-dimensional (pseudo-)Riemannian smooth manifold $(\mathcal{M}, \boldsymbol{g})$. We shall make use of the abstract index notation of [PR84]. Upstairs and downstairs lower case Roman indices will refer to vector fields and 1-forms on $\mathcal{M}$ respectively, e.g. V^a and α_a, and similarly for more general tensor fields, e.g. $T_{ab}{}^c{}_d$. Symmetrization will be denoted by round brackets around a set of indices, and skew-symmetrization by squared brackets, e.g. $A_{(ab)} = \frac{1}{2}\left(A_{ab} + A_{ba}\right)$ and $B_{[ab]} = \frac{1}{2}\left(B_{ab} - B_{ba}\right)$. Indices will be lowered and raised by the metric g_{ab} and its inverse g^{ab} whenever needs arise, e.g. $V_a = V^b g_{ba}$ and $\alpha^a = g^{ab}\alpha_b$, etc. Bold font will be used for vectors, forms and tensors whenever the index notation is suspended.

The space of sections of a given vector bundle E, say, over $\mathcal{M}$, will be denoted $\Gamma(E)$. The Lie bracket of two vector fields $\boldsymbol{V}$ and $\boldsymbol{W}$ will be denoted by $[\boldsymbol{V}, \boldsymbol{W}]$.

The orientation on $\mathcal{M}$ will be given by a volume form e_{abc} on $\mathcal{M}$ satisfying the normalization conditions

$$
\begin{aligned}
e_{abc}e^{def} &= (-1)^q 6\, g_{[a}^{d} g_{b}^{e} g_{c]}^{f}, \qquad & e_{abe}e^{cde} &= (-1)^q 2\, g_{[a}^{c} g_{b]}^{d}, \\
e_{acd}e^{bcd} &= (-1)^q 2\, g_{a}^{b}, \qquad & e_{abc}e^{abc} &= (-1)^q 6,
\end{aligned} \tag{2.1}
$$

where q is the number of negative eigenvalues of the metric g_{ab}. We can then eliminate 2-forms in favour of 1-forms by means of the Hodge duality operation, i.e.

$$
(*\,\alpha)_a := \frac{1}{2} e_a{}^{bc}\alpha_{bc},
$$

for any 2-form α_{ab}.

The Levi-Civita connection of g_{ab}, i.e. the unique torsion-free connection preserving g_{ab}, will be denoted ∇_a. The Riemann curvature tensor associated with ∇ is defined by

$$
R_{abd}{}^c V^d := 2\, \nabla_{[a}\nabla_{b]} V^c.
$$

In three dimensions, the Riemann tensor decomposes as

$$R_{abcd} = 4\, g_{[a|[c}\Phi_{d]|b]} + \frac{1}{3} R\, g_{[a|[c}g_{d]|b]}, \tag{2.2}$$

where $\Phi_{ab} := R_{ab} - \frac{1}{3}Rg_{ab}$ and R are tracefree part of the Ricci tensor $R_{ab} := R_{acb}{}^{c}$ and the Ricci scalar respectively. In three dimensions, the Bianchi identity $\nabla_{[a}R_{bc]de} = 0$ is equivalent to the contracted Bianchi identity

$$\nabla^{b}\Phi_{ba} - \frac{1}{6}\nabla_{a}R = 0. \tag{2.3}$$

For future use, we define the *Schouten* or *Rho* tensor

$$\mathrm{P}_{ab} := -\Phi_{ab} - \frac{1}{12}Rg_{ab}.$$

To eliminate the use of fractions, we also set

$$S := \frac{1}{12}R.$$

With this convention, the *Cotton tensor* takes the form

$$A_{abc} := 2\, \nabla_{[b}\mathrm{P}_{c]a} = -2\, \nabla_{[b}\Phi_{c]a} + 2\, g_{a[b}\, \nabla_{c]}S. \tag{2.4}$$

By construction, the Cotton tensor A_{abc} satisfies the symmetry $A_{[abc]} = 0$ and $A^{a}{}_{ab} = 0$. It also satisfies the condition

$$\nabla^{a}A_{abc} = 0, \tag{2.5}$$

since commuting the covariant derivatives and using (2.3) and (2.2) give

$$\nabla^{a}A_{abc} = -2\, \nabla^{a}\nabla_{[b}\Phi_{c]a} + \cancel{2\, \nabla_{[b}\nabla_{c]}S} = \cancel{-4\, \nabla_{[b}\nabla_{c]}S} + \cancel{2\, R^{a}{}_{[bc]}{}^{d}\Phi_{da}} + \cancel{2\, \Phi_{[b}{}^{d}\Phi_{c]d}} = 0.$$

It is more convenient to Hodge-dualize A_{abc} to obtain the tensor of valence 2

$$(*A)_{ab} := \frac{1}{2}e_{b}{}^{cd}A_{acd}. \tag{2.6}$$

Dualizing a second time over ab yields $e_{a}{}^{bc}(*A)_{bc} = A^{b}{}_{ba} = 0$, which means that $(*A)_{ab} = (*A)_{(ab)}$. The Cotton tensor can then be expressed by

$$(*A)_{ab} = -e_{(a|}{}^{cd}\,\nabla_{c}\Phi_{d|b)}.$$

2.1. Null structures in dimension three

For the time being, we shall keep the discussion general, by considering an oriented three-dimensional (pseudo-)Riemannian manifold $(\mathcal{M}, \boldsymbol{g})$ of any signature. We shall denote the complexification of the tangent bundle by $\mathrm{T}^{\mathbb{C}}\mathcal{M}$. It will be often convenient to consider a complex-valued metric $\boldsymbol{g}^{\mathbb{C}}$ by extending $\boldsymbol{g}$ via

$$\boldsymbol{g}^{\mathbb{C}}(\boldsymbol{X} + \mathrm{i}\boldsymbol{Y}, \boldsymbol{Z} + \mathrm{i}\boldsymbol{W}) = \boldsymbol{g}(\boldsymbol{X}, \boldsymbol{Z}) - \boldsymbol{g}(\boldsymbol{Y}, \boldsymbol{W}) + \mathrm{i}(\boldsymbol{g}(\boldsymbol{X}, \boldsymbol{W}) + \boldsymbol{g}(\boldsymbol{Y}, \boldsymbol{Z})),$$

for all $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}, \boldsymbol{W} \in \Gamma(\mathrm{T}\mathcal{M})$. For the remaining part of the paper, we shall omit the $^{\mathbb{C}}$ on $\boldsymbol{g}^{\mathbb{C}}$. It should be clear from the context whether we are using $\boldsymbol{g}$ or $\boldsymbol{g}^{\mathbb{C}}$.

Let $\mathcal{N}$ be a line subbundle of the complexified tangent bundle $\mathrm{T}^{\mathbb{C}}\mathcal{M} := \mathbb{C} \otimes \mathrm{T}\mathcal{M}$, null with respect to the complexified metric, i.e. for any section k^{a} of $\mathcal{N}$, $g_{ab}k^{a}k^{b} = 0$, and $\mathcal{N}^{\perp}$ the orthogonal complement of $\mathcal{N}$ with respect to g_{ab}, i.e. at any point p in $\mathcal{M}$,

$$\mathcal{N}_p^{\perp} := \left\{ V^a \in \mathrm{T}_p^{\mathbb{C}}\mathcal{M} \colon V^a k^b g_{ab} = 0, \text{ for all} k^a \in \mathcal{N}_p \right\}.$$

We thus have a filtration $\mathcal{N} \subset \mathcal{N}^{\perp}$ on $\mathrm{T}^{\mathbb{C}}\mathcal{M}$.

Definition 2.1. We shall refer to a real or complex null line distribution on $(\mathcal{M}, \boldsymbol{g})$ as a *null structure*.

This is a three-dimensional specialization of the notion of *almost null structures* presented in [TC12a] (also referred to as *γ-plane distributions* in [TC13]). Since $\mathcal{N}$ is one-dimensional, $\mathcal{N}$ is clearly automatically formally integrable, i.e. $[\Gamma(\mathcal{N}), \Gamma(\mathcal{N})] \subset \Gamma(\mathcal{N})$, and we can unambiguously dispense with the word 'almost' in definition 2.1.

2.2. Rudiments of Newman–Penrose formalism

For most of the paper, it will be convenient to use the Newman–Penrose formalism as described in details in appendix B. To do this we introduce a frame of $\mathrm{T}^{\mathbb{C}}\mathcal{M}$ as follows. We fix a section k^a of a given null structure $\mathcal{N}$, and choose a vector field ℓ^a such that $k^a \ell^b g_{ab} = 1$. Clearly, ℓ^a is transversal to $\mathcal{N}^{\perp}$. We also choose a section n^a of $\mathcal{N}^{\perp}$, not in $\mathcal{N}$, which we may normalize as $n^a n_a = -\frac{1}{2}$.

Definition 2.2. We say that a frame (k^a, ℓ^a, n^a) is *adapted to a null structure $\mathcal{N}$ on $(\mathcal{M}, \boldsymbol{g})$* if and only if k^a generates $\mathcal{N}$ and the metric can be expressed as

$$g_{ab} = 2\, k_{(a}\ell_{b)} - 2\, n_a n_b. \tag{2.7}$$

In fact, we have a class of frames adapted to the null structure, and any two frames in that class are related via the transformation

$$k^a \mapsto a k^a, \qquad n^a \mapsto b\left(n^a + z k^a\right), \qquad \ell^a = a^{-1}\left(\ell^a + 2\, z n^a + z^2 k^a\right). \tag{2.8}$$

for some functions a, z and b where a is non-vanishing and $b^2 = 1$.

We can expand the covariant derivatives of the adapted frame vectors as follows

$$\nabla_a k^b = 2\,\gamma\, k_a k^b + 2\,\epsilon\, \ell_a k^b - 4\,\alpha\, n_a k^b - 2\,\tau\, k_a n^b - 2\,\kappa\, \ell_a n^b + 4\,\rho\, n_a n^b, \tag{2.9}$$

$$\nabla_a \ell^b = -2\,\epsilon\, \ell_a \ell^b - 2\,\gamma\, k_a \ell^b + 4\,\alpha\, n_a \ell^b + 2\,\pi\, \ell_a n^b + 2\,\nu\, k_a n^b - 4\,\mu\, n_a n^b, \tag{2.10}$$

$$\nabla_a n^b = -\kappa\, \ell_a \ell^b + \nu k_a k^b + \pi\, \ell_a k^b - \tau\, k_a \ell^b + 2\,\rho\, n_a \ell^b - 2\,\mu\, n_a k^b, \tag{2.11}$$

where α, γ, ϵ, κ, μ, ν, π, ρ, and τ are the connection coefficients, also known as *Newman–Penrose coefficients*. We shall also introduce the following notation for the frame derivatives:

$$D := k^a \nabla_a, \qquad \Delta := \ell^a \nabla_a, \qquad \delta := n^a \nabla_a.$$

The curvature components of ∇_a and the Bianchi identies can then be expressed in terms of these coefficients and derivatives thereof, and their full description, also given in appendix B, is known as the *Newman–Penrose equations*.

2.3. Geometric properties

While a null structure $\mathcal{N}$ is always integrable, the following definition gives additional geometric conditions that $\mathcal{N}$ can satisfy—these are related to the notion of *intrinsic torsion* examined in [TC13].

Definition 2.3. Let $\mathcal{N}$ be a null structure on $(\mathcal{M}, \boldsymbol{g})$. We say that

- $\mathcal{N}$ is *co-integrable* if its orthogonal complement $\mathcal{N}^\perp$ is formally integrable, i.e.

$$\left[\Gamma\left(\mathcal{N}^\perp\right), \Gamma\left(\mathcal{N}^\perp\right)\right] \subset \Gamma\left(\mathcal{N}^\perp\right); \tag{2.12}$$

- $\mathcal{N}$ is *co-geodetic* if its orthogonal complement $\mathcal{N}^\perp$ is formally totally geodetic, i.e.

$$\boldsymbol{g}\left(\nabla_{\boldsymbol{X}}\boldsymbol{Y}, \boldsymbol{Z}\right) = 0, \text{ for all } \boldsymbol{X}, \boldsymbol{Y} \in \Gamma\left(\mathcal{N}^\perp\right), \boldsymbol{Z} \in \Gamma(\mathcal{N}); \tag{2.13}$$

- $\mathcal{N}$ is *parallel* if $\nabla_{\boldsymbol{Y}}\boldsymbol{X} \in \Gamma(\mathcal{N})$ for all $\boldsymbol{X} \in \Gamma(\mathcal{N})$, $\boldsymbol{Y} \in \Gamma(\mathrm{T}\mathcal{M})$.

Using the standard formula $[\boldsymbol{X}, \boldsymbol{Y}] = \nabla_{\boldsymbol{X}}\boldsymbol{Y} - \nabla_{\boldsymbol{Y}}\boldsymbol{X}$ for any vector fields $\boldsymbol{X}, \boldsymbol{Y} \in \Gamma(\mathrm{T}^{\mathbb{C}}\mathcal{M})$, one can prove the following lemma [TC13].

Lemma 2.4. *Let $\mathcal{N}$ be a null structure on $(\mathcal{M}, \boldsymbol{g})$. Then*

$$\mathcal{N} \text{ is parallel} \Rightarrow \mathcal{N} \text{ is co-geodetic} \Rightarrow \mathcal{N} \text{ is co-integrable.}$$

Remark 2.5. Note that in three dimensions, unlike in higher odd dimensions, a null structure $\mathcal{N}$ automatically satisfies $\nabla_{\boldsymbol{Y}}\boldsymbol{X} \in \Gamma(\mathcal{N}^\perp)$ for all $\boldsymbol{X} \in \Gamma(\mathcal{N})$, $\boldsymbol{Y} \in \Gamma(\mathrm{T}\mathcal{M})$, as can be read off from (2.9).

Remark 2.6. Of the three geometric properties listed in definition 2.3, only the property that $\mathcal{N}$ be co-integrable is conformally invariant since it depends only on the Lie bracket. The remaining properties break conformal invariance—see [TC13] for details.

It is convenient to re-express condition (2.12) and (2.13) in terms of the Levi-Civita connection as given in the following proposition.

Proposition 2.7. *Let $\mathcal{N}$ be a null structure on $(\mathcal{M}, \boldsymbol{g})$, and let k^a be a generator of $\mathcal{N}$. Then*

$$\mathcal{N} \text{ is co-integrable} \iff k_{[a}\nabla_b k_{c]} = 0 \iff \left(k^b\nabla_b k^{[a}\right)k^{b]} = 0, \tag{2.14}$$

$$\mathcal{N} \text{ is co-geodetic} \iff k^a\nabla_b k^b - k^b\nabla_b k^a = 0 \iff k_{[a}\left(\nabla_{b]}k_{[c}\right)k_{d]} = 0. \tag{2.15}$$

Remark 2.8. Conditions (2.14) tell us that any generator of $\mathcal{N}$ is formally geodetic, or equivalently in three dimensions, formally twist-free, or equivalently, formally shear-free, i.e. $\mathcal{L}_k g_{ab} \propto g_{ab} (\mathrm{mod} \quad k_{(a}\alpha_{b)})$. Conditions (2.15) tell us that any generator of $\mathcal{N}$ is formally geodetic and divergence-free.

Proof. Let k^a be a generator of $\mathcal{N}$. In terms of the Newman–Penrose coefficients, we have

$$\mathcal{N} \text{ is co-integrable} \quad \Longleftrightarrow \quad \kappa = 0,$$

which follows from

$$[D, \delta] = 2\,\rho\delta + (\pi - 2\,\alpha)D - \kappa\Delta. \tag{B.4}$$

On the other hand, comparison with

$$\left(Dk^{[a}\right)k^{b]} = -4\,\kappa\, n^{[a}k^{b]}, \qquad k_{[a}\nabla_b k_{c]} = -4\,\kappa\, k_{[a}\ell_b n_{c]},$$

establishes the equivalence (2.14).

Further, since (2.13) can be expressed as $n^a D k_a = 2\,\kappa$ and $n^a \delta k_a = 2\,\rho$ in our null basis, we have

$$\mathcal{N} \text{ is co-geodetic} \quad \Longleftrightarrow \quad \kappa = \rho = 0.$$

Comparison with

$$k^a \nabla_b k^b - Dk^a = 4\left(\kappa\, n^a - \rho\, k^a\right),$$

establishes the equivalence (2.15). □

Another way to express the condition for a null structure to be co-geodetic is given by the following proposition.

Proposition 2.9. *Locally, there is a one-to-one correspondence between closed and co-closed complex-valued 1-forms and co-geodetic null structures.*

Proof. We note that a co-integrable null structure $\mathcal{N}$ is equivalent to its generator k^a, say, satisfying $k_{[a}\nabla_b k_{c]} = 0$ since k_a is also the annihilator of $\mathcal{N}^\perp$. We can always rescale k^a to make it closed, i.e. $\nabla_{[b}k_{c]} = 0$–for details, see lemma 5.1 in [HN09]. In this case, k^a also satisfies $k^a \nabla_a k^b = 0$. If in addition $\mathcal{N}$ is co-geodetic, then using the equivalence (2.15), we have $\nabla^a k_a = 0$, i.e. k^a is coclosed. The converse is also true. □

A generalization to higher dimensions applicable to higher valence spinor fields in the analytic case is given in [TC13].

As we shall see in the next section, the differential conditions on a null structure $\mathcal{N}$ will yield quite different geometric interpretations depending on the signature of $\boldsymbol{g}$.

Relation to harmonic morphisms

Definition 2.10 **[BW88, BW95].** Let $\varphi\colon \mathcal{M} \to \mathbb{C}$ be a complex-valued smooth map on $\mathcal{M}$. We say that φ is *horizontal conformal* if it satisfies $(\nabla^a\varphi)(\nabla_a\varphi) = 0$, and a *harmonic morphism* if it satisfies $(\nabla^a\varphi)(\nabla_a\varphi) = \nabla^a\nabla_a\varphi = 0$.

With reference to the proof of proposition 2.9, we can rescale a generator k^a of a co-integrable null structure $\mathcal{N}$ such that $k_a = \nabla_a\varphi$. Since such a k^a is null, φ is a horizontal conformal map.

If $\mathcal{N}$ is also co-geodetic, k_a is also co-closed and φ must be a harmonic morphism. Summarizing,

Corollary 2.11. *On a three-dimensional (pseudo-)Riemannian manifold, locally, there is a one-to-one correspondence between*

- *horizontal conformal maps and co-integrable null structures;*
- *harmonic morphisms and co-geodetic null structures.*

3. Real metrics

As before, $(\mathcal{M}, \boldsymbol{g})$ will denote an oriented three-dimensional (pseudo-)Riemannian manifold. So far the discussion has been independent of the signature of the metric $\boldsymbol{g}$. Different metric signatures will induce different reality conditions on $\mathrm{T}^{\mathbb{C}}\mathcal{M}$, and consequently, different geometric interpretations of a null stucture $\mathcal{N}$. As is standard, the complex conjugate of $\mathcal{N}$ will be denoted $\overline{\mathcal{N}}$. We start with the following definition.

Definition 3.1 **[KT92, Kop97].** The *real index* of a null structure $\mathcal{N}$ on $(\mathcal{M}, \boldsymbol{g})$ at a point p is the dimension of the intersection $\mathcal{N}_p \cap \overline{\mathcal{N}}_p$.

Lemma 3.2 **[KT92, Kop97].** *At any point, the real index of a null structure $\mathcal{N}$ on $(\mathcal{M}, \boldsymbol{g})$ must be*

- *0 when $\mathbf{g}$ has signature (3,0);*
- *either 0 or 1 when $\mathbf{g}$ has signature (2,1).*

3.1. Lorentzian signature

Assume that $(\mathcal{M}, \boldsymbol{g})$ is an oriented Lorentzian manifold with signature (2,1). The geometrical features of a null structure of real index 0 or real index 1 are discussed separately.

3.1.1. Real index 0. We shall exhibit the relation between null structures of real index 0 and congruences of timelike curves. In particular, we shall also show that a *co-geodetic* null structure is equivalent to the existence of a *shear-free* congruence of timelike *geodesics*.

Assume that $\mathcal{N}$ is of real index 0. Then $\mathcal{N}$ is a complex distribution, whose real and imaginary parts span a spacelike distribution $\Re(\mathcal{N} \oplus \overline{\mathcal{N}})$. The orthogonal complement of $\Re(\mathcal{N} \oplus \overline{\mathcal{N}})$ is necessarily timelike and orthogonal to both $\mathcal{N}$ and $\overline{\mathcal{N}}$. Let u^a be a unit timelike vector field generating this timelike distribution, i.e. u^a satisfies $u^a u_a = -1$. There is a sign ambiguity in the definition of u^a, which can be fixed by a choice of time-orientation. In this case, the metric takes the form

$$g_{ab} = h_{ab} - u_a u_b. \tag{3.1}$$

where h_{ab} is annhilated by u^a, i.e. $h_{ab} u^a = 0$. The orientation of $(\mathcal{M}, \boldsymbol{g})$ given by the volume form e_{abc} normalized as in (2.1) with $q = 1$ allows us to Hodge-dualize the timelike vector field u^a to produce a 2-form ω_{ab}, i.e.

$$\omega_{ab} := -e_{abc} u^c.$$

This in turns yields an endomorphism

$$J_a{}^b = \omega_{ac} g^{cb},$$

on $\mathrm{T}\mathcal{M}$, which can be seen to satisfy

$$J_a{}^c J_c{}^b = -h_a{}^b, \qquad u^b J_b{}^a = 0. \tag{3.2}$$

This yields a splitting of the complexified tangent bundle

$$\mathrm{T}^{\mathbb{C}}\mathcal{M} = \mathrm{T}^{(1,0)} \oplus \mathrm{T}^{(0,1)} \oplus \mathrm{T}^{(0,0)},$$

where $\mathrm{T}^{(1,0)}$, $\mathrm{T}^{(0,1)}$ and $\mathrm{T}^{(0,0)}$ are the $-$i-, $+$i- and 0-eigenbundles of $J_a{}^b$ respectively. In particular, a null structure of index 0 yields a CR structure with a preferred splitting as described in [HN09].

In an adapted frame (k^a, ℓ^a, n^a), we have $u^a = \sqrt{2} n^a$ with the following reality conditions

$$\left(k^a, \ell^a, n^a\right) \mapsto \left(\overline{k^a}, \overline{\ell^a}, \overline{n^a}\right) = \left(\ell^a, k^a, n^a\right).$$

Thus ℓ^a is the complex conjugate of k^a, and n^a is real. If our orientation is chosen such that

$$e_{abc} = 6\mathrm{i} k_{[a} \ell_b u_{c]},$$

then k^a and $\bar{k}^a := \ell^a$ satisfy

$$k^b J_b{}^a = \mathrm{i} k^a, \qquad \bar{k}^b J_b{}^a = -\mathrm{i} \bar{k}^a.$$

At this stage, we remark that a unit timelike vector field determines a congruence of oriented timelike geodesics, and conversely, given such a congruence, we can *always* find a unit timelike vector field u^a tangent to the curves of the congruence. Further, the effect of changing the orientation of u^a will have the effect of interchanging the null structure $\mathcal{N}$ and its complex conjugate $\overline{\mathcal{N}}$. We can therefore summarize our results in the following way.

Proposition 3.3. *On an oriented and time-oriented three-dimensional Lorentzian manifold, there is a one-to-one correspondence between null structures and congruences of oriented timelike curves.*

It is convenient to encode the geometric properties of $\mathcal{N}$ and $\mathcal{N}^\perp$ in the covariant derivative of u^a. To characterize these properties, we decompose the covariant derivative of u^a into irreducibles under the stabilizer of u^a in SO(2, 1) as recorded in the following definition.

Definition 3.4. Let u^a be a unit timelike vector field on $(\mathcal{M}, \boldsymbol{g})$ and $\mathcal{U}$ its associated congruence of oriented (timelike) curves. Then $\mathcal{U}$ is

- *geodetic* if and only if

$$u^b \nabla_b u^a = 0; \tag{3.3}$$

- *twist-free* if and only if

$$\left(\nabla_{[a} u_b\right) u_{c]} = 0; \tag{3.4}$$

- *divergence-free* if and only if

$$\nabla^c u_c = 0; \tag{3.5}$$

- *shear-free* if and only if

$$\nabla_{(a} u_{b)} - \frac{1}{2} h_{ab} \nabla^c u_c + \left(u^c \nabla_c u_{(a} \right) u_{b)} = 0. \tag{3.6}$$

Using the standard formula for the Lie derivative

$$\mathcal{L}_u h_{ab} = 2\ \nabla_{(a} u_{b)} + 2\left(u^c \nabla_c u_{(a} \right) u_{b)}, \tag{3.7}$$

we can re-express the shear-free condition (3.6) as

$$\mathcal{L}_u h_{ab} = \Omega^2\ h_{ab}, \tag{3.8}$$

for some function Ω.

These properties of the congruence $\mathcal{U}$ do not depend on the orientation of u^a, and thus also apply to congruences of unoriented curves.

Proposition 3.5. *Let $\mathcal{N}$ be a null structure of real index 0 on $(\mathcal{M}, \boldsymbol{g})$ equipped with a time-orientation, and let $\mathcal{U}$ be its associated congruence of oriented timelike curves. Then*

- *$\mathcal{N}$ is co-integrable if and only if $\mathcal{U}$ is shear-free;*
- *$\mathcal{N}$ is co-geodetic if and only if $\mathcal{U}$ is shear-free and geodetic.*

Proof. In terms of the Newman–Penrose formalism, and with suitable reality conditions, we have

$$\begin{aligned} \nabla_a u_b = &-\sqrt{2}\left(\bar{\kappa} k_a k_b + \kappa \bar{k}_a \bar{k}_b \right) + 2\left(\bar{\rho} u_a k_b + \rho u_a \bar{k}_b \right) \\ &+ \frac{\mathrm{i}}{\sqrt{2}} (\tau - \bar{\tau}) \omega_{ab} - \frac{1}{\sqrt{2}} (\tau + \bar{\tau}) h_{ab}, \end{aligned} \tag{3.9}$$

so that taking the irreducible components, we obtained

$$\begin{aligned} u^b \nabla_b u_a &= -2\left(\rho\ \bar{k}_a + \bar{\rho}\ k_a \right), \\ \left(\nabla_{[a} u_b \right) u_{c]} &= \frac{\mathrm{i}}{\sqrt{2}} (\tau - \bar{\tau}\ ) \omega_{[ab} u_{c]}, \\ \nabla^a u_a &= -\sqrt{2}\,(\tau - \bar{\tau}), \\ 2\nabla_{(a} u_{b)} - h_{ab} \nabla^c u_c + 2\left(u^c \nabla_c u_{(a} \right) u_{b)} &= -2\sqrt{2}\left(\kappa\ \bar{k}_a \bar{k}_b + \bar{\kappa}\ k_a k_b \right). \end{aligned}$$

The conclusion of the proof now follows from (the proof of) proposition 2.7. □

Local forms of metrics. We now give the normal form of a metric admitting a shear-free congruence of timelike geodesics.

Proposition 3.6. *Let $(\mathcal{M}, \boldsymbol{g})$ be an oriented and time-oriented three-dimensional Lorentzian manifold admitting a co-integrable null structure $\mathcal{N}$, k^a a section of $\mathcal{N}$ and u^a the associated* *unit timelike vector field. Then, around each point, there exist coordinates* $(t, z, \bar{z})$ *such that the metric takes the form*

$$\begin{aligned} \boldsymbol{g} &= -\left(\mathrm{d}t - \bar{p}\,\mathrm{d}z - p\,\mathrm{d}\bar{z}\right)^2 + 2h^2 \mathrm{d}z\,\mathrm{d}\bar{z}, \\ \boldsymbol{k} &= h^{-1}\left(\partial_{\bar{\zeta}} + p\partial_t\right), \qquad \boldsymbol{u} = \partial_t, \end{aligned} \tag{3.10}$$

where $h = h(z, \bar{z}, t)$ *and* $p = p(z, \bar{z}, t)$.

If $\mathcal{N}$ *is co-geodetic, we have in addition*

$$\partial_t p = 0. \tag{3.11}$$

Proof. We first note that the complex-valued 1-form k_a satisfies $k_{[a} \nabla_b k_{c]} = 0$. By a lemma of [HLN08], k_a can be put in the form $k_a = h \nabla_a \zeta$ for some real function h and complex function ζ such that $\mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta} \neq 0$. We can therefore use ζ and its complex conjugate $\bar{\zeta}$ as complex coordinates on $(\mathcal{M}, \boldsymbol{g})$. We can also choose a real coordinate t such that $u^a \nabla_a = \dfrac{\partial}{\partial t}$. The metric must therefore take the form (3.10).

Further, the property that u^a be geodetic can be expressed as

$$0 = -u^b \nabla_b u_a = u^b \left(\nabla_a u_b - \nabla_b u_a \right),$$

which leads to (3.11). □

Remark 3.7. One can check that a null structure $\mathcal{N}$ of real index 0 on a three-dimensional Lorentzian manifold $(\mathcal{M}, \boldsymbol{g})$ is also known as an *almost contact Lorentzian structure*—see [Cal11] for details and generalization to higher odd dimensions. The contact distribution is precisely annihilated by the timelike vector field u^a. When the null structure is co-geodetic, the almost contact Lorentzian structure is said to be *normal*.

3.1.2. Real index 1. A null structure $\mathcal{N}$ of real index 1 satisfies $\dim(\mathcal{N}_p \cap \overline{\mathcal{N}_p}) = 1$ at every point $p \in \mathcal{M}$. In particular, since $\mathcal{N}$ is one-dimensional, it must be totally real.

We can therefore introduce a totally real basis (k^a, ℓ^a, u^a) of $\mathrm{T}\mathcal{M}$, where k^a is a generator of $\mathcal{N}$, ℓ^a a null vector field transversal to $\mathcal{N}^\perp$ such that $k^a \ell^b g_{ab} = 1$, and u^a a unit spacelike vector field in $\mathcal{N}^\perp$, i.e. $u^a u_a = 1$, and thus complementary to k^a and ℓ^a. The Lorentzian metric then takes the form

$$g_{ab} = 2\, k_{(a} \ell_{b)} + u_a u_b. \tag{3.12}$$

Setting $n^a = \dfrac{\mathrm{i}}{\sqrt{2}} u^a$, an adapted frame (k^a, ℓ^a, n^a) can be recovered from (k^a, ℓ^a, u^a), in which case it satisfies the reality condition

$$\left(k^a, \ell^a, n^a\right) \mapsto \left(\overline{k^a}, \overline{\ell^a}, \overline{n^a}\right) = \left(k^a, \ell^a, -n^a\right).$$

In what follows, we shall not be concerned with the orientation of k^a, i.e. whether it is past-pointing or future-pointing.

Definition 3.8. Let $\mathcal{N}$ be a null structure of real index 1 on $(\mathcal{M}, \boldsymbol{g})$, and k^a a section of $\mathcal{N}$. Let $\mathcal{K}$ be the congruence of null curves generated by k^a. Then $\mathcal{K}$ is

- *geodetic* if and only if

$$\left(k^b \nabla_b k^{[a}\right) k^{b]} = 0; \tag{3.13}$$

- *divergence-free geodetic* if and only if

$$k^a \nabla_b k^b - k^b \nabla_b k^a = 0. \tag{3.14}$$

Definition 3.9. A three-dimensional Lorentzian manifold equipped with a divergence-free congruence of null geodesics is called a *Kundt spacetime*.

Remark 3.10. In dimensions greater than three, congruences of null geodesics are also characterized by their *shear* and *twist*, and a Kundt spacetime is usually defined as a Lorentzian manifold equipped with a shear-free, twist-free and divergence-free congruence of null geodesics. However, in three dimensions, any congruence of null geodesics has vanishing shear and twist.

From proposition 2.7, we now obtain the geometric interpretation of a null structure of real index 1.

Proposition 3.11. *Let $\mathcal{N}$ be a null structure $\mathcal{N}$ of real index 1 on $(\mathcal{M}, \boldsymbol{g})$. Then*

- *$\mathcal{N}$ is co-integrable if and only if it generates a congruence of null geodesics.*
- *$\mathcal{N}$ is co-geodetic if and only if it generates a divergence-free congruence of null geodesics.*

3.2. Euclidean signature

Assume now $(\mathcal{M}, \boldsymbol{g})$ has Euclidean signature. Then a null structure is necessarily of real index 0, and an adapted frame $\{k^a, \ell^a, n^a\}$ will satisfy the reality conditions

$$\left\{k^a, \ell^a, n^a\right\} \mapsto \left\{\overline{k^a}, \overline{\ell^a}, \overline{n^a}\right\} = \left\{\ell^a, k^a, -n^a\right\}.$$

Set $\bar{k}^a := \ell^a$ and $u^a := \sqrt{2}\,\mathrm{i}\, n^a$, so that u^a is a real spacelike vector of unit norm, i.e. $u^a u_a = 1$. We can then write the metric (2.7) as

$$g_{ab} = 2\, k_{(a} \bar{k}_{b)} + u_a u_b. \tag{3.15}$$

Clearly, this setting is almost identical to the case where $(\mathcal{M}, \boldsymbol{g})$ has Lorentzian signature and is equipped with a null structure of real index 0. The only difference is that now u^a is spacelike, rather than timelike. Real tensors ω_{ab} and $J_a{}^b$ are defined in exactly the same manner as in the Lorentzian case, and we now have $h_{ab} = 2\, k_{(a} \bar{k}_{b)} = g_{ab} - u_a u_b$.

The reader is invited to go through section 3.1.1 with u^a now spacelike.

Remark 3.12. Following on from remark 3.7, a null structure $\mathcal{N}$ on a three-dimensional Riemannian manifold $(\mathcal{M}, \boldsymbol{g})$ can be shown to be equivalent to an *almost contact Riemannian (or metric) structure*—see [CG90] and references therein for details.

4. Algebraic classification of the tracefree Ricci tensor

A special feature of the Riemann curvature of the Levi-Civita connection on a three-dimensional (pseudo-)Riemannian manifold $(\mathcal{M}, \boldsymbol{g})$ is that it is entirely determined by the Ricci tensor. Its tracefree part Φ_{ab} belongs to a five-dimensional irreducible representation of $\mathrm{SO}(3, \mathbb{C})$, and as for the Weyl tensor in four dimensions, we can introduce the notion of principal null structure to classify Φ_{ab}.

Definition 4.1. Let $\mathcal{N}$ be a null structure. We say that it is *principal* at a point p of $\mathcal{M}$ if a null vector ξ^a generating $\mathcal{N}_p$ satisfies

$$\Phi_{ab}\xi^a\xi^b = 0, \tag{4.1}$$

at p.

Now, the space of all complex null vectors at a point is parametrized by points of the Riemann sphere $S^2 \cong \mathbb{CP}^1$. To be precise, using a standard chart on $\mathbb{CP}^1$, an arbitrary null vector, written in an adapted frame (k^a, ℓ^a, n^a), is of the form

$$\xi^a(z) = k^a + 2\,z\,n^a + z^2\ell^a, \tag{4.2}$$

for some $z \in \mathbb{C}$. The other standard chart on $\mathbb{CP}^1$ is simply obtained by sending z to z^{-1} in (4.2). Thus, to determine all the principal null structures at a point, it suffices to plug (4.2) into (4.1) to form the quartic polynomial

$$0 = \Phi(z) := \frac{1}{2}\Phi_{ab}\xi^a(z)\xi^b(z) = \Phi_0 + 4\,\Phi_1 z + 6\,\Phi_2 z^2 + 4\,\Phi_3 z^3 + \Phi_4 z^4, \tag{4.3}$$

in $\mathbb{C}$, where

$$\Phi_0 := \frac{1}{2}\Phi_{ab}k^a k^b, \qquad \Phi_1 := \frac{1}{2}\Phi_{ab}k^a n^b, \qquad \Phi_2 := \frac{1}{2}\Phi_{ab}k^a\ell^a = \frac{1}{2}\Phi_{ab}n^a n^b,$$
$$\Phi_3 := \frac{1}{2}\Phi_{ab}\ell^a n^b, \qquad \Phi_4 := \frac{1}{2}\Phi_{ab}\ell^a\ell^b.$$

Thus, a root z of (4.3) determines a principal null structure $\mathcal{N}$ where $\xi^a(z)$ generates $\mathcal{N}$ at a point. Conversely, any principal null structure determines a unique root (up to multiplicity) of (4.3) at a point. In particular, the algebraic classification of the tracefree Ricci tensor boils down to the classification of the roots of (4.3) and their multiplicities. A full review of the classification of the Weyl tensor in four dimensions is given in [GHN10], and we shall only recall their results closely following their terminology.

Definition 4.2. Let $\mathcal{N}$ be a principal null structure determined by a root z of the associated polynomial (4.3) at a point. We say that $\mathcal{N}$ is *multiple* at that point if z is multiple.

Definition 4.3. We say that Φ_{ab} is *algebraically special* at a point if it admits a multiple principal null structure at that point.

Rather than considering the quartic polynomial (4.3), it is convenient to consider the quartic *homogeneous* polynomial

$$0 = \Phi(z) = \Phi_{ABCD}\xi^A\xi^B\xi^C\xi^D, \tag{4.4}$$

where ξ^A are now complex homogeneous coordinates on $\mathbb{CP}^1$, and Φ_{ABCD} is an element of $\odot^4(\mathbb{C}^2)^*$, with the understanding that the upper case Roman indices take the values 0 and 1. We can then recover (4.3) by setting $\xi^A(z) = o^A + z\,\iota^A$ where $\{o^A, \iota^A\}$ is a basis of $\mathbb{C}^2$. The roots of (4.4) then determine a unique factorization (up to permutation of the factors)

$$0 = \Phi(z) = \left(\xi^A\alpha_A\right)\left(\xi^B\beta_B\right)\left(\xi^C\gamma_C\right)\left(\xi^D\delta_D\right),$$

where α_A, β_A, γ_A and δ_A are elements $(\mathbb{C}^2)^*$ defined up to scale. In this case, we can write

$$\Phi_{ABCD} \propto \alpha_{(A}\beta_B\gamma_C\delta_{D)}.$$

Multiplicities of the roots of (4.4) will be mirrored by some of the corresponding α_A, β_A, γ_A and δ_A being proportional to each other.

Remark 4.4. The above identification is Φ_{ab} with Φ_{ABCD} is a consequence of the local isomorphism of Lie groups $SO(3, \mathbb{C}) \cong SL(2, \mathbb{C})$ as explained in appendix A, where $\mathbb{C}^2$ is identified with the spinor representation of $SO(3, \mathbb{C})$. This is virtually identical to the treatment of the (anti-)self-dual part of the Weyl tensor in four dimensions [Pen60, Pet00, Wit59].

4.1. Complex case

If the metric is complex with no preferred reality condition imposed on it, the coefficients $(\Phi_0, \Phi_1, \Phi_2, \Phi_3, \Phi_4)$ are generically complex, and the polynomial (4.3) has generically four distinct roots, and thus four distinct principal null structures. Following the notation of [PR86], we can encode the multiplicities of the roots of (4.3) in a partition $\{a_1, a_2, a_3, a_4\}$ of the integer 4, where $a_1 + a_2 + a_3 + a_4 = 4$. We shall omit those a_i from the partition whenever $a_i = 0$. We thus obtain a Petrov classification of Φ_{ab}:

Petrovtype	$\{a_1, a_2, a_3, a_4\}$	Φ_{ABCD}
I	$\{1, 1, 1, 1\}$	$\alpha_{(A}\beta_B\gamma_C\delta_{D)}$
II	$\{2, 1, 1\}$	$\alpha_{(A}\alpha_B\beta_C\gamma_{D)}$
III	$\{3, 1\}$	$\alpha_{(A}\alpha_B\alpha_C\beta_{D)}$
N	$\{4\}$	$\alpha_A\alpha_B\alpha_C\alpha_D$
D	$\{2, 2\}$	$\alpha_{(A}\alpha_B\beta_C\beta_{D)}$
O	$\{-\}$	0

Petrov types II, II, III and N single out a multiple principal null structure, and Petrov type D a pair of distinct multiple principal null structures.

Remark 4.5. Suppose Φ_{ab} is of Petrov type D so that the polynomial (4.3) has two distinct roots of multiplicity two. Then we can always arrange that these roots are 0 and ∞ in $\mathbb{CP}^1$, which is done by some suitable change of frame (2.8) by assuming, with no loss, that one of these roots is 0. In this case, we have a distinguished frame (k^a, ℓ^a, n^a) adapted to *both* multiple principal null structures, namely k^a and ℓ^a.

Let k^a be a generator of a null structure $\mathcal{N}$. To verify whether $\mathcal{N}$ is a (multiple) principal null structure, it suffices to check whether any of the following algebraic relations holds:

Petrov type I:	$k^a \Phi_{ab} k^b = 0,$	$\Phi_0 = 0,$
Petrov type II:	$k_{[a} \Phi_{b]c} k^c = 0,$	$\Phi_0 = \Phi_1 = 0,$
Petrov type D:	$k_{[a} \Phi_{b]c} k^c = 0$ and $\ell_{[a} \Phi_{b]c} \ell^c = 0,$	$\Phi_0 = \Phi_1 = 0 = \Phi_3 = \Phi_4,$
Petrov type III:	$k^a \Phi_{ab} = 0$ or $k_{[a} \Phi_{b][c} k_{d]} = 0,$	$\Phi_0 = \Phi_1 = \Phi_2 = 0,$
Petrov type N:	$k_{[a} \Phi_{b]c} = 0,$	$\Phi_0 = \Phi_1 = \Phi_2 = \Phi_3 = 0,$

where ℓ^a, in the case of Petrov type D, determines a multiple principal structure distinct from $\mathcal{N}$.

Because of the importance of the Goldberg–Sachs theorem, we highlight the algebraically special condition of the tracefree Ricci tensor by means of the following proposition. In particular, the proofs of the various versions of the Goldberg–Sachs theorem in section 6 will impinge on it.

Proposition 4.6. *In an adapted frame, the tracefree Ricci tensor is algebraically special if and only if*

$$\Phi_0 = \Phi_1 = 0.$$

4.2. Real case

4.2.1. Euclidean signature. In Euclidean signature, the four roots of the polynomial (4.3) come in two complex conjugate pairs. Thus, we distinguish only two algebraic types: the generic type G, where the conjugate pairs of complex roots are distinct, and the special type D, where the pairs coincide. Notationally, we shall bracket a conjugate pair of complex roots, i.e. $\{1^{\mathbb{C}}, \overline{1^{\mathbb{C}}}\}$, and where the $^{\mathbb{C}}$ indicates that the root is complex.

Petrovtype	$\{a_1, a_2, a_3, a_4\}$	Φ_{ABCD}
G	$\left\{ \left\{ 1^{\mathbb{C}}, \overline{1^{\mathbb{C}}} \right\}, \left\{ 1^{\mathbb{C}}, \overline{1^{\mathbb{C}}} \right\} \right\}$	$\xi_{(A} \hat{\xi}_B \eta_C \hat{\eta}_{D)}$
D	$\left\{ \left\{ 1^{\mathbb{C}}, \overline{1^{\mathbb{C}}} \right\}^2 \right\}$	$\xi_{(A} \hat{\xi}_B \xi_C \hat{\xi}_{D)}$
O	$\{-\}$	0

Here, a $\hat{}$ denotes a reality condition on $(\mathbb{C}^2)^*$ defined as follows: if $\xi_A = (\xi_0, \xi_1)$, then $\hat{\xi}_A = (-\bar{\xi}_1, \bar{\xi}_0)$.

Since a null structure determines a unit vector u^a (up to sign) and an endomorphism $J_a{}^b$ as described in section 3.2, we can characterize principal null structures as follows

Petrov type G:	$J_{(a}{}^c \Phi_{b)c} = 0$ or $2u_{[a} \Phi_{b][c} u_{d]} + u_{[a} g_{b][c} u_{d]} u^e u^f \Phi_{ef} = 0,$
Petrov type D:	$J_{(a}{}^c \Phi_{b)c} = 0$ and $u_{[a} \Phi_{b]c} u^c = 0.$

Remark 4.7. In the context of almost contact metric manifolds (see remarks 3.7 and 3.12), the type D condition is equivalent to the manifold being *η-Einstein*, i.e. the Ricci tensor takes the form $R_{ab} = a\, g_{ab} + b\, u_a u_b$ for some unit vector u^a and functions a and b, i.e. $R = 3\,a - b$ and $\Phi_{ab} = \frac{b}{3}\left(g_{ab} + 3u_a u_b\right)$.

4.2.2. Lorentzian signature. In Lorentzian signature, the situation is a little more complex, and one distinguishes ten Petrov types including type O. Here, a root of (4.3) can either be real or complex. We distinguish the following cases, excluding type O:

- If all roots are real, we obtain a totally real analogue of the complex Petrov types with five Petrov types denoted G_r, II_r, III_r, N_r and D_r. Petrov types II_r, III_r and N_r single out a multiple principal null structure of real index 1, and Petrov type D_r a pair of distinct multiple principal null structures of real index 1.
- If all the roots are complex, they come in conjugate pairs, and we have two Petrov types, G and D as in the Euclidean case. Petrov type D singles out a complex conjugate pair of multiple principal null structures of real index 0.
- The remaining types, denoted SG and II, occur when $\Phi(z)$ has two real roots and one conjugate pair of complex roots. Petrov Type II singles out a multiple principal null structure of real index 1.

Using the same notation as above to describe the degeneracy and reality of the roots of (4.3), we obtain the following Petrov types of Φ_{ab}:

Petrovtype	$\{a_1, a_2, a_3, a_4\}$	Φ_{ABCD}
G	$\left\{\left\{1^{\mathbb{C}}, \overline{1}^{\mathbb{C}}\right\}, \left\{1^{\mathbb{C}}, \overline{1}^{\mathbb{C}}\right\}\right\}$	$\xi_{(A}\hat{\xi}_B\eta_C\hat{\eta}_{D)}$
SG	$\left\{1, 1, \left\{1^{\mathbb{C}}, \overline{1}^{\mathbb{C}}\right\}\right\}$	$\alpha_{(A}\beta_B\eta_C\hat{\eta}_{D)}$
II	$\left\{2, \left\{1^{\mathbb{C}}, \overline{1}^{\mathbb{C}}\right\}\right\}$	$\alpha_{(A}\alpha_B\eta_C\hat{\eta}_{D)}$
D	$\left\{\left\{1^{\mathbb{C}}, \overline{1}^{\mathbb{C}}\right)^2\right\}$	$\xi_{(A}\hat{\xi}_B\eta_C\hat{\eta}_{D)}$
G_r	$\{1, 1, 1, 1\}$	$\alpha_{(A}\beta_B\gamma_C\delta_{D)}$
II_r	$\{2, 1, 1\}$	$\alpha_{(A}\alpha_B\beta_C\gamma_{D)}$
III_r	$\{3, 1\}$	$\alpha_{(A}\alpha_B\alpha_C\beta_{D)}$
N_r	$\{4\}$	$\alpha_A\alpha_B\alpha_C\alpha_D$
D_r	$\{2, 2\}$	$\alpha_{(A}\alpha_B\beta_C\beta_{D)}$
O	$\{-\}$	0

Characterization of the Petrov types in terms of their principal null structures can be done as in the previous cases in the obvious way.

5. Curvature conditions

Before we move to our main results on the Goldberg–Sachs theorem, we remark that in three dimensions, unlike in higher dimensions [TC13], the conformally invariant condition

$$\left[\Gamma\left(\mathcal{N}^{\perp}\right), \Gamma\left(\mathcal{N}^{\perp}\right)\right] \subset \Gamma\left(\mathcal{N}^{\perp}\right),$$

for a null structure $\mathcal{N}$, imposes no constraint on the curvature. Non-trivial constraints on the Ricci curvature are expected to arise from non-conformally invariant conditions on $\mathcal{N}$ [TC13]. In particular, we have the following proposition.

Proposition 5.1. *Let $\mathcal{N}$ be a null structure on an oriented three-dimensional (pseudo-) Riemannian manifold $(\mathcal{M}, \boldsymbol{g})$.*

- *Suppose that $\mathcal{N}$ is co-geodetic. Then $\mathcal{N}$ is a principal null structure, i.e.*

$$k^a k^b \Phi_{ab} = 0, \tag{5.1}$$

for any generator k^a of $\mathcal{N}$.

- *Suppose $\mathcal{N}$ is parallel. Then Φ_{ab} is algebraically special, i.e.*

$$k^c \Phi_{c[a} k_{b]} = 0. \tag{5.2}$$

for any generator k^a of $\mathcal{N}$.

Proof. We use the Newman–Penrose formalism of appendix B adapted to $\mathcal{N}$.

- By assumption, $\kappa = \rho = 0$. Then, by equation (B.7), we have $\Phi_0 = 0$.
- By assumption, $\kappa = \rho = \tau = 0$. Then, (B.6), (B.7) and (B.8) give $\Phi_1 = 0$, $\Phi_0 = 0$ and $\Phi_2 = S$ respectively. This completes the proof. □

Remark 5.2. In anticipation of the Goldberg–Sachs theorem, which will be concerned with the relation between algebraically special tracefree Ricci tensors and co-geodetic null structures, the above proposition tells that the existence of a co-geodetic null structure $\mathcal{N}$ already imposes algebraic constraints relating the curvature and $\mathcal{N}$.

6. Main results

Throughout this section, $(\mathcal{M}, \boldsymbol{g})$ will denote an oriented three-dimensional (pseudo-)Riemannian manifold. As before, the tracefree Ricci tensor will be denoted by Φ_{ab} and the Ricci scalar by R, and its scalar multiple $S := \frac{1}{12}R$. We also recall the definitions of the Cotton tensor and its Hodge dual:

$$A_{abc} := 2\, \nabla_{[b} \mathrm{P}_{c]a} = -2\, \nabla_{[b} \Phi_{c]a} + 2 g_{a[b} \nabla_{c]} S, \qquad (*A)_{ab} := \frac{1}{2} e_b{}^{cd} A_{acd} = -e_{(a|}{}^{cd} \nabla_c \Phi_{d|b)}.$$

In particular, the components of A_{abc} with respect to the frame (k^a, ℓ^a, n^a) will be denoted

$$\begin{aligned} &A_0 := 2\, A_{abc} k^a k^b n^c, \quad A_1 := A_{abc} k^a k^b \ell^c, \; A_2 := 2\, A_{abc} k^a n^b \ell^c, \\ &A_3 := A_{abc} \ell^a k^b \ell^c, \quad A_4 := 2\, A_{abc} \ell^a n^b \ell^c. \end{aligned} \tag{6.1}$$

The results of sections 6.1, 6.2 and 6.3 will be stated for an arbitrary complex-valued metric with no reality conditions imposed.

6.1. Obstructions to the existence of multiple co-geodetic null structures

We first present results concerning curvature obstructions to the existence of multiple co-geodetic null structures.

Proposition 6.1. *Let $\mathcal{N}$ be a null structure on $(\mathcal{M}, \boldsymbol{g})$, and let k^a be any generator of $\mathcal{N}$. Suppose that $\mathcal{N}$ is co-geodetic and multiple principal.*

- *If Φ_{ab} is of Petrov type II, then*

$$k^a k^b \Big(A_{abc} + 3\, g_{bc} \nabla_a S \Big) = 0. \tag{6.2}$$

- *If Φ_{ab} is of Petrov type III, then*

$$k^a \Big(A_{abc} - 2\, g_{a[b} \nabla_{c]} S \Big) = 0, \quad k^a k^b A_{abc} = 0, \quad k^a \nabla_a S = 0. \tag{6.3}$$

- *If Φ_{ab} is of Petrov type N, then*

$$k^c \Big(A_{abc} - g_{ca} \nabla_b S \Big) = 0, \quad k^a A_{abc} = 0, \quad k_{[a} \nabla_{b]} S = 0. \tag{6.4}$$

Proof. With reference to the Newman–Penrose formalism, and in a frame adapted to $\mathcal{N}$, we first note that

$$\text{Condition (6.2)} \Longleftrightarrow \begin{cases} A_0 \equiv 0, \\ A_1 + 3\, DS \equiv 0, \end{cases}$$

$$\text{Condition (6.3)} \Longleftrightarrow \begin{cases} A_0 = A_1 \equiv 0, \quad DS = 0 \\ A_2 + 2\, \delta S \equiv 0. \end{cases}$$

$$\text{Condition (6.4)} \Longleftrightarrow \begin{cases} A_0 = A_1 = A_2 \equiv 0, \quad DS = \delta S = 0 \\ A_3 + \Delta S \equiv 0. \end{cases}$$

The assumption that $\mathcal{N}$ is co-geodetic is simply $\kappa \equiv 0$ and $\rho \equiv 0$. We now deal with each case separately, referring to the Newman–Penrose equations given in appendix B.

- Assuming the type II condition, i.e. $\Phi_0 = \Phi_1 \equiv 0$, we have

$$\begin{aligned} \text{(B.18)}: &\quad A_0 \equiv 0, \\ 3 \times \text{(B.15)} + \text{(B.19)}: &\quad A_1 + 3\, DS \equiv 0. \end{aligned}$$

- Assuming the type III condition, i.e. $\Phi_0 = \Phi_1 = \Phi_2 \equiv 0$, we have

$$\begin{aligned} \text{(B.15)}: &\quad DS = 0, \\ \text{(B.18)}: &\quad A_0 = 0, \\ \text{(B.19)}: &\quad A_1 = 0, \\ 2 \times \text{(B.16)} + \text{(B.20)}: &\quad A_2 + 2\, \delta S \equiv 0. \end{aligned}$$

- Assuming the type N condition, i.e. $\Phi_0 = \Phi_1 = \Phi_2 = \Phi_3 \equiv 0$, we have

$$\begin{aligned} \text{(B.15)}: &\quad DS = 0, \\ \text{(B.16)}: &\quad \delta S = 0, \\ \text{(B.18)}: &\quad A_0 = 0, \\ \text{(B.19)}: &\quad A_1 = 0, \\ \text{(B.20)}: &\quad A_2 = 0, \\ \text{(B.17)} + \text{(B.21)}: &\quad A_3 + \Delta S \equiv 0. \end{aligned}$$

Comparison with the frame components (6.1) completes the proof. □

Remark 6.2. Proposition 6.1 also applies to tracefree Ricci tensors of Petrov type D, in which case one has a pair of distinct multiple principal null structures as described in remark 4.5.

6.2. Algebraic speciality implies co-geodetic null structures

We are now in the position of formulating the Goldberg–Sachs theorems (theorems 6.3, 6.4 and 6.5), along lines similar to Kundt and Thompson [KT62] and Robinson and Schild [RS63]. Note however that unlike the versions of these authors, the following theorems are *not* conformally invariant.

Theorem 6.3 **(Petrov type II).** *Let $\mathcal{N}$ be a multiple principal null structure on $(\mathcal{M}, \boldsymbol{g})$. Assume that Φ_{ab} is of Petrov type II and does not degenerate further. Suppose further that, for any generator k^a of $\mathcal{N}$,*

$$k^a k^b \left(A_{abc} + 3\, g_{bc} \nabla_a S \right) = 0. \tag{6.2}$$

Then $\mathcal{N}$ is co-geodetic.

Proof. Assume that Φ_{ab} is of Petrov type II, i.e. $\Phi_0 = \Phi_1 = 0$ in an adapted frame. In this case, the Newman–Penrose equations give

$$\begin{aligned} \text{(B.18)}: \qquad & A_0 = -12\, \kappa \Phi_2, \\ 3 \times \text{(B.15)} + \text{(B.19)}: \qquad & A_1 + 3DS = -12\, \rho \Phi_2. \end{aligned}$$

The assumption (6.2) in an adapted frame tells us that the lhs of the above set of equations are precisely zero. Now, since Φ_{ab} does not degenerate further, $\Phi_2 \neq 0$, so we conclude $\kappa \equiv 0$ and $\rho \equiv 0$. □

Theorem 6.4 **(Petrov type III).** *Let $\mathcal{N}$ be a multiple principal null structure on $(\mathcal{M}, \boldsymbol{g})$. Assume that Φ_{ab} is of Petrov type III and does not degenerate further. Suppose further that, for any generator k^a of $\mathcal{N}$,*

$$k^a \left(A_{abc} - 2\, g_{a[b} \nabla_{c]} S \right) = 0, \qquad k^a k^b A_{abc} = 0, \qquad k^a \nabla_a S = 0, \tag{6.3}$$

Then $\mathcal{N}$ is co-geodetic.

Proof. Assume that Φ_{ab} is of Petrov type III, i.e. $\Phi_0 = \Phi_1 = \Phi_2 = 0$ in an adapted frame. In this case, the Newman–Penrose equations give

$$\begin{aligned} \text{(B.15)}: \qquad & DS = 2\, \kappa \Phi_3, \\ \text{(B.18)}: \qquad & A_0 = 0, \\ \text{(B.19)}: \qquad & A_1 = -6\, \kappa \Phi_3, \\ 2 \times \text{(B.16)} + \text{(B.20)}: \qquad & A_2 + 2\, \delta S = -8\, \rho \Phi_3. \end{aligned}$$

The assumption (6.3) in an adapted frame tells us that the lhs of the above set of equations are precisely zero. Now, since Φ_{ab} does not degenerate further, $\Phi_3 \neq 0$, so we conclude $\kappa \equiv 0$ and $\rho \equiv 0$. □

Theorem 6.5 **(Petrov type N).** *Let $\mathcal{N}$ be a multiple principal null structure on $(\mathcal{M}, \boldsymbol{g})$. Assume that Φ_{ab} is of Petrov type N and does not degenerate further. Suppose further that, for* *any generator k^a of $\mathcal{N}$,*

$$k^c\left(A_{abc} - g_{ca}\nabla_b S\right) = 0, \quad k^a A_{abc} = 0, \quad k_{[a}\nabla_{b]}S = 0, \tag{6.4}$$

Then $\mathcal{N}$ is co-geodetic.

Proof. Assume that Φ_{ab} is of Petrov type N, i.e. $\Phi_0 = \Phi_1 = \Phi_2 = \Phi_3 = 0$ in an adapted frame. In this case, the Newman–Penrose equations give

$$\begin{aligned}
&\text{(B.15)}: && DS = 0,\\
&\text{(B.16)}: && \delta S = \kappa\Phi_4,\\
&\text{(B.18)}: && A_0 = 0,\\
&\text{(B.19)}: && A_1 = 0,\\
&\text{(B.20)}: && A_2 = -2\,\kappa\Phi_4,\\
&\text{(B.17) + (B.21)}: && \Delta S + A_3 = -4\,\rho\Phi_4.
\end{aligned}$$

The assumption (6.4) in an adapted frame tells us that the lhs of the above set of equations are precisely zero. Now, since Φ_{ab} does not degenerate further, $\Phi_4 \neq 0$, so we conclude $\kappa \equiv 0$ and $\rho \equiv 0$. □

Theorem 6.6 (Petrov type D). *Assume that Φ_{ab} is of Petrov type D with multiple principal null structures $\mathcal{N}$ and $\mathcal{N}'$ on $(\mathcal{M}, \boldsymbol{g})$, and does not degenerate further. Let k^a and ℓ^a be any generators of $\mathcal{N}$ and $\mathcal{N}'$ respectively, and suppose further that*

$$k^a k^b\left(A_{abc} + 3\,g_{bc}\nabla_a S\right) = 0, \quad \ell^a \ell^b\left(A_{abc} + 3\,g_{bc}\nabla_a S\right) = 0. \tag{6.5}$$

Then both $\mathcal{N}$ and $\mathcal{N}'$ are co-geodetic.

Proof. Assume that Φ_{ab} is of Petrov type D, i.e. $\Phi_0 = \Phi_1 = 0 = \Phi_3 = \Phi_4$ in a frame adapted to both $\mathcal{N}$ and $\mathcal{N}'$ as explained in remark 4.5. In this case, we see that the additional constraints $\Phi_3 = \Phi_4 = 0$ do not affect any of the argument of the proof of theorem 6.6, which impinges on the condition $\Phi_2 \neq 0$, and we can conclude $\mathcal{N}$ is geodetic.

To show that $\mathcal{N}'$ is integrable, we have to show that in an adapted frame, and with reference to the covariant derivative (2.10) of ℓ^a, the Newman–Penrose coefficients μ and ν should also be zero. But the Newman–Penrose equations give

$$\begin{aligned}
&\text{(B.22)}: && A_4 = -12\,\nu\Phi_2,\\
&3\times\text{(B.17)} - \text{(B.21)}: && A_3 - 3\Delta S = -12\,\mu\Phi_2.
\end{aligned}$$

The assumption (6.5) in an adapted frame tells us that the lhs of the above set of equations are precisely zero. Now, since Φ_{ab} does not degenerate further, $\Phi_2 \neq 0$, so we conclude $\kappa \equiv 0$ and $\rho \equiv 0$. □

6.3. Co-geodetic null structures implies algebraic speciality

We now state and prove the converse to theorems 6.3, 6.4 and 6.5.

Theorem 6.7. *Let $\mathcal{N}$ be a co-geodetic null structure on $(\mathcal{M}, \boldsymbol{g})$. Suppose that, for any generator k^a of $\mathcal{N}$,*

$$k^a k^b\left(A_{abc} + 3\,g_{bc}\nabla_a S\right) = 0. \tag{6.2}$$

Then Φ_{ab} is algebraically special with $\mathcal{N}$ as multiple principal null structure.

Proof. As always we work in an adapted frame and use the Newman–Penrose formalism, in which $\kappa = \rho \equiv 0$ means that $\mathcal{N}$ is co-geodetic. Then, assuming $\kappa = \rho \equiv 0$, we know by proposition 5.1 that $\Phi_0 \equiv 0$. In this case, we have the following components of the Bianchi identity

$$D\Phi_2 - 2\,\delta\Phi_1 - DS = (2\,\pi - 4\,\tau - 4\,\alpha)\Phi_1, \tag{B.15}$$

and of the Cotton tensor

$$-4\,D\Phi_1 = A_0 - 8\,\epsilon\Phi_1, \tag{B.18}$$

$$2\,\delta\Phi_1 - 3D\Phi_2 = A_1 + (4\,\alpha - 4\,\tau - 6\,\pi)\Phi_1, \tag{B.19}$$

We proceed by steps:

- first, computing $3 \times$(B.15) + (B.19) yields
$$-4\delta\Phi_1 = \left(A_1 + 3\,DS\right) - 8(2\tau + \alpha)\Phi_1; \tag{6.6}$$

- then δ(B.18)–D(6.6) gives
$$\begin{aligned} 4\,[D,\,\delta]\Phi_1 = {}& \delta\,A_0 - D\left(A_1 + 3D\,S\right) - 8(\delta\epsilon - D\alpha - 2\,D\tau)\Phi_1 \\ & - 8\,\epsilon\,\delta\Phi_1 + 8\,(\alpha + 2\tau)D\Phi_1; \end{aligned} \tag{6.7}$$

- at this stage, we can substitute the commutation relation
$$[D,\,\delta] = (\pi - 2\,\alpha)D, \tag{B.4}$$
into the lhs of (6.7), and the following components of the Ricci identity
$$D\tau = -2\,\Phi_1, \tag{B.6}$$
$$D\alpha - \delta\epsilon = -2\,\epsilon\alpha + \pi\epsilon - \Phi_1, \tag{B.10}$$
together with (6.6) and (B.18) into the rhs of (6.7), to get
$$\begin{aligned} 4\,(\pi - 2\,\alpha)D\Phi_1 = {}& \delta\,A_0 - D\left(A_1 + 3D\,S\right) + 2\,\epsilon\,\left(A_1 + 3\,DS\right) - 2\,(\alpha + 2\tau)A_0 \\ & - 8\left(2\,\epsilon\alpha - \pi\epsilon + 5\,\Phi_1\right)\Phi_1 - 16\,\epsilon\,(2\tau + \alpha)\Phi_1 + 16\,(\alpha + 2\tau)\epsilon\Phi_1; \end{aligned} \tag{6.8}$$

- subsituting (B.18) into the rhs of (6.8) and expanding yields
$$40\left(\Phi_1\right)^2 = \delta\,A_0 - D\left(A_1 + 3D\,S\right) + 2\,\epsilon\,\left(A_1 + 3\,DS\right) + (\pi - 4\tau - 4\alpha)A_0; \tag{6.9}$$

- finally, by condition (6.2), the rhs of (6.9) vanishes identically and we conclude $\Phi_1 \equiv 0$. In summary, $\kappa = \rho \equiv 0$ implies $\Phi_0 = \Phi_1 \equiv 0$, i.e. co-geodetic $\mathcal{N}$ implies algebraic speciality of Φ_{ab} with $\mathcal{N}$ multiple principal null structure. □

6.4. Topological massive gravity

Next, we consider the equations governing topological massive gravity. These are none other than Einstein's equations with cosmological constant Λ in which the energy–momentum tensor is proportional to the Hodge-dual of the Cotton tensor, i.e.

$$R_{ab} - \frac{1}{2}R\, g_{ab} + \Lambda\, g_{ab} = \frac{1}{m}(*A)_{ab}. \tag{6.10}$$

Here, $m \neq 0$ is a parameter of topological massive gravity theory. Substituting the expression for the tracefree Ricci tensor and tracing yield the expressions

$$\Phi_{ab} = \frac{1}{m}(*A)_{ab}, \tag{6.11}$$

$$R = 6\,\Lambda = \text{constant}, \tag{6.12}$$

equivalent to (6.10).

Remark 6.8. It is in fact sufficient to consider only (6.11) since (6.12) follows from it. To see this, we note that $\nabla^a(*A)_{ab} = 0$ which follows from (2.5). So, by (6.11), $\nabla^a\Phi_{ab} = 0$. Now, the Bianchi identity (2.3) gives $\nabla_a R = 0$, i.e. (6.12).

In an adapted frame, and with reference to (B.2), equations (6.11) read as

$$\begin{aligned} &\Phi_0 = -\frac{\mathrm{i}^q}{2\sqrt{2}\,m}A_0, \quad \Phi_1 = -\frac{\mathrm{i}^q}{2\sqrt{2}\,m}A_1, \quad \Phi_2 = -\frac{\mathrm{i}^q}{2\sqrt{2}\,m}A_2, \\ &\Phi_3 = -\frac{\mathrm{i}^q}{2\sqrt{2}\,m}A_3, \quad \Phi_4 = -\frac{\mathrm{i}^q}{2\sqrt{2}\,m}A_4, \end{aligned} \tag{6.13}$$

where $q = 0$ in Euclidean signature, and $q = 1$ in Lorentzian signature.

Lemma 6.9. *Suppose $(\mathcal{M}, \boldsymbol{g})$ is a solution of the topological massive gravity equations (6.11) and (6.12).*

- *If the tracefree Ricci tensor is of Petrov type II, then*

$$k^a k^b \left(A_{abc} + 3\, g_{bc}\, \nabla_a S\right) = 0. \tag{6.2}$$

- *If the tracefree Ricci tensor is of Petrov type III, then*

$$k^a \left(A_{abc} - 2\, g_{a[b}\, \nabla_{c]} S\right) = 0, \quad k^a k^b A_{abc} = 0, \quad k^a\, \nabla_a S = 0. \tag{6.3}$$

- *If the tracefree Ricci tensor is of Petrov type N, then*

$$k^c \left(A_{abc} - g_{ca}\, \nabla_b S\right) = 0, \quad k^a A_{abc} = 0, \quad k_{[a}\, \nabla_{b]} S = 0. \tag{6.4}$$

Proof. We first note that $S = \frac{1}{2}\Lambda$ is constant by virtue of the topological massive gravity equations (6.12). Consequently, equations (6.2), (6.3) and (6.4), which we need to assert, are reduced to algebraic conditions on the Cotton tensor. More precisely, with respect to an adapted frame, we must now show that

$$\begin{aligned} &\text{Petrov type II:} && \Phi_0 = \Phi_1 \equiv 0 && \Rightarrow && A_0 = A_1 = 0, \\ &\text{Petrov type III:} && \Phi_0 = \Phi_1 = \Phi_2 \equiv 0 && \Rightarrow && A_0 = A_1 = A_2 = 0, \\ &\text{Petrov type N:} && \Phi_0 = \Phi_1 = \Phi_2 = \Phi_3 \equiv 0 && \Rightarrow && A_0 = A_1 = A_2 = A_3 = 0. \end{aligned}$$

But the veracity of these statements follows from the topological massive gravity equations (6.11), which are (6.13) in an adapted frame. □

Combining theorems 6.3, 6.4, 6.5, 6.6 and 6.7 leads to our main result.

Theorem 6.10. *Let $(\mathcal{M}, \boldsymbol{g})$ be an oriented three-dimensional (pseudo-)Riemannian manifold that is a solution of the topological massive gravity equations, and assume that the Petrov type of the tracefree Ricci tensor Φ_{ab} does not change in an open subset of $\mathcal{M}$. Then Φ_{ab} is algebraically special if and only if $(\mathcal{M}, \boldsymbol{g})$ admits a co-geodetic null structure.*

6.5. Real versions

All the theorems given in sections 6.1, 6.2, 6.3 and 6.4 can easily be adapted to the case where the metric is real-valued. The crucial points here are that

- each of the algebraically special Petrov types of the tracefree Ricci tensor, as given in sections 4.2.1 and 4.2.2, singles out multiple principal null structure of a particular real index, and
- the real index r of this null structure yields a particular *real* geometric interpretation as given in section 3, i.e. a congruence of null curves when $r = 1$, or a congruence of timelike curves when $r = 0$.

We shall only give real versions of theorem 6.10 in the context of the topological massive gravity equations.

Theorem 6.11 (Lorentzian Goldberg–Sachs theorem for topological massive gravity). *Let $(\mathcal{M}, \boldsymbol{g})$ be an oriented three-dimensional Lorentzian manifold that is a solution of the topological massive gravity equations. Then*

- *$(\mathcal{M}, \boldsymbol{g})$ admits a divergence-free congruence of null geodesics (i.e. is a Kundt spacetime) if and only if its tracefree Ricci tensor is of Petrov type II, II_r, D_r, III_r or N_r;*
- *$(\mathcal{M}, \boldsymbol{g})$ admits two distinct divergence-free congruences of null geodesics if and only if its tracefree Ricci tensor is of Petrov type D_r;*
- *$(\mathcal{M}, \boldsymbol{g})$ admits a shear-free congruence of timelike geodesics if and only if its tracefree Ricci tensor is of Petrov type D.*

In fact, parts of theorem 6.11 were proved in reference in [CPS10b]: namely, that every Kundt spacetime that is a solution of the topological massive gravity equations must be algebraically special. By theorem 6.11, this exhausts all solutions of Petrov types II, II_r, D_r, III_r or N_r. All Petrov type D solutions of the topological massive gravity equations are also given in [CPS10a]. Therefore, theorem 6.11 tells us that these are the only possible algebraically special solutions of the topological massive gravity equations.

For the sake of completeness, we state the Riemannian version of theorem 6.12.

Theorem 6.12 (Riemannian Goldberg–Sachs theorem for topological massive gravity). *Let $(\mathcal{M}, \boldsymbol{g})$ be an oriented three-dimensional Riemannian manifold that is a solution of the topological massive gravity equations. Then $(\mathcal{M}, \boldsymbol{g})$ admits a shear-free congruence of geodesics if and only if its tracefree Ricci tensor is algebraically special, i.e. of Petrov type D.*

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Appendix A. Spinor calculus in three dimensions

Let $(\mathcal{M}, \boldsymbol{g})$ be a three-dimensional (pseudo-)Riemannian manifold, which we shall also assume to be oriented and equipped with a spin structure. To make the discussion signature-independent, we shall as before complexify both the tangent bundle $\mathrm{T}\mathcal{M}$ and $\boldsymbol{g}$, and shall not assume the existence of any particular reality structure on $\mathrm{T}^{\mathbb{C}}\mathcal{M}$ preserving $\boldsymbol{g}$. With these considerations, the spinor bundle $\mathcal{S}$ over $\mathcal{M}$ is a complex rank-2 vector bundle, whose sections will carry upstairs upper-case Roman indices, e.g. $\alpha^A \in \Gamma(\mathcal{S})$. Similarly, sections of the dual spinor bundle $\mathcal{S}^*$ will carry downstairs upper-case Roman indices, e.g. $\beta_A \in \Gamma(\mathcal{S}^*)$. The bundles $\mathcal{S}$ and $\mathcal{S}^*$ are equipped with non-degenerate skew-symmetric 2-spinors ε_{AB} and ε^{AB} respectively, which we shall choose to satisfy the normalization condition

$$\varepsilon_{AC}\varepsilon^{BC} = \delta_A^B,$$

where δ_A^B is the identity on $\mathcal{S}$. These bilinear forms establish an isomorphism between $\mathcal{S}$ and its dual $\mathcal{S}^*$, and we shall then raise and lower indices on spinors and dual spinors according to the convention

$$\alpha_A = \alpha^B\varepsilon_{BA}, \qquad \beta^A = \varepsilon^{AB}\beta_B.$$

This spinor calculus is almost identical to the two-spinor calculus in four dimensions, except for the absence of chirality (i.e. of 'primed' spinor indices).

We can consider the tensor product of any number copies of $\mathcal{S}$. Since the fibers of $\mathcal{S}$ are two-dimensional, any skew-symnetric 2-spinor must be pure trace, i.e. $\phi_{[AB]} = \frac{1}{2}\varepsilon_{AB}\phi_C{}^C$. In particular, there is a natural isomorphism between $\odot^2\mathcal{S}$ and $\mathrm{T}^{\mathbb{C}}\mathcal{M}$, and, by Hodge duality, $\wedge^2\mathrm{T}^{\mathbb{C}}\mathcal{M}$. This means that vector fields can be represented by a symmetric 2-spinor, i.e.

$$V^a = V^{AB},$$

where $V^{AB} = V^{(AB)}$. Here, we are employing the abstract index notation of [PR84]. For those uncomfortable with this approach, one can introduce γ-matrices $\gamma_a{}^{AB}$ to convert spinorial indices into tensorial ones and vice versa, i.e.

$$V^{AB} = V^a\gamma_a{}^{AB}, \quad V^a = \gamma^a{}_{AB}V^{AB}.$$

These γ-matrices satisfy $2\,\gamma_{(aA}{}^C\gamma_{b)BC} = g_{ab}\varepsilon_{AB}$.

As in four dimensions, the metric g_{ab} can be reinterpreted as the outer product of two copies ε_{AB}, which we find to be

$$g_{ab} = g_{ABCD} = g_{(AB)(CD)} = -\varepsilon_{A(C}\varepsilon_{D)B}. \tag{A.1}$$

It follows that the norm of any vector field V^a at any point equals the Pfaffian of its corresponding spinor V^{AB}, i.e.

$$V^a V_a = \varepsilon_{AC}\varepsilon_{BD} V^{AB} V^{CD}.$$

Hence, a non-zero vector field V^a is null if and only if its corresponding spinor V^{AB} has vanishing Pfaffian. Hence, V^{AB} must be of rank 1, and we can write $V^{AB} = \alpha^A \beta^B$ for some spinors α^A and β^A. In fact, using (A.1) once more, we see that $\alpha^A \beta_A = 0$, and so β^A must be proportional to α^A. The constant of proportionality can be absorbed by the spinor so that

Lemma A.1. *Any null vector field k^a can be written in the form*

$$k^a = k^{AB} = \xi^A \xi^B,$$

for some spinor field ξ^A.

In particular, there is a one-to-one correspondence between null line subbundle of $\mathrm{T}^{\mathbb{C}}\mathcal{M}$ *and lines of spinor fields.*

Decomposition of a 2-form. Any 2-form on $\mathcal{M}$ can be expressed as

$$\phi_{ab} = \phi_{[ab]} = \phi_{ABCD} = \phi_{(AB)(CD)} = \phi_{ABCD} = 2\, \varepsilon_{(A|(C}\phi_{D)|B)}, \tag{A.2}$$

where $\phi_{AB} = \phi_{(AB)} = \frac{1}{2}\phi_{ACB}{}^C$.

Curvature spinors. The decomposition rule (A.2) allows us to write the Riemann tensor as

$$R_{abcd} = R_{(AB)(CD)(EF)(GH)} = 4\, \varepsilon_{(A|(C}X_{D)|B)(E|(G}\varepsilon_{H)|F)},$$

where $X_{ABCD} = X_{(AB((CD)}$ satisfies $X_{ABCD} = X_{CDAB}$. Writing

$$2\, \Phi_{ABCD} = \Phi_{ab}, \qquad R = R_a{}^a =: 12\, S,$$

for the tracefree Ricci tensor and the Ricci scalar respectively. Here the factor 2 preceding Φ_{ABCD} has been added for later convenience. We can show

$$X_{ABCD} = \Phi_{ABCD} + S\, \varepsilon_{A(C}\varepsilon_{D)B},$$

where $\Phi_{ABCD} = \Phi_{(ABCD)}$ and $X_{ACB}{}^C = 3\, S\, \varepsilon_{AB}$ and $X_{AB}{}^{AB} = 6\, S$. Contracting yields

$$R_{ab} = 2X_{(A|(CD)|B)} - 2\, \varepsilon_{(A|(C}X_{D)}{}^E{}_{E|B)}.$$

Spinor geometry. Applying the decomposition (A.2) to the commutator yields

$$2\, \nabla_{[a}\nabla_{b]} = \nabla_{AB}\nabla_{CD} - \nabla_{CD}\nabla_{AB} = 2\, \varepsilon_{(A|(C}\square_{D)|B)}, \tag{A.3}$$

where $\square_{AB} := \nabla_{C(A}\nabla_{B)}{}^C$, which on any spinor α^A, acts as

$$\square_{AB}\alpha^C = -X_{ABD}{}^C\alpha^D, \tag{A.4}$$

where X_{ABCD} is the curvature spinor.

The contracted Bianchi identitiy (2.3) in spinorial form reads

$$\nabla^{CD}\Phi_{CDAB} - \nabla_{CD}S = 0, \tag{A.5}$$

while the Cotton tensor (2.4) or (2.6) reads

$$A_{ABCD} = 4\, \nabla_{(A}{}^E\Phi_{BCD)E}. \tag{A.6}$$

As a matter of interest, we record the topological massive gravity equations (6.11) and (6.12) in spinorial form

$$\Phi_{ABCD} = -\frac{\mathrm{i}^q}{2\sqrt{2}m}A_{ABCD}, \quad 2\,S = \Lambda = \text{constant},$$

where $q = 0$ in Euclidean signature and $q = 1$ in Lorentzian signature, and m is a constant.

A useful formula. We can convenient split the image of Φ_{ABCD} under the Dirac operator into irreducibles, in the sense of

$$\nabla_A{}^E\Phi_{BCDE} = \frac{1}{4}\Big(-3\,\varepsilon_{A(B}\nabla^{EF}\Phi_{CD)EF} + 4\,\nabla_{(A}{}^E\Phi_{BCD)E}\Big).$$

Now, by the Bianchi identity (A.5) and the definition of the Cotton 'spinor' (A.6), we obtain the useful identity

$$\nabla_A{}^E\Phi_{BCDE} = \frac{1}{4}\Big(-3\,\varepsilon_{A(B}\nabla_{CD)}S + A_{ABCD}\Big). \tag{A.7}$$

Reality conditions. When $\boldsymbol{g}$ has signature (3,0) the spin group is isomorphic to $\mathrm{SU}(2)$, while when $\boldsymbol{g}$ has signature (2, 1), the spin group is isomorphic to $\mathrm{SL}(2, \mathbb{R})$. Both are real forms of the complex Lie group $\mathrm{SL}(2, \mathbb{C})$.

Appendix B. A Newman–Penrose formalism in three dimensions

Our starting point will be a spin dyad (o^A, ι^A) normalized to $o_A\iota^A = 1$. We shall adopt the convention that

$$o^A = \delta_0^A, \quad \iota^A = \delta_1^A, \quad \iota_A = -\delta_A^0, \quad o_A = \delta_A^1,$$

where we think of δ_A^B as a Kronecker delta. Thus, to take the components of a spinor S_{ABC}, say, with respect to this dyad, we shall write

$$S_{ABC}o^Ao^Bo^C = S_{000}, \quad S_{ABC}o^A\iota^Bo^C = S_{010}, \quad \ldots$$

and so on. The spin-invariant bilinear form then takes the form $\varepsilon_{AB} = 2\,o_{[A}\iota_{B]}$. With respect to the spin dyad, the components of ε_{AB} and its inverse ε^{AB} are given by

$$\varepsilon_{01} = -\varepsilon_{10} = 1, \quad \varepsilon^{01} = -\varepsilon^{10} = 1.$$

This normalized spin dyad determines a null triad

$$k^a := o^Ao^B, \quad \ell^a := \iota^A\iota^B, \quad n^a := o^{(A}\iota^{B)},$$

so that $k^a\ell_a = 1$ and $n^an_b = -\frac{1}{2}$, and all other contractions vanish. The metric then takes the form

$$g_{ab} = 2\,k_{(a}\ell_{b)} - 2\,n_an_b. \tag{B.1}$$

Spin coefficients. As before, we let ∇_{AB} denote the Levi-Civita connection preserving g_{ab}, and by extension its lift to the spinor bundle. We introduce a connection ∂_{AB}, which preserves g_{ab} together with the spin dyad $\{o^A, \iota^A\}$. Then the difference between ∇_{AB} and ∂_{AB} on any spinor ξ^A will be given by

$$\nabla_{AB}\xi^C = \partial_{AB}\xi^C + \gamma_{ABD}{}^C\xi^D,$$

for some spinor $\gamma_{ABCD} = \gamma_{(AB)(CD)}$. This spinor can then be interpreted as the connection 1-form of the Levi-Civita connection.

We define the following differential operators

$$\begin{pmatrix} D \\ \Delta \\ \delta \end{pmatrix} := \begin{pmatrix} o^A o^B \nabla_{AB} \\ \iota^A \iota^B \nabla_{AB} \\ o^A \iota^B \nabla_{AB} \end{pmatrix} = \begin{pmatrix} k^a \nabla_a \\ \ell^a \nabla_a \\ n^a \nabla_a \end{pmatrix}.$$

Then, we can express the components of the connection 1-form $\gamma_{ABC}{}^D$

$$\begin{pmatrix} \kappa & \rho & \tau \\ \epsilon & \alpha & \gamma \\ \pi & \mu & \nu \end{pmatrix} := \begin{pmatrix} \gamma_{0000} & \gamma_{0100} & \gamma_{1100} \\ \gamma_{0001} & \gamma_{0101} & \gamma_{1101} \\ \gamma_{0011} & \gamma_{0111} & \gamma_{1111} \end{pmatrix} = \begin{pmatrix} o^B D o_B & o^B \delta o_B & o^B \Delta o_B \\ \iota^B D o_B & \iota^B \delta o_B & \iota^B \Delta o_B \\ \iota^B D \iota_B & \iota^B \delta \iota_B & \iota^B \Delta \iota_B \end{pmatrix}$$

$$= \begin{pmatrix} n^b D k_b & n^b \delta k_b & n^b \Delta k_b \\ \frac{1}{2} \ell^b D k_b & \frac{1}{2} \ell^b \delta k_b & \frac{1}{2} \ell^b \Delta k_b \\ -n^b D \ell_b & -n^b \delta \ell_b & -n^b \Delta \ell_b \end{pmatrix}.$$

Expanding the covariant derivatives of k^a, ℓ^a and n^a in terms of the spin coefficients yield

$$\nabla_a k^b = 2\,\gamma\; k_a k^b + 2\,\epsilon\; \ell_a k^b - 4\,\alpha\; n_a k^b - 4\,\tau\; k_a n^b - 4\,\kappa\; \ell_a n^b + 8\,\rho\; n_a n^b, \tag{2.9}$$

$$\nabla_a \ell^b = -2\,\epsilon\; \ell_a \ell^b - 2\,\gamma\; k_a \ell^b + 4\,\alpha\; n_a \ell^b + 4\,\pi\; \ell_a n^b + 4\,\nu\; k_a n^b - 8\,\mu\; n_a n^b, \tag{2.10}$$

$$\nabla_a n^b = -2\,\kappa\; \ell_a \ell^b + 2\,\nu k_a k^b + 2\,\pi\; \ell_a k^b - 2\,\tau\; k_a \ell^b + 4\,\rho\; n_a \ell^b - 4\,\mu\; n_a k^b. \tag{2.11}$$

Curvature coefficients. Similarly, the components of the tracefree Ricci tensor are given by

$$\begin{pmatrix} \Phi_0 \\ \Phi_1 \\ \Phi_2 \\ \Phi_3 \\ \Phi_4 \end{pmatrix} := \begin{pmatrix} \Phi_{0000} \\ \Phi_{0001} \\ \Phi_{0011} \\ \Phi_{0111} \\ \Phi_{1111} \end{pmatrix} = \begin{pmatrix} \Phi_{ABCD} o^A o^B o^C o^D \\ \Phi_{ABCD} o^A o^B o^C \iota^D \\ \Phi_{ABCD} o^A o^B \iota^C \iota^D \\ \Phi_{ABCD} o^A \iota^B \iota^C \iota^D \\ \Phi_{ABCD} \iota^A \iota^B \iota^C \iota^D \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \Phi_{ab} k^a k^b \\ \frac{1}{2} \Phi_{ab} k^a n^b \\ \frac{1}{2} \Phi_{ab} k^a \ell^a = \frac{1}{2} \Phi_{ab} n^a n^b \\ \frac{1}{2} \Phi_{ab} \ell^a n^b \\ \frac{1}{2} \Phi_{ab} \ell^a \ell^b \end{pmatrix},$$

while those of the Cotton tensor by

$$\begin{pmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix} := \begin{pmatrix} A_{0000} \\ A_{0001} \\ A_{0011} \\ A_{0111} \\ A_{1111} \end{pmatrix} = \begin{pmatrix} A_{ABCD} o^A o^B o^C o^D \\ A_{ABCD} o^A o^B o^C \iota^D \\ A_{ABCD} o^A o^B \iota^C \iota^D \\ A_{ABCD} o^A \iota^B \iota^C \iota^D \\ A_{ABCD} \iota^A \iota^B \iota^C \iota^D \end{pmatrix} = \begin{pmatrix} 2\,A_{abc} k^a k^b n^c \\ A_{abc} k^a k^b \ell^c \\ 2\,A_{abc} k^a n^b \ell^c \\ A_{abc} \ell^a k^b \ell^c \\ 2\,A_{abc} \ell^a n^b \ell^c \end{pmatrix} = \begin{pmatrix} -\sqrt{2}\,(-\mathrm{i})^q\; (*A)_{ab} k^a k^b \\ -\sqrt{2}\,(-\mathrm{i})^q\; (*A)_{ab} k^a n^b \\ -\sqrt{2}\,(-\mathrm{i})^q\; (*A)_{ab} k^a \ell^b \\ -\sqrt{2}\,(-\mathrm{i})^q\; (*A)_{ab} \ell^a n^b \\ -\sqrt{2}\,(-\mathrm{i})^q\; (*A)_{ab} \ell^a \ell^b \end{pmatrix}, \tag{B.2}$$

where $q = 0$ in Euclidean signature and $q = 1$ in Lorentzian signature, and we have assumed that the volume form is given by

$$e_{abc} = \mathrm{i}^q 6 \sqrt{2}\, k_{[a} \ell_b n_{c]}.$$

Commutation relations. The commutator of the Levi-Civita connection

$$\left[\nabla_{AB},\ \nabla_{CD}\right] = 2\,\gamma_{AB(C}{}^{E}\nabla_{D)E} - 2\,\gamma_{CD(A}{}^{E}\nabla_{B)E}$$

has components given by

$$[D,\ \Delta] = 2\,(\pi + \tau)\delta - 2\,\gamma D - 2\,\epsilon\Delta, \tag{B.3}$$

$$[D,\ \delta] = 2\,\rho\delta + (\pi - 2\,\alpha)D - \kappa\Delta, \tag{B.4}$$

$$[\Delta,\ \delta] = -2\,\mu\delta + \nu D + (-\tau + 2\,\alpha)\Delta. \tag{B.5}$$

Ricci identity. The Ricci identity (A.3) together with (A.4) can be re-expressed as

$$\begin{aligned}
&\partial_{AB}\gamma_{CDE}{}^{F} - \partial_{CD}\gamma_{ABE}{}^{F} = \gamma_{ABE}{}^{G}\gamma_{CDG}{}^{F} - \gamma_{CDE}{}^{G}\gamma_{ABG}{}^{F} - \gamma_{CDA}{}^{G}\gamma_{GBE}{}^{F}\\
&\quad + \gamma_{ABC}{}^{G}\gamma_{GDE}{}^{F} - \gamma_{CDB}{}^{G}\gamma_{GAE}{}^{F} + \gamma_{ABD}{}^{G}\gamma_{GCE}{}^{F}\\
&- \frac{1}{2}\Big(\varepsilon_{AC}\Phi_{DBE}{}^{F} + \varepsilon_{AD}\Phi_{CBE}{}^{F} + \varepsilon_{BC}\Phi_{DAE}{}^{F} + \varepsilon_{BD}\Phi_{CAE}{}^{F}\Big)\\
&\quad - \frac{1}{4}S\Big(\varepsilon_{AC}\varepsilon_{ED}\varepsilon_{B}{}^{F} + \varepsilon_{AC}\varepsilon_{EB}\varepsilon_{D}{}^{F} + \varepsilon_{AD}\varepsilon_{EC}\varepsilon_{B}{}^{F} + \varepsilon_{AD}\varepsilon_{EB}\varepsilon_{C}{}^{F}\\
&\quad + \varepsilon_{BC}\varepsilon_{ED}\varepsilon_{A}{}^{F} + \varepsilon_{BC}\varepsilon_{EA}\varepsilon_{D}{}^{F} + \varepsilon_{BD}\varepsilon_{EC}\varepsilon_{A}{}^{F} + \varepsilon_{BD}\varepsilon_{EA}\varepsilon_{C}{}^{F}\Big).
\end{aligned}$$

Taking the various components with respect to the spin dyad $\{o^A,\ \iota^A\}$ yields

$$D\tau - \Delta\kappa = -4\,\gamma\kappa + 2\,\pi\rho + 2\,\tau\rho - 2\,\Phi_1, \tag{B.6}$$

$$D\rho - \delta\kappa = 2\,\epsilon\rho - 4\,\alpha\kappa + 2\,\rho\rho + \pi\kappa - \kappa\tau - \Phi_0, \tag{B.7}$$

$$\Delta\rho - \delta\tau = 2\,\gamma\rho - 2\,\mu\rho + \nu\kappa - \tau\tau + \Phi_2 - S, \tag{B.8}$$

$$D\gamma - \Delta\epsilon = -4\,\epsilon\gamma - \kappa\nu + \tau\pi + 2\,\pi\alpha + 2\,\tau\alpha - 2\,\Phi_2 - S, \tag{B.9}$$

$$D\alpha - \delta\epsilon = -\,2\,\epsilon\alpha - \kappa\mu + \rho\pi + \pi\epsilon + 2\,\rho\alpha - \kappa\gamma - \Phi_1, \tag{B.10}$$

$$\Delta\alpha - \delta\gamma = 2\,\gamma\alpha - \tau\mu + \rho\nu - 2\,\mu\alpha + \nu\epsilon - \tau\gamma + \Phi_3, \tag{B.11}$$

$$D\nu - \Delta\pi = -\,4\,\nu\epsilon + 2\,\pi\mu + 2\,\tau\mu - 2\,\Phi_3, \tag{B.12}$$

$$D\mu - \delta\pi = -\,2\,\mu\epsilon + \pi\pi + 2\,\rho\mu - \kappa\nu - \Phi_2 + S, \tag{B.13}$$

$$\Delta\mu - \delta\nu = 4\,\nu\alpha - 2\,\mu\gamma - 2\,\mu\mu + \nu\pi - \tau\nu + \Phi_4. \tag{B.14}$$

Bianchi identity. The Bianchi identity (A.5) can be re-expressed as

$$\begin{aligned}
&\varepsilon^{AC}\varepsilon^{BD}\big(\partial_{AB}\Phi_{CDEF} - \ \partial_{EF}S\big)\\
&\quad = \varepsilon^{AC}\varepsilon^{BD}\Big(\gamma_{ABC}{}^{G}\Phi_{DEFG} + \gamma_{ABD}{}^{G}\Phi_{EFCG} + \gamma_{ABE}{}^{G}\Phi_{FCDG} + \gamma_{ABF}{}^{G}\Phi_{CDEG}\Big),
\end{aligned}$$

so that taking components with respect to the spin dyad yields

$$D\Phi_2 + \Delta\Phi_0 - 2\,\delta\Phi_1 - DS = (2\,\pi - 4\,\tau - 4\,\alpha)\Phi_1 - 2\,\kappa\Phi_3 + (4\,\gamma - 2\,\mu)\Phi_0 + 6\,\rho\Phi_2, \tag{B.15}$$

$$\begin{aligned}
&D\Phi_3 + \Delta\Phi_1 - 2\,\delta\Phi_2 - \delta S = (3\,\pi - 3\,\tau)\Phi_2 + (4\,\rho - 2\,\epsilon)\Phi_3 - \kappa\Phi_4\\
&\quad + \nu\Phi_0 + (2\,\gamma - 4\,\mu)\Phi_1,
\end{aligned} \tag{B.16}$$

$$D\Phi_4 + \Delta\Phi_2 - 2\,\delta\Phi_3 - \Delta S = (4\,\pi - 2\,\tau + 4\,\alpha)\Phi_3 + (2\,\rho - 4\,\epsilon)\Phi_4 + 2\,\nu\Phi_1 - 6\,\mu\Phi_2. \tag{B.17}$$

Cotton tensor. Finally, from the definition (A.6) of the Cotton tensor, one obtains

$$4\,\varepsilon^{EF}\partial_{(A|F|}\Phi_{BCD)E} = A_{ABCD} + \varepsilon^{EF}\Big(12\,\gamma_{(A|F|B}{}^{G}\Phi_{CD)EG} + 4\,\gamma_{(A|FE}{}^{G}\Phi_{|BCD)G}\Big),$$

with components

$$4\big(\delta\Phi_0 - D\Phi_1\big) = A_0 + 4(4\,\alpha - \pi)\Phi_0 - 4(4\,\rho + 2\,\epsilon)\Phi_1 + 12\,\kappa\Phi_2, \tag{B.18}$$

$$\begin{aligned}\Delta\Phi_0 + 2\,\delta\Phi_1 - 3D\Phi_2 = A_1 &+ (2\,\mu + 4\,\gamma)\Phi_0 + (4\,\alpha - 4\,\tau - 6\,\pi)\Phi_1\\ &- 6\,\rho\Phi_2 + 6\,\kappa\Phi_3,\end{aligned} \tag{B.19}$$

$$2\Delta\Phi_1 - 2D\Phi_3 = A_2 + 2\,\nu\Phi_0 + 2\,\kappa\Phi_4 + 4\,\gamma\Phi_1 + 4\,\epsilon\Phi_3 - 6(\pi + \tau)\Phi_2, \tag{B.20}$$

$$-D\Phi_4 - 2\,\delta\Phi_3 + 3\Delta\Phi_2 = A_3 + (2\,\rho + 4\,\epsilon)\Phi_4 + (4\,\alpha - 4\,\pi - 6\,\tau)\Phi_3 - 6\,\mu\Phi_2 + 6\,\nu\Phi_1, \tag{B.21}$$

$$4\big(-\delta\Phi_4 + \Delta\Phi_3\big) = A_4 + 4(4\,\alpha - \tau)\Phi_4 - 4(4\,\mu + 2\,\gamma)\Phi_3 + 12\,\nu\Phi_2. \tag{B.22}$$

B.1. Reality conditions

There remain to impose suitable reality conditions on the null basis (k^a, ℓ^a, n^a) so that the metric (B.1) has the correct signature. These are listed together with their effects on the spin coefficients and the components of the tracefree Ricci tensor and Cotton tensor.

- Signature (3, 0): $\{k^a, \ell^a, n^a\} \mapsto \{\overline{k^a}, \overline{\ell^a}, \overline{n^a}\} = \{\ell^a, k^a, -n^a\}$

$$\begin{pmatrix}\kappa & \rho & \tau\\ \epsilon & \alpha & \gamma\\ \pi & \mu & \nu\end{pmatrix} \mapsto \begin{pmatrix}\bar\kappa & \bar\rho & \bar\tau\\ \bar\epsilon & \bar\alpha & \bar\gamma\\ \bar\pi & \bar\mu & \bar\nu\end{pmatrix} = \begin{pmatrix}\nu & -\mu & \pi\\ -\gamma & \alpha & -\epsilon\\ \tau & -\rho & \kappa\end{pmatrix},$$

$$\begin{pmatrix}\Phi_0\\ \Phi_1\\ \Phi_2\\ \Phi_3\\ \Phi_4\end{pmatrix} \mapsto \begin{pmatrix}\overline{\Phi_0}\\ \overline{\Phi_1}\\ \overline{\Phi_2}\\ \overline{\Phi_3}\\ \overline{\Phi_4}\end{pmatrix} = \begin{pmatrix}\Phi_4\\ -\Phi_3\\ \Phi_2\\ -\Phi_1\\ \Phi_0\end{pmatrix}, \qquad \begin{pmatrix}A_0\\ A_1\\ A_2\\ A_3\\ A_4\end{pmatrix} \mapsto \begin{pmatrix}\overline{A_0}\\ \overline{A_1}\\ \overline{A_2}\\ \overline{A_3}\\ \overline{A_4}\end{pmatrix} = \begin{pmatrix}A_4\\ -A_3\\ A_2\\ -A_1\\ A_0\end{pmatrix}.$$

- Signature (2,1):

◊ Real index 0: $\{k^a, \ell^a, n^a\} \mapsto \{\overline{k^a}, \overline{\ell^a}, \overline{n^a}\} = \{\ell^a, k^a, n^a\}$

$$\begin{pmatrix}\kappa & \rho & \tau\\ \epsilon & \alpha & \gamma\\ \pi & \mu & \nu\end{pmatrix} \mapsto \begin{pmatrix}\bar\kappa & \bar\rho & \bar\tau\\ \bar\epsilon & \bar\alpha & \bar\gamma\\ \bar\pi & \bar\mu & \bar\nu\end{pmatrix} = \begin{pmatrix}-\nu & -\mu & -\pi\\ -\gamma & -\alpha & -\epsilon\\ -\tau & -\rho & -\kappa\end{pmatrix},$$

$$\begin{pmatrix} \Phi_0 \\ \Phi_1 \\ \Phi_2 \\ \Phi_3 \\ \Phi_4 \end{pmatrix} \mapsto \begin{pmatrix} \overline{\Phi_0} \\ \overline{\Phi_1} \\ \overline{\Phi_2} \\ \overline{\Phi_3} \\ \overline{\Phi_4} \end{pmatrix} = \begin{pmatrix} \Phi_4 \\ \Phi_3 \\ \Phi_2 \\ \Phi_1 \\ \Phi_0 \end{pmatrix}, \qquad \begin{pmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix} \mapsto \begin{pmatrix} \overline{A_0} \\ \overline{A_1} \\ \overline{A_2} \\ \overline{A_3} \\ \overline{A_4} \end{pmatrix} = \begin{pmatrix} -A_4 \\ -A_3 \\ -A_2 \\ -A_1 \\ -A_0 \end{pmatrix}.$$

◊ Real index 1: $\{k^a, \ell^a, n^a\} \mapsto \{\overline{k^a}, \overline{\ell^a}, \overline{n^a}\} = \{k^a, \ell^a, -n^a\}$

$$\begin{pmatrix} \kappa & \rho & \tau \\ \epsilon & \alpha & \gamma \\ \pi & \mu & \nu \end{pmatrix} \mapsto \begin{pmatrix} \bar\kappa & \bar\rho & \bar\tau \\ \bar\epsilon & \bar\alpha & \bar\gamma \\ \bar\pi & \bar\mu & \bar\nu \end{pmatrix} = \begin{pmatrix} -\kappa & \rho & -\tau \\ \epsilon & -\alpha & \gamma \\ -\pi & \mu & -\nu \end{pmatrix},$$

$$\begin{pmatrix} \Phi_0 \\ \Phi_1 \\ \Phi_2 \\ \Phi_3 \\ \Phi_4 \end{pmatrix} \mapsto \begin{pmatrix} \overline{\Phi_0} \\ \overline{\Phi_1} \\ \overline{\Phi_2} \\ \overline{\Phi_3} \\ \overline{\Phi_4} \end{pmatrix} = \begin{pmatrix} \Phi_0 \\ -\Phi_1 \\ \Phi_2 \\ -\Phi_3 \\ \Phi_4 \end{pmatrix}, \qquad \begin{pmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix} \mapsto \begin{pmatrix} \overline{A_0} \\ \overline{A_1} \\ \overline{A_2} \\ \overline{A_3} \\ \overline{A_4} \end{pmatrix} = \begin{pmatrix} -A_0 \\ A_1 \\ -A_2 \\ A_3 \\ -A_4 \end{pmatrix}.$$

Appendix C. A spinorial approach to the Goldberg–Sachs theorem

The aim of this appendix is to give alternative proofs for the results in the main text using the spinor calculus of appendix A. Throughout, $(\mathcal{M}, \boldsymbol{g})$ will denote an oriented three-dimensional (pseudo-)Riemannian manifold. Recall that there is a one-to-one correspondence between projective spinor fields and null complex line distributions, i.e. null structures, on $\mathcal{M}$. Some of the following results are already given and generalized to arbitrary dimensions in [TC12b, TC13].

The following proposition is a spinorial version of proposition 2.7, and relates the integrability properties of a null structure to differential conditions on its associated spinor field.

Proposition C.1. *Let ξ^A be a spinor field on $(\mathcal{M}, \boldsymbol{g})$ with associated null structure $\mathcal{N}$. Then*

$$\begin{aligned} \mathcal{N} \text{ is co-integrable} &\iff \xi^A \xi^B \xi^C \nabla_{AB} \xi_C = 0, \\ \mathcal{N} \text{ is co-geodetic} &\iff \xi^B \xi^C \nabla_{AB} \xi_C = 0, \\ \mathcal{N} \text{ is parallel} &\iff \xi^C \nabla_{AB} \xi_C = 0. \end{aligned}$$

Proof. The above result is already given in [TC13]. We can however use the NP formalism of appendix B by taking $o^A := \xi^A$. Then

$$o^C \nabla_{AB} o_C = \kappa\, \iota_A \iota_B - 2\, \rho\, o_{(A} \iota_{B)} + \tau\, o_A o_B.$$

Contracting the free indices with instances of o^A give conditions on κ, ρ and τ, which, with reference to the the expression for $\left(\nabla_a k_{[b}\right) k_{c]}$, yields the result. □

Having translated the Lie bracket conditions of a null line and its orthogonal complement into spinorial differential equations, we can now re-express some of the results of sections 4, 5 and 6. In particular, with reference to remark 4.4, the Petrov classification of the Weyl tensor can be expressed in the following terms.

Lemma C.2. *Let ξ^A be a pure spinor field on $(\mathcal{M}, \boldsymbol{g})$ with associated null structure $\mathcal{N}$. Then*

- *$\mathcal{N}$ is a principal null structure if and only if $\Phi_{ABCD}\xi^A\xi^B\xi^C\xi^D = 0$;*
- *Φ_{ab} is algebraically special, i.e. of Petrov type II with $\mathcal{N}$ as multiple principal null structure if and only if $\Phi_{ABCD}\xi^A\xi^B\xi^C = 0$;*
- *Φ_{ab} is of Petrov type III with $\mathcal{N}$ as multiple principal null structure if and only if $\Phi_{ABCD}\xi^A\xi^B = 0$;*
- *Φ_{ab} is of Petrov type N with $\mathcal{N}$ as multiple principal null structure if and only if $\Phi_{ABCD}\xi^A = 0$.*

With this lemma, it is easy to compare the remaining results with those of sections 5 and 6.

Proposition C.3. ***(Integrability condition).*** *Let ξ^A be a spinor field on $(\mathcal{M}, \boldsymbol{g})$, and suppose it satisfies*

$$\xi^B\xi^C\nabla_{AB}\xi_C = 0. \tag{C.1}$$

Then

$$\Phi_{ABCD}\xi^A\xi^B\xi^C\xi^D = 0. \tag{C.2}$$

Our first aim is to reformulate the obstruction to the existence of a co-geodetic multiple principal null structure of the tracefree Ricci tensor. Proposition 6.1 gave conditions on the components of the Cotton tensor A_{abc} and the derivative of the Ricci scalar R. It turns out that the spinorial formalism gives very concise expressions for conditions (6.2), (6.3) and (6.4). Indeed using equation (A.7), we obtain

$$\begin{aligned}
4\,\xi^B\xi^C\xi^D\nabla_A{}^E\Phi_{BCDE} &= A_{ABCD}\xi^B\xi^C\xi^D + 3\,\xi_A\xi^B\xi^C\nabla_{BC}S,\\
4\,\xi^C\xi^D\nabla_A{}^E\Phi_{BCDE} &= A_{ABCD}\xi^C\xi^D + 2\,\xi_A\xi^C\nabla_{BC}S - \varepsilon_{AB}\xi^C\xi^D\nabla_{CD}S,\\
4\,\xi^D\nabla_A{}^E\Phi_{BCDE} &= A_{ABCD}\xi^D + \xi_A\nabla_{BC}S - 2\,\varepsilon_{A(B}\xi^D\nabla_{C)D}S.
\end{aligned}$$

We can now re-express proposition 6.1 as

Proposition C.4. *Let ξ^A be a spinor field on $(\mathcal{M}, \boldsymbol{g})$. Suppose ξ^A satisfies*

$$\xi^B\xi^C\nabla_{AB}\xi_C = 0. \tag{C.1}$$

Then,

$$\xi^B\xi^C\xi^D\Phi_{ABCD} = 0 \implies \xi^B\xi^C\xi^D\nabla_A{}^E\Phi_{BCDE} = 0, \tag{C.3}$$

$$\xi^C \xi^D \Phi_{ABCD} = 0 \Longrightarrow \quad \xi^C \xi^D \nabla_A{}^E \Phi_{BCDE} = 0, \tag{C.4}$$

$$\xi^D \Phi_{ABCD} = 0 \quad \Longrightarrow \quad \xi^D \nabla_A{}^E \Phi_{BCDE} = 0. \tag{C.5}$$

Proof. Assume ξ^A satisfies (C.1). We first differentiate $\Phi_{ABCD}\xi^B\xi^C\xi^D = 0$ so that

$$0 = \nabla_A{}^E \left(\Phi_{BCDE} \xi^B \xi^C \xi^D \right) = \left(\nabla_A{}^E \Phi_{BCDE} \right) \xi^B \xi^C \xi^D + 3\, \Phi_{BCDE} \left(\nabla_A{}^E \xi^B \right) \xi^C \xi^D .$$

The condition on Φ_{ABCD} can be rewritten as $\Phi_{ABCD}\xi^C\xi^D = \phi\, \xi_A \xi_B$ for some ϕ. The second term then becomes $3\, \phi\, \xi^B \xi^C \nabla_{AB} \xi_C$, but this must vanish since ξ^A satisfies (C.1). This proves (C.3).

The remaining cases are similar and left to the reader. □

Remark C.5. Using the useful identity (A.7), it is straightforward to see that the conditions on the rhs of (C.3), (C.4) and (C.5) are equivalent to the tensorial expression (6.2), (6.3) and (6.4).

For conciseness, we combine the statements of theorems 6.3, 6.4 and 6.5 into a single theorem.

Theorem C.6. *Let ξ^A be a spinor field on $(\mathcal{M}, \boldsymbol{g})$. Suppose ξ^A satisfies any of the following conditions*

1. $\xi^B\xi^C\xi^D\Phi_{ABCD} = 0$, $\xi^C\xi^D\Phi_{ABCD} \neq 0$, *and* $\xi^B\xi^C\xi^D \nabla_A{}^E \Phi_{BCDE} = 0$;
2. $\xi^C\xi^D\Phi_{ABCD} = 0$, $\xi^D\Phi_{ABCD} \neq 0$ *and* $\xi^C\xi^D \nabla_A{}^E \Phi_{BCDE} = 0$;
3. $\xi^D\Phi_{ABCD} = 0$, $\Phi_{ABCD} \neq 0$ *and* $\xi^D \nabla_A{}^E \Phi_{BCDE} = 0$.

Then ξ^A satisfies

$$\xi^B \xi^C \nabla_{AB} \xi_C = 0. \tag{C.1}$$

Proof. We assume the conditions given in case 1. We first differentiate $\Phi_{ABCD}\xi^B\xi^C\xi^D = 0$ so that

$$0 = \nabla_A{}^E \left(\Phi_{BCDE} \xi^B \xi^C \xi^D \right) = \left(\nabla_A{}^E \Phi_{BCDE} \right) \xi^B \xi^C \xi^D + 3\, \Phi_{BCDE} \left(\nabla_A{}^E \xi^B \right) \xi^C \xi^D.$$

Since Φ_{ABCD} does not degenerate further with respect to ξ^A, we can write $\Phi_{ABCD}\xi^C\xi^D = \phi\, \xi_A \xi_B$ for some non-vanishing ϕ. Hence,

$$0 = \xi^B \xi^C \xi^D \nabla_A{}^E \Phi_{BCDE} + 3\, \phi\, \xi^B \xi^C \nabla_{AB} \xi_C.$$

By assumption, the first term vanishes, and since ϕ is non-vanishing, we conclude $\xi^B\xi^C\nabla_{AB}\xi_C = 0$.

We omit the proofs of the remaining cases, which are similar. □

Finally, theorem 6.7 reads

Theorem C.7. *Let ξ^A be a spinor field on $(\mathcal{M}, \boldsymbol{g})$. Suppose ξ^A satisfies*

$$\xi^B \xi^C \nabla_{AB} \xi_C = 0, \tag{C.1}$$

and

$$\xi^B \xi^C \xi^D \nabla_A{}^E \Phi_{BCDE} = 0 \; . \tag{C.6}$$

Then the tracefree Ricci tensor is algebraically special, i.e.

$$\Phi_{ABCD} \xi^B \xi^C \xi^D = 0 \; ,$$

Proof. Assume ξ^A satisfies (C.1). Then

- we can write

$$\xi^E \nabla_{AE} \xi_B = \eta_A \xi_B, \quad \xi^C \nabla_{AB} \xi_C = \lambda \; \xi_A \xi_B \tag{C.7}$$

for some η_A and λ. Using the identity $\xi_C \nabla_{AB} \xi^C - \xi_A \nabla_{CB} \xi^C = -\xi^C \nabla_{CB} \xi_A$ tells us that

$$\lambda \; \xi_A = \eta_A - \nabla_{AB} \xi^B \tag{C.8}$$

- by proposition C.3, the tracefree Ricci tensor satisfies

$$\phi \; \xi_A = \Phi_{ABCD} \xi^B \xi^C \xi^D \tag{C.9}$$

for some function ϕ.

Take the covariant derivative of (C.9) and use the Leibniz rule to get

$$\xi_E \nabla_A{}^E \phi + \phi \nabla_A{}^E \xi_E = 3 \; \Phi_{BCDE} \Big(\nabla_A{}^E \xi^B \Big) \xi^C \xi^D, \tag{C.10}$$

where we have made use of the curvature assumption (C.6).

We shall now suppose that ϕ does not vanish, and divide (C.10) through by ϕ. Then using (C.1), (C.2), (C.7) and (C.8) yields

$$\xi^B \nabla_{AB} \ln \phi = 6 \eta_A - 4 \nabla_{AB} \xi^B =: \alpha_A. \tag{C.11}$$

The consistency condition for (C.11) to be locally integrable can be obtained by applying $\xi^B \nabla_{AB}$ to (C.11) and commuting the derivatives: we find

$$\eta^A \alpha_A = \xi^B \nabla_B{}^A \alpha_A \; . \tag{C.12}$$

We proceed by checking that (C.12) is indeed satisfied. Plugging the definition of α_A in the rhs of (C.11) into (C.12) yields

$$-6 \; \xi^A \nabla_{AB} \eta^B + 4 \; \xi^A \nabla_{AB} \nabla^B{}_C \xi^C = -4 \; \eta^A \nabla_{AB} \xi^B \; . \tag{C.13}$$

By commuting the covariant derivatives, the second term on the lhs of (C.13) becomes

$$\begin{aligned} \xi^A \nabla_{AB} \nabla^B{}_C \xi^C &= -\xi^A \nabla_{CB} \nabla^B{}_A \xi^C + 2 \; \xi^B \square_{BC} \xi^C \\ &= -\nabla_{CB} \Big(\xi^A \nabla^B{}_A \xi^C \Big) + \Big(\nabla_{CB} \xi^A \Big) \Big(\nabla^B{}_A \xi^C \Big) \; , \end{aligned}$$

where we have made use of the fact that $\xi^B \square_{BC} \xi^C = 0$ in the first line, and the Leibniz rule in the second line. The last term in the second line vanishes by symmetry consideration. Hence, using the definition of η^A in (C.7), we are left with $\xi^A \nabla_{AB} \nabla^B{}_C \xi^C = -\nabla_{CB} \Big(\eta^B \xi^C \Big)$, which on substitution into (C.13) leads to $\xi^B \nabla_{BA} \eta^A = 0$. But now observe that

$$0 = \xi_A \xi^B \nabla_{BC} \eta^C = \xi^B \nabla_{BC} \left(\eta^C \xi_A \right)$$
$$= \xi^B \nabla_{BC} \left(\xi^D \nabla_D{}^C \xi_A \right) = \xi^B \xi^D \nabla_{BC} \nabla_D{}^C \xi_A = \Phi_{ABCD} \xi^B \xi^C \xi^D,$$

which shows that Φ_{ABCD} is algebraically special in contradiction to our assumption that ϕ is non-vanishing. Hence the result. □

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