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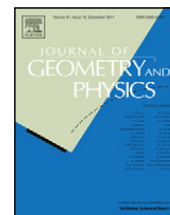
Conformal field theories in six-dimensional twistor space

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ABSTRACT

This article gives a study of the higher-dimensional Penrose transform between conformally invariant massless fields on space–time and cohomology classes on twistor space, where twistor space is defined to be the space of projective pure spinors of the conformal group. We focus on the six-dimensional case in which twistor space is the 6-quadric Q in $\mathbb{CP}^7$ with a view to applications to the self-dual $(0, 2)$-theory. We show how spinor-helicity momentum eigenstates have canonically defined distributional representatives on twistor space (a story that we extend to arbitrary dimension). These yield an elementary proof of the surjectivity of the Penrose transform. We give a direct construction of the twistor transform between the two different representations of massless fields on twistor space (H^2 and H^3) in which the H^3s arise as obstructions to extending the H^2s off Q into $\mathbb{CP}^7$.

We also develop the theory of Sparling's '$\varXi$-transform', the analogous totally real split signature story based now on real integral geometry where cohomology no longer plays a role. We extend Sparling's $\varXi$-transform to all helicities and homogeneities on twistor space and show that it maps kernels and cokernels of conformally invariant powers of the ultrahyperbolic wave operator on twistor space to conformally invariant massless fields on space–time. This is proved by developing the six-dimensional analogue of the half-Fourier transform between functions on twistor space and momentum space. We give a treatment of the elementary conformally invariant $\varPhi^3$ amplitude on twistor space and finish with a discussion of conformal field theories in twistor space.

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1. Introduction

Twistor methods have come to constitute a powerful tool in the study of four-dimensional gauge theories [1–4] and the question arises as to how they might interact with other important techniques such as AdS/CFT and reductions of $(2, 0)$ self-dual gerbe theory in six dimensions. This article is a study of the appropriate higher-dimensional twistor theory that applies to such higher-dimensional theories with a focus on six dimensions. Ultimately we would like to encode interacting field theories on twistor space in higher dimensions, but this paper will focus mostly on the twistor correspondence for linear fields (with the exception of a discussion of the ϕ^3 vertex for scalar fields).

The Penrose transform in higher dimensions, and indeed for general classes of homogeneous spaces, was developed in the 1980s with the general framework summarized in [5]. In higher even dimension $2m$, twistor space can be taken to be the space of totally null self-dual m-planes, so that twistor space is the projective self-dual chiral pure spinor space of the conformal group [6]. There is a Penrose transform from conformally invariant massless fields on space–time to cohomology classes on regions in twistor space. In six dimensions the conformal group is a real form of $SO(8, \mathbb{C})$ and we have a version of triality such that the chiral spinor representations for the conformal group are eight dimensional and are endowed with

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a quadratic form like the fundamental one. This is the first dimension in which the purity condition is non-trivial, being the condition that the chiral spinors should be null with respect to the quadratic form. Thus, twistor space turns out to be the six-dimensional complex quadric Q in $\mathbb{CP}^7$. As a manifold it is the same as complexified space–time, but the representation of the conformal group and its correspondence with space–time is quite different. Conformally invariant massless fields on space–time are represented on Q both as H^2 and as H^3 cohomology classes (i.e., closed Dolbeault $(0, 2)$-forms or $(0, 3)$-forms modulo exact ones). As far as massless fields are concerned, Maxwell and linearized gravity are no longer conformally invariant, but the wave equation and symmetric spinor fields satisfying higher spin versions of the massless Dirac equation are, and it is these fields of a fixed chirality that are most straightforwardly represented on twistor space. The abelian version of the $(2, 0)$ self-dual gerbe theory can be built from such ingredients and we focus on these fields in this paper. The Penrose transform for H^3s, which we will describe as the direct transform, is the most straightforward, being easily obtained from an integral formula (3), and was the first to be studied in [7–10]. The Penrose transform for H^2s is not direct, although we give an integral formula in (24). This is the case that naturally gives rise to a holomorphic gerbe on twistor space [11] and so may well be the most geometrically natural for making contact with the $(0, 2)$ gerbe theory.[1] In [12] a supertwistor space was introduced and some speculations were made concerning the Penrose transform of the $(0, 2)$-theory (in H^3 form).

Much recent progress in the study of scattering amplitudes in four dimensions has exploited the simplicity arising from spinor-helicity methods and these have recently been extended to six dimensions and higher [13,14]. The spinor-helicity method is essentially a method for efficiently encoding the polarization information of momentum eigenstates in terms of spinors. It is a useful starting point for exploring the twistor space formulation of scattering amplitudes and thereby gives insight into the structure of the twistor formulation of the theory; it is straightforward to construct the twistor space representation of an amplitude given the twistor representation of momentum eigenstates and the momentum space amplitude in spinor-helicity form. This has been instrumental in recent progress, both for expressing known amplitudes in twistor space and for obtaining amplitudes on momentum space from their counterparts in twistor space; see for example [15–17].

In this paper we obtain distributional Dolbeault representatives on twistor space for such spinor-helicity representations of momentum eigenstates. We do so in a way that naturally extends to higher dimensions and indeed gives some explanation of the structure of the Penrose transform in higher dimensions: the fact that in dimension $2m$, the relevant cohomology is obtained in H^{m-1} and $H^{m(m-1)/2}$, and the fact that the helicities obtained for these different cohomology degrees are the same for m odd, and opposite for m even. In brief, the spinor-helicity data for a momentum eigenstate consist of a null momentum P and polarization data ξ in some irreducible representation of the 'little group', i.e., the $SO(2m - 2)$ inside the stabilizer of P. For our massless fields, the polarization data will be irreducible symmetric spinor representations of this little group and these can be obtained either as holomorphic sections of line bundles, or dually as $H^{(m-1)(m-2)/2}$s on the chiral projective pure spinor spaces for the little group. These can then be combined with distributional Dolbeault forms supported on the little group spin space inside twistor space to generate the spinor-helicity Dolbeault representatives. In four dimensions, with $m = 2$, the projective pure spin space for the little group is trivial (just a point) and so this feature does not play a role, but in six dimensions it is the Riemann sphere and so this is the first non-trivial case where this comes in. It is the dimension of this little group pure spin space that gives the difference between the cohomology degrees in the Penrose transform.

These twistor representatives associated with spinor-helicity forms of the momentum eigenstates constitute a useful tool that allows us to understand key features of the Penrose transform. For example, they give an elementary proof of the Penrose transform isomorphism without the spectral sequences of [5]. One feature that is peculiar to six dimensions is that the same massless fields are represented both as H^2s and H^3s. We will see that the best way to understand the correspondence between these two representations is to consider the problem of extending the cohomology classes off Q into $\mathbb{CP}^7$. The H^2s turn out to have a unique extension to the order given by the helicity, but any further extension is obstructed with obstruction given by the corresponding H^3. This understanding of the extrinsic behaviour of the cohomology class allows us to write an integral formula in this indirect case. The proof here is expedited by the spinor-helicity representatives introduced earlier. Although such features are not encountered in four-dimensional twistor space $\mathbb{CP}^3$, they are reminiscent of the extensions off ambitwistor space that arise in the description of four-dimensional physics [18–24] although in that case it is the H^1s that are extended with obstructions in H^2s as opposed to the H^2s being extended with obstructions in H^3.

Another tool that has proved important in four dimensions is the half-Fourier transform introduced by Witten [1] and developed and exploited in [25]. The Penrose transform is replaced here by an integral transform first studied by Sparling [26,27] for the wave equation and referred to as the $\varXi$-transform. In four dimensions this is known as the X-ray transform and takes functions on real twistor space, $\mathbb{RP}^3$, to functions on split signature space–time by integrating over lines. This can then be concatenated with the Fourier transform which then gives functions on the momentum light-cone. This is the half-Fourier transform which involves Fourier transforming in two of the four non-projective variables. In six dimensions, twistor space is now six dimensional whereas the momentum light-cone is only five dimensional and so there must be some loss of information. The spinor-helicity momentum eigenstates have a natural representation in this split signature also, and lead to a natural generalization of the half-Fourier transform from functions on twistor space to the momentum light-cone augmented by the pure spinor space for the little group at each point. This is also a 6-manifold and hence can be, and is, an isomorphism.

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[1] We remind the reader that, loosely speaking, a gerbe is the extension of the concept of a line bundle with connection 1-form (or $(0, 1)$-form deforming a $\bar{\partial}$-operator in the holomorphic case) replaced by a connection 2-form (or $(0, 2)$-form in the holomorphic case).

We make a preliminary investigation of amplitudes. However, for the self-dual gerbe theory, Huang and Lipstein [28,29] have observed, using the spinor-helicity formalism, that there are no good candidates for amplitudes that are local in space–time. This is in keeping with the idea that the self-dual gerbe theory only exists as a strongly coupled theory in six dimensions with no parameters that can be taken as a coupling constant to expand in to obtain perturbative amplitudes. Because we have restricted our attention to conformally invariant chiral fields in this paper, the only non-trivial interaction that we are able to consider is the basic conformally invariant ϕ^3 vertex, but we see that it has a natural formulation on twistor space.

In this article, we find new features important to higher-dimensional twistor theory that are absent in the four-dimensional case. This may account for some of the remarkable features of six-dimensional quantum theories. One of the motivations for studying higher-dimensional twistor space was the realization that twistor theory is the natural framework in which to study extended supersymmetry [30]. It is tempting to suggest that a study of higher-dimensional supersymmetric physics from a twistorial perspective may shed light on many of the mysterious properties that higher-dimensional supersymmetric theories exhibit [31–38]. We hope to return to this in a subsequent paper.

The format of this paper is as follows. Section 2 gives an overview of the geometry of six-dimensional twistor space and the Penrose transforms that relate cohomology classes in twistor space to the solutions of zero-rest-mass field equations. Section 3 presents a higher-dimensional formalism for relating spinor-helicity methods and twistor theory that generalizes the four-dimensional approach outlined in [15] to six dimensions. Although the focus is on the six-dimensional case, in Section 4 we show how our formalism works in arbitrary dimension. Section 5 considers two integral transforms that exist in split signature. The first is a six-dimensional analogue of the half-Fourier transform introduced by Witten in [1] which relates objects on null momentum space to objects in twistor space. The second is the Ξ-transform introduced by Sparling in [26,27]. We find an analogue of the half-Fourier transform in six dimensions and extend the Ξ-transform to include fields of arbitrary spin. We give a brief discussion of interacting theories in six dimensions focusing on the conformal Φ^3 scalar theory. We show how the three-point amplitude for this theory may be constructed from six-dimensional twistor space and close with some discussion of the formulation of conformal field theories in twistor space. In an Appendix, we give the indirect Penrose transform for a gerbe.

Whilst preparing this manuscript we learnt of the work [39] which overlaps with some sections of this article.

2. The six-dimensional twistor correspondence

In this section we review the structure of six-dimensional twistor space and the Penrose transform. For reviews of the four-dimensional case see [40,41,4]. In order to deal with a variety of different signatures, we will start by working on complexified Minkowski space, $\mathbb{M}^I = \mathbb{C}^6$. We can extend $\mathbb{M}^I$ to the compactified, complexified Minkowski space $\mathbb{M}$ by adding a light-cone at infinity to give a 6-quadric $\mathbb{M}$ in $\mathbb{CP}^7$. For the most part, we shall be working with some region in $\mathbb{M}$ or one of its real slices and its associated twistor space. Introducing spinor indices from the start, we can coordinatize $\mathbb{M}^I$ with coordinates $x^{AB} = x^{[AB]}$, $A, B = 1, \ldots, 4$, equipped with metric

$$\mathrm{d}s^2 = \frac{1}{2}\varepsilon_{ABCD}\,\mathrm{d}x^{AB}\,\mathrm{d}x^{CD}, \quad \varepsilon_{ABCD} = \varepsilon_{[ABCD]}, \qquad \varepsilon_{0123} = 1.$$

The two four-dimensional chiral spinor representations, dual to one another, are denoted as $\mathbb{S}^A$ and $\mathbb{S}_A$ with the given index structure and have structure group $\mathrm{SL}(4, \mathbb{C})$ in the complex.

The *real* slices $\mathbb{M}^I_{p,q}$ of $\mathbb{M}^I$ are those $\mathbb{R}^6 \subset \mathbb{C}^6$ on which the metric has signature (p, q). On Euclidean, Lorentzian and split signature real slices, the spin group reduces to the real subgroups $\mathrm{SU}(4)$, $\mathrm{SL}(2, \mathbb{H})$ and $\mathrm{SL}(4, \mathbb{R})$ respectively. These can be characterized as follows:

- $(p, q) = (6, 0) \rightsquigarrow \mathrm{SU}(4)$, the subgroup of $\mathrm{SL}(4, \mathbb{C})$ that commutes with the conjugation $\pi_A \to \bar{\pi}^A$ with $\pi_A\bar{\pi}^A$ positive definite.
- $(p, q) = (5, 1) \rightsquigarrow \mathrm{SL}(2, \mathbb{H})$ is the subgroup commuting with a quaternionic conjugation$\hat{} : \pi_A \mapsto \widehat{\pi_A} =: \hat{\pi}_A$ on $\mathbb{S}^A$ (and on $\mathbb{S}_A$) where

$$\hat{\pi}_A = (-\bar{\pi}_1, \bar{\pi}_0, -\bar{\pi}_3, \bar{\pi}_2).$$

  We have $\hat{\hat{\pi}}_A = -\pi_A$, so there are no real spinors.
- $(p, q) = (4, 2) \rightsquigarrow \mathrm{SU}(2, 2)$, the subgroup of $\mathrm{SL}(4, \mathbb{C})$ that commutes with the conjugation $\pi_A \to \bar{\pi}^A$, but this time, $\mathbb{S}^A$ divides into three parts according to the definiteness of $\pi_A\bar{\pi}^A$.
- $(p, q) = (3, 3) \rightsquigarrow \mathrm{SL}(4, \mathbb{R})$, the group commuting with a conjugation $\pi_A \to \bar{\pi}_A$, and we can take spinors to be real; $\pi_A = \bar{\pi}_A$.

There is no natural way to raise or lower individual spin indices; however, skew-symmetric *pairs* of indices can be raised and lowered by means of $\frac{1}{2}\varepsilon_{ABCD}$, i.e. $v_{AB} = \frac{1}{2}\varepsilon_{ABCD}v^{CD}$. We have the useful identities $v_{AC}v^{BC} = \frac{1}{4}\delta_A^B v^2$ for any vector v^{AB}.

A null vector v^{AB} satisfies $v^{[AB}v^{CD]} = 0$, and so has vanishing determinant as a 4×4 matrix. Since skew matrices have even rank, if it is to be non-trivial, this rank must be 2, so it can be decomposed as $v_{AB} = \lambda^0_{[A}\lambda^1_{B]}$ for some spinors $\lambda^a_A = (\lambda^0_A, \lambda^1_A)$. This decomposition is not unique, but is subject to the $\mathrm{SL}(2, \mathbb{C})$ freedom $\lambda^a_A \mapsto \Lambda^a{}_b\lambda^b_A$ for some $\Lambda^a{}_b \in \mathrm{SL}(2, \mathbb{C})$. For a real null vector in Lorentz signature we can take $\lambda^1_A = \hat{\lambda}^0_A$ and in split signature the λ^a_A can be taken to be real.
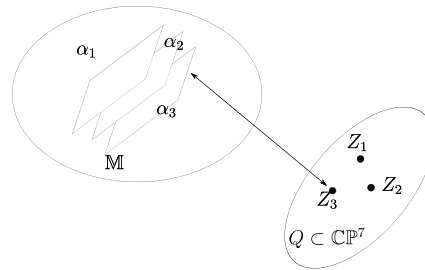
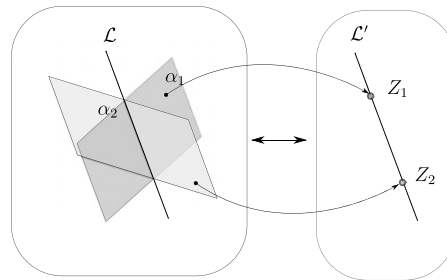**Fig. 1.** Self-dual null planes are represented by points in twistor space.

Fig. 2. Two α-planes always intersect in a null line in $\mathbb{M}$.

2.1. Twistor space

The twistor space Q is the six-dimensional projective quadric in $\mathbb{CP}^7$ [6]. We introduce the homogeneous coordinates $Z^\alpha = (\omega^A, \pi_A) \in \mathbb{S}^A \oplus \mathbb{S}_A$ on $\mathbb{CP}^7$, and define the quadric as

$$Q := \{[Z^\alpha] \in \mathbb{CP}^7 | Z^2 = 0\} \quad \text{where } Z^2 = 2\omega^A \pi_A. \tag{1}$$

The relationship with space–time $\mathbb{M}^I$ follows from the incidence relations

$$\omega^A = x^{AB} \pi_B, \tag{2}$$

where we assume $\pi_B \neq 0$ for now. Since x^{AB} is skew-symmetric, we must have the condition $\omega^A \pi_A = 0$, i.e., the quadric condition $Z^2 = 0$. Holding x^{AB} fixed in Eq. (2) and varying $[\omega^A, \pi_A]$, this defines a three-dimensional linear subspace $\mathbb{CP}^3 \subset Q$, which we shall denote as S_x. The converse statement, of what a point in twistor space corresponds to in space–time, is found by holding $[\omega^A, \pi_A]$ fixed and allowing x^{AB} to vary. This determines a totally null self-dual 3-plane $x^{AB} = x_0^{AB} + \varepsilon^{ABCD} \pi_A \alpha_B$ for some α_B defined modulo π_B. This is an α-plane (see Fig. 1).

We have so far ignored the case where $\pi_A = 0$, but we can easily extend the geometric interpretation of twistor space to compact complex space–time $\mathbb{M}$ by simply identifying $[\omega^A, 0]$ as the $\mathbb{CP}^3$ corresponding to the point added to $\mathbb{M}^I$ at infinity. Notationally, Q and Q^I will refer to the twistor spaces of $\mathbb{M}$ and $\mathbb{M}^I$ respectively.

In contrast to the four-dimensional case where a null line in space–time is a point in twistor space, null lines in six-dimensional space–time correspond to null lines in six-dimensional twistor space. Generically two α-planes, α_1 and α_2, do not intersect. However, if $Z_1 \cdot Z_2 = 0$, then they do and indeed they must do in a null line $\mathcal{L} = \alpha_1 \cap \alpha_2$ [42]. In twistor space, this configuration is given by the line joining the two twistors, Z_1 and Z_2, connected by the null line $\mathcal{L}'$ corresponding to the null line $\mathcal{L}$ in space–time (if $Z_1 \cdot Z_2 \neq 0$, then the line joining Z_1 to Z_2 in $\mathbb{CP}^7$ will not lie in Q)—see Fig. 2.

Twistor space is a quadric in the projectivization of the positive chiral spinor representation $\mathbb{S}^\alpha$ of $SO(8, \mathbb{C})$. Primed twistor space Q' is the quadric in the projectivization of the negative chiral spinor representation $\mathbb{S}^{\alpha'}$ of $SO(8, \mathbb{C})$ which we can coordinatize as $W^{\alpha'} = (\mu_A, \lambda^A)$ with inner product $W \cdot W = 2\lambda^A \mu_A$. Primed twistor space Q' is the space of all anti-self-dual null planes, β-planes, in $\mathbb{M}$ given by $\mu_A = x_{AB} \lambda^B$. In analogy to Q^I, the primed twistor space of $\mathbb{M}^I$ will be denoted as Q'^I.

This primed twistor space is the six-dimensional analogue of the *dual twistor space* from four dimensions. The latter is also a primed twistor space, but happens to be isomorphic to the dual of twistor space, a feature of dimensions 0 mod 4. However, in dimensions 2 mod 4, twistor space and primed twistor space have canonically defined inner products and so are dual to themselves. Thus important aspects of the Penrose transform are different in these cases as we shall see.

As a final remark, we note that one of the features of six-dimensional twistor theory is the *triality* between the three six-dimensional complex quadrics $\mathbb{M}$, Q and Q'. This can be described by means of the generators of the Clifford algebra for $SO(8, \mathbb{C})$, but here, it suffices to say that the geometric correspondences between $\mathbb{M}$ and Q, and $\mathbb{M}$ and Q' are also mirrored by one between Q and Q', e.g. an α-plane in Q' corresponds to a point in Q, and so on. Triality holds in split signature, but is broken by other choices of signature (it holds also for $SO(8, \mathbb{R})$ but then the real quadrics are empty).

2.2. Massless fields

Our main area of interest in this article will be conformally invariant chiral theories. Unlike in four dimensions, Yang–Mills theories cannot be chiral in six dimensions, nor are they conformally invariant. We will instead focus on symmetric spinor fields $\phi_{A_1 \cdots A_k} = \phi_{(A_1 \cdots A_k)}$ and $\psi^{A_1 \cdots A_k} = \psi^{(A_1 \cdots A_k)}$ describing chiral fields of helicity $\frac{k}{2}$ and $-\frac{k}{2}$ respectively. In linearized, or non-interacting theories, such fields with $k > 0$ satisfy the zero-rest-mass equations of motion

$$\nabla^{BA_1}\phi_{A_1 A_2 \cdots A_k} = 0, \qquad \nabla_{BA_1}\psi^{A_1 A_2 \cdots A_k} = 0, \quad \text{where } \nabla_{AB} = \frac{\partial}{\partial x^{AB}}.$$

For scalar fields ($k = 0$) we have

$$\Box\phi = 0.$$

For any $k \in \mathbb{Z}$, the space of massless fields of helicity $k/2$ on a region U in space–time will be denoted as $\Gamma_k(U)$, although we will often drop the dependence on U.

2.3. The Penrose transform

In six dimensions, the Penrose transform [43,5] relates the space of massless fields of non-negative helicity on a region $U \subset \mathbb{M}$ to the cohomology classes with values in certain holomorphic line bundles $\mathcal{O}(m)$ over the corresponding region Q^U swept out by the S_x with $x \in U$ in twistor space. For most of the paper, we shall in fact take U to be $\mathbb{M}^I$, complexified space–time, so the corresponding twistor space Q^U will simply be Q^I, the six-dimensional quadric with a $\mathbb{CP}^3$ removed. The line bundles $\mathcal{O}(m)$ are the restriction of the corresponding line bundles from $\mathbb{CP}^7$ whose sections can be identified with holomorphic functions on the non-projective space $\mathbb{C}^8$ that are homogeneous of degree m. We denote these cohomology classes by $H^\bullet(Q^U; \mathcal{O}(m))$.

We note that one can obtain massless fields of negative helicity from cohomology of holomorphic line bundles over some subset of *primed* twistor space. This is in contradistinction to four-dimensional twistor theory, where fields of any integral helicity can be obtained from twistor space. This dichotomy corresponds to that of whether twistor space is canonically isomorphic to itself or to its primed counterpart.

There are two ways in which one can obtain Γ_k for $k \geq 0$ from cohomology classes in Q. The first, the *direct Penrose transform*, is

$$\mathcal{P} : H^3(Q^U, \mathcal{O}(-k-4)) \xrightarrow{\simeq} \Gamma_k(U).$$

This is direct in the sense that it follows explicitly by means of an integral formula:

$$F_{A_1 \cdots A_k}(x) = \int_{S_x} \pi_{A_1} \cdots \pi_{A_k} f(x \cdot \pi, \pi) \mathrm{D}^3\pi, \tag{3}$$

where $f = f(\omega, \pi)$ is a $(0, 3)$-form homogeneous of degree $-k - 4$ representing the cohomology class[2] on twistor space, and $\mathrm{D}^3\pi = \varepsilon^{ABCD}\pi_A \,\mathrm{d}\pi_B \,\mathrm{d}\pi_C \,\mathrm{d}\pi_D$ is the projective volume form on $S_x = \mathbb{CP}^3$.

The second transform will be referred to as the *indirect Penrose transform*

$$\widetilde{\mathcal{P}} : H^2(Q^U, \mathcal{O}(k-2)) \xrightarrow{\simeq} \Gamma_k(U).$$

In this case, the cohomology classes in $H^2(Q^U, \mathcal{O}(k-2))$ have trivial restriction on $\mathbb{CP}_3$ embedded in Q^U. For this reason, one cannot obtain the value of a massless field directly. The transform most naturally yields a potential modulo gauge description[3] of the field, but this is nevertheless equivalent locally to the given field.

3. Spinor-helicity representatives

In this section we turn to the problem of how to relate the six-dimensional spinor-helicity methods proposed in [13] to the six-dimensional twistor theory discussed in the preceding section. We first review the spinor-helicity formalism.

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[2] We focus on Dolbeault cohomology here but it is sometimes more elegant to express the cohomology class in terms of Čech cohomology, by use of the Čech–Dolbeault isomorphism

$$\check{H}^\bullet(Q; \mathcal{O}(m)) \cong H^\bullet_{\bar{\partial}}(Q; \mathcal{O}(m))$$

so that massless space–time fields can be described by holomorphic functions on regions in twistor space. In this case (3) is interpreted as a contour integral surrounding poles of f.

[3] We elaborate on this construction in Section 6.

3.1. The spinor-helicity formalism

For the fields Γ_k, a momentum eigenstate is a field $\Phi_{A_1 \cdots A_k} = \Phi^0_{A_1 \cdots A_k} e^{iP \cdot x}$ with $\Phi^0_{A_1 \cdots A_k}$ constant. The massless field equation then implies that $P^{BA_1} \Phi^0_{A_1 \cdots A_k} = 0$. We can solve these algebraic equations as follows. Firstly, if P^{AB} were invertible then $\Phi^0_{A_1 \cdots A_k}$ would have to vanish. Thus its rank must be less than 4 for a non-trivial solution. However, a skew matrix necessarily has even rank, so if P is to be non-zero, it must have rank 2. This certainly implies $P^2 = 0$ and furthermore means that we can write

$$P_{AB} = \varepsilon_{ab} \lambda^a_A \lambda^b_B, \quad a, b = 0, 1,$$

for some λ^a_A. We refer to a, b as 'little group' indices. The little group is $SU(2) \times SU(2)$, the maximally compact part of the stabilizer of P in $Spin(1, 5)$. The representations of this maximal compact subgroup constitute the possible polarization data for massless fields. The a, b are indices for the first of these $SU(2)$ factors and we will introduce a', b' indices below for the second. Little group indices will be raised and lowered by means of the invariant skew forms ε^{ab} and ε_{ab} respectively, satisfying $\varepsilon_{ac} \varepsilon^{bc} = \delta^b_a$. The reduction from $SL(2, \mathbb{C})$ to $SU(2)$ is effected by choosing $\lambda^1_A = \hat{\lambda}^0_A$.

It is easily seen from the definition of λ^a_A that $P^{AB} \lambda^a_B = 0$ and in fact that λ^a_A, $a = 0, 1$, span the kernel of P^{AB}. Thus $\Phi^0_{A_1 \cdots A_k} = \lambda^{a_1}_{A_1} \cdots \lambda^{a_k}_{A_k} \xi_{a_1 \cdots a_k}$ for some $\xi_{a_1 \cdots a_k} = \xi_{(a_1 \cdots a_k)}$. The irreducible polarization information is therefore contained in the polarization spinor $\xi_{a_1 \cdots a_k}$ which is a spin $k/2$ representation of the little group (which is the spin group two dimensions down, here a four-dimensional spin group). We therefore have the spinor-helicity representation of a momentum eigenstate as

$$\Phi_{A_1 \cdots A_k}(x) = \lambda^{a_1}_{A_1} \cdots \lambda^{a_k}_{A_k} \xi_{a_1 \cdots a_k} e^{iP \cdot x}$$

with data $(\lambda^a_A, \xi_{a_1 \cdots a_k})$. Massless fields of the opposite chirality and of mixed type can be treated similarly[4] but we will not consider them here although we will return to these in [44]; they require the use of cohomology with values in a vector bundle rather than a line bundle, and in the mixed case require also the breaking of conformal invariance.

3.2. Twistor representatives for momentum eigenstates

We now find canonical twistor cohomology classes for such momentum eigenstates. Using Dolbeault cohomology, the twistor representatives will be $\bar{\partial}$-closed $(0, 3)$-forms (6) of weight $-k - 4$ for the direct transform (3) or a $(0, 2)$-form (7) of homogeneity $k - 2$ for the indirect case. We first establish some machinery and notation.

We can encode the polarization information as a cohomology class on a Riemann sphere $\mathbb{CP}^1_u$ acted on by the unprimed $SU(2)$ of the little group. We will use homogeneous coordinates u_a, $a = 0, 1$, on $\mathbb{CP}^1_u$ (hence the subscript). The polarization information can be expressed as a holomorphic function homogeneous of degree k on $\mathbb{CP}^1_u$:

$$\xi(u) = \xi_{a_1 \cdots a_k} u^{a_1} \cdots u^{a_k} \in H^0(\mathbb{CP}^1; \mathcal{O}(k)).$$

Alternatively, by Serre duality, there is an $\alpha_\xi \in H^1(\mathbb{CP}^1, \mathcal{O}(-2 - k))$ such that

$$\xi_{a_1 \cdots a_k} = \int_{\mathbb{CP}^1_u} u_{a_1} \cdots u_{a_k} \alpha_\xi \wedge Du, \quad \text{e.g. } \alpha_\xi = \xi_{a_1 \cdots a_k} \hat{u}^{a_1} \cdots \hat{u}^{a_k} D\hat{u},$$

where

$$\hat{u}^a = \frac{1}{\left(|u_0|^2 + |u_1|^2\right)} \begin{pmatrix} \bar{u}_0 \\ \bar{u}_1 \end{pmatrix}, \quad \text{and} \quad Du = u_a \, du^a, \quad D\hat{u} = \hat{u}_a \, d\hat{u}^a.$$

---

[4] For the massless fields of the opposite chirality we introduce $\tilde{\lambda}^{Aa'}$ such that

$$\Phi^{A_1 \cdots A_k}(x) = \tilde{\lambda}^{A_1}_{a'_1} \cdots \tilde{\lambda}^{A_k}_{a'_k} \xi^{a'_1 \cdots a'_k} e^{iP \cdot x} \quad \text{where } P^{AB} = \varepsilon_{a'b'} \tilde{\lambda}^{Aa'} \tilde{\lambda}^{Bb'}, \ a', b' = 0, 1.$$

Linearized gravity and Maxwell theory are of mixed type. Maxwell theory is described by using a photon with (traceless) field strength F^A_B satisfying $\nabla_{AB} F^B_C = 0 = \nabla^{AB} F^C_B$ with momentum eigenstates

$$F^A_B(x) = \tilde{\lambda}^A_{a'} \lambda^b_B \zeta^{a'}_{\ b} e^{iP \cdot x},$$

for some little group spinor $\zeta^{a'}_{\ b}$. Similarly, linearized gravity is described by a linearized Weyl tensor Ψ^{CD}_{AB} symmetric in each pair, subject to $\nabla_{AB} \Psi^{BC}_{DE} = 0 = \nabla^{AB} \Psi^{DE}_{BC}$ with momentum eigenstates

$$\Psi^{AB}_{CD}(x) = \tilde{\lambda}^A_{a'} \tilde{\lambda}^B_{b'} \lambda^c_C \lambda^d_D \zeta^{a'b'}_{\ cd} e^{iP \cdot x},$$

for some little group spinor $\zeta^{a'b'}_{cd} = \zeta^{(a'b')}_{(cd)}$. The corresponding polarization data $\zeta^{a'}_{\ b}$ and $\zeta^{a'b'}_{cd}$ have little group indices of both types.

Here the $\hat{u}_a$ coordinates can also be invariantly defined as $\hat{u}_a = \frac{\bar{u}_a}{u_b \bar{u}^b}$, where $\bar{u}_a = (-\bar{u}_1, \bar{u}_0)$ are anti-holomorphic coordinates on $\mathbb{CP}^1_u$, so that it is normalized, i.e. $u_a \hat{u}^a = 1$, and is thus homogeneous of weight -1 in u_a.

The little group spinor u defines two full spinors

$$\lambda_A(u) := \lambda_A^a u_a, \qquad \widehat{\lambda}_A(u) := \lambda_A^a \hat{u}_a$$

and we will have that $P_{AB} = \lambda_{[A} \widehat{\lambda}_{B]}$. We will often suppress the indices and dependence on u, just writing λ and $\widehat{\lambda}$ for these quantities in what follows, and make the dependence explicit with $\lambda \cdot u$ where confusion might otherwise arise.

The twistor representative associated with the momentum space eigenstate of null momentum P will turn out to be supported on the line in the 'π_A' spin space spanned by the two λ_A^a, $a = 1, 2$. This can be done with holomorphic delta functions on $\mathbb{CP}^3$. To define these, we first introduce, for $z = x + iy$,

$$\bar{\delta}(z) = \delta(x)\delta(y)\, \mathrm{d}\bar{z}.$$

These can be multiplied together to give delta functions on $\mathbb{C}^4$: $\bar{\delta}^4(\mu_A) = \prod_{A=1}^4 \bar{\delta}(\mu_A)$. To give a delta function on the projective space $\mathbb{CP}^3$, for two points with homogeneous coordinates μ_A, ν_A we define

$$\bar{\delta}_k^3(\mu, \nu) = \int_{\mathbb{C}} \frac{\mathrm{d}s}{s^{k+1}} \wedge \bar{\delta}^4(\mu + s\nu)$$

which has weight $-k - 4$ in μ and k in ν.

Given spinor-helicity data $(\lambda_A^a, \xi_{a_1 \cdots a_k})$, we can now write the two formulae for the twistor representatives. We define the corresponding representative $\phi \in H^3(Q; \mathcal{O}(-k-4))$ to be

$$\phi(\omega, \pi) = \mathrm{e}^{iP \cdot x} \int_{\mathbb{CP}^1_u} \alpha_\xi \wedge \bar{\delta}_k^3(\pi, \lambda) \wedge \mathrm{D}u, \tag{4}$$

and the representative $\psi \in H^2(Q; \mathcal{O}(k-2))$ to be

$$\psi(\omega, \pi) = \mathrm{e}^{iP \cdot x} \int_{\mathbb{CP}^1_u} \xi(u)\, \bar{\delta}_{-2-k}^3(\pi, \lambda \cdot u) \wedge \mathrm{D}u. \tag{5}$$

Throughout, we take $k \geq 0$.

In these formulae, $P \cdot x$ is not manifestly a twistor function; however, on the support of the delta function it is in the sense that it satisfies $\pi_A \partial^{AB}(P \cdot x) = 0$ as $P^{AB} \pi_B = 0$. We can make this more explicit by observing that $P_{AB} = \lambda_{[A} \widehat{\lambda}_{B]}$ and that on the support of the delta function $\bar{\delta}^4(s\pi - \lambda)$ we can take $s\pi_A = \lambda_A$. Thus, using the incidence relations $x \cdot P = -s\omega \cdot \widehat{\lambda}$. We therefore obtain as our definitive formulae for our twistor representatives of momentum eigenstates

$$\phi(\omega, \pi) = \int_{\mathbb{C} \times \mathbb{CP}^1_u} \mathrm{e}^{-is\omega \cdot \widehat{\lambda}}\, \alpha_\xi \wedge \bar{\delta}^4(s\pi - \lambda \cdot u) \wedge s^{k+3}\, \mathrm{d}s \wedge \mathrm{D}u, \tag{6}$$

$$\psi(\omega, \pi) = \int_{\mathbb{C} \times \mathbb{CP}^1_u} \mathrm{e}^{-is\omega \cdot \widehat{\lambda}}\, \xi(u)\, \bar{\delta}^4(s\pi - \lambda \cdot u) \wedge \frac{\mathrm{d}s}{s^{k-1}} \wedge \mathrm{D}u. \tag{7}$$

In both cases it is easy to check that all the weights and form degrees match appropriately. The $\bar{\partial}$-closure on $Z \cdot Z = 0$ can be seen by checking $\bar{\partial}$-closure of the integrands of (7) and (6), as $\bar{\partial}$-closure is preserved by integration. In the H^3 case $\bar{\partial}$-closure follows by virtue of its holomorphic dependence on ω and being a form of maximal degree on the $\mathbb{CP}^3 \times \mathbb{CP}^1$ parametrized by (π, u). In the H^2 case, $\xi(u)\, \bar{\delta}_{-2-k}^3(\pi, \lambda \cdot u) \wedge \mathrm{D}u$ is $\bar{\partial}$-closed. Operating with $\bar{\partial}$ on the exponential factor we obtain, on the support of the delta function,

$$\bar{\partial}\big(\omega \cdot \widehat{\lambda}\big) = \omega^A \lambda_A^a \bar{\partial}\hat{u}_a = -\omega^A \lambda_A^a u_a \mathrm{D}\hat{u} \tag{8}$$

where we have used the fact that $\bar{\partial}\hat{u}_a = u_a \mathrm{D}\hat{u}$. On the support of the delta function, $\omega^A \lambda_A^a u_a = \omega \cdot \pi$, and so (8) vanishes on Q.

4. Twistor spinor-helicity states in higher dimensions

The spinor-helicity formalism and the corresponding twistor space representatives that we have presented in six dimensions have straightforward extensions to higher dimensions. One new issue that we meet in higher dimensions is the fact that purity conditions come in for spinors in dimensions greater than 6 and symmetric spinors are then no longer an irreducible representation of the Lorentz group, so further irreducibility conditions must be imposed. The other is that the canonical structures and identifications amongst spinors change from dimension to dimension. We will work in the complex to avoid further changes in character between the various signatures.

The purity condition on a spinor $\pi^{A'}$ can be expressed in many ways, but the most convenient for our purposes will be as follows. We first establish notation. Let $\mu = 1, \ldots, 2m$ be the space–time indices, and $A = 1, \ldots, 2^{m-1}$ and

$A' = 1', \ldots, 2^{m-1'}$ be the primed and the unprimed spinor indices, and decompose the gamma matrices into their chiral parts, so that we have

$$\Gamma_\mu = \begin{pmatrix} 0 & \gamma_{\mu A}{}^{B'} \\ \gamma_{\mu A'}{}^{B} & 0 \end{pmatrix}, \tag{9}$$

and the Clifford algebra relations become

$$\gamma_{(\mu|A'}{}^{A} \gamma_{|\nu)A}{}^{B'} = -\eta_{\mu\nu} \delta_{A'}^{B'}, \qquad \gamma_{(\mu|A}{}^{A'} \gamma_{|\nu)A'}{}^{B} = -\eta_{\mu\nu} \delta_{A}^{B}, \tag{10}$$

where $\eta_{\mu\nu}$ is the metric on $\mathbb{C}^{2m}$. The purity condition on a spinor π_A is the condition

$$\pi^{A'} \pi^{B'} \gamma_{\mu A'}{}^{A} \gamma^{\mu}{}_{B'}{}^{B} = 0. \tag{11}$$

This guarantees that the vector fields of the form $V^\mu = \gamma^\mu{}_{A'}{}^{A} \pi^{A'} \alpha_A$ for arbitrary α_A span a totally null plane which can be of dimension at most m and will, for non-zero $\pi^{A'}$, be an m-dimensional one.

4.1. Spinor helicity in higher dimensions

We shall only be interested in the elementary conformally invariant symmetric spinor fields that we have been discussing in six dimensions; massless fields such as Maxwell and linearized gravity are never conformally invariant in dimensions greater than 4 with their standard second-order field equations. Thus we will take our massless fields to be symmetric spinors $\phi^{A'_1 \cdots A'_k} = \phi^{(A'_1 \cdots A'_k)}$ and $\psi_{A_1 \cdots A_k} = \psi_{(A_1 \cdots A_k)}$. This choice allows us to deal with both cases of m even and odd at once. But when m is odd, there will be some redundancy in this notation since primed indices can be eliminated by means of the isomorphism between primed spinor space and dual spinor space. This is consistent with the fact that the Penrose transform, as we describe it in this paper, produces massless fields of both positive and negative helicities when m is even, but only massless fields of positive helicity when m is odd. Further distinctions between these cases will be pointed out in the course of this section and the next. In dimensions greater than 6 these spinors are also subject to a further irreducibility condition. This can be expressed in the form

$$\phi^{A'_1 \cdots A'_k} \gamma_{\mu A'_1}{}^{A} \gamma^{\mu}{}_{A'_2}{}^{B} = 0, \qquad \psi_{A_1 \cdots A_k} \gamma_{\mu A}{}^{A_1} \gamma^{\mu}{}_{B'}{}^{A_2} = 0. \tag{12}$$

The zero-rest-mass equations on such fields are then

$$\gamma_{\mu A'_1}{}^{A} \nabla^\mu \phi^{A'_1 \cdots A'_k} = 0, \qquad \gamma_{\mu A'}{}^{A_1} \nabla^\mu \psi_{A_1 \cdots A_k} = 0. \tag{13}$$

We can now obtain the spinor-helicity formalism for such fields. We shall assume that our momentum eigenstates take the form $\phi^{A'_1 \cdots A'_k} = e^{iP \cdot x} \phi_0^{A'_1 \cdots A'_k}$ and $\psi_{A_1 \cdots A_k} = e^{iP \cdot x} \psi^0_{A_1 \cdots A_k}$ with $\phi_0^{A'_1 \cdots A'_k}$ and $\psi^0_{A_1 \cdots A_k}$ constant, and so

$$P^\mu \gamma_{\mu A'_1}{}^{A} \phi^{A'_1 \cdots A'_k} = 0, \qquad P^\mu \gamma_{\mu A'}{}^{A_1} \psi_{A_1 \cdots A_k} = 0. \tag{14}$$

The Clifford algebra relations imply that $P^\mu \gamma_{\mu A'}{}^{B}$ is invertible unless P is null, in which case it is standard[5] that $P^\mu \gamma_{\mu A'}{}^{B}$ is nilpotent with rank 2^{m-2}. We can, as before, introduce bases $\lambda_a^{A'}$ and λ_A^a ($a = 1, \ldots, 2^{m-2}$), respectively, of the kernel of $P \cdot \gamma$ and deduce that we must have

$$\phi_0^{A'_1 \cdots A'_k} = \xi^{a_1 \cdots a_k} \lambda_{a_1}^{A'_1} \cdots \lambda_{a_k}^{A'_k}, \qquad \psi^0_{A_1 \cdots A_k} = \eta_{a_1 \cdots a_k} \lambda_{A_1}^{a_1} \cdots \lambda_{A_k}^{a_k}, \tag{15}$$

for some symmetric $\xi^{a_1 \cdots a_k}$ and $\eta_{a_1 \cdots a_k}$. However, we must now also implement the irreducibility conditions (12) on $\xi^{a_1 \cdots a_k}$ and $\eta_{a_1 \cdots a_k}$. To do this we note that a is a spinor index for the group $SO(2m - 2, \mathbb{C})$, the spin group for the space–time of two dimensions lower, which is the semi-simple part of the stabilizer of P acting on $P^\perp / P$. The irreducibility conditions

$$\xi^{a_1 a_2 \cdots a_k} \lambda_{a_1}^{A'_1} \lambda_{a_2}^{A'_2} \gamma_{\mu A'_1}{}^{A_1} \gamma^{\mu}{}_{A'_2}{}^{A_2} = 0, \qquad \eta_{a_1 a_2 \cdots a_k} \lambda_{A_1}^{a_1} \lambda_{A_2}^{a_2} \gamma^\mu{}_{A'_1}{}^{A_1} \gamma^\mu{}_{A'_2}{}^{A_2} = 0 \tag{16}$$

are then simply the analogues of (12) for symmetric spinors $\xi^{a_1 \cdots a_k}$ and $\eta_{a_1 \cdots a_k}$ for $SO(2m - 2, \mathbb{C})$. This is first non-trivial for dimension $2m = 10$, when it is simply a trace-free condition on $\xi_{a_1 \cdots a_k}$. Here again, we remark that when m is odd, the little

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[5] This follows by choosing another null vector Q with $P \cdot Q = \frac{1}{2}$ and observing that $P \cdot \gamma \cdot Q \cdot \gamma + Q \cdot \gamma \cdot P \cdot \gamma = 1$ and this algebra has a standard representation with

$$P \cdot \gamma = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}, \qquad Q \cdot \gamma = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix},$$

where I is the identity matrix on $\mathbb{C}^{2^{m-2}}$.

group spinor space is isomorphic to its dual, and the little group spinor indices can be raised and lowered. Thus, in this case, the identification of $\lambda_a^{A'}$ with λ_A^a is consistent with the fact that only massless fields of positive helicity are treated here. On the other hand, when m is even, the little group spinor space and its dual are not isomorphic to one another, but correspond to distinct chiral spinor spaces for $\mathrm{SO}(2m-2, \mathbb{C})$.

Thus we are left with momentum eigenstates for solutions to (12) and (13) given by the formula

$$\phi^{A'_1 \cdots A'_k} = \xi^{a_1 \cdots a_k} \lambda_{a_1}^{A'_1} \cdots \lambda_{a_k}^{A'_k} e^{iP \cdot x}, \qquad \psi_{A_1 \cdots A_k} = \eta_{a_1 \cdots a_k} \lambda_{A_1}^{a_1} \cdots \lambda_{A_k}^{a_k} e^{iP \cdot x}. \tag{17}$$

This gives the chiral spinor-helicity description for these fields.

4.2. Twistor representatives for spinor-helicity states

Following the last chapter of [6], in arbitrary even dimension $2m$, twistor space will be defined to be the projective pure spinors for the conformal group $\mathrm{SO}(2m+2, \mathbb{C})$. Here we work in the complex and make no restriction on m, so we have to work independently of the special structures that arise in different dimensions modulo 8. The space of projective positive pure spinors for $\mathrm{SO}(2m, \mathbb{C})$ will be denoted as $\mathbb{PPS}_m$. The space $\mathbb{PPS}_m$ has dimension $m(m-1)/2$ and can be represented as the space of α-planes, the homogeneous space $\mathrm{SO}(2m)/\mathrm{U}(m)$ or the complex subvariety of the $(2^{m-1} - 1)$-dimensional projective spin space $\mathbb{PS}_m$ cut out by the purity conditions

$$\pi^{A'} \pi^{B'} \gamma_{\mu A'}{}^A \gamma^\mu{}_{B'}{}^B = 0. \tag{11}$$

This condition guarantees that $V^\mu = \gamma^\mu{}_{A'}{}^A \pi^{A'} \alpha_A$ is a null vector for any choice of α_A and, with the purity condition, this will be maximal, so $\pi^{A'}$ will determine a totally null self-dual m-plane through the origin. We take $\mathbb{PT} := \mathbb{PPS}_{m+1}$ as our definition of (projective) twistor space, the space of such totally null self-dual m-planes but not necessarily through the origin in $\mathbb{C}^{2m}$. For example, in dimension 6 ($m = 3$) the twistor space is $\mathbb{PT} = Q = \mathbb{PPS}_4$, the space of projective pure spinors in eight dimensions. These null self-dual planes, often called α-planes in this context, can be represented by the incidence relation

$$\omega^A = x^\mu \gamma_{\mu A'}{}^A \pi^{A'}, \tag{18}$$

as is familiar in dimension 4 and as was seen for dimension 6 in (2) where primed indices are eliminated in favour of the unprimed ones. However, we note that (18) will be inconsistent unless $(\omega^A, \pi^{A'})$ is a pure $\mathrm{SO}(2m+2)$ spinor.

If we now wish to proceed analogously to how we developed the six-dimensional theory presented in previous sections, we must describe the physical degrees of freedom in terms of cohomology representatives on spaces of projective pure spinors, i.e. on twistor spaces. Let $\mathbb{PPS}_{m-1}$ be the $(m-1)(m-2)/2$-dimensional space of projective pure spinors for $\mathrm{SO}(2m-2, \mathbb{C})$. We will take u^a to be homogeneous coordinates on the projective spin space $\mathbb{PS}_{m-1} = \mathbb{CP}^{2^{m-2}-1}$ and impose the purity conditions

$$u^{a_1} u^{a_2} \lambda_{a_1}^{A'_1} \lambda_{a_2}^{A'_2} \gamma_{\mu A'_1}{}^A \gamma^\mu{}_{A'_2}{}^B = 0.$$

It is a standard consequence of Bott–Borel–Weyl theory [5] that the representations defined by (16) can be represented by

$$\alpha_\xi \in H^{\text{top}}(\mathbb{PPS}_{m-1}, \mathcal{O}(4 - 2m - k)) \quad \text{and} \quad \eta(u) \in H^0(\mathbb{PPS}_{m-1}^*, \mathcal{O}(k)) \tag{19}$$

respectively, where top $= (m-1)(m-2)/2$ is the dimension of $\mathbb{PPS}_{m-1}$; these are related by Serre duality, as the canonical bundle on $\mathbb{PPS}_{m-1}$ is $\mathcal{O}(4 - 2m)$ (see [5]). The second of these is simply given by $\eta(u) = \eta_{a_1 \cdots a_k} u^{a_1} \cdots u^{a_k}$ whereas the first will not in general have a canonical representative, and we will denote such a representative simply by α_ξ satisfying

$$\xi^{a_1 a_2 \cdots a_k} = \int_{\mathbb{PPS}_{m-1}} u^{a_1} u^{a_2} \cdots u^{a_k} \alpha_\xi \, \mathrm{D}u,$$

where $\mathrm{D}u$ is the projective holomorphic volume form of weight $2m - 4$ on $\mathbb{PPS}_{m-1}$. When m is odd, one has $\xi_{a_1 \cdots a_k} = \eta_{a_1 \cdots a_k}$ since the little group spinor space is isomorphic to its dual, and we thus have two ways of representing the polarization spinor of a given massless field of positive helicity.

We must also introduce the projective holomorphic volume form $\mathrm{D}\pi$ of weight $2m - 2$ on $\mathbb{PPS}_m$ and the weighted delta function $\bar{\delta}(\pi, \rho) \in \Omega^{0, \frac{m(m-1)}{2}}(2 - 2m - k)$ on $\mathbb{PPS}_m$. The former exists simply via the identification of the canonical bundle of $\mathbb{PPS}_m$ as the restriction of $\mathcal{O}(2 - 2m)$, as follows from the Bott–Borel–Weyl theory as described in [5] (explicit formulae in terms of the ambient projective coordinates in $\mathbb{PS}_m$ are obtained in [10]). The delta function is defined tautologically by

$$f(\pi) = \int f(\rho) \, \bar{\delta}(\pi, \rho) \, \mathrm{D}\rho \tag{20}$$

where f is a function on $\mathbb{PPS}_m$ of weight k, $\mathrm{D}\rho$ is the canonical holomorphic volume form of weight $2m - 2$ and $\bar{\delta}(\pi, \rho)$ has appropriate weights in each of its arguments for the formula to make sense, i.e., of weight k in π and weight $2 - 2m - k$ in ρ.

We can now define the twistor representatives for the spinor-helicity states (17) as

$$\phi(\omega, \pi) = \int e^{iP \cdot x} \alpha_\xi \wedge \bar{\delta}(\pi, \lambda \cdot u) \, Du$$

$$\psi(\omega, \pi) = \int e^{iP \cdot x} \eta(u) \, \bar{\delta}(\pi, \lambda \cdot u) \, Du \tag{21}$$

and it can be seen that these are respectively $\bar{\partial}$-closed $(0, \frac{1}{2}m(m-1))$-forms of weight $2 - 2m - k$ and $(0, m-1)$-forms of weight $k - 2$. This follows from the weights k of $\eta(u)$, $4 - 2m - k$ of α_ξ and $2m - 4$ of the canonical holomorphic volume form Du on $\mathbb{PPS}_{m-1}$. It can be checked that $e^{iP \cdot x}$ is indeed a function on twistor space when restricted to the support of the delta function as before, as it will be annihilated by $\pi^{A'} \gamma^\mu{}_{A'}{}^A \partial_\mu$ on the support of the delta function.

5. Formal neighbourhoods and the twistor transform

In four dimensions a field of a given helicity is represented by the $H^1(\mathbb{PT}^I; \mathcal{O}(m))$ cohomology classes on twistor space and the $H^1(\mathbb{PT}^{*I}; \mathcal{O}(m-4))$ classes on dual twistor space. The direct map between these two representations is known as the twistor transform. In six dimensions, the situation is rather different, as the fields of positive helicity $k/2$ correspond to classes on the same twistor space either as an H^2 of homogeneity $k - 2$ or an H^3 of homogeneity $-k - 4$. There is no description of such fields simply in terms of homogeneous functions on dual twistor space. We will identify the direct map between these representatives, the twistor transform, $\mathcal{T}$, on twistor space in this section via an obstruction to the problem of extending the H^2s off the quadric $Q \subset \mathbb{CP}^7$. The twistor transform can be written schematically as

$$
\begin{array}{ccc}
 & \Phi_{A_1 \cdots A_k}(x) & \\
\mathscr{P}^{-1} \swarrow & & \searrow \widetilde{\mathscr{P}}^{-1} \\
g(Z) \longleftarrow & \mathcal{T} \longrightarrow & f(Z)
\end{array}
$$

where $\Phi_{A_1 \cdots A_k}(x) \in \Gamma_k$, $g(Z) \in H^3(Q^I; (-k-4))$ and $f(Z) \in H^2(Q^I; (k-2))$.

The obstruction problem can also be motivated by the task of writing integral formulae for the indirect transform. In four dimensions, there are integral formulae for the space–time fields associated with classes in $H^1(\mathbb{PT}^I; \mathcal{O}(k-2))$ involving derivatives of the twistor function; for example at $k = 2$, the self-dual photon, we have

$$F_{a'b'}(x) = \int_{\mathbb{CP}^1_x} D\lambda \, \frac{\partial^2}{\partial \mu^{a'} \partial \mu^{b'}} f(x \cdot \lambda, \lambda)$$

where $f \in H^1(\mathbb{PT}^I; \mathcal{O})$. Such a formula was proposed for dimension 6 [10] with (μ, λ) replaced by (ω, π). However, there are a number of problems: in six dimensions, the cohomology classes are, a priori, only defined on Q rather than $\mathbb{CP}^7$, and straightforward differentiation with respect to ω^A takes a derivative in directions off Q into $\mathbb{CP}^7$, and so they are not immediately meaningful. We will however show in the following that certain such derivatives can be canonically defined in the positive homogeneity case (although there are other problems with the formula in [10], as the homogeneity weights and cohomology degrees are not right either).

We will therefore consider the task of constructing extensions of the cohomology classes off the quadric as an expansion in powers of Z^2. In the H^3 cases we will see that classes can be extended off the quadric to all orders in Z^2, but that there is no way to fix the ambiguity that arises at each order and so the derivatives do not have any invariant meaning. In the H^2 case, we will show:

Proposition 5.1. *Every $f \in H^2(Q^I, \mathcal{O}(k-2))$ has a canonical extension to the kth-order formal neighbourhood around Q, so, in particular, its kth derivative transverse to Q is canonically defined; however, any further extension is obstructed, the obstruction being the corresponding $g \in H^3(Q^I, \mathcal{O}(-4-k))$ to which it corresponds under the twistor transform.*

Thus in examining the obstruction theory we find the canonical map from $H^2(Q^I, \mathcal{O}(k-2))$ to $H^3(Q^I, \mathcal{O}(-4-k))$ that we know must exist via the Penrose transform to space–time, intrinsically in twistor space. This also enables us to write integral formulae for the H^2 case.

Such extensions will be examined explicitly below for our representatives above. We will first examine the problem abstractly of extending cohomology classes to formal neighbourhoods of the quadric $Z \cdot Z = 2\omega^A \pi_A = 0$. The subsequent explicit calculations will then demonstrate that certain maps are indeed isomorphisms as claimed (see Fig. 3).
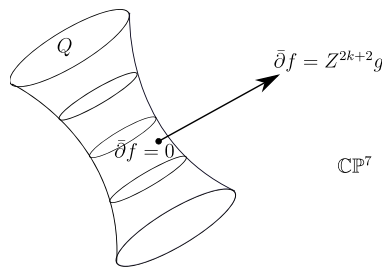
Fig. 3. The extension of $f \in H^2$ off Q is obstructed by $g \in H^3$.

5.1. Formal neighbourhoods

We are interested in determining to what extent the cohomology classes $H^2(Q^I; \mathcal{O}(k-2))$ and $H^3(Q^I; \mathcal{O}(-k-4))$ can be extended off the quadric. The natural language in which to approach this question is that of formal neighbourhoods [20,18,45,23,24]: a step by step process in which we consider the task of extending a class defined to rth order to the $(r+1)$th order about $Q \subset \mathbb{CP}^7$. The starting point is a twistor cohomology class $f(Z) \in H^\bullet(Q^I; \mathcal{O}(n))$ defined only on $Z^2 = 0$. The extension to the first formal neighbourhood is given by introducing the commuting variable ξ such that $\xi^4 = 0$ and allowing the twistor function to now depend on (Z^α, ξ), where $Z^2 = \xi^2$. Similarly, we can think of the twistors associated with the rth formal neighbourhood as the usual twistor Z^α with an additional variable ξ, for which $\xi^{2(r+1)} = 0$, subject to

$$(Z^\alpha, \xi) \sim (tZ^\alpha, t\xi) \qquad Z^2 = \xi^2 \qquad \xi^{2(r+1)} = 0.$$

This leads to a *thickening* of twistor space $Q \to Q_{[r]}$ [22] where $Q_{[r]}$ can be thought of as Q, but with the enlarged sheaf of holomorphic functions $\mathcal{O}_{[r]}$ that contain the information of the first r-derivatives off Q into $\mathbb{CP}^7$ encoded in the dependence on Z and ξ.

More formally, we make use of the long exact sequence of cohomology groups that follows from the short exact sequence:

$$0 \to \mathcal{O}_Q(n-2r) \xrightarrow{\times (Z \cdot Z)^r} \mathcal{O}_{[r]}(n) \longrightarrow \mathcal{O}_{[r-1]}(n) \to 0.$$

So $\mathcal{O}_{[r]}$ is the sheaf of functions on the rth formal neighbourhood which can be thought of as local functions on a neighbourhood of Q in $\mathbb{CP}^7$ modulo local functions of the form $(Z \cdot Z)^{r+1} \times g$ where g is another local function, with $\mathcal{O}_Q = \mathcal{O}_{[0]}$. We have, in terms of the ideal sheaf $\mathcal{I}$ of Q, $\mathcal{O}_{[r]} = \sum_{i=0}^{r} \mathcal{I}^r / \mathcal{I}^{r+1}$. In the following we abbreviate $\mathcal{O}_Q$ to $\mathcal{O}$.

This short exact sequence leads to long exact sequences of cohomology groups in the usual way. We know from [5] that the cohomology groups are only non-zero at the $H^2(Q^I; \mathcal{O}(n))$ level for $n \geq -2$ and $H^3(Q^I; \mathcal{O}(n))$ for $n \leq -4$. To give an idea of the process we first consider the easy case of extending $H^3(\mathcal{O}(n))$ for $n \leq -4$. The long exact sequence gives

$$0 \to H^3(\mathcal{O}(n-2r)) \to H^3(\mathcal{O}_{[r]}(n)) \to H^3(\mathcal{O}(n)) \xrightarrow{\delta} 0.$$

The zeros arise because there are no H^2s in these homogeneity degrees and there are never any H^4s. Thus the obstruction δ to extending a class from one formal neighbourhood to a higher one always vanishes. However, there is always the freedom arising from adding the choice of an element of $H^3(\mathcal{O}(n-2r))$ to some original choice. So, we can extend any class in $H^3(\mathcal{O}(n))$ to all orders but with much ambiguity, with a new non-trivial term arising at each term in its Taylor series.

The problem of extending $H^2(Q^I, \mathcal{O}(n))$ for $n \geq -2$ is much less trivial. For the extension to the first formal neighbourhood $\mathcal{O}_{[1]}$, the long exact sequence gives

$$0 \to H^2(\mathcal{O}(n-2)) \to H^2(\mathcal{O}_{[1]}(n)) \to H^2(\mathcal{O}(n)) \xrightarrow{\delta} H^3(\mathcal{O}(n-2)) \to \cdots.$$

The case $n = -2$ is exceptional here as $H^3(\mathcal{O}(n-2)) = H^3(\mathcal{O}(-4))$ is only non-vanishing in this case and by the Penrose transform is given by solutions to the wave equation and so is indeed isomorphic to $H^2(\mathcal{O}(-2))$. In fact we will see by direct computation that δ is an isomorphism, so it is impossible to extend an element of $H^2(\mathcal{O}(-2))$ to $H^2(\mathcal{O}_{[1]}(-2))$. In fact this sequence then implies that $H^2(\mathcal{O}_{[1]}(-2)) = 0$ as $H^2(\mathcal{O}(-4)) = 0$.

For $n > -2$, $H^3(\mathcal{O}(n)) = 0$, so the obstruction map δ always vanishes, and we can always extend subject to an ambiguity in $H^2(\mathcal{O}(n-2))$. This ambiguity is trivial for $n = -1$ and in that case there is a unique extension $H^2(\mathcal{O}_{[1]}(-1)) = H^2(\mathcal{O}(-1))$, but in all other cases there is some ambiguity. Finally for $n \geq 0$, we now have no obstruction to extension as the corresponding $H^3(\mathcal{O}(n-2))$ vanishes, but we have an ambiguity of $H^2(\mathcal{O}(n-2))$ in the choice of extension.

At the higher orders we obtain

$$0 \to H^2(\mathcal{O}(n-2r)) \to H^2(\mathcal{O}_{[r]}(n)) \to H^2(\mathcal{O}_{[r-1]}(n)) \xrightarrow{\delta} H^3(\mathcal{O}(n-2r)) \to \cdots.$$

For $n = -1$ and $r = 2$ we now see that there is a possible obstruction to extension as $H^2(\mathcal{O}_{[1]}(-1)) = H^2(\mathcal{O}(-1)) = H^3(\mathcal{O}(-5))$ and indeed, as we shall see later, the map δ is again an isomorphism. Thus $H^2(\mathcal{O}_{[2]}(-1)) = 0$ and there is no extension to the second formal neighbourhood in this homogeneity.

The other case that we will be interested in is that of homogeneity $n = 0$. As we saw earlier, an $\alpha \in H^2(\mathcal{O}(0))$ could always be extended to an $H^2(\mathcal{O}_{[1]}(0))$ but with freedom in $H^2(\mathcal{O}(-2))$. Here at the next order $r = 2$, the obstruction group is now $H^3(\mathcal{O}(-4))$ and indeed this map can be non-zero because, in $H^2(\mathcal{O}_{[1]}(0))$, we had the summand (the freedom in extension to $H^2(\mathcal{O}_{[1]})$) consisting of $H^2(\mathcal{O}(-2))$ which is isomorphic to $H^3(\mathcal{O}(-4))$. We will see again later that δ is an isomorphism from this summand and so we can always choose an extension of α to $\alpha_{[1]} \in H^1(\mathcal{O}_{[1]}(0))$ such that $\delta\alpha_{[1]} = 0$. This condition fixes the ambiguity in the first step completely and there exists a unique $\alpha_{[2]} \in H^2(\mathcal{O}_{[2]})$ that maps onto $\alpha_{[1]}$ because at this homogeneity, $H^2(\mathcal{O}(-4)) = 0$. Now, at $r = 3$ the obstruction to further extension is $\delta(\alpha_{[2]}) \in H^3(\mathcal{O}(-6))$. This latter space is now isomorphic to the space $H^2(\mathcal{O})$ that we started with and in fact the map δ will be seen to be an isomorphism.

5.1.1. An explicit example: momentum eigenstates

Our task here is to explicitly compute the 'connecting homomorphism' δ using the spinor-helicity based twistor representatives and to show that it is an isomorphism whenever the cohomology groups it connects are related by the twistor transform. In fact the action of δ on Dolbeault representatives is clear. If we take the first non-trivial case δ : $H^2(Q^I; \mathcal{O}(-2)) \to H^3(Q^I; \mathcal{O}(-4))$, with $\varphi \in H^2(Q^I; \mathcal{O}(-2))$ we should, abusing notation, redefine φ to be an arbitrary smooth $(0, 2)$-form on a *neighbourhood* of $Z^2 = 0$ that agrees with the originally chosen φ. Then, because the original φ is a cohomology class, we will have $\bar{\partial}\varphi = 0$ on $Z^2 = 0$, so on a neighbourhood we must have

$$\bar{\partial}\varphi = Z^2 \delta\varphi$$

for some $(0, 3)$-form $\delta\varphi$. If we change the extension of φ to $\varphi + Z^2 \Lambda$, then $\delta\varphi \to \delta\varphi + \bar{\partial}\Lambda|_{Z^2=0}$, so the cohomology class of $\delta\varphi$ is well defined.

We first show that for $\varphi \in H^2(\mathcal{O}(-2))$, $\delta\phi \in H^3(\mathcal{O}(-4))$ is precisely the twistor transform of the class that we started with, corresponding to the same solution to the wave equation Φ on space–time. We have that

$$\varphi = \int_{\mathbb{CP}^1} \mathrm{D}u \, \bar{\delta}^3_{-2,-2}(\pi, \lambda \cdot u) \, \mathrm{e}^{-i\omega \cdot \hat{\lambda}}, \qquad \phi = \int_{\mathbb{CP}^1} \mathrm{D}u \, \mathrm{D}\hat{u} \, \bar{\delta}^3_{-4,0}(\pi, \lambda \cdot u) \, \mathrm{e}^{-i\omega \cdot \hat{\lambda}}.$$

The classes clearly extend off $Z^2 = 0$ and, using (8), we see that ϕ is $\bar{\partial}$-closed. There is clearly much freedom in extending these representatives off the quadric as we can add on Z^2 multiplied by any form of the same degree and with homogeneity two lower; however, the form ϕ is not closed away from $Z^2 = 0$. Following the calculation in (8) but not imposing the condition $Z^2 = 0$ we find

$$\bar{\partial}\varphi = -i\omega \cdot \pi \int_{\mathbb{CP}^1} \mathrm{D}u \, \mathrm{D}\hat{u} \, \mathrm{e}^{-i\omega \cdot \hat{\lambda}} \, \bar{\delta}^3(\pi, \lambda) = -iZ^2\phi.$$

Thus $\delta\varphi = -i\phi$ and this gives the non-triviality of the map δ.

Similar calculations can be done with other homogeneities. We shall be particularly interested in twistor representatives corresponding to $k = 0, 1$ and 2 and so we only deal with these cases here.

For $k = 1$ we have representatives $\chi \in H^2(Q^I, \mathcal{O}(-1))$ and $\psi \in H^3(Q^I, \mathcal{O}(-5))$ corresponding to momentum eigenstates of the spin-1/2 field $\Psi_A(x)$ given by

$$\chi = \int_{\mathbb{CP}^1} \mathrm{D}u \, \xi \cdot u \, \bar{\delta}^3_{-1,-3}(\pi, \lambda) \, \mathrm{e}^{-i\omega \cdot \hat{\lambda}}, \qquad \psi = \int_{\mathbb{CP}^1} \mathrm{D}u \, \mathrm{D}\hat{u} \, \xi \cdot \hat{u} \, \bar{\delta}^3_{-5,1}(\pi, \lambda) \, \mathrm{e}^{-i\omega \cdot \hat{\lambda}}.$$

As before we calculate the anti-holomorphic derivative of the H^2 representative

$$\bar{\partial}\chi = -i\omega \cdot \pi \int_{\mathbb{CP}^1} \mathrm{D}u \, \mathrm{D}\hat{u} \, \xi \cdot u \, \bar{\delta}^3(\pi, \lambda) \, \mathrm{e}^{-i\omega \cdot \hat{\lambda}}.$$

All this does not vanish to first order in $Z^2 = \omega \cdot \pi$. However, to first order in Z^2 we must have that this is exact by the abstract arguments above. To see this, we observe that

$$\bar{\partial} \int_{\mathbb{CP}^1} \mathrm{D}u \, \xi \cdot \hat{u} \, \bar{\delta}^3(\pi, \lambda) \, \mathrm{e}^{-i\omega \cdot \hat{\lambda}} = \int_{\mathbb{CP}^1} \mathrm{D}u \, \mathrm{D}\hat{u} \, (\xi \cdot u) \, \bar{\delta}^3(\pi, \lambda) \mathrm{e}^{-i\omega \cdot \hat{\lambda}} - i\omega \cdot \pi \, \psi$$

with the first term arising from $\bar{\partial}\xi \cdot \hat{u}$ and the second from $\bar{\partial}$ of the exponential. Thus, if we redefine

$$\chi = \int_{\mathbb{CP}^1} \mathrm{D}u \, \left(\xi \cdot u \, \bar{\delta}^3_{-1,-3}(\pi, \lambda) + i(\xi \cdot \hat{u}) \, (\omega \cdot \pi) \, \bar{\delta}^3_{-3,-1}(\pi, \lambda) \right) \mathrm{e}^{-i\omega \cdot \hat{\lambda}}$$

we obtain the desired relation

$$\bar{\partial}\chi = -iZ^4\psi.$$

The final case of interest to us is the case corresponding to spin-1 fields, described in twistor space by $b \in H^2(Q^I, \mathcal{O})$ and $h \in H^3(Q^I, \mathcal{O}(-6))$. Following the above strategy, we now see that if we adopt the redefinition

$$b = \int_{\mathbb{CP}^1} \mathrm{D}u \, \mathrm{e}^{-i\omega \cdot \hat{\lambda}} \left(\xi_{ab} u^a u^b \, \bar{\delta}^3_{0,-4}(\pi, \lambda) - \xi_{ab} u^a \hat{u}^b \, \omega \cdot \pi \, \bar{\delta}^3_{-2,-2}(\pi, \lambda) + \xi_{ab} \hat{u}^a \hat{u}^b (\omega \cdot \pi)^2 \bar{\delta}^3_{-4,0}(\pi, \lambda) \right)$$

and

$$h = \int_{\mathbb{CP}^1} \mathrm{D}u \, \mathrm{D}\hat{u} \, \xi_{ab} \hat{u}^a \hat{u}^b \, \bar{\delta}^3_{-6,2}(\pi, \lambda) \, \mathrm{e}^{-\mathrm{i}\omega \cdot \hat{\lambda}},$$

then we will have

$$\bar{\partial} b = Z^6 h \tag{22}$$

as desired. By the abstract arguments of the previous subsection, the final forms of these representatives are unique. In general we have

$$\bar{\partial} f = Z^{2k+2} g \tag{23}$$

where $g(Z) \in H^3(Q^I; (-k-4))$ and $f(Z) \in H^2(Q^I; (k-2))$.

5.2. Integral formulae in the $k-2$ homogeneity case

The above uniquely extended representatives in the $k-2$ homogeneity case, together with their connection to the negative homogeneity case, allow us to define integral formulae. We recall the form of the integral expression for the direct Penrose transform:

$$F_{A_1 \cdots A_k}(x) = \int_{S_x} \mathrm{D}^3\pi \, \pi_{A_1} \cdots \pi_{A_k} \, g(\omega, \pi)$$

where $g(Z) \in H^3(Q^I; (-k-4))$. Combining this with the expression (23), we have

$$F_{A_1 \cdots A_k}(x) = \int_{S_x} \mathrm{D}^3 \pi Z^{-2k-2} \, \pi_{A_1} \cdots \pi_{A_k} \, \bar{\partial} f(\omega, \pi) \tag{24}$$

where $f(Z) \in H^2(Q^I; (k-2))$. This integral formula hides the fact that we have to perform the onerous construction of the canonical extension of f off Q, so its practical use may well be rather limited.

6. $\varXi$ and half-Fourier transforms in split signature

As described in Section 2, in split signature, the components of the twistor Z^α can be taken to be real and twistor space to be the real quadric of signature $(4, 4)$ inside $\mathbb{RP}^7$, $Q = (S^3 \times S^3)/\mathbb{Z}_2$. In this signature, the integral formula for solutions to the massless fields equations

$$F(x)_{A_1 \cdots A_k} = \int_{S_x} \mathrm{D}^3\pi \, \pi_{A_1} \cdots \pi_{A_k} f\left(x^{AB}\pi_B, \pi_A\right), \tag{25}$$

can be taken as an integral now over $S_x = \mathbb{RP}^3$ with f being a straightforward smooth function on Q of weight $-4-k$ rather than as a representative of some H^3 cohomology class. Thus we have the benefit in split signature that Čech and Dolbeault representatives are replaced by real functions on the real slice where all twistor coordinates and space–time coordinates are real. Tree amplitudes generally are rational functions and so extend to split signature real slices. This real approach simplifies matters significantly and allows one to exploit standard tools such as Fourier analysis. Sparling [26,27] referred to the $k = 0$ version of this transform as the $\varXi$-transform, and we will follow his terminology here. We will use Fourier analysis to identify the kernel of this map on twistor space and to extend it to the 'indirect' cases of weight $k-2$.

In four dimensions with split signature the Penrose transform similarly has a non-cohomological analogue, the X-ray transform, that maps functions on the real twistor space, $\mathbb{RP}^3$, to solutions to the massless field equations on space–time by straightforward integration along lines in the real twistor space. This can be combined with the Fourier transform to give a map from functions on twistor space to functions on the light-cone in momentum space. This yields what has become known as the 'half-Fourier transform' [1]. Both the momentum light-cone and twistor space are three-dimensional and the map is a Fourier transform along a natural family of two-dimensional fibres.

In this section we derive the analogue of this construction for six-dimensional space–time with split signature. However, the real momentum space light-cone is five dimensional whereas the real twistor space is six dimensional and we will find that this leads to new features in the correspondence as it can no longer be one to one. For this context, the transforms have been studied by Sparling [26,27] for the wave equation and homogeneities -2 and -4; he referred to them as $\varXi$-transforms. He discovered that the -4 case leads to solutions to the wave equation but the kernel of the map consists of those twistor functions F that are in the image of the conformally invariant wave operator $\square$ on *twistor space*, $f = \square g$ for some g (recall that twistor space, being a quadric, is canonically a conformal manifold in six dimensions). Using triality he was able to follow this around the correspondences between twistor space, primed twistor space and space–time, with them all on an equal footing. Here we will extend this to all other weights, both positive and negative, and see how the conformally invariant powers of the Laplacian of [46,47] play a role in characterizing the twistor data in this case.

Twistor space is a conformal manifold and so admits conformally invariant powers of the ultrahyperbolic wave operator [47]

$$\Box^{k+1} : \mathcal{E}(-k-2) \to \mathcal{E}(-k-4).$$

Here, $\mathcal{E}(k)$ denotes the sheaf of smooth sections over $\mathcal{Q}$ homogeneous of degree k.

We will show:

Proposition 6.1. *The kernel of the Ξ-transform* (25) *for functions on twistor space of weight* $-k-4$ *is the image of*

$$\Box^{k+1} : \Gamma(Q, \mathcal{E}(-k-2)) \to \Gamma(Q, \mathcal{E}(-k-4)).$$

The Ξ-transform therefore gives an isomorphism

$$\Gamma_k(\mathbb{M}) \simeq \Gamma(Q, \mathcal{E}(-k-4))/\{\text{Im}\,\Box^{k+1} : \Gamma(Q, \mathcal{E}(-k-2)) \to \Gamma(Q, \mathcal{E}(-k-4))\}$$

and we will show, as the analogue of the indirect Penrose transform:

Proposition 6.2. *For $k > 0$, there is a one to one correspondence*

$$\{h \in \Gamma(Q, \mathcal{E}(k))|\Box^{k+1}h = 0\} \simeq \Gamma_k(\mathbb{M}).$$

Our main tool will be the six-dimensional analogue of the half-Fourier transform. However, before we embark on that, we remark that the spinor-helicity representatives for cohomology classes that we obtained earlier have totally real analogues that we write down directly here as

$$\phi(\omega, \pi) = \int_{\mathbb{R} \times \mathbb{RP}^1_u} e^{-is\omega \cdot \widehat{\lambda}} \alpha_\xi \delta^4(s\pi - \lambda \cdot u) s^{k+3}\, ds \wedge Du, \tag{26}$$

$$\psi(\omega, \pi) = \int_{\mathbb{R} \times \mathbb{RP}^1_u} e^{-is\omega \cdot \widehat{\lambda}} \xi(u)\, \delta^4(s\pi - \lambda \cdot u) \frac{ds}{s^{k-1}} \wedge Du \tag{27}$$

where now Z, λ_A^a, u_a and s are real, $\hat{u}_a = (u_1, -u_0)/(u_0^2 + u_1^2)$ as before but is real and α_ξ is a smooth function of the u_a of homogeneity $-k-2$ satisfying

$$\int_{\mathbb{RP}^1_u} u_{a_1} \cdots u_{a_k} \alpha_\xi Du = \xi_{a_1 \cdots a_k}.$$

Here, α_ξ can still be thought of as a representative of a class in $H^1(\mathbb{CP}^1, \mathcal{O}(-k-2))$, but now as a Čech cocycle defined on $\mathbb{RP}^1 \subset \mathbb{CP}^1$. It is easily seen that substituting (26) into (25) gives the spinor-helicity momentum eigenstate as expected.

Analogues of these representatives can be found in higher dimensions also as before.

6.1. The Fourier and half-Fourier transforms

We first formulate the Fourier transform from space–time for a field $F(x)_{A_1 \cdots A_k}$. Since it satisfies the massless field equation $\nabla^{AB} F_{B \cdots D} = 0$, the transform to momentum space

$$\widetilde{F}(P)_{A_1 \cdots A_k} = \int d^6x\, F(x)_{A_1 \cdots A_k}\, e^{iP \cdot x}$$

will satisfy $P^{AB}\widetilde{F}_{B \cdots D} = 0$. Thus $\widetilde{F}$ only has support when P^{AB} has rank 2. In particular P is null and F is supported on the momentum light-cone

$$M_0 = \{P^{AB}|P^2 = 0\} = \{P_{AB}|P_{AB} = \varepsilon_{ab}\lambda_A^a \lambda_B^b\}$$

for some λ_A^a defined up to $SL(2, \mathbb{R})$ on the a index. As before, we obtain the spinor-helicity representation

$$\widetilde{F}_{A_1 \cdots A_k} = \widetilde{F}(\lambda)_{a_1 \cdots a_k} \lambda_{A_1}^{a_1} \cdots \lambda_{A_k}^{a_k} \delta(P^2),$$

and define $\widetilde{F}_{a_1 \cdots a_k}$ to be the Fourier transform of the field $F(x)_{A_1 \cdots A_k}$.

Our next task is to establish the following.

Proposition 6.3. *We have the following direct formula for $\widetilde{F}_{a_1 \cdots a_k}$ in terms of the twistor function f that gives rise to it via* (25) *in split signature:*

$$\widetilde{F}(\lambda^a)_{a_1 \cdots a_k} = \int Du\, u_{a_1} \cdots u_{a_k} \int d^4\omega\, \delta(\omega^A \lambda_A) f(\omega^A, \lambda_A)\, e^{-i\widehat{\lambda}_A \omega^A} \tag{28}$$

where $\lambda_A = \lambda_A^a u_a$ where u^a are now homogeneous coordinates on $\mathbb{RP}^1$ and $\widehat{\lambda}_A$ is any spinor chosen such that $\lambda_{[A}\widehat{\lambda}_{B]} = P_{AB}$.

For definiteness we will often choose $\widehat{\lambda}_A = \lambda_A^a \hat{u}_a$ where $\hat{u}_a = (-u_1, u_0)/(u_0^2 + u_1^2)$ but our formulae will be invariant under $\widehat{\lambda} \to \widehat{\lambda} + \alpha\lambda$ for all α.

Proof. In split signature all quantities, $P, x, \lambda, u, \hat{u}, \omega$ and π can be taken real and, at least for homogeneity $-k - 4$, the twistor cohomology class can be replaced by a smooth function f of the real twistor variables (ω^A, π_A) of projective weight $-k - 4$. The direct Penrose transform gives a space–time massless field

$$F(x)_{A_1 \cdots A_k} = \int_{S_x} \mathrm{D}^3\pi \, \pi_{A_1} \cdots \pi_{A_k} f\left(x^{AB}\pi_B, \pi_A\right),$$

where now the integral is over $S_x = \mathbb{RP}^3$. We can Fourier transform this to get a function on momentum space and, substituting in its above form, we must obtain

$$\widetilde{F}(\lambda)_{a_1 \cdots a_k} \lambda_{A_1}^{a_1} \cdots \lambda_{A_k}^{a_k} \delta(P^2) = \int \mathrm{d}^6 x \, \mathrm{D}^3\pi \, \pi_{A_1} \cdots \pi_{A_k} f\left(x^{AB}\pi_B, \pi_A\right) \, \mathrm{e}^{\mathrm{i}P \cdot x}.$$

We now reverse the order of integration performing the x-integral first. We can reparametrize x with ω^A together with a parameter χ_A defined up to the addition of multiples of π by

$$x_{AB} = \varepsilon_{ABCD}\omega^C \alpha^D + \chi_{[A}\pi_{B]}.$$

We can choose α^A such that $\alpha \cdot \pi = 1$ and fix the freedom in χ such that $\alpha \cdot \chi = 0$. This is then consistent with the incidence relation $\omega^A = x^{AB}\pi_B$ and χ_A has three independent components (and projective weight -1). We may now write

$$P \cdot x = 2P_{AB}\omega^A \alpha^B + P^{AB}\chi_A\pi_B.$$

The measure on space–time can then be written as

$$\mathrm{d}^6 x = \mathrm{d}^4\omega \, \mathrm{D}^3\chi \, \delta(\omega^A \pi_A)$$

where $\mathrm{D}^3\chi = \varepsilon^{ABCD}\pi_A \, \mathrm{d}\chi_B \, \mathrm{d}\chi_C \, \mathrm{d}\chi_D (= \mathrm{d}^4\chi \, \delta(\alpha^A\chi_A))$. Thus we have

$$\widetilde{F}(\lambda)_{A_1 \cdots A_k} \delta(P^2) = \int \mathrm{d}^4\omega \, \mathrm{D}^3\chi \, \mathrm{D}^3\pi \, \delta(\omega \cdot \pi) \, \pi_{A_1} \cdots \pi_{A_k} f(\omega^A, \pi_A) \, \mathrm{e}^{2\mathrm{i}P_{AB}\omega^A\alpha^B + \mathrm{i}P^{AB}\chi_A\pi_B}.$$

Performing the χ integration we have[6]

$$\int \mathrm{D}^3\chi \, \mathrm{e}^{\mathrm{i}P^{AB}\chi_A\pi_B} = \delta(P^2) \int_{\mathbb{RP}^1} \mathrm{D}u \, \delta_{-2}^3(\pi_A, \lambda_A^a u_a).$$

On the support of this δ-function all the π_A factors can be replaced by $\lambda_A^a u_a$ and we can now remove the λ_A^a and $\delta(P^2)$ factors from both sides of the equation to obtain

$$\widetilde{F}(\lambda)_{a_1 \cdots a_k} = \int \mathrm{D}u \, u_{a_1} \cdots u_{a_k} \int \mathrm{d}^4\omega \, \mathrm{D}^3\pi \, \delta(\omega^A\pi_A) \, \delta^3(\pi_A; \lambda_A \cdot u) \, f(\omega^A, \pi_A) \, \mathrm{e}^{2\mathrm{i}P_{AB}\omega^A\alpha^B}.$$

Using $\varepsilon_{ab} = (u_a\hat{u}_b - u_b\hat{u}_a)$, on the support of the delta functions, the exponent may be written as

$$2\mathrm{i}P_{AB}\omega^A\alpha^B = -\mathrm{i}\widehat{\lambda}_A\omega^A$$

where $\lambda_A = \lambda_A^a u_a$ and $\widehat{\lambda}_A = 2\lambda_A^a \hat{u}_a$. With this, and using the $\delta^3(\pi_A, \lambda_A)$ to do the $\mathrm{D}^3\pi$ integration, the map from functions on twistor space to scalar fields on the null-cone becomes

$$\widetilde{F}(\lambda^a)_{a_1 \cdots a_k} = \int \mathrm{D}u \, u_{a_1} \cdots u_{a_k} \int \mathrm{d}^4\omega \, \delta(\omega^A\lambda_A) f(\omega^A, \lambda_A) \, \mathrm{e}^{-\mathrm{i}\widehat{\lambda}_A\omega^A} \tag{29}$$

as required. $\square$

This map factors into a *half-Fourier transform* followed by an integral over u (see Fig. 4). The half-Fourier transform part of the map, being a Fourier transform, is necessarily one to one. This takes functions on twistor space Q to functions on an extension of the momentum light-cone M_0 by $\mathbb{RP}^1$ to include the real projective coordinate u^a giving a six-dimensional

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[6] To see this, choose α such that $\alpha \cdot \chi = \chi_1 = 0$ so that $\int \mathrm{D}^3\chi \, \mathrm{e}^{\mathrm{i}\chi_I P^{IA}\pi_A} = \prod_{I=2,3,4} \delta^3(P^{IA}\pi_A)$. On $P_{12} \neq 0$, we can choose $\lambda_1^0 = 1 \, \lambda_2^0 = \lambda_1^1 = 0$ and $\lambda_2^1 = P_{12}$ and we find

$$|P_{12}|\delta(P^{3A}\pi_A)\delta(P^{4A}\pi_A) = \delta^2(\pi_A, \lambda_A^a).$$

Consider now $\delta(P^2)\delta^2(\pi_A; \lambda^+, \lambda^-) = |P_{12}|\delta(P^{12}P^{34} + P^{14}P^{23} + P^{13}P^{42})\delta(P^{3A}\pi_A)\delta(P^{4A}\pi_A)$. On the support of the second and third delta functions, the first can be rewritten and we find $\delta(P^2)\delta^2(\pi_A; \lambda^a) = \delta^3(P^{IA}\pi_A)$ and therefore $\int \mathrm{D}^3\chi \, \mathrm{e}^{\mathrm{i}P^{AB}\chi_A\pi_B} = \delta(P^2)\delta^2(\pi_A, \lambda_A^a)$.
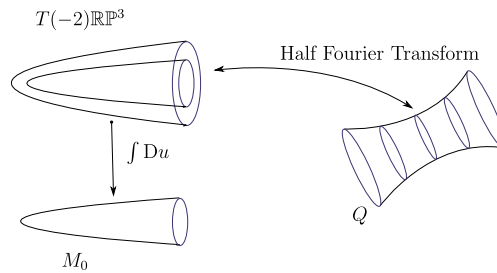
Fig. 4. The half-Fourier transform relates a bundle over M_0 to Q.

auxiliary space with coordinates $\{\lambda_A^a, u^a\}/\{SL(2, \mathbb{R}) \times \mathbb{R}^*\}$ with the $SL(2, \mathbb{R})$ acting on the a-index and the $\mathbb{R}^*$ on u^a. We identify this space with the bundle $T(-2)\mathbb{RP}^3$ of tangent vectors of homogeneity (-2) over $\mathbb{RP}^3$. This follows by rewriting the data (λ_A^a, u^a) as

$$\left(\lambda_A, \widehat{\lambda}_B\right) := \left(\lambda \cdot u, \lambda \cdot \hat{u}\right) \longrightarrow \left(\lambda_A \cdot u, \lambda_A \cdot \hat{u}\frac{\partial}{\partial \lambda_A}\right) \in \mathbb{RP}^3 \times T_\lambda \mathbb{RP}^3$$

where $\widehat{\lambda}$ is only defined up to the addition of multiples of λ but $\lambda \cdot \partial/\partial\lambda$ is zero in $T_\lambda \mathbb{RP}^3$.

Definition 6.1. The half-Fourier transform takes a homogeneous function $f(\omega, \pi)$ of weight $-4 - k$ on twistor space to a homogeneous function $K(\lambda, \widehat{\lambda})$ on $T(-2)\mathbb{RP}^3$ of weight $2 - k$ through

$$K(\lambda, \hat{\lambda}) = \int d^4\omega \, \delta(\omega^A \lambda_A) f(\omega^A, \lambda_A) \, e^{-i\widehat{\lambda}_A \omega^A}. \tag{30}$$

We remark that in this definition, k can be any integer.

The delta function in the integrand implies that $K(\lambda, \widehat{\lambda} + \alpha\lambda) = K(\lambda, \widehat{\lambda})$ for all α. This gives an analogue of the half-Fourier transform familiar from four dimensions. It is invertible with inverse

$$f(\omega^A, \lambda_A) \, \delta(\omega^A \lambda_A) = \frac{1}{(2\pi)^4} \int d^4\widehat{\lambda} \, K(\lambda_A, \widehat{\lambda}_A) \, e^{i\widehat{\lambda}_A \omega^A}$$

where we think of $f(\omega^A, \lambda_A)\delta(\omega^A \lambda_A)$ as a distribution on $\mathbb{RP}^7$ with support on the twistor quadric, the delta function arising because $K(\lambda, \widehat{\lambda}) = K(\lambda, \widehat{\lambda} + r\lambda)$ for all r.

6.2. The extended direct Ξ-transform and half-Fourier transform

Our proposition above therefore gives the Fourier transform of the massless field as the following integral over $\mathbb{RP}^1$ of the half-Fourier transform K of f:

$$\widetilde{F}(\lambda_A^a)_{a_1 \cdots a_k} = \int_{\mathbb{RP}^1} Du \, u_{a_1} \cdots u_{a_k} K(\lambda, \widehat{\lambda}). \tag{31}$$

The combination cannot be one to one since the dimension of twistor space is 6 and the dimension of the momentum null-cone is 5. There must therefore be a kernel of the map (25) from twistor functions to massless fields which we now seek to identify. Clearly, since the half-Fourier transform is 1:1, the kernel is that of the integration in (31), i.e., those K for which the integrand is exact. We will identify this and its counterpart on twistor space. We will also in the process see how to extend this correspondence to the case of homogeneity $k - 2$ which will work dually.

It is clear that the integration (31) has as kernel those K for which the integrand is exact for the $\mathbb{RP}^1$ integration. In order to understand this kernel, we need to review the theory of the $\eth^k$ operators on $\mathbb{RP}^1$. These are operators defined in the first instance on functions on $\mathbb{RP}^1$, for $k \geq 0$,

$$\eth^{k+1} : \mathcal{E}(k) \to \mathcal{E}(-k-2),$$

that are invariant under Mobius transformations. Following [48] we observe that for $f(u)$ of weight k, $\frac{\partial^k f}{\partial u^{a_1} \cdots \partial u^{a_k}}$ has weight 0 and so

$$u^{a_0} \frac{\partial^k f}{\partial u^{a_0} \cdots \partial u^{a_k}} = 0.$$

Thus

$$\frac{\partial^k f}{\partial u^{a_0} \cdots \partial u^{a_k}} = u_{a_0} \cdots u_{a_k} \eth^{k+1} f$$

for some $\eth^{k+1}f$ of weight $-k-2$. This defining equation can be written as

$$d\left(\frac{\partial^k f}{\partial u^{a_1} \cdots \partial u^{a_k}}\right) = u_{a_1} \cdots u_{a_k} \eth^{k+1}f \, Du, \tag{32}$$

thus showing that K is in the kernel of the integral (31) iff $K = \eth^{k+1}J$ for some J.

This gives a complete characterization of the kernel of the integration in (31) where now $\eth^{k+1}$ is understood fibrewise for the fibres of the fibration $T(-2)\mathbb{RP}^3 \to M_0$. In order to understand what this means on twistor space, we first express $\eth$ in terms of the $(\lambda, \widehat{\lambda})$ coordinates on $T(-2)\mathbb{RP}^3$. We first observe that if we are not concerned to preserve manifest projective invariance on $\mathbb{RP}^1$, then we can define $\eth^{k+1} = (\eth)^{k+1}$ where on a function f of weight k we define the action of a single $\eth : \mathcal{E}(k) \to \mathcal{E}(k-2)$ for all integral values of k by

$$\eth f = (u_0^2 + u_1^2)^{k/2} \eth\left((u_0^2 + u_1^2)^{-k/2}f\right).$$

It is now possible to see that, with this definition, $(\eth)^{k+1}$ is independent of the choice of quadratic form (here $(u_0^2 + u_1^2)$) and reduces to the invariant form given above. As an operator on $K(\lambda, \widehat{\lambda})$ with $K(\lambda, \widehat{\lambda} + \alpha\lambda)$ we find, using the chain rule, together with $\eth u_a = \hat{u}_a$ and $\eth \hat{u}_a = u_a$, that

$$\eth K(\lambda, \widehat{\lambda}) = \widehat{\lambda} \cdot \frac{\partial K}{\partial \lambda}.$$

It is now straightforward to see that, since under the half-Fourier transform, $\widehat{\lambda} \leftrightarrow \partial/\partial\omega^A$, we have

$$\eth K \leftrightarrow \frac{\partial^2 f}{\partial\omega^A \partial\pi_A}.$$

Thus $\eth$ corresponds to the ultrahyperbolic wave operator $\square$ on the non-projective $\mathbb{R}^8$ associated with the $\mathbb{RP}^7$ in which Q lives. Homogeneous functions on $\mathbb{RP}^7$ do simply correspond to ordinary functions on this $\mathbb{R}^8$, but this is in general not well defined on functions on Q as the derivative off the quadric Q is not given. However, for weights $k \geq 0$ the following operators are well defined on homogeneous functions on Q:

$$\square^{k+1} : \mathcal{E}(k-2) \to \mathcal{E}(-k-4)$$

as defined in precisely this context in [47] (see also [46,49] for antecedents). Thus we have:

Proposition 6.4. *The kernel of the Ξ-transform (31) for functions on twistor space of weight $-k-4$ is the image of*

$$\square^{k+1} : \Gamma(Q, \mathcal{E}(-k-2)) \to \Gamma(Q, \mathcal{E}(-k-4)).$$

The Ξ-transform therefore gives an isomorphism

$$\Gamma_k(\mathbb{M}) \simeq \Gamma(Q, \mathcal{E}(-k-4))/\{\text{Im}\,\square^{k+1} : \Gamma(Q, \mathcal{E}(-k-2)) \to \Gamma(Q, \mathcal{E}(-k-4))\}.$$

6.3. The indirect Ξ-transform

The question remains as to how the Ξ-transform works for homogeneity $k - 2$. The half-Fourier transform is an isomorphism from functions h of weight $k - 2$ on twistor space to functions J of weight k on $T(-2)\mathbb{RP}^3$. The map from a momentum space representative $\widetilde{F}(\lambda_A^a)_{a_1 \cdots a_k}$ to such a function J of weight k is clear: it should be

$$J = \widetilde{F}(\lambda_A^a)_{a_1 \cdots a_k} u^{a_1} \cdots u^{a_k}.$$

This is inverted by setting

$$\widetilde{F}(\lambda_A^a)_{a_1 \cdots a_k} = \frac{\partial^k}{\partial u^{a_1} \cdots \partial u^{a_k}} J(\lambda, \widehat{\lambda})$$

but if J has been chosen arbitrarily, this will only give a sensible momentum space representative if the right hand side is independent of u_a. Using (32) we see that this will follow iff $\eth^{k+1}J = 0$. Using the above argument, we can see that this is equivalent for the function h on twistor space to the vanishing of the conformally invariant power of the ultrahyperbolic wave equation $\square^{k+1}h = 0$. Thus we have the positive chirality Ξ-transform:

Proposition 6.5. *For $k > 0$, there is a one to one correspondence*

$$\Gamma_k(\mathbb{M}) \simeq \{h \in \Gamma(Q, \mathcal{E}(k)) | \square^{k+1}h = 0\}.$$

Thus the Ξ-transform maps solutions of differential equations on one space to solutions on another in this case. We remark also that in split signature we have triality, so twistor space, primed twistor space and (compactified) space–time are all on an equivalent footing and the Ξ-transform applies between any two of these three spaces in either direction.

We finally note the integral formula in this case:

$$\phi_{A_1 \cdots A_k}(x) = \int_{\gamma \subset S_x} \frac{\partial^{k+1} h}{\partial \omega^{A_1} \cdots \partial \omega^{A_k} \partial \omega^{A_{k+1}}} \varepsilon^{A_{k+1} BCD} \pi_B \, \mathrm{d}\pi_C \, \mathrm{d}\pi_D, \tag{33}$$

where the integral is over some two-dimensional contour γ cohomologous to $\mathbb{RP}^2 \subset S_x = \mathbb{RP}^3$. In order to make sense of the $\partial/\partial\omega^A$ derivatives, h must be extended off Q to kth order in $\mathbb{RP}^7$; the $k+1$th derivative is skewed with π_A and so is acting only tangent to Q. This is precisely what is done in [47] with the extension determined by the condition that $\square_{\mathbb{R}^8} h = 0$ in the ambient non-projective space to the appropriate order; the operator $\square_Q^{k+1}$ on h is obtained in [47] as the obstruction to extending h as a solution to $\square_{\mathbb{R}^8} h = 0$ at $k + 1$th order, although there is in any case some ambiguity at that order. The result is independent of the chosen contour by virtue of $\square^{k+1} h = 0$ which in particular implies that the kth-order extension is annihilated by $\square_{\mathbb{R}^8}$. We remark that it is not sufficient to check this with the spinor-helicity representative (27) except for the $k = 0$ case, as for general k (27) has not been extended appropriately off Q into $\mathbb{RP}^7$ and formulae more analogous to those used in the formal neighbourhood discussion are required.

7. Interactions and scattering amplitudes

There has been considerable work on the spinor-helicity construction of scattering amplitudes for $(1, 1)$ super Yang–Mills theory in six dimensions [13,50–54]. The invariances of the theory appear to uniquely fix the form of the three-photon amplitude and higher-point amplitudes have been derived using the BCFW construction [13].[7] Despite a clear demonstration that spinor-helicity methods are useful in six dimensions, there is little understanding of how to construct scattering amplitudes from six-dimensional twistor space. We shall return to the question of how to describe non-chiral theories such as the Yang–Mills one in future work. In this article we are chiefly concerned with conformally invariant and chiral theories. Because the amplitudes for the $(0, 2)$-theory are believed to be trivial, we are rather limited as to the amplitudes that we can consider and we focus on the Φ^3 vertex. (The only fields of spin less than 1 that we can consider are scalars and spin-1/2 fermions and it does not seem possible to construct an interaction involving spin-1/2 fields without introducing a spin-1/2 field $\Psi^A(x)$ of opposite chirality.)

We can write down a classically conformally invariant Lagrangian for an interacting scalar field:

$$\mathcal{L} = \frac{1}{2} \partial_\mu \Phi \, \partial^\mu \Phi + \frac{\kappa}{6} \, \Phi^3 \tag{34}$$

where κ is a dimensionless coupling constant. On applying the Fourier transform, the three-scalar amplitude corresponding to the Φ^3 vertex takes the simple form

$$\mathcal{A}(P_1, P_2, P_3) = \frac{\kappa}{6} \, \delta^6 \left(\sum_{i=1}^{3} P_{AB}^i \right).$$

A natural candidate for the three-point scalar amplitude in terms of indirect Penrose transform functions φ_i, $i = 1, 2, 3$, of weight -2 is

$$\mathcal{A}(P_1, P_2, P_3) = \int_{\mathbb{RP}^7} \mathrm{D}^7 Z \, \delta(Z \cdot Z) \, \varphi_1 \varphi_2 \varphi_3.$$

We insert the twistor representatives for momentum eigenstates:

$$\varphi_i(\omega, \pi) = \frac{1}{2} \int \mathrm{D} u_i \, \mathrm{d} k_i \, k_i \, \delta^4(\pi_A - k_i \lambda_{iA}) \, \mathrm{e}^{-\frac{i}{k} \widehat{\lambda}_{iA} \omega^A}$$

where a factor of 1/2 has been introduced for later convenience.

The scattering amplitude is then

$$\mathcal{A}(P_1, P_2, P_3) = \frac{1}{8} \int \mathrm{D}^3 \pi \, \mathrm{d}^4 \omega \, \delta(\omega \cdot \pi) \left(\prod_{i=1}^{3} \int \mathrm{D} u_i \, \mathrm{d} k_i \, k_i \, \delta^4(\pi_A - k_i \lambda_{iA}) \right) \exp \left(-\mathrm{i} \sum_{i=1}^{3} \frac{\omega^A \widehat{\lambda}_{iA}}{k_i} \right).$$

Writing the quadric delta function as an integral,

$$\delta(\omega \cdot \pi) = \int \mathrm{d} t \, \mathrm{e}^{\mathrm{i} t \, \omega \cdot \pi},$$

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[7] Loop amplitudes have also been constructed by unitarity methods in [50,52].

and then doing the four ω^A integrations, gives

$$\mathcal{A}(P_1, P_2, P_3) = \frac{1}{8} \int D^3\pi \; dt \left(\prod_{i=1}^{3} \int Du_i \; dk_i \; k_i \; \delta^4(\pi_A - k_i \lambda_{iA}) \right) \delta^4 \left(t\,\pi_A - \sum_{i=1}^{3} \frac{\widehat{\lambda}_{iA}}{k_i} \right).$$

We can already see that this expression has the support of the 6-momentum conserving delta functions. Consider the skew product of the argument of the delta function $\delta^4\left(t\pi_A - \sum_i \frac{1}{k_i}\widehat{\lambda}_{iA}\right)$ with π:

$$t\pi_{[A}\pi_{B]} - \sum_i \frac{1}{k_i} \pi_{[A|}\widehat{\lambda}_{i|B]} = 0.$$

The first term vanishes and on the support of the $\delta^4(\pi_A - k_i\lambda_{iA})$, and the second term may be written as

$$0 = \sum_i \lambda_{i[A|}\widehat{\lambda}_{i|B]} = \sum_i P_{AB}^i,$$

which gives the expected momentum conservation. Thus

$$\mathcal{A}(P_1, P_2, P_3) = \mathcal{K}\delta^6 \left(\sum_{i=1}^{3} P_{AB}^{(i)} \right) \tag{35}$$

for some $\mathcal{K}$. Since there are no Lorentz invariants of three null momenta that add up to zero, we see that $\mathcal{K}$ must be constant and so we have the correct amplitude. A more laborious argument can be used to obtain $\mathcal{K}$ explicitly.

8. Discussion: conformal theories in twistor space

One of the triumphs of the twistor programme was the elegant description of self-dual Yang–Mills theory in four dimensions in terms of the Penrose–Ward correspondence [55]. This correspondence relates a holomorphic vector bundle on projective twistor space without connection to a holomorphic bundle over space–time with self-dual connection. In six dimensions, a connection on a fibre bundle cannot be self-dual; however, a gerbe can have self-dual connection. In this section we shall be particularly interested in six-dimensional linearized physical theories in which self-dual gerbes play a role. These objects play an important role in string theory and it is doubtful that a full understanding of *M*-theory will be possible without at least a partial understanding of the conjectured non-linear versions of such theories.

8.1. Self-dual gerbes in twistor space

An abelian gerbe on space–time is usually thought of as a generalization of a connection in which the connection 1-form is replaced by a 2-form, B, defined modulo the addition of the exterior derivative of a 1-form, and so the 'curvature' dB is now a 3-form. On twistor space, we have the two descriptions: as $b \in H^2(\mathcal{O})$ and $g \in H^3(\mathcal{O}(-6))$, both as forms modulo $\bar{\partial}$-exact forms. Since these are both in a potential modulo gauge format, the form degree of the latter does not naturally fit the concept of the gerbe, but the H^2 case does. Indeed this case has already been studied as a route to defining a holomorphic gerbe on twistor space by Hitchin [56] and Chatterjee [11] who present the theory as a generalization of the Čech description of line bundles; this is outlined in the Appendix for Čech cohomology along with the Penrose transform in the indirect case leading to a potential modulo gauge description for the space–time field.

The Dolbeault route gives a picture slightly different to that arising from the Čech approach. In the Dolbeault picture, we can think of the $b \in H^2(\mathcal{O})$ as defining a class of local $\bar{\partial}$-operators $\bar{\partial}_{a_i}$ on a fixed complex line bundles over a covering U_i of Q^I for which $\bar{\partial}_{a_i}^2 = b$. In the simplest case these can be $\bar{\partial}$-operators on the same trivial line bundle and b is the obstruction to extending the local $\bar{\partial}_{a_i}$ to a global $\bar{\partial}$-operator.

It is straightforward to write down action principles for the linear theories on twistor space. We first define the holomorphic volume form $\Omega \in \Gamma(Q, \Omega^6(6))$ by

$$\int_Q (\cdot)\Omega = \int_{\mathbb{CP}^7} (\cdot) \, D^7 Z \, \bar{\delta}^1(Z^2).$$

In Euclidean signature, for $\mathbb{M}^I = $ Euclidean $\mathbb{R}^6$, the S_x sweep out Q^I. Then an action for a pair $(g, f) \in \Omega^{0,2}(k-2) \times \Omega^{0,3}(-k-4)$ to define Dolbeault cohomology classes is

$$S[g, f] = \int_Q g \wedge \bar{\partial}f \wedge \Omega,$$

since solutions to field equations mod gauge give

$$([g], [f]) \in H^2(\mathcal{O}(k-2)) \times H^3(\mathcal{O}(-k-4)) = \Gamma_k \times \Gamma_k.$$

For $k = 2$ this is an action for a pair of self-dual gerbe fields and for $k = 4$ it gives an action for a pair of the spin-2 fields discussed below. Such an action for the $k = 2$ case is echoed on space–time by the action

$$S(H, B) = \int_{\mathbb{M}^l} H \wedge \mathrm{d}B, \quad (H, B) \in \Omega^{3+} \times \Omega^2,$$

since self-dual 3-forms annihilate self-dual 3-forms under the wedge product in six dimensions. Thus it is not so surprising.

In Lorentz signature we can write action formulations that do not seem to have a space–time analogue. Recall that we have in Lorentz signature the quaternionic reality structure $\pi_A \to \hat{\pi}_A$ etc, $Z \to \hat{Z}$ with $\hat{\hat{Z}} = -Z$. In this signature, the S_x that are invariant under the quaternionic conjugation only sweep out the real codimension-1 set

$$Q_0 = \{Z \in Q | Z \cdot \hat{Z} = 0\}.$$

If we now choose data $(g, \tilde{g}) \in \Omega^{0,2}(k - 2) \times \Omega^{0,2}(-k - 4)$ we obtain

$$S[g, \tilde{g}] = \int_{Q_0} \tilde{g} \wedge \bar{\partial} g \wedge \Omega.$$

The field equations lead to a pair of cohomology classes but now

$$([g], [\tilde{g}]) \in H^2(Q_0, \mathcal{O}(k - 2)) \times H^2(Q_0, \mathcal{O}(-k - 4)) = \Gamma_k \times 0.$$

Because of the vanishing of H^2s for sufficiently negative homogeneity, we only obtain one helicity-$k/2$ field. Thus an action corresponding to a linear self-dual gerbe theory on Minkowski space is given by

$$S[b] = \int_{Q_0} \mathrm{D}^7 Z \wedge \bar{\delta}(Z^2) \wedge \bar{\delta}(Z \cdot \hat{Z}) \wedge \tilde{b} \wedge \bar{\partial} b \tag{36}$$

for some twistor field $\tilde{b}$ of homogeneity -4. The equations of motion imply that $b \in H^2(Q; \mathcal{O})$ as it should be, and $\tilde{b} \in H^2(Q; \mathcal{O}(-4))$; however, $H^2(Q; \mathcal{O}(-4)) = 0$ and so $\tilde{b}$ has no on-shell degrees of freedom and acts simply as a Lagrange multiplier that vanishes on the shell, constraining b to lie in $H^2(Q; \mathcal{O})$.

The scattering amplitude for the scalar theory in split signature suggests that there is a twistor action for the space–time Lagrangian (34), at least in Euclidean and split signatures. However, it is difficult to make this work coherently. One is tempted to write

$$S[\varphi, \phi] = \int_{\mathbb{CP}^7} \mathrm{D}^7 Z \wedge \bar{\delta}(Z^2) \wedge \left(\phi \wedge \bar{\partial} \varphi + \frac{1}{6} \varphi \wedge \varphi \wedge \varphi \right) \tag{37}$$

where $\bar{\delta}(Z^2)$ is a $(0, 1)$-form of weight -2 and $\mathrm{D}^7 Z = \varepsilon_{\alpha_0 \alpha_1 \cdots \alpha_7} Z^{\alpha_0} \partial Z^{\alpha_1} \wedge \cdots \wedge \partial Z^{\alpha_7} = \mathrm{D}^3 \pi \, \mathrm{d}^4 \omega$ is the natural projective $(7, 0)$-form on $\mathbb{CP}^7$ of weight $+8$. Here φ is a $(0, 2)$-form of weight -2 and ϕ is a $(0, 3)$-form of weight -4. The form degrees and weights match and so this action makes sense as an action functional. In split signature, the interaction term gives the expression for the three-point amplitude (35). This action is well defined on the quadric Q and, under variation with respect to the representatives, gives

$$\bar{\partial} \varphi = 0, \qquad \bar{\partial} \phi + \frac{1}{2} \varphi \wedge \varphi = 0.$$

So φ corresponds simply to a solution of the wave equation whereas ϕ seems likely to correspond to a solution to an imhomogeneous wave equation sourced by φ^2.

It therefore remains difficult to piece these together to give an action for the Φ^3 theory on twistor space. The Lorentzian formulation seems to be no better as it is not clear how to encode the cubic interaction as a $(0, 5)$-form.

8.2. Non-geometric gravitational theories

Up to now we have been concerned with spin-1 gerbes; here we extend our considerations to spin-2 fields. The on-shell graviton is given by a field strength Ψ_{AB}^{CD} with spinor-helicity polarization data $\Psi_{a'b'}^{ab}$, having nine degrees of freedom; however, the spin-2 field strength G_{abcd} arising from the direct Penrose transform

$$G_{ABCD} = \oint_{S_x} \mathrm{D}^3 \pi \, \pi_A \pi_B \pi_C \pi_D \, g(\omega, \pi) \tag{38}$$

has five on-shell degrees of freedom. Furthermore G_{abcd} is a *chiral* field, whereas the graviton is not. This spin-2 field appearing from twistor space is clearly not describing linearized Einstein gravity, but a more exotic six-dimensional relative. It is conjectured that there exists a superconformal $(4, 0)$ theory in six dimensions [32,57,58,33] which includes just such a field; however, we can consider this field in a bosonic context. A discussion of the supersymmetric theory will be presented in [44]. The novelty of this theory is that the spin-2 field is not a graviton in the conventional sense and is not thought

to give rise to a conventional, geometric, theory of gravitation. Rather, the spin-2 field is given by a tensor $C_{\mu\nu\lambda\rho}$ with the symmetries of the Riemann tensor and field strength

$$G_{\mu\nu\lambda\rho\sigma\eta} = 3\partial_\mu\partial_{[\nu}C_{\lambda\rho]\sigma\eta} + 3\partial_\eta\partial_{[\nu}C_{\lambda\rho]\mu\sigma} + 3\partial_\sigma\partial_{[\nu}C_{\lambda\rho]\eta\mu}$$

which is self-dual:

$$G_{\mu\nu\lambda\rho\sigma\eta} = \frac{1}{3!}\varepsilon_{\rho\sigma\eta}{}^{\kappa\xi\zeta}G_{\mu\nu\lambda\kappa\xi\zeta}.$$

In terms of spinor notation, the field can be encoded into a potential field C_{AB}^{CD} symmetric in each pair of indices with (linearized) manifestly self-dual field strength

$$G_{ABCD} = \nabla_{(A|M}\nabla_{|B|N}C_{|CD)}{}^{MN}.$$

This field, with five on-shell 'gravi-gerbe' degrees of freedom, is the highest spin member of the $(4, 0)$ multiplet. At the linearized level, a dimensional reduction on a circle to five dimensions yields the linearized form of the conventional Einstein maximal supergravity in five dimensions and it is conjectured that there exists a non-linear $(4, 0)$ theory in six dimensions which gives rise to the full Einstein supergravity in five dimensions [32]. It is not clear what the full non-linear $(4, 0)$ theory should look like but it is expected that the interactions will not be of a conventional field-theoretic type but rather should be based on yet to be identified M-theoretic principles. In this section we consider only the linearized form of the $(4, 0)$ theory in supertwistor space. The spin-2 field may be described in terms of the conventional Penrose transform (38) where $g(\omega, \pi) \in H^3(Q^I; \mathcal{O}(-8))$ and it is straightforward to generalize the arguments above for the spin-1 gerbe to a representative of $H^2(Q^I; \mathcal{O}(+2))$ to get a description of this field in terms of a potential C_{ABCD}, modulo gauge invariance, from the indirect Penrose transform.

In a subsequent paper we will turn to the supersymmetric formulations, non-conformally invariant and non-chiral theories, and reductions to lower dimensions.

Acknowledgements

Appendix. The indirect Penrose transform for a self-dual gerbe

It is perhaps simplest to understand what a holomorphic gerbe is by comparing its definition with that of a holomorphic line bundle [56]. A line bundle may be understood in terms of a set of transition functions between open sets $g_{ij} : U_i \cap U_j \to S^1$ with $g_{ij} = g_{ji}^{-1}$ and a triviality condition on the overlap of three open sets: $g_{ij}g_{jk}g_{ki} = 1$ on $U_i \cap U_j \cap U_k$. The bundle is holomorphic if the g_{ij} are holomorphic functions. By contrast a gerbe is defined by functions on a *triple* intersection

$$b_{ijk} : U_i \cap U_j \cap U_k \to S^1,$$

with $b_{ijk} = b_{jik}^{-1} = b_{ikj}^{-1} = b_{kji}^{-1}$ and the triviality condition on the overlap of four open sets

$$b_{jkl}b_{ikl}^{-1}b_{ijl}b_{ijk}^{-1} = 1, \quad \text{on } U_i \cap U_j \cap U_k \cap U_l.$$

Crucially a gerbe, unlike a fibre bundle, is *not* a manifold.[8] A *holomorphic gerbe* is one for which the b_{ijk} are holomorphic functions.

We can also do differential geometry on gerbes. We can define a connection on a line bundle by $A_i - A_j = g_{ij}^{-1}\, dg_{ij}$ and a field strength, defined over the whole bundle, $F = dA_i = dA_j$. On a gerbe we may define a connection in a similar way:

$$H = dB_i = dB_j, \qquad B_i - B_j = dA_{ij}, \qquad A_{ij} + A_{jk} + A_{ki} = b_{ijk}^{-1}\, db_{ijk},$$

where B_i is the connection[9] and H is a globally defined closed 3-form field strength. A connection on a gerbe is (anti-)self-dual if $H = \pm * H$. The six-dimensional analogue of the Penrose–Ward correspondence, which we sketch below, relates cohomology classes $[b] = \{b_{ijk}\}$ on twistor space to self-dual connections B on space–time.

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[8] An equivalent way to define a gerbe is in terms of transition *line bundles* on $U_i \cap U_j$, as opposed to the transition *functions* that define a line bundle.

[9] Following the physics literature, we shall also refer to B_i as 'the gerbe'.

The simplest twistor representation for such a self-dual gerbe is via the direct Penrose transform given by

$$H_{AB}(x) = \int_{S_x} D^3\pi \ \pi_A\pi_B \ h(\omega, \pi),$$

where $h \in \check{H}^3(Q^I; \mathcal{O}(-6))$ and the integral is taken over the $\mathbb{CP}^3$ picked out by the incidence relation. However, this represents the gerbe as a $(0, 3)$-form potential modulo gauge which is not appropriate for defining a gerbe on twistor space.

The most natural association between twistor cohomology and connections of a gerbe in space–time comes from representatives of $H^2(Q^I, \mathcal{O})$ or more geometrically its exponentiation $H^2(Q^I; \mathcal{O}^*)$ given by the indirect Penrose transform. This defines a holomorphic gerbe on twistor space. Following [11], this can be understood via Čech coholomogy: let $[b]$ be a representative of $\check{H}^2(Q^I; \mathcal{O}^*)$ and $\{U_i\}$ a Leray cover[10] of Q. We then have a family of functions of homogeneity degree zero, $[b] = \{b_{ijk}\}$, defined on the triple intersection:

$$b_{ijk} : U_i \cap U_j \cap U_k \to \mathbb{C}^*, \quad b_{ijk}b_{jkl}b_{kli}b_{lij} = 1.$$

Here b_{ijk} is a cohomology representative and is defined on Q and may be lifted to the correspondence space $\mathbb{F} = \{(x, [\pi]) \in \mathbb{M} \times \mathbb{CP}^3\}$ where, by virtue of being pulled back from twistor space, it satisfies

$$\mu^*\left(\pi_B\nabla^{AB}b_{ijk}\right) = 0.$$

Restricting to $S_x = \mathbb{CP}^3$, $\check{H}^2(S_x; \mathcal{O}) = 0$ and so we can write

$$b_{ijk} = a_{ij}a_{jk}a_{ki}. \tag{39}$$

Here $a_{ij} = a_{ij}(x, \pi)$ is not pulled back from twistor space (assuming $[b]$ was not trivial) and so $\pi_B\nabla^{AB}a_{ij} \neq 0$. We can define $a_{ij}{}^A = \pi_B\nabla^{AB}\log a_{ij}$, and differentiating (39) we obtain

$$a_{ij}{}^A + a_{jk}{}^A + a_{ki}{}^A = 0.$$

It is also the case that $\check{H}^1(S_x; \mathcal{O}) = 0$ and so

$$a_{ij}{}^A = f_i^A - f_j^A. \tag{40}$$

Furthermore, since partial derivatives commute, $\pi_C\nabla^{C[A}a_{ij}^{B]} = 0$. Thus, differentiating (40) we obtain

$$\pi_C\nabla^{C[A}f_i^{B]} = \pi_C\nabla^{C[A}f_j^{B]} = S^{AB}.$$

The left and right hand sides of the first equality are defined on different patches (U_i and U_j) but are equal. From this we infer the existence of the globally defined field s^{AB} which is homogeneous of degree 2 in π_A. We can therefore express the π dependence explicitly as $s^{AB} = s^{ABCD}\pi_C\pi_D$. We note that skew symmetry implies that $\pi_A\pi_B\nabla^{AB} = 0$, so $\pi_A a_{ij}^A = 0 = \pi_A f_{ij}^A$ and $\pi_A S^{AB} = 0$. Thus s^{ABCD} can be expressed in terms of some B_A^B by

$$s^{ABCD} = \frac{1}{2}\varepsilon^{ABE(C}B_E^{D)}.$$

This potential can be taken to be traceless, corresponding to a 2-form, and is defined modulo gauge:

$$B_A^B \sim B_A^B + \nabla_{AC}A^{BC} - \frac{1}{4}\delta_A^B\nabla \cdot A$$

for a 1-form $A^{BC} = -A^{CB}$ because f_i^A was defined up to $f_i^A \to f_i^A + \varepsilon^{ABCD}\pi_B A_{CD}$. B_A^B can therefore be interpreted as a gerbe connection on space–time. We have that $\pi_A\nabla^{A[B}S^{CD]} = 0$ from its definition, and this gives the field equation

$$\nabla^{A(B}B_A^{C)} = 0.$$

The gauge-invariant field strength for the gerbe is therefore

$$H_{AB} = \nabla_{(A|C}B_{|B)}^C,$$

corresponding to a self-dual 3-form which is closed, being the exterior derivative of the 2-form corresponding to B.

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