Wilson Coefficients for Flavour-Changing Processes

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1. **Effective theory for flavour-changing processes**

2. **Classification of interactions and coupling constants**
   [Wilson Coefficients (WC)]

3. **Calculational methods**

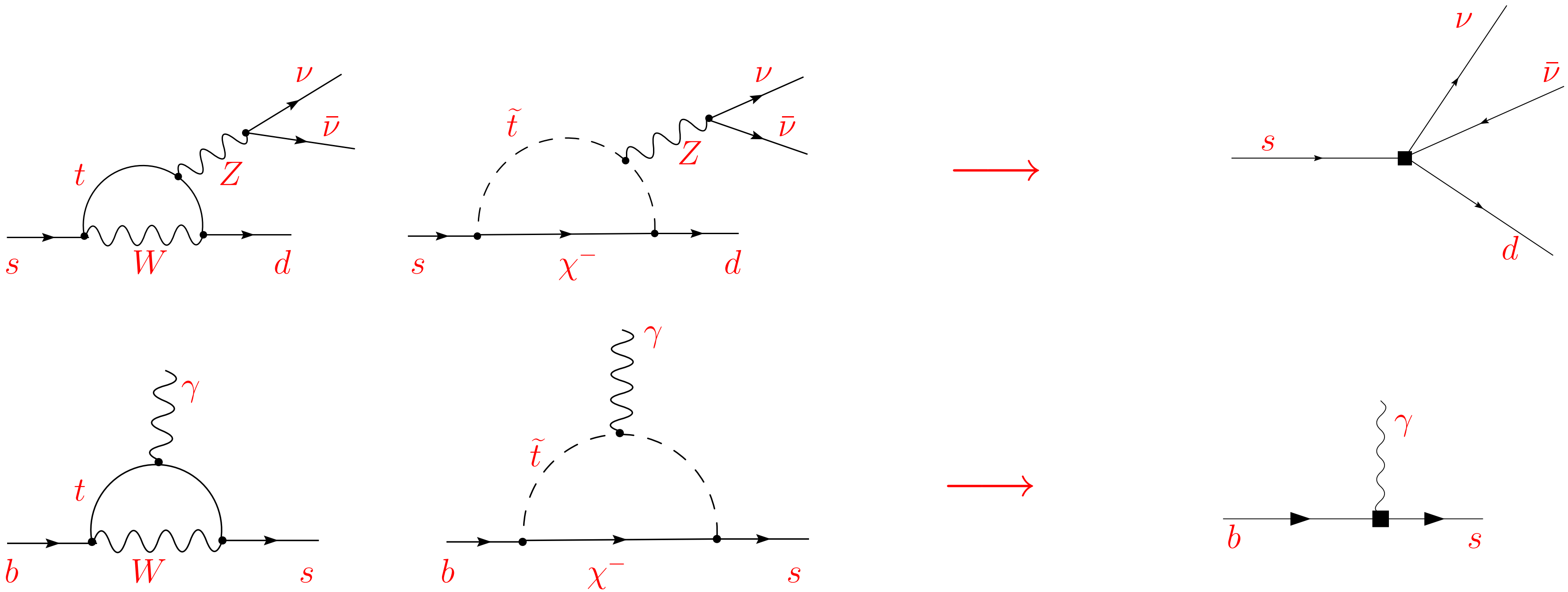
4. **Current status of the WC evaluation**

Flavour-changing processes that we are interested in at the LHCb occur at low energies, at scales $\mu \ll M_W$. It is then convenient to pass from the full theory of electroweak interactions to an effective theory by removing the high-energy degrees of freedom, i.e. integrating out the W-boson and all the other particles with $m \sim M_W$.

$$\mathcal{L}_{\text{(full EW}\times\text{QCD)}} \longrightarrow \mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{QED}\times\text{QCD}}\left(\begin{smallmatrix}\text{quarks} \neq t \\ \& \text{ leptons}\end{smallmatrix}\right) + \sum_n C_n(\mu)\, Q_n$$

Q_n **– local interaction terms (operators),** C_n **– coupling constants (Wilson coefficients)**

Information on the electroweak-scale physics is encoded in the values of $C_i(\mu)$ **, e.g.,**



This is a modern version of the Fermi theory for weak interactions. It is "nonrenormalizable" in the traditional sense but actually renormalizable. It is also predictive because all the C_i are calculable, and only a finite number of them is necessary at each given order in the (external momenta)/M_W expansion.

Advantages: **Resummation of** $\left(\alpha_s \ln \frac{M_W^2}{\mu^2}\right)^n$ **using renormalization group, easier account for symmetries.**

The leading flavour-changing effects originate from the lowest dimension operators Q_i, i.e. dim-5 and dim-6 ones. Higher-dimensional ones are suppressed by powers of μ^2/M_W^2, where $\mu \lesssim m_b$, which corresponds to permile-level corrections – an accuracy hardly accessible in flavour physics.

Quark-flavour-violating dim-5 and dim-6 operators:

$\left(\bar{q}_1\,\sigma^{\alpha\beta} P_{L,R}\, q_2\right) F_{\alpha\beta}, \qquad \left(\bar{q}_1\,\sigma^{\alpha\beta} P_{L,R}\, T^a q_2\right) G^a_{\alpha\beta}$ – dipole-type, the only dim-5 ones, chirality-suppressed,

$\left(\bar{q}_1\,\gamma^\alpha P_L\, q_2\right)\left(\bar{l}\,\gamma_\alpha \nu\right)$ – charged-current quark-lepton,

$\left(\bar{q}_1\,\gamma^\alpha P_L\, q_2\right)\left(\bar{l}\,\gamma_\alpha P_{L,R}\, l\right), \quad \left(\bar{q}_1\,\gamma^\alpha P_L\, q_2\right)\left(\bar{\nu}\,\gamma_\alpha \nu\right)$ – neutral-current quark-lepton,

$\left(\bar{q}_1\,\gamma^\alpha P_L\, q_2\right)\left(\bar{q}_3\,\gamma^\alpha P_L\, q_4\right), \quad \left(\bar{q}_1\,\gamma^\alpha P_L\, T^a q_2\right)\left(\bar{q}_3\,\gamma^\alpha P_L\, T^a q_4\right)$ – charged-current four-quark,

$\left(\bar{q}_1\,\gamma^\alpha P_L\, q_2\right)\left(\bar{q}\,\gamma^\alpha P_{L,R}\, q\right), \quad \left(\bar{q}_1\,\gamma^\alpha P_L\, T^a q_2\right)\left(\bar{q}\,\gamma^\alpha P_{L,R}\, T^a q\right)$ – neutral-current four-quark $\Delta F = 1$,

$\left(\bar{q}_1\,\gamma^\alpha P_L\, q_2\right)\left(\bar{q}_1\,\gamma^\alpha P_L\, q_2\right)$ – neutral-current four-quark $\Delta F = 2$,

We have not listed:

- operators with four down-type quarks of three different flavours, like $\left(\bar{s}\gamma^\alpha P_L b\right)\left(\bar{s}\gamma^\alpha P_L d\right)$,
- four-fermion operators that get chirality-suppressed in the SM,
- lepton-flavour-violating four-fermion operators

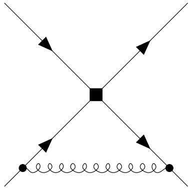
Our ability to observe or constrain new physics depends on the accuracy of determining the SM "background". Thus, precise evaluation of $C_i(\mu)$ in the SM is particularly important.

Two steps of the WC calculation:

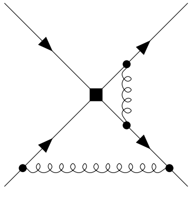
Matching: Evaluating $C_i(\mu_0)$ at $\mu_0 \sim M_W$ by requiring equality of the SM and the effective theory Green's functions.

Mixing: Deriving the effective theory Renormalization Group Equations (RGE) from the renormalization constant matrices (the operators mix under renormalization).
Next, using the RGE to evolve C_i from μ_0 to μ ∼(external momenta).

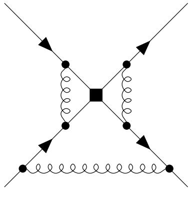
Example — Four-quark charged-current operator mixing:



Gaillard, Lee, 1974,
Altarelli, Maiani, 1974

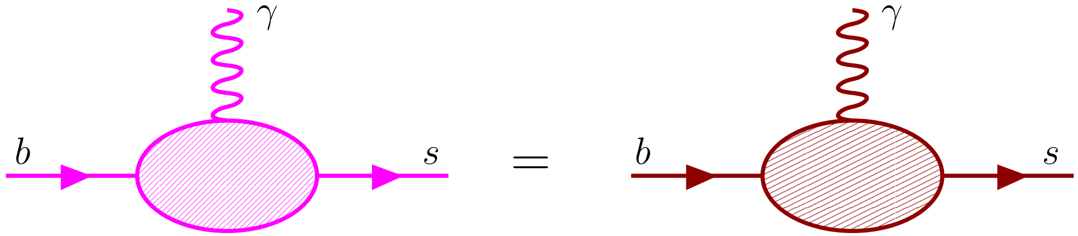


Altarelli et al, 1981,
Buras, Weisz, 1990



Gorbahn, Haisch, 2004

The matching conditions are most easily found by requiring equality of the full SM and the effective theory 1PI off-shell Green's functions that are expanded in external momenta and light masses prior to loop-momentum integration.



$$+ \mathcal{O}\left(\frac{1}{M_W^4}\right)$$

Full EW theory | **Effective Theory**

UV counterterms included | **Loop diagrams vanish**

Spurious IR $\frac{1}{\epsilon^n}$ remain | **UV $\frac{1}{\epsilon^n}$ remain**

The $\frac{1}{\epsilon^n}$ poles cancel in the matching equation.

The only Feynman integrals to calculate: partly-massive tadpoles.

Calculation of 3-loop single-scale partly-massive tadpoles has been fully automatized long time ago (the code `MATAD` by M. Steinhauser).

Differences among the simultaneously decoupled heavy particle masses can be taken into account by Taylor expanding around the equal-mass point.

Renormalization constant calculation using masses as IR regulators

MM, Münz, 1995	2-loop	dipole operator mixing
van Ritbergen, Vermaseren, Larin, 1997	4-loop	$\beta_{\rm QCD}$
Chetyrkin, MM, Münz, 1997	3-loop	(4-quark) $\rightarrow$ dipole
(...)		
Gambino, Gorbahn, Haisch, 2003	3-loop	(4-quark) $\rightarrow$ (quark-lepton)
Gorbahn, Haisch, 2004	3-loop	four-quark operator mixing
Czakon, 2004	4-loop	$\beta_{\rm QCD}$
Gorbahn, Haisch, MM, 2005	3-loop	dipole operator mixing
Czakon, Haisch, MM, 2006	4-loop	(4-quark) $\rightarrow$ dipole

Exact decomposition of a propagator denominator:

$$\underbrace{\frac{1}{(q+p)^2-M^2}}_{\Delta D=-2} = \underbrace{\frac{1}{q^2-m^2}}_{\Delta D=-2} + \underbrace{\frac{M^2-p^2-2qp-m^2}{q^2-m^2}\,\frac{1}{(q+p)^2-M^2}}_{\Delta D=-3}$$

q– linear combination of loop momenta, p– linear combination of external momenta,
M– mass of the considered particle, m– IR regulator mass (arbitrary)

After applying this identity sufficiently many times, the last term can be dropped in each propagator. The only Feynman integrals to perform then are single-scale massive tadpoles.

Up to three loops, explicit expressions for pole parts of all the single-scale massive tadpoles are available in terms of solved recurrences [Chetyrkin, MM, Münz, 1997].

At four loops:

T. van Ritbergen – ??

M. Czakon – The integration-by-parts method and the Laporta algorithm [hep-ph/0102033] are used for reduction to 19 master integrals [hep-ph/0411261].

See also S. Laporta, hep-ph/0210336
Y. Schröder, hep-ph/0211288
Y. Schröder and A. Vuorinen, hep-ph/0311323, hep-ph/0503209.

Current status of the WC evaluation:

$(\bar{q}_1\, \sigma^{\alpha\beta} P_{L,R}\, q_2)\, F_{\alpha\beta}, \qquad (\bar{q}_1\, \sigma^{\alpha\beta} P_{L,R}\, T^a q_2)\, G^a_{\alpha\beta}$ – **dipole-type, the only dim-5 ones, chirality-suppressed**

$\mathcal{O}(\alpha_s^2, \alpha_{\rm em})$ **known**

$(\bar{q}_1\, \gamma^\alpha P_L\, q_2)\, (\bar{l}\, \gamma_\alpha \nu)$ – **charged-current quark-lepton**

$\mathcal{O}(\alpha_{\rm em})$ **known,** $\mathcal{O}(\alpha_s^n) = 0$

$(\bar{q}_1\, \gamma^\alpha P_L\, q_2)\, (\bar{l}\, \gamma_\alpha P_{L,R}\, l), \quad (\bar{q}_1\, \gamma^\alpha P_L\, q_2)\, (\bar{\nu}\, \gamma_\alpha \nu)$ – **neutral-current quark-lepton**

$\mathcal{O}(\alpha_s, \alpha_{\rm em})$ **known,** $\mathcal{O}(\alpha_s^2)$ **would be possible**

$(\bar{q}_1\, \gamma^\alpha P_L\, q_2)\, (\bar{q}_3\, \gamma^\alpha P_L\, q_4), \quad (\bar{q}_1\, \gamma^\alpha P_L\, T^a q_2)\, (\bar{q}_3\, \gamma^\alpha P_L\, T^a q_4)$ – **charged-current four-quark**

$\mathcal{O}(\alpha_s^2, \alpha_{\rm em})$ **known,** $\mathcal{O}(\alpha_s^3)$ **would be possible**

$(\bar{q}_1\, \gamma^\alpha P_L\, q_2)\, (\bar{q}\, \gamma^\alpha P_{L,R}\, q), \quad (\bar{q}_1\, \gamma^\alpha P_L\, T^a q_2)\, (\bar{q}\, \gamma^\alpha P_{L,R}\, T^a q)$ – **neutral-current four-quark** $\Delta F = 1$

$\mathcal{O}(\alpha_s^2, \alpha_{\rm em})$ **known,** $\mathcal{O}(\alpha_s^3)$ **would be possible**

$(\bar{q}_1\, \gamma^\alpha P_L\, q_2)\, (\bar{q}_1\, \gamma^\alpha P_L\, q_2)$ – **neutral-current four-quark** $\Delta F = 2$

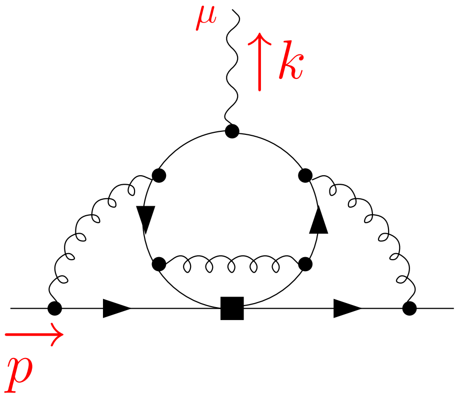
$\mathcal{O}(\alpha_s^2, \alpha_{\rm em})$ **known,** $\mathcal{O}(\alpha_s^3)$ **would be possible**

Concluding question:

Do we need more precise Wilson coefficients for the LHCb or SuperB?

BACKUP SLIDES

Calculation of the bare 4-loop penguins (M. Czakon)



There are **21986** diagrams like this (both for $b \to s\gamma$ and $b \to sg$).

Their degree of divergence $D = 2$, so they must be expanded to the second order in external momenta and masses. However, momentum-independent terms can be neglected.

Each of the diagrams, when evaluated off-shell, gives a linear combination of:

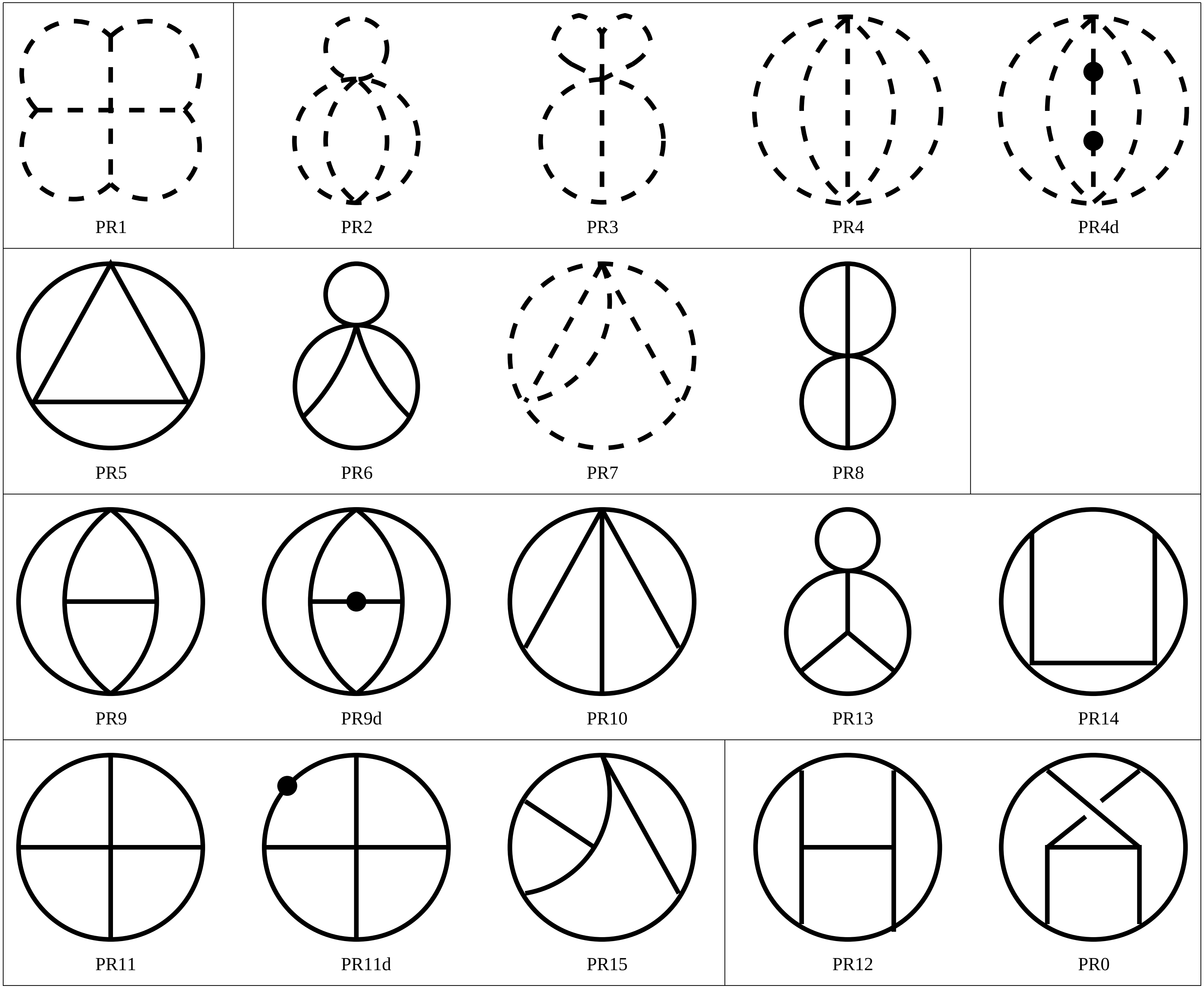
$\gamma_\mu \not{p}\not{k},\ \gamma_\mu\,(p\cdot k),\ \gamma_\mu p^2,\ \gamma_\mu k^2,\ \not{p} k_\mu,\ \not{p} p_\mu,\ \not{k} p_\mu,\ \not{k} k_\mu,\ M_b \not{k}\gamma_\mu,\ M_b \gamma_\mu \not{k},\ M_b \not{p}\gamma_\mu,\ M_b \gamma_\mu \not{p}.$

Coefficients at these structures are linear combinations of scalar Feynman integrals, i.e. fully massive single-scale 4-loop tadpoles. Computing these coefficients (while keeping the integrals as symbols) was the most computer time consuming part of the calculation (FORM programs):

$\sim$**5 months running on $\sim$10 processors (Würzburg)**
$+\sim$**1 month running on $\sim$100 processors (FNAL and DESY Zeuthen).**

The final result turns out to contain no new scalar integrals with respect to the 4-loop $\beta_{\rm QCD}$ calculation [M. Czakon, hep-ph/0411261] that took $\sim$2 weeks on $\sim$10 processors, including the reduction to master integrals. In that calculation, the reduction of $\sim 2 \times 10^5$ integrals required resolving $\sim 2 \times 10^6$ ones.

Master integrals



Sample analytical expressions for the pole parts of the master integrals:

$$\mathbf{PR11} = \frac{5}{\epsilon}\,\zeta_5 + \mathcal{O}(\epsilon^0),$$

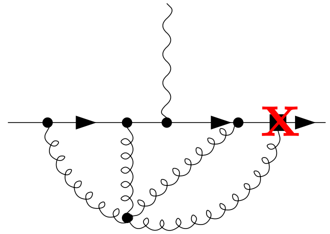
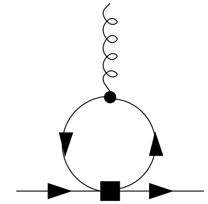
$$\mathbf{PR14} = \frac{1}{\epsilon^4}\,\frac{3}{4} + \frac{1}{\epsilon^3}\,\frac{25}{4} + \frac{1}{\epsilon^2}\,\left(\frac{137}{4} - \frac{81}{2}\,\mathbf{S2} + \frac{3}{2}\,\zeta_2\right)$$
$$+\frac{1}{\epsilon}\left(\frac{363}{4} - 162\,\mathbf{S2} - 3\,\mathbf{T1ep} - \zeta_2 - \frac{27}{2}\,\zeta_3\right) + \mathcal{O}(\epsilon^0),$$

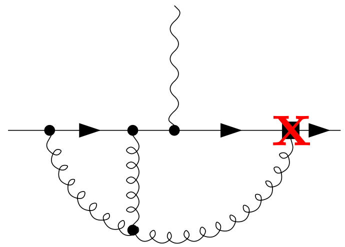
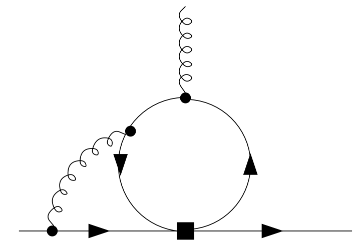
$$\mathbf{PR15} = \frac{1}{\epsilon^2}\,\frac{3}{2}\,\zeta_3 + \frac{1}{\epsilon}\,\left(\mathbf{D6} + \frac{3}{2}\,\zeta_3 - \frac{3}{4}\,\zeta_4\right) + \mathcal{O}(\epsilon^0),$$

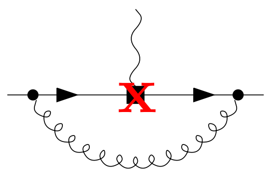
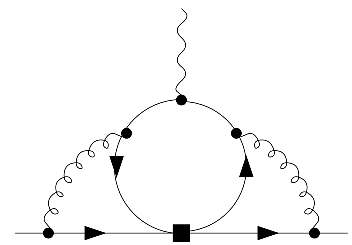
The quantities S2, T1ep and D6 cancel out in the pole parts after adding lower-loop diagrams with counterterms. Consequently, the anomalous dimensions depend only on rationals and ζ_k with $k \leq 5$.

Subtraction of subdivergences

Examples of diagrams with operator counterterms (Wilson coefficient renormalization):

3-loop diagrams  **with 1-loop counterterms** 

2-loop diagrams  **with 2-loop counterterms** 

1-loop diagrams  **with 3-loop counterterms** 

Types of off-shell operator counterterms (apart from the physical operators $O_1,\ldots,O_8$)

- **gauge-invariant EOM-vanishing operators, e.g.,** $(\bar{s}_L \gamma^\mu T^a b_L)\left[D^\nu G^a_{\mu\nu} \;+\; g \sum_q (\bar{q}\gamma^\mu T^a q)\right]$,
- **gauge-variant EOM-vanishing operators, e.g.,** $\bar{s}_L \left[-i \overleftarrow{\not{D}} \not{G} + \not{G}(i\not{D} - M_b) \right] b_R$
- **BRS-exact operator** $\delta_{\rm BRS}\left[(\bar{s}_L \gamma^\mu T^a b_L)\partial_\mu \bar{\eta}^a\right] \;=\; (\bar{s}_L \gamma^\mu T^a b_L)\left[\partial_\mu \partial^\nu G^a_\nu - g f^{abc}(\partial_\mu \bar{\eta}^b)\eta^c\right]$
- **evanescent operators, e.g.,** $(\bar{s}_L \gamma^{[\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}\gamma^{\mu_4}\gamma^{\mu_5]} c_L)(\bar{c}_L \gamma_{[\mu_1}\gamma_{\mu_2}\gamma_{\mu_3}\gamma_{\mu_4}\gamma_{\mu_5]} b_L)$
- **so-called m^2-operators (due to propagator numerator simplification) e.g.,** $m^2 \bar{s}_L \not{G} b_L$

Cross-checks of the calculation

The sum of n-loop diagrams (with counterterms for $n < 4$) has the form

$$\mathbf{SUM}_n = \mu^{2n\epsilon} \sum_{k=1}^{4} \frac{1}{\epsilon^k} X_{nk} \qquad (D = 4 - 2\epsilon)$$

Since the tree-level counterterm must be local ($\Rightarrow$ polynomial in external momenta), no logarithms of μ can remain in the pole part of $\mathbf{SUM}_1 + \mathbf{SUM}_2 + \mathbf{SUM}_3 + \mathbf{SUM}_4$. Consequently, the following relations must be satisfied:

$$X_{14} = -\frac{2}{3} X_{24} = X_{34} = -4 X_{44}$$
$$X_{23} + 3X_{33} + 6X_{43} = 0$$
$$X_{13} - 3X_{33} - 8X_{43} = 0$$
$$X_{12} + 2X_{22} + 3X_{32} + 4X_{42} = 0$$

Additional cross-checks are possible thanks to the fact that all the 12 structures

$$\gamma_\mu \not{p}\not{k},\ \gamma_\mu (p \cdot k),\ \gamma_\mu p^2,\ \gamma_\mu k^2,\ \not{p} k_\mu,\ \not{p} p_\mu,\ \not{k} p_\mu,\ \not{k} k_\mu,\ M_b \not{k} \gamma_\mu,\ M_b \gamma_\mu \not{k},\ M_b \not{p} \gamma_\mu,\ M_b \gamma_\mu \not{p}.$$

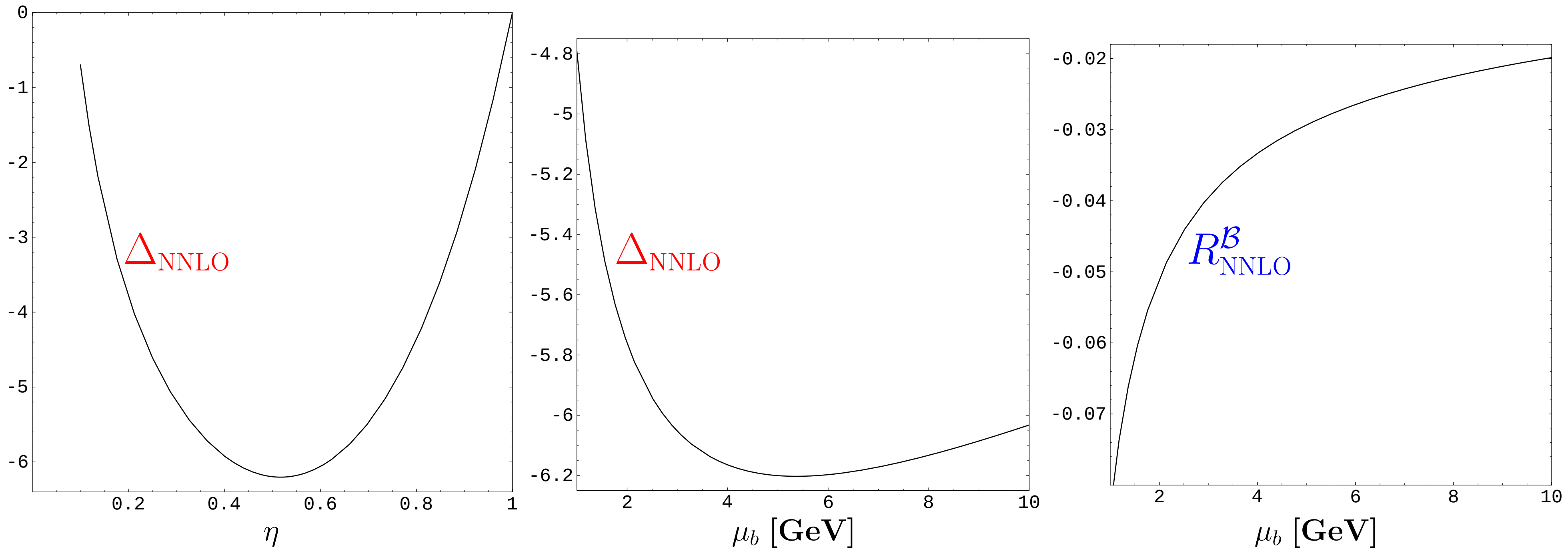
are calculated for the counterterm diagrams. Since the number of available $b \to s\gamma$ operators is only 5,

$$\bar{s}_L \sigma^{\mu\nu} b_R F_{\mu\nu}, \qquad \bar{s}_L \not{D}\not{D} b_R, \qquad \bar{s}_L \not{D}\not{D}\not{D} b_L,$$

$$\bar{s}_L \left[i \overleftarrow{\not{D}} \sigma^{\mu\nu} F_{\mu\nu} - F_{\mu\nu} \sigma^{\mu\nu} (i\not{D} - M_b) \right] b_L, \qquad \bar{s}_L \gamma_\mu b_L \partial_\nu F^{\mu\nu},$$

we get 7 relations between the 12 coefficients at the considered structures (after summing up the bare diagrams and all the counterterms). The number of relations for $b \to sg$ reduces to 3 due to the relevance of gauge-variant operators in that case.

Numerical effect of the 4-loop mixing at the NNLO

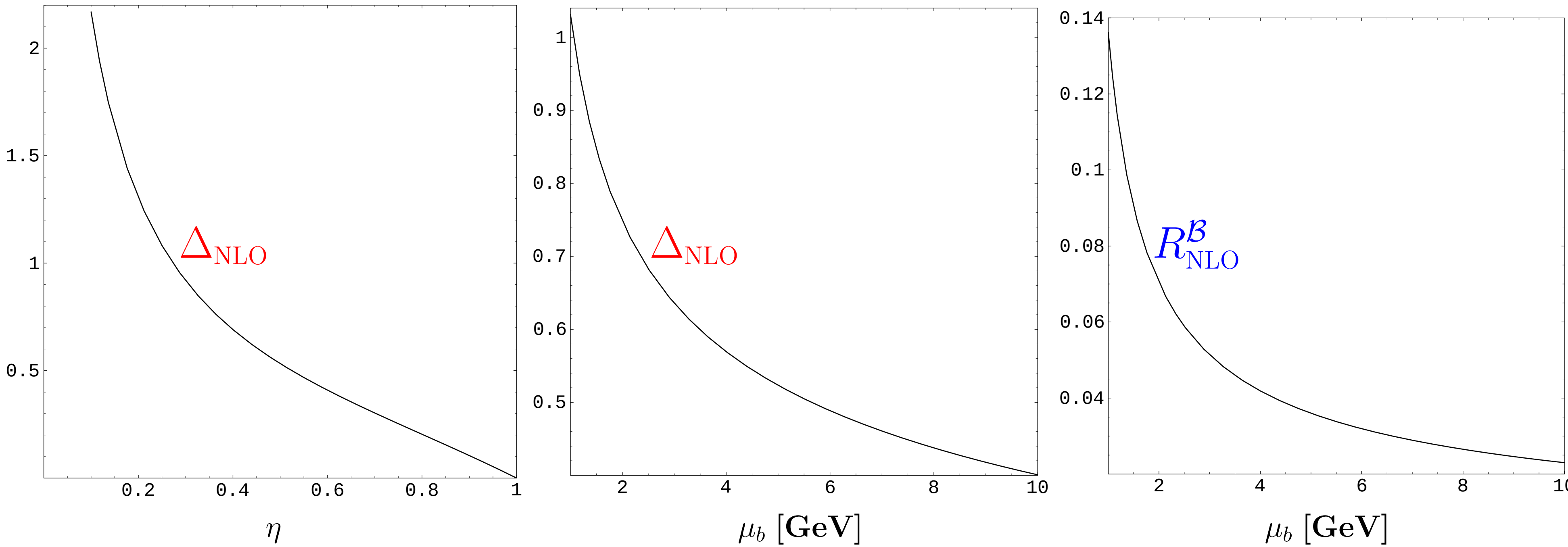


$$R^{\mathcal{B}}_{\rm NNLO} \equiv \frac{\mathcal{B}_{\rm NNLO} - \mathcal{B}^{\rm 4L\to 0}_{\rm NNLO}}{\mathcal{B}_{\rm LO}} = \left(\frac{\alpha_s(\mu_b)}{\pi}\right)^2 \Delta_{\rm NNLO}$$

$$\Delta_{\rm NNLO} = \frac{C_7^{(2)\rm eff}(\mu_b) - \left[C_7^{(2)\rm eff}(\mu_b)\right]^{\rm 4L\to 0}}{8\, C_7^{(0)\rm eff}(\mu_b)} = \frac{h_1^{(2)}\eta^{a_1+2} + h_2^{(2)}\eta^{a_2+2} + \sum_{i=3}^{8} h_i^{(2)}\eta^{a_i}}{\eta^{a_2} C_7^{(0)}(\mu_0) + \frac{8}{3}\left(\eta^{a_1} - \eta^{a_2}\right) C_8^{(0)}(\mu_0) + \sum_{i=1}^{8} h_i^{(0)}\eta^{a_i}},$$

$$\eta = \alpha_s(\mu_0)/\alpha_s(\mu_b)$$

Numerical effect of the 3-loop mixing at the NLO



$$R^{\mathcal{B}}_{\rm NLO} \equiv \frac{\mathcal{B}_{\rm NLO} - \mathcal{B}^{\rm 3L\to 0}_{\rm NLO}}{\mathcal{B}_{\rm LO}} = \frac{\alpha_s(\mu_b)}{\pi} \, \Delta_{\rm NLO}$$

$$\Delta_{\rm NLO} = \frac{C_7^{(1)\rm eff}(\mu_b) - \left[C_7^{(1)\rm eff}(\mu_b)\right]^{\rm 3L\to 0}}{2\, C_7^{(0)\rm eff}(\mu_b)} = \frac{h_1^{(1)}\eta^{a_1+1} + h_2^{(1)}\eta^{a_2+1} + \sum_{i=3}^{8} h_i^{(1)}\eta^{a_i}}{\eta^{a_2}C_7^{(0)}(\mu_0) + \frac{8}{3}\left(\eta^{a_1} - \eta^{a_2}\right) C_8^{(0)}(\mu_0) + \sum_{i=1}^{8} h_i^{(0)}\eta^{a_i}}.$$

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