Mechanika Klasyczna 2019/2020
1. seria zadań domowych

Zad. 1

Znaleźć wersory i składowe prędkości i przyspieszenia we współrzędnych paraboloidalnych (ξ, η, ϕ), $x = \sqrt{\xi\eta}\cos\phi$, $y = \sqrt{\xi\eta}\sin\phi$, $z = (\xi - \eta)/2$ wyrażone przez ξ, η, ϕ i ich kolejne pochodne po czasie.

Rozwiązanie – Beata Siorek

Współrzędne paraboloidalne:

$$\begin{cases} x = \sqrt{\xi\eta}\cos\phi \\ y = \sqrt{\xi\eta}\sin\phi \\ z = (\xi - \eta)/2 \end{cases}$$

$$\vec{r} = x\vec{\mathrm{e}}_x + y\vec{\mathrm{e}}_y + z\vec{\mathrm{e}}_z$$

Znajdźmy wersory $\vec{\mathrm{e}}_\xi$, $\vec{\mathrm{e}}_\eta$, $\vec{\mathrm{e}}_\phi$ w tych współrzędnych:

$$\tilde{\vec{\mathrm{e}}}_\xi = \frac{\partial\vec{r}}{\partial\xi} = \left(\frac{\eta\cos\phi}{2\sqrt{\xi\eta}},\ \frac{\eta\sin\phi}{2\sqrt{\xi\eta}},\ \frac{1}{2}\right)$$

$$\tilde{\vec{\mathrm{e}}}_\eta = \frac{\partial\vec{r}}{\partial\eta} = \left(\frac{\xi\cos\phi}{2\sqrt{\xi\eta}},\ \frac{\xi\sin\phi}{2\sqrt{\xi\eta}},\ -\frac{1}{2}\right)$$

$$\tilde{\vec{\mathrm{e}}}_\phi = \frac{\partial\vec{r}}{\partial\phi} = \left(-\sqrt{\xi\eta}\sin\phi,\ \sqrt{\xi\eta}\cos\phi,\ 0\right)$$

$$|\tilde{\vec{\mathrm{e}}}_\xi| = \sqrt{\left(\frac{\eta\cos\phi}{2\sqrt{\xi\eta}}\right)^2 + \left(\frac{\eta\sin\phi}{2\sqrt{\xi\eta}}\right)^2 + \frac{1}{4}} = \frac{1}{2}\sqrt{\frac{\eta+\xi}{\xi}}$$

$$|\tilde{\vec{\mathrm{e}}}_\eta| = \sqrt{\left(\frac{\xi\sin\phi}{2\sqrt{\xi\eta}}\right)^2 + \left(\frac{\xi\cos\phi}{2\sqrt{\xi\eta}}\right)^2 + \frac{1}{4}} = \frac{1}{2}\sqrt{\frac{\eta+\xi}{\eta}}$$

$$|\tilde{\vec{\mathrm{e}}}_\phi| = \sqrt{\left(-\sqrt{\xi\eta}\sin\phi\right)^2 + \left(\sqrt{\xi\eta}\cos\phi\right)^2} = \sqrt{\xi\eta}$$

Po unormowaniu otrzymujemy ostatecznie:

$$\vec{\mathrm{e}}_\xi = \frac{2\sqrt{\xi}}{\sqrt{\eta+\xi}}\left(\frac{\eta\cos\phi}{2\sqrt{\xi\eta}},\ \frac{\eta\sin\phi}{2\sqrt{\xi\eta}},\ \frac{1}{2}\right) = \left(\sqrt{\frac{\eta}{\eta+\xi}}\cos\phi,\ \sqrt{\frac{\eta}{\eta+\xi}}\sin\phi,\ \sqrt{\frac{\xi}{\eta+\xi}}\right)$$

$$\vec{\mathrm{e}}_\eta = \frac{2\sqrt{\eta}}{\sqrt{\eta+\xi}}\left(\frac{\xi\cos\phi}{2\sqrt{\xi\eta}},\ \frac{\xi\sin\phi}{2\sqrt{\xi\eta}},\ -\frac{1}{2}\right) = \left(\sqrt{\frac{\xi}{\eta+\xi}}\cos\phi,\ \sqrt{\frac{\xi}{\eta+\xi}}\sin\phi,\ -\sqrt{\frac{\eta}{\eta+\xi}}\right)$$

$$\vec{e}_\phi = \frac{1}{\sqrt{\xi\eta}}\Big(-\sqrt{\xi\eta}\sin\phi,\ \sqrt{\xi\eta}\cos\phi,\ 0\Big) = (-\sin\phi,\ \cos\phi,\ 0)$$

Obliczmy pochodne wersorów $\vec{e}_\xi$, $\vec{e}_\eta$, $\vec{e}_\phi$ po czasie:

$$\dot{\vec{e}}_\xi = \left(\frac{\xi\dot{\eta}-\dot{\xi}\eta}{2\sqrt{\eta}(\eta+\xi)^{3/2}}\cos\phi - \sqrt{\frac{\eta}{\eta+\xi}}\dot{\phi}\sin\phi,\ \frac{\xi\dot{\eta}-\dot{\xi}\eta}{2\sqrt{\eta}(\eta+\xi)^{3/2}}\sin\phi + \sqrt{\frac{\eta}{\eta+\xi}}\dot{\phi}\cos\phi,\ \frac{\dot{\xi}\eta-\xi\dot{\eta}}{2\sqrt{\xi}(\eta+\xi)^{3/2}}\right)$$

$$\dot{\vec{e}}_\xi = \frac{\xi\dot{\eta}-\dot{\xi}\eta}{2\sqrt{\eta}(\eta+\xi)^{3/2}}\sqrt{\frac{\eta}{\xi}}\sqrt{\frac{\eta+\xi}{\eta}}\ \vec{e}_\eta + \sqrt{\frac{\eta}{\eta+\xi}}\dot{\phi}\ \vec{e}_\phi$$

$$\dot{\vec{e}}_\xi = \frac{\xi\dot{\eta}-\dot{\xi}\eta}{2\sqrt{\xi\eta}(\eta+\xi)}\ \vec{e}_\eta + \sqrt{\frac{\eta}{\eta+\xi}}\dot{\phi}\ \vec{e}_\phi$$

$$\dot{\vec{e}}_\eta = \left(\frac{\dot{\xi}\eta-\xi\dot{\eta}}{2\sqrt{\xi}(\eta+\xi)^{3/2}}\cos\phi - \sqrt{\frac{\xi}{\eta+\xi}}\dot{\phi}\sin\phi,\ \frac{\dot{\xi}\eta-\xi\dot{\eta}}{2\sqrt{\xi}(\eta+\xi)^{3/2}}\sin\phi + \sqrt{\frac{\xi}{\eta+\xi}}\dot{\phi}\cos\phi,\ \frac{\dot{\xi}\eta-\xi\dot{\eta}}{2\sqrt{\eta}(\eta+\xi)^{3/2}}\right)$$

$$\dot{\vec{e}}_\eta = \frac{\dot{\xi}\eta-\xi\dot{\eta}}{2\sqrt{\xi}(\eta+\xi)^{3/2}}\sqrt{\frac{\xi}{\eta}}\sqrt{\frac{\eta+\xi}{\xi}}\ \vec{e}_\xi + \sqrt{\frac{\xi}{\eta+\xi}}\dot{\phi}\ \vec{e}_\phi$$

$$\dot{\vec{e}}_\eta = \frac{\dot{\xi}\eta-\xi\dot{\eta}}{2\sqrt{\xi\eta}(\eta+\xi)}\ \vec{e}_\xi + \sqrt{\frac{\xi}{\eta+\xi}}\dot{\phi}\ \vec{e}_\phi$$

$$\dot{\vec{e}}_\phi = \Big(-\dot{\phi}\cos\phi,\ -\dot{\phi}\sin\phi,\ 0\Big)$$

$$\dot{\vec{e}}_\phi = -\dot{\phi}\sqrt{\frac{\xi\eta}{\eta+\xi}}\sqrt{\frac{1}{\xi}}\ \vec{e}_\xi - \dot{\phi}\sqrt{\frac{\xi\eta}{\eta+\xi}}\sqrt{\frac{1}{\eta}}\ \vec{e}_\eta$$

$$\dot{\vec{e}}_\phi = -\dot{\phi}\sqrt{\frac{\eta}{\eta+\xi}}\ \vec{e}_\xi - \dot{\phi}\sqrt{\frac{\xi}{\eta+\xi}}\ \vec{e}_\eta$$

Otrzymaliśmy:

$$\begin{cases} \dot{\vec{e}}_\xi = \frac{\xi\dot{\eta}-\dot{\xi}\eta}{2\sqrt{\xi\eta}(\eta+\xi)}\ \vec{e}_\eta + \sqrt{\frac{\eta}{\eta+\xi}}\dot{\phi}\ \vec{e}_\phi \\ \dot{\vec{e}}_\eta = \frac{\dot{\xi}\eta-\xi\dot{\eta}}{2\sqrt{\xi\eta}(\eta+\xi)}\ \vec{e}_\xi + \sqrt{\frac{\xi}{\eta+\xi}}\dot{\phi}\ \vec{e}_\phi \\ \dot{\vec{e}}_\phi = -\dot{\phi}\sqrt{\frac{\eta}{\eta+\xi}}\ \vec{e}_\xi - \dot{\phi}\sqrt{\frac{\xi}{\eta+\xi}}\ \vec{e}_\eta \end{cases}$$

Możemy zatem przejść do obliczenia składowych prędkości i przyspieszenia we współrzędnych paraboloidalnych.

Wektor położenia:

$$\vec{r} = \frac{1}{2}\sqrt{(\eta+\xi)\xi}\;\vec{e}_\xi + \frac{1}{2}\sqrt{(\eta+\xi)\eta}\;\vec{e}_\eta$$

Wektor prędkości:

$$\dot{\vec{r}} = \frac{\dot{\xi}\eta + \xi(\dot{\eta}+2\dot{\xi})}{4\sqrt{(\eta+\xi)\xi}}\;\vec{e}_\xi + \frac{1}{2}\sqrt{(\eta+\xi)\xi}\;\dot{\vec{e}}_\xi + \frac{\xi\dot{\eta} + \eta(2\dot{\eta}+\dot{\xi})}{4\sqrt{(\eta+\xi)\eta}}\;\vec{e}_\eta + \frac{1}{2}\sqrt{(\eta+\xi)\eta}\;\dot{\vec{e}}_\eta$$

$$\dot{\vec{r}} = \frac{\dot{\xi}\sqrt{\eta+\xi}}{2\sqrt{\xi}}\;\vec{e}_\xi + \frac{\dot{\eta}\sqrt{\eta+\xi}}{2\sqrt{\eta}}\;\vec{e}_\eta + \sqrt{\xi\eta}\dot{\phi}\;\vec{e}_\phi$$

Wektor przyspieszenia:

$$\ddot{\vec{r}} = \frac{2\ddot{\xi}\xi(\eta+\xi) + \dot{\xi}\xi\dot{\eta} - (\dot{\xi})^2\eta}{4(\xi)^{3/2}\sqrt{\eta+\xi}}\;\vec{e}_\xi + \frac{\dot{\xi}\sqrt{\eta+\xi}}{2\sqrt{\xi}}\;\dot{\vec{e}}_\xi +$$

$$+\frac{2\ddot{\eta}\eta(\eta+\xi) + \dot{\xi}\dot{\eta}\eta - \xi(\dot{\eta})^2}{4(\eta)^{3/2}\sqrt{\eta+\xi}}\;\vec{e}_\eta + \frac{\dot{\eta}\sqrt{\eta+\xi}}{2\sqrt{\eta}}\;\dot{\vec{e}}_\eta +$$

$$+\left(\frac{\dot{\xi}\sqrt{\eta}\dot{\phi}}{2\sqrt{\xi}} + \frac{\sqrt{\xi}\dot{\eta}\dot{\phi}}{2\sqrt{\eta}} + \sqrt{\xi\eta}\ddot{\phi}\right)\;\vec{e}_\phi + \sqrt{\xi\eta}\dot{\phi}\;\dot{\vec{e}}_\phi$$

$$\ddot{\vec{r}} = \frac{2\ddot{\xi}\xi\eta^2 + 2\ddot{\xi}\xi^2\eta + 2\dot{\xi}\xi\dot{\eta}\eta - (\dot{\xi})^2\eta^2 - \xi^2(\dot{\eta})^2 - 4\xi^2\eta^2(\dot{\phi})^2}{4\xi\eta\sqrt{(\eta+\xi)\xi}}\;\vec{e}_\xi +$$

$$+\frac{2\xi^2\ddot{\eta}\eta + 2\xi\ddot{\eta}\eta^2 + 2\dot{\xi}\xi\dot{\eta}\eta - \xi^2(\dot{\eta})^2 - (\dot{\xi})^2\eta^2 - 4\xi^2\eta^2(\dot{\phi})^2}{4\xi\eta\sqrt{(\eta+\xi)\eta}}\;\vec{e}_\eta +$$

$$+\frac{\dot{\xi}\eta\dot{\phi} + \xi\dot{\eta}\dot{\phi} + \xi\eta\ddot{\phi}}{\sqrt{\xi\eta}}\;\vec{e}_\phi$$

Zad. 2

Napisać macierz obrotu wokół osi $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$ o kąt $\pi/3$ przeciwnie do ruchu wskazówek zegara, patrząc z kierunku osi.

Rozwiązanie - Filip Rękawek

Najpierw znajdź macierz obrotu (nazwijmy ją A) przenoszącego wektor $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ na wektor $(1,0,0)$. Zadana oś przechodzi wówczas na oś x, wokół, której obroty da się łatwo wyrazić za pomocą macierzy. Zatem

$$A^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} \\ 0 & \sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{bmatrix} A$$

jest poszukiwaną przez nas macierzą. Sformułowanie "patrząc z kierunku osi" rozumiem jako "oś obrotu kieruje się w stronę obserwatora".

Aby znaleźć macierz A możemy pomnożyć macierz obrotu wokół, osi y o kąt $\frac{\pi}{4}$ przeciwnie do ruchu wskazówek zegara (patrząc z kierunku osi) i macierz obrotu wokół, osi z o kąt θ, zawarty pomiędzy obrazem wektora $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ i osią x, zgodnie z ruchem wskazówek zegara (patrząc z kierunku osi).

$$\begin{bmatrix} \cos\frac{\pi}{4} & 0 & \sin\frac{\pi}{4} \\ 0 & 1 & 0 \\ -\sin\frac{\pi}{4} & 0 & \cos\frac{\pi}{4} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} \sqrt{2} \\ 1 \\ 0 \end{bmatrix}, \text{ a więc } \cos\theta = \frac{\sqrt{2}}{\sqrt{3}} \text{ oraz } \sin\theta = \frac{1}{\sqrt{3}}$$

W istocie, okazuje się, że $\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \frac{1}{\sqrt{3}} \begin{bmatrix} \sqrt{2} \\ 1 \\ 0 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Stąd, $A = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\frac{\pi}{4} & 0 & \sin\frac{\pi}{4} \\ 0 & 1 & 0 \\ -\sin\frac{\pi}{4} & 0 & \cos\frac{\pi}{4} \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} \sqrt{2} & 1 & 0 \\ -1 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{3} \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ -1 & 0 & 1 \end{bmatrix} =$
$\frac{1}{\sqrt{6}} \begin{bmatrix} \sqrt{2} & -1 & -\sqrt{3} \\ \sqrt{2} & 2 & 0 \\ \sqrt{2} & -1 & \sqrt{3} \end{bmatrix}$

Szukana macierz to:

$$\frac{1}{\sqrt{6}} \begin{bmatrix} \sqrt{2} & -1 & -\sqrt{3} \\ \sqrt{2} & 2 & 0 \\ \sqrt{2} & -1 & \sqrt{3} \end{bmatrix} \cdot \frac{1}{2} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & -\sqrt{3} \\ 0 & \sqrt{3} & 1 \end{bmatrix} \cdot \frac{1}{\sqrt{6}} \begin{bmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 1 & 2 & -1 \\ -\sqrt{3} & 0 & \sqrt{3} \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 & -1 & 2 \\ 2 & 2 & -1 \\ -1 & 2 & 2 \end{bmatrix}$$

Zad. 3

Wykazać, że

$$(\boldsymbol{a} \times \boldsymbol{b}) \times (\boldsymbol{c} \times \boldsymbol{d}) = [\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{d}]\boldsymbol{c} - [\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]\boldsymbol{d}$$

Niech $\boldsymbol{f} = \boldsymbol{a} \times \boldsymbol{b}$ Wtedy

$$\boldsymbol{f} \times (\boldsymbol{c} \times \boldsymbol{d}) = \boldsymbol{c}(\boldsymbol{f} \cdot \boldsymbol{d}) - \boldsymbol{d}(\boldsymbol{f} \cdot \boldsymbol{c})$$

a jednocześnie $\boldsymbol{f} \cdot \boldsymbol{d} = (\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{d} = [\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{d}]$ a potem zamieniamy $\boldsymbol{d}$ na $\boldsymbol{c}$.