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Several examples of nonholonomic mechanical systems

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Abstract. A unified geometric approach to nonholonomic constrained mechanical systems is applied to several concrete problems from the classical mechanics of particles and rigid bodies. In every of these examples the given constraint conditions are analysed, a corresponding constraint submanifold in the phase space is considered, the corresponding constrained mechanical system is modelled on the constraint submanifold, the reduced equations of motion of this system (i.e. equations of motion defined on the constraint submanifold) are presented. Finally, solvability of these equations is discussed and general solutions in explicit form are found.

1 Introduction

In some mechanical and engineering problems one encounters different kinds of additional conditions, constraining and restricting motions of mechanical systems. Such conditions are called *constraints*. Constraints may be given by algebraic equations connecting coordinates (holonomic or geometric constraints), or by differential equations, which restrict coordinates and components of velocities (kinematic constraints). Nonintegrable kinematic constraints, which cannot be reduced to holonomic ones, are called *nonholonomic constraints*.

Classical theoretical mechanics deals with nonholonomic constraints only marginally, mostly in a form of short remarks about the existence of such constraints, or mentioning some problems where simple nonholonomic constraints occur. Only rarely, for example, in textbook [2] one can find sections where nonholonomic constraints are discussed in more detail and a few examples of simple mechanical systems subjected to a nonholonomic constraint are solved. However, these books deal only with semiholonomic or linear nonholonomic constraints (constraints linear in components of velocities), arising for example in the connection with rolling

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of rigid bodies. Discussion is usually concluded by a remark that more complicated nonholonomic constraints (when the dependence on velocities is nonlinear) are not mastered by means of classical methods and motion equations of mechanical systems subjected to such constraints are not known.

A significant contribution to the study of problems of nonholonomic mechanics represents an extensive monograph [22] which contains various application problems, mostly problems concerning rolling of rigid bodies on a horizontal plane or on an absolutely rough surface where typically nonholonomic constraints linear in velocities occur. This monograph serves as a classical collection of solved problems of nonholonomic dynamics. However, it does not give a unified and consistent approach applicable to arbitrary nonholonomic mechanical systems. Equations of motion of the considered nonholonomic systems are mostly derived on the basis of a heuristic analogy with holonomic systems. On the other hand their solutions agree with experience and experiments.

During the last 20 years the problems of nonholonomic mechanics have been intensively studied in many papers, e.g. [3], [4], [5], [7], [8], [9], [10], [13], [14], [20], [21], [23] and there have been proposed several alternative geometric concepts, appropriate in different situations, applicable to Lagrangian systems in tangent bundles or in jet bundles. Equations of motion of nonholonomic systems are investigated also in the monographs [1], [6], where a number of concrete application problems is discussed and numerical aspects of solutions are presented. However, it should be stressed, that almost all the work on nonholonomic systems is concerned with the case of constraints linear in components of velocities.

A geometric theory covering general nonholonomic systems has been proposed and developed by Krupková in [14], [15], [16], [17] (see also [18] for review). Her approach is suitable for study of all kinds of mechanical systems – without restricting to Lagrangian, time-independent, or regular ones, and is applicable to arbitrary constraints (holonomic, semiholonomic, linear, nonlinear or general nonholonomic). The theory gives motion equations for constrained mechanical systems in a form of reduced equations defined on the constraint submanifold (without Lagrange multipliers), provides a nonholonomic variational principle [17], [24] from which one can obtain reduced equations as corresponding "nonholonomic Euler-Lagrange equations", enables one to study constraint symmetries and the corresponding conservation laws, etc. In particular, a new treatment of concrete examples of nonholonomic systems is at hand, suitable for either systems with linear constraints [11], [12], [25], [26], [27], or even with nonlinear constraints [19], [25] and providing new methods for explicit studies and solutions.

The aim of this paper is to apply Krupková's geometric theory of nonholonomic mechanical systems to study concrete problems in both linear and nonlinear nonholonomic dynamics. In all the cases we analyse the given constraint conditions, consider the corresponding constraint submanifold in the phase space, we construct the corresponding constrained mechanical system on the constraint submanifold, present the reduced equations of motion of this system, and finally discuss the solvability of these equations. In most cases we are able to obtain general solutions in an explicit form. It turns out that reduced equations indeed represent an effective method for solving concrete mechanical and engineering problems of nonholonomic mechanics.

The paper contains complete and comprehensive solutions of seven problems from the classical mechanics of particles and rigid bodies where nonholonomic constraints appear. Three of them (5.1, 5.4 and 5.5) concern dynamics of a free particle or a particle in a homogeneous gravitational field subject to a *nonlinear nonholonomic constraint*. We find general solutions in an explicit form, with respect to appropriate initial conditions. Problem 5.2 (a dog pursues a man) is formulated in [2]; we study it as a mechanical system modelled on a nonholonomic submanifold and provide the reduced equation of motion. A solution in an explicit form is found by eliminating the time parameter from Chetaev equations. The next problem (5.3) is then a generalization of the previous one. The last two problems belong to the mechanics of rigid bodies (a disc rolling without sliding on a horizontal plane and a ball rolling without sliding on a horizontal plane) and as examples of nonholonomic systems are discussed in the monograph [22]. We study them in a different way, again using the geometric model leading to reduced equations. In particular, compared with [22] where a solution of the last problem 5.7 for the case of constant angular velocity of rotation of the horizontal plane is given, dealing with reduced equations we provide a procedure of solution applicable in the case of constant angular velocity as well as of nonconstant angular velocity.

2 Lagrangian systems on fibered manifolds

Throughout the paper we consider a fibered manifold $\pi\colon Y \to X$ with a one-dimensional base space X and $(m+1)$-dimensional total space Y. We use jet prolongations $\pi_1\colon J^1Y \to X$ and $\pi_2\colon J^2Y \to X$ and jet projections $\pi_{1,0}\colon J^1Y \to Y$ and $\pi_{2,1}\colon J^2Y \to J^1Y$. Configuration space at a fixed time is represented by a fiber of the fibered manifold π and a corresponding phase space is then a fiber of the fibered manifold π_1. Local fibered coordinates on Y are denoted by (t, q^σ), where $1 \le \sigma \le m$. The associated coordinates on J^1Y and J^2Y are denoted by $(t, q^\sigma, \dot q^\sigma)$ and $(t, q^\sigma, \dot q^\sigma, \ddot q^\sigma)$, respectively. In calculations we use either a canonical basis of one forms on J^1Y, $(dt, dq^\sigma, d\dot q^\sigma)$, or a basis adapted to the contact structure, $(dt, \omega^\sigma, d\dot q^\sigma)$, where

$$\omega^\sigma = dq^\sigma - \dot q^\sigma\ dt, \quad 1 \le \sigma \le m.$$

Whenever possible, the summation convention is used. If $f(t, q^\sigma, \dot q^\sigma)$ is a function defined on an open set of J^1Y we write

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q^\sigma}\,\dot q^\sigma + \frac{\partial f}{\partial \dot q^\sigma}\,\ddot q^\sigma, \quad \frac{\bar d f}{\bar d t} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q^\sigma}\,\dot q^\sigma.$$

A (local) section δ of π_1 is called *holonomic* if $\delta = J^1\gamma$ for a section γ of π.

A vector field ξ defined on J^1Y is called π_1-*vertical* (or simply *vertical*) if $T\pi_1 \cdot \xi = 0$, where T is the tangent functor. Similarly, a vector field ξ is called $\pi_{1,0}$-*vertical* if $T\pi_{1,0} \cdot \xi = 0$.

A differential form ρ is called *contact* if $J^1\gamma^*\rho = 0$ for every section γ of π. A differential form ρ is called *horizontal* if $i_\xi \rho = 0$ for every vertical vector field ξ. We denote by h the operator assigning to ρ its horizontal part. Every 2-form on J^1Y is contact and admits a *unique decomposition* $\pi^*_{2,1}\rho = \rho_1 + \rho_2$, where ρ_1 is a 1-*contact* form on J^2Y (i.e. for every vertical vector field ξ, $i_\xi \rho_1$ is a horizontal form), and ρ_2 is a 2-*contact* form (i.e. for every vertical vector field ξ, $i_\xi \rho_2$ is a 1-contact form). We denote by p_1, and p_2 operators assigning to ρ its 1-contact and 2-contact part, respectively.

By a *distribution* on J^1Y we shall mean a mapping D assigning to every point $z \in J^1Y$ a vector subspace $D(z)$ of the vector space T_zJ^1Y. A distribution can be spanned by a system of (local) vector fields. If D is a distribution, we denote by D^0 its annihilator, i.e. the set of all 1-forms η_κ on J^1Y such that $i_{\xi_\iota}\eta_\kappa = 0$ for every vector field ξ_ι belonging to D. In this sense, every distribution can be defined by a system of (local) 1-forms. For a distributions of a constant rank, i.e. that $\dim D(z)$ does not depend on z, the description by means of vector fields is completely equivalent with that by means of 1-forms. Recall that a section δ of π_1 is called an *integral section* of D if $\delta^*\eta = 0$ for every 1-form η belonging to D^0.

If λ is a Lagrangian on J^1Y, we denote by θ_λ its *Lepage equivalent* or *Cartan form* and E_λ its *Euler-Lagrange form*, respectively. Recall that $E_\lambda = p_1\, d\theta_\lambda$. In fibered coordinates where $\lambda = L(t, q^\sigma, \dot{q}^\sigma)\, dt$, we have

$$\theta_\lambda = L\, dt + \frac{\partial L}{\partial \dot{q}^\sigma}\, \omega^\sigma, \tag{1}$$

and $E_\lambda = E_\sigma(L)\omega^\sigma \wedge dt$, where the components

$$E_\sigma(L) = \frac{\partial L}{\partial q^\sigma} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^\sigma} \tag{2}$$

are the *Euler-Lagrange expressions.* Since the functions E_σ are affine in the second derivatives we write

$$E_\sigma = A_\sigma + B_{\sigma\nu}\, \ddot{q}^\nu,$$

where

$$A_\sigma = \frac{\partial L}{\partial q^\sigma} - \frac{\partial^2 L}{\partial t \partial \dot{q}^\sigma} - \frac{\partial^2 L}{\partial q^\nu \partial \dot{q}^\sigma}\dot{q}^\nu, \quad B_{\sigma\nu} = -\frac{\partial^2 L}{\partial \dot{q}^\sigma \partial \dot{q}^\nu}. \tag{3}$$

A section γ of π is called a *path* of the Euler-Lagrange form E_λ if

$$E_\lambda \circ J^2\gamma = 0. \tag{4}$$

In fibered coordinates this equation represents a system of m second-order ordinary differential equations

$$A_\sigma\Big(t, \gamma^\nu, \frac{d\gamma^\nu}{dt}\Big) + B_{\sigma\rho}\Big(t, \gamma^\nu, \frac{d\gamma^\nu}{dt}\Big)\frac{d^2\gamma^\rho}{dt^2} = 0 \tag{5}$$

for components $\gamma^\nu(t)$ of a section γ, where $1 \le \nu \le m$. These equations are called *Euler-Lagrange equations* or *motion equations* and their solutions are called *paths.*

Euler-Lagrange equations (4) or (5) can be written either in an intrinsic form as follows

$$J^1\gamma^* i_\xi d\theta_\lambda = 0,$$

where ξ runs over all π_1-vertical vector fields on J^1Y, or equivalently in the form

$$J^1\gamma^* i_\xi \alpha = 0,$$

where α is any 2-form defined on an open subset $W \subset J^1Y$, such that $p_1\alpha = E_\lambda$. Apparently $\alpha = d\theta_\lambda + F$, where F runs over $\pi_{1,0}$-horizontal 2-contact 2-forms. In fibered coordinates we have $F = F_{\sigma\nu}\,\omega^\sigma \wedge \omega^\nu$, where $F_{\sigma\nu}(t, q^\rho, \dot{q}^\rho)$ are arbitrary functions. Recall from [14] that the family of all such (local) 2-forms:

$$\alpha = d\theta_\lambda + F = A_\sigma \omega^\sigma \wedge dt + B_{\sigma\nu}\omega^\sigma \wedge d\dot{q}^\nu + F$$

is called a *first order Lagrangian system*, and is denoted by $[\alpha]$.

It is important to note that motion equations (5) of a Lagrangian system $[\alpha]$ need not be affine with respect to the second derivatives. If they posses this property, i.e. if

$$\det(B_{\sigma\rho}) = \det\left(\frac{\partial^2 L}{\partial\dot{q}^\sigma \partial\dot{q}^\nu}\right) \neq 0,$$

then the Lagrangian system $[\alpha]$ is called *regular*.

3 Constraints

From the physical point of view, constraints on a mechanical system are conditions restricting possible geometrical positions of the mechanical system or limiting its motion. We distinguish between geometric and kinematic constraints.

Constraints are called *geometric* or *holonomic* if they are expressed by equations of the form

$$f^i(t, q^1, \ldots, q^m) = 0, \quad 1 \leq i \leq k,$$

where m is a dimension of the configuration space and k is a given number (the number of constraint equations). Functions f^i are defined on the configuration space. Holonomic constraints are called *skleronomic* if they do not depend explicitly on time

$$f^i(q^1, \ldots, q^m) = 0, \quad 1 \leq i \leq k.$$

From the geometric point of view holonomic constraints represent submanifolds in the configuration space-time Y.

Constraints are called *kinematic* if they are expressed by

$$f^i(t, q^1, \ldots, q^m, \dot{q}^1, \ldots, \dot{q}^m) = 0, \quad 1 \leq i \leq k. \tag{6}$$

Now f^i are functions on the “phase space” J^1Y. Kinematic constraints are said to be *integrable* if the corresponding system of differential equations (6) is integrable. Integrable kinematic constraints are geometric constraints, since after integration they represent a restriction in the configuration space. Nonintegrable kinematic constraints (6), which cannot be reduced to geometric ones are called *nonholonomic* constraints.

Holonomic or nonholonomic constraints which depend explicitly on time are called *rheonomic*.

Nonholonomic constraints (6) are called *affine* or *linear in velocities* if they can be expressed by

$$\mathcal{A}_i(t,q^\nu) + \mathcal{B}_{i\sigma}(t,q^\nu)\,\dot{q}^\sigma = 0, \quad 1 \le \sigma,\nu \le m,\ 1 \le i \le k. \tag{7}$$

In particular, if the left-hand sides of (7) can be written in the form of total time derivatives of some functions defined on the configuration space, say $\frac{d\psi^i(t,q^\nu)}{dt} = 0$, then instead of equations (7) we write

$$\psi^i(t,q^\nu) - C^i = 0, \quad 1 \le i \le k,$$

where C^i are constants determined by initial conditions. In this case constraints (7) are called *linear integrable* or *semiholonomic* and the following identities hold

$$\mathcal{A}_i = \frac{\partial \psi^i}{\partial t}, \quad \mathcal{B}_{i\sigma} = \frac{\partial \psi^i}{\partial q^\sigma}.$$

Nonholonomic constraints (6) are called *affine of degree n in velocities* if they can be expressed by

$$f^i \equiv \mathcal{A}_i(t,q^\nu) + \mathcal{B}_{i\sigma}(t,q^\nu)\,(\dot{q}^\sigma)^n = 0, \quad 1 \le \sigma,\nu \le m,\ 1 \le i \le k.$$

For example, a relativistic particle in space-time $\mathbb{R}^4$ with Minkowski metric can be considered as mechanical system subjected to one nonholonomic constraint

$$-(\dot{q}^1)^2 - (\dot{q}^2)^2 - (\dot{q}^3)^2 + (\dot{q}^4)^2 - 1 = 0,$$

see [19], which is simple affine of degree 2 in velocities.

A geometric meaning of nonholonomic constraints is such that they represent submanifolds in the jet space J^1Y.

4 Nonholonomic Lagrangian systems

Following [14] we introduce general nonholonomic constraints (6) as submanifolds of J^1Y canonically endowed with a distribution.

Let $k < m$ be an integer. By a *constraint submanifold* in J^1Y we mean a fibered submanifold $\pi_{1,0}|_Q \colon Q \to Y$ of the fibered manifold $\pi_{1,0} \colon J^1Y \to Y$. We denote by ι the canonical embedding of Q into J^1Y, and suppose $\operatorname{codim} Q = k < m$ (cf. for example [14], [15], [21], [23]). Locally, Q can be given by equations

$$f^i(t,q^1,\dots,q^m,\dot{q}^1,\dots,\dot{q}^m) = 0, \quad 1 \le i \le k,$$

where

$$\operatorname{rank}\left(\frac{\partial f^i}{\partial \dot{q}^\sigma}\right) = k, \tag{8}$$

or, equivalently in an explicit form

$$\dot{q}^{m-k+i} = g^i(t,q^\sigma,\dot{q}^1,\dot{q}^2,\dots,\dot{q}^{m-k}), \quad 1 \le i \le k. \tag{9}$$

Equations (9) are called a *system of k nonholonomic constraints in normal form.*

The presence of a constraint submanifold in J^1Y gives rise to a concept of a *constrained section* as a local section $\bar{\delta}$ of the fibered manifold π_1 such that $\bar{\delta}(x) \in Q$ for every $x \in \operatorname{dom} \bar{\delta}$ and a *Q-admissible* section as a section $\bar{\gamma}$ of the fibered manifold π such that $J^1\bar{\gamma}(x) \in Q$ for every $x \in \operatorname{dom} \bar{\gamma}$.

The submanifold Q is naturally endowed with a distribution, called the *canonical distribution* [14], or *Chetaev bundle* [21], and denoted by C. It is annihilated by a system of k linearly independent (local) 1-forms

$$\varphi^i = \iota^* \phi^i, \quad \text{where} \quad \phi^i = f^i dt + \frac{\partial f^i}{\partial \dot{q}^\sigma} \omega^\sigma, \quad 1 \le i \le k,$$

called *canonical constraint 1-forms*. More frequently we shall use equations of a constraint submanifold Q in the form (9), i.e. $f^i = \dot{q}^{m-k+i} - g^i$. In this case canonical contact 1-forms $\bar{\omega}^\sigma = \iota^* \omega^\sigma$, $1 \le \sigma \le m$, restricted on Q split into two kinds of forms $\bar{\omega}^l = dq^l - \dot{q}^l dt$, $1 \le l \le m-k$, and $\bar{\omega}^{m-k+i} = dq^{m-k+i} - g^i dt$, $1 \le i \le k$, and we obtain the following local coordinate representation of canonical constraint 1-forms

$$\varphi^i = -\sum_{l=1}^{m-k} \frac{\partial g^i}{\partial \dot{q}^l} \, \bar{\omega}^l + \bar{\omega}^{m-k+i}, \quad 1 \le i \le k. \tag{10}$$

The ideal in the exterior algebra of forms on Q generated by canonical constraint 1-forms is called the *constraint ideal*, and denoted by I; its elements are called *constraint forms*. The pair (Q, C) is then called a *(nonholonomic) constraint structure* on the fibered manifold π [14], [15].

Remark 1. *From the point of view of physics, the rank of the canonical distribution C has the meaning of the number of (generalized, or "phase space") degrees of freedom of systems constrained to Q, and the canonical distribution itself represents possible (generalized) displacements. Its π_1-vertical and $\pi_{1,0}$-vertical subdistribution then has the meaning of virtual (generalized) displacements and virtual velocities, respectively.*

Now we will recall the concept of a nonholonomic Lagrangian system. Consider on J^1Y an unconstrained Lagrangian system $[\alpha] = [d\theta_\lambda]$. With help of the nonholonomic constraint structure (Q, C) one can construct a new mechanical system directly on the constraint submanifold Q of J^1Y. In keeping with [14], [15], by a related *(nonholonomic) constrained system* we shall mean an equivalence class of 2-forms on Q elements of which are of the form

$$\alpha_Q = \iota^* d\theta_\lambda + \bar{F} + \varphi_{(2)},$$

where $\bar{F}$ and $\varphi_{(2)}$ run over all 2-contact $\pi_{1,0}$-horizontal 2-forms and constraint 2-forms defined on Q, respectively. For the constrained system we use notation $[\alpha_Q]$. Equations of motion of the constrained system $[\alpha_Q]$, then have the following intrinsic form:

$$J^1\bar{\gamma}^* i_\xi \iota^* d\theta_\lambda = 0 \quad \text{for every vertical vector field } \xi \in C, \tag{11}$$

where $\bar{\gamma}$ is a Q-admissible section of π. These equations are sometimes called *reduced equations of motion* of the constrained system $[\alpha_Q]$, since they are restricted to the constraint submanifold Q.

Let us find a coordinate expression of a representative of the class $[\alpha_Q]$ and an explicit expression of reduced equations of motion of the constrained system $[\alpha_Q]$ arising from the Lagrangian system $[\alpha]$ and a nonholonomic constraint structure (Q, C). Let $\lambda = L(t, q^\sigma, \dot{q}^\sigma)\, dt$ be a (local) Lagrangian for an unconstrained Lagrangian system $[\alpha] = [d\theta_\lambda]$, where θ_λ is its Cartan form coordinate representation of which is given by (1), and consider the constraint submanifold Q locally given by equations (9) in normal form. We introduce Lagrange function $\bar{L}$ on the constraint submanifold Q as the restriction of the original unconstrained Lagrange function L on Q, i.e. $\bar{L} = L \circ \iota$, thus $\bar{L}(t, q^\sigma, \dot{q}^l) = L\left(t, q^\sigma, \dot{q}^l, g^i(t, q^\sigma, \dot{q}^l)\right)$. Computing the coordinate expression of $\iota^* d\theta_\lambda$ we get that a representative of the class $[\alpha_Q]$ takes the form

$$\alpha_Q = \sum_{l=1}^{m-k} A'_l \omega^l \wedge dt + \sum_{l,s=1}^{m-k} B'_{ls} \omega^l \wedge d\dot{q}^s + \bar{F} + \varphi_{(2)},$$

where the components A'_l are given by

$$A'_l = \frac{\partial \bar{L}}{\partial q^l} + \frac{\partial \bar{L}}{\partial q^{m-k+i}} \frac{\partial g^i}{\partial \dot{q}^l} - \frac{\bar{d}_c}{dt} \frac{\partial \bar{L}}{\partial \dot{q}^l} + \\ + \left(\frac{\partial L}{\partial \dot{q}^{m-k+j}}\right)_\iota \left[\frac{\bar{d}_c}{dt}\left(\frac{\partial g^j}{\partial \dot{q}^l}\right) - \frac{\partial g^j}{\partial q^l} - \frac{\partial g^j}{\partial q^{m-k+i}} \frac{\partial g^i}{\partial \dot{q}^l}\right], \quad (12)$$

where

$$\frac{\bar{d}_c}{dt} = \frac{\partial}{\partial t} + \dot{q}^s \frac{\partial}{\partial q^s} + g^i \frac{\partial}{\partial q^{m-k+i}}.$$

Components $B'_{l,s}$ are of the form

$$B'_{ls} = -\frac{\partial^2 \bar{L}}{\partial \dot{q}^l \partial \dot{q}^s} + \left(\frac{\partial L}{\partial \dot{q}^{m-k+i}}\right)_\iota \frac{\partial^2 g^i}{\partial \dot{q}^l \partial \dot{q}^s}. \quad (13)$$

Finally, reduced equations of motion of the constrained system $[\alpha_Q]$ (11) in fibered coordinates take the form

$$\frac{\partial \bar{L}}{\partial q^l} + \frac{\partial \bar{L}}{\partial q^{m-k+i}} \frac{\partial g^i}{\partial \dot{q}^l} - \frac{d_c}{dt}\left(\frac{\partial \bar{L}}{\partial \dot{q}^l}\right) + \\ + \left(\frac{\partial L}{\partial \dot{q}^{m-k+j}}\right)_\iota \left[\frac{d_c}{dt}\left(\frac{\partial g^j}{\partial \dot{q}^l}\right) - \frac{\partial g^j}{\partial q^l} - \frac{\partial g^j}{\partial q^{m-k+i}} \frac{\partial g^i}{\partial \dot{q}^l}\right] = 0\,,$$

where

$$\frac{d_c}{dt} = \frac{\bar{d}_c}{dt} + \ddot{q}^s \frac{\partial}{\partial \dot{q}^s}.$$

Notice that the above system of equations can be viewed as 2nd order equations

$$\left(A'_l + \sum_{s=1}^{m-k} B'_{ls} \ddot{q}^s\right) \circ J^2\bar{\gamma} = 0, \quad (14)$$

for components $\gamma^1(t), \gamma^2(t), \ldots, \gamma^{m-k}(t)$ of a Q-admissible section $\bar{\gamma}$ dependent on time t and parameters $q^{m-k+1}, q^{m-k+2}, \ldots, q^m$, which have to be determined as functions $\gamma^{m-k+1}(t), \gamma^{m-k+2}(t), \ldots, \gamma^m(t)$ from the equations (9) of the constraint

$$\frac{dq^{m-k+i}}{dt} = g^i\left(t, q^\sigma, \frac{dq^1}{dt}, \frac{dq^2}{dt}, \ldots, \frac{dq^{m-k}}{dt}\right), \quad 1 \le i \le k.$$

A nonholonomic constraint system $[\alpha_Q]$ is called *regular* if the matrix $(B'_{l,s})$ is regular, i.e.

$$\det\left(\frac{\partial \bar{L}}{\partial \dot{q}^l \partial \dot{q}^s} - \left(\frac{\partial L}{\partial \dot{q}^{m-k+i}}\right)_\iota \frac{\partial^2 g^i}{\partial \dot{q}^l \partial \dot{q}^s}\right) \neq 0.$$

For more details on concepts and results in this section the reader is referred e.g. to the survey article [18].

5 Examples of nonholonomic mechanical systems

5.1 Decelerated motion of a free particle

Consider a "free particle" in $\mathbb{R}^3$ moving in such a way, that the square of its speed decreases proportionally to the reciprocal value of time passed from the beginning of the motion. (See [14], p. 5123, Example 1.)

We denote by (t) the coordinate on $X = \mathbb{R}$, by (t, q^1, q^2, q^3) fibered coordinates on $Y = \mathbb{R} \times \mathbb{R}^3$, and $(t, q^1, q^2, q^3, \dot{q}^1, \dot{q}^2, \dot{q}^3)$ the associated coordinates on $J^1Y = \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3$.

Lagrangian of a free particle has the standard form

$$\lambda = L\, dt = \frac{1}{2} m \left((\dot{q}^1)^2 + (\dot{q}^2)^2 + (\dot{q}^3)^2\right) dt,$$

where m is the mass of the particle. We consider a first order mechanical system $[\alpha]$

$$\alpha = d\theta_\lambda + F = -m\left(\omega^1 \wedge d\dot{q}^1 + \omega^2 \wedge d\dot{q}^2 + \omega^3 \wedge d\dot{q}^3\right) + F \tag{15}$$

on the fibered manifold $\mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$, related with the Euler–Lagrange form

$$E = \sum_{\sigma=1}^{3} -m\ddot{q}^\sigma\, dq^\sigma \wedge dt.$$

The motion of the mechanical system $[\alpha]$ is for $t > 0$ subject to the following nonholonomic constraint Q

$$f(t, q^\sigma, \dot{q}^\sigma) \equiv \left[\left(\dot{q}^1\right)^2 + \left(\dot{q}^2\right)^2 + \left(\dot{q}^3\right)^2\right] - 1/t = 0, \tag{16}$$

meaning that the particle's speed decreases proportionally to $1/\sqrt{t}$. This nonholonomic constraint is rheonomic and is affine of degree 2 in components of velocity. In a neighbourhood of the submanifold Q

$$\operatorname{rank}\left(\frac{\partial f^i}{\partial \dot{q}^\sigma}\right) = 2t(\dot{q}^1, \dot{q}^2, \dot{q}^3) = 1,$$

i.e. condition (8) is satisfied.

Let $U \subset J^1Y$ be the set of all points, where $\dot{q}^3 > 0$, and consider on U canonical coordinates and the adapted coordinates $(t, q^1, q^2, q^3, \dot{q}^1, \dot{q}^2, \bar{f})$, where $\bar{f} = \dot{q}^3 - g$, $g = \sqrt{1/t - (\dot{q}^1)^2 - (\dot{q}^2)^2}$ is the equation of the constraint (16) in normal form. Notice that $g > 0$ on U.

The constrained system $[\alpha_Q]$ related to the mechanical system $[\alpha]$ (15) and the constraint Q (16) is the equivalence class of the 2-form

$$\alpha_Q = \sum_{l=1,2} A'_l\, \omega^l \wedge dt + \sum_{l,s=1,2} B'_{ls}\, \omega^l \wedge d\dot{q}^s + \bar{F} + \varphi_{(2)}$$

on Q, where

$$A'_l = \left[-\frac{m\dot{q}^l}{2t(\dot{q}^3)^2}\left(\left(\dot{q}^1\right)^2 + \left(\dot{q}^2\right)^2 + \left(\dot{q}^3\right)^2\right)\right]_\iota = -\frac{m\dot{q}^l}{2t^2g^2}, \qquad 1 \le l \le 2,$$

$$B'_{ls} = \left[-m\left(\delta_{ls} + \frac{\dot{q}^l\dot{q}^s}{(\dot{q}^3)^2}\right)\right]_\iota = -m\left(\delta_{ls} + \frac{\dot{q}^l\dot{q}^s}{g^2}\right), \qquad 1 \le l, s \le 2,$$

and $\bar{F}$ is any 2-contact 2-form and $\varphi_{(2)}$ is any constraint 2-form defined on Q. The matrix $(-B'_{ls})$ is on $Q \cap U$ equivalent to the matrix

$$\begin{pmatrix} g^2 + (\dot{q}^1)^2 & \dot{q}^1\dot{q}^2 \\ \dot{q}^1\dot{q}^2 & g^2 + (\dot{q}^2)^2 \end{pmatrix},$$

hence

$$\begin{pmatrix} g^2 + (\dot{q}^1)^2 & \dot{q}^1\dot{q}^2 \\ 0 & \frac{g^2}{t} \end{pmatrix},$$

which is obviously regular at each point of $Q \cap U$. This means that the constrained system $[\alpha_Q]$ is regular on $Q \cap U$.

Reduced equations of motion of the constrained system are as follows

$$\left[\frac{m\dot{q}^1}{2t^2g^2} + m\left(1 + \frac{(\dot{q}^1)^2}{g^2}\right)\ddot{q}^1 + m\frac{\dot{q}^1\dot{q}^2}{g^2}\,\ddot{q}^2\right] \circ J^2\bar{\gamma} = 0\,,$$

$$\left[\frac{m\dot{q}^2}{2t^2g^2} + m\left(1 + \frac{(\dot{q}^2)^2}{g^2}\right)\ddot{q}^2 + m\frac{\dot{q}^1\dot{q}^2}{g^2}\,\ddot{q}^1\right] \circ J^2\bar{\gamma} = 0\,,$$

where $\bar{\gamma} = (t, q^1(t), q^2(t), q^3(t))$ is a Q-admissible section, i.e. a section satisfying the constraint equation $f \circ J^1\gamma = 0$. After arrangements we obtain equations of motion of the constrained system in the following simple form:

$$\ddot{q}^1(t) = -\frac{1}{2t}\dot{q}^1(t)\,,$$

$$\ddot{q}^2(t) = -\frac{1}{2t}\dot{q}^2(t)\,,$$

$$\dot{q}^3(t) = \sqrt{\frac{1}{t} - (\dot{q}^1)^2 - (\dot{q}^2)^2}\,.$$

Solution of these equations is

$$q^1(t) = C_1^1\sqrt{t} + C_2^1 \,,$$
$$q^2(t) = C_1^2\sqrt{t} + C_2^2 \,,$$
$$q^3(t) = C_1^3\sqrt{t} + C_2^3 \,,$$

where C_j^i are constants connected by the relation $C_1^3 = \sqrt{4 - (C_1^1)^2 + (C_1^2)^2}$. Analogous results are obtained if one considers the other adapted charts belonging to an atlas covering Q.

5.2 A dog pursuing a man

Consider a man and a dog moving in the plane. The man starts from the origin O of the coordinate system Oxy and moves along the y-axis with a constant velocity c. His dog starts at the same moment from the point $[x_0, y_0]$, $x_0 \geq 0$, $y_0 \neq 0$ and runs in such a way, that its velocity at each moment is given by the line connecting its instantaneous position and the instantaneous position of the man. We shall find the trajectory of the dog. (See [2], pp. 236–239.)

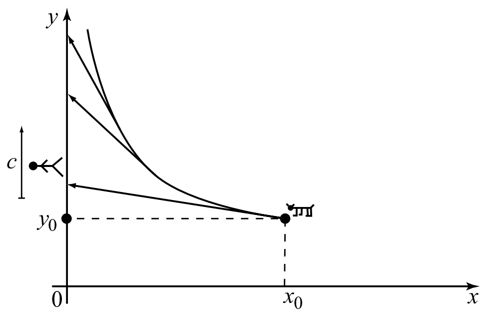

Figure 1

We denote by (t) the coordinate on $X = \mathbb{R}$, by (t, x, y) the canonical coordinates on $Y = \mathbb{R}\times\mathbb{R}^2$ and by $(t, x, y, \dot{x}, \dot{y})$ the associated coordinates on $J^1Y = \mathbb{R}\times\mathbb{R}^2\times\mathbb{R}^2$.

The Lagrangian of this problem is

$$\lambda = L\,dt = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)\,dt$$

and defines a first order mechanical system $[\alpha]$ on the fibered manifold $\mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}$ represented by the Lepage 2-form

$$\alpha = d\theta_\lambda + F = -m\,\omega^1 \wedge d\dot{x} - m\,\omega^2 \wedge d\dot{y} + F, \tag{17}$$

where m denotes the mass of the dog, $\omega^1 = dx - \dot{x}\,dt$, $\omega^2 = dy - \dot{y}\,dt$ are corresponding contact 1-forms and F is any 2-contact 2-form. This mechanical system is related to the dynamical form

$$E = -m\ddot{x}\,dx \wedge dt - m\ddot{y}\,dy \wedge dt.$$

The constraint is given by the requirement that at each moment the direction of the motion of the dog is known. For the angular coefficient of the dog's trajectory it holds

$$\frac{dy}{dx} = G(t, x, y). \tag{18}$$

This equation can be written in the equivalent form

$$G(t, x, y)\, \dot{x} - \dot{y} = 0 \tag{19}$$

which is a rheonomic nonholonomic constraint affine in components of velocity. On the other hand, the instantaneous direction of the motion of the dog at a time t and at a point $[x, y]$ is given by the line connecting this point with the point $[0, c\,t]$ where the man is at this moment. Hence the angular coefficient of the trajectory at a time t and at a point $[x, y]$ is given by

$$G(t, x, y) = \frac{y - c\,t}{x}, \quad x \neq 0. \tag{20}$$

Consequently, the nonholonomic constraint (19) has the form

$$\dot{y} = \frac{y - c\,t}{x}\, \dot{x}. \tag{21}$$

This equation defines a constraint submanifold $Q \subset J^1Y$, since the rank condition (8)

$$\operatorname{rank}\left(\frac{y - c\,t}{x},\ -1\right) = 1$$

is satisfied. The canonical constraint 1-form (10) reads

$$\varphi = -(y - c\,t)\, dx + x\, dy.$$

The constrained system $[\alpha_Q]$ related to the mechanical system $[\alpha]$ (17) and the constraint Q given by (21) is the equivalence class of the 2-form

$$\alpha_Q = A'_1\, \omega^1 \wedge dt + B'_{11}\, \omega^1 \wedge d\dot{x} + \bar{F} + \varphi_{(2)},$$

where

$$A'_1 = \frac{mc\dot{x}\,(y - c\,t)}{x^2}, \quad B'_{11} = -\,m\left(1 + \frac{(y - c\,t)^2}{x^2}\right),$$

and $\bar{F}$ is any 2-contact 2-form and $\varphi_{(2)}$ is any constraint 2-form defined on this constraint submanifold Q. Since

$$\det B'_{11} = -\,m\left(\frac{x^2 + (y - c\,t)^2}{x^2}\right) \neq 0,$$

the constrained system $[\alpha_Q]$ is regular.

The reduced equation of motion of the constrained system is

$$\left[\frac{mc\,(y - c\,t)}{x^2}\, \dot{x} - m\left(\frac{x^2 + (y - c\,t)^2}{x^2}\right) \ddot{x}\right] \circ J^2\bar{\gamma} = 0, \tag{22}$$

where $\bar{\gamma} = (t, x(t), y(t))$ is a Q-admissible section satisfying the constraint equation (21).

In [2] the dynamics is obtained by solving Chetaev equations of motion (equations with Lagrange multipliers), which take a very simple form

$$\ddot{x} = \mu^* G(x, y, t)\,,$$
$$\ddot{y} = -\mu^*\,.$$

The symbol $\mu^* = \mu/m$ denotes a (reduced) Lagrange multiplier and G is the function given by (20). Now, multiplying the first equation by $\dot{x}$ and the second one by $\dot{y}$ and adding these equations we get

$$\frac{d}{dt}\left[\frac{1}{2}m(\dot{x}^2 + \dot{y}^2)\right] = \mu^*[G(x, y, t)\,\dot{x} - \dot{y}].$$

Since the constraint equation (19) holds we obtain a first integral

$$\dot{x}^2 + \dot{y}^2 = v^2 = const. \tag{23}$$

This means that the dog moves with a constant speed. This fact together with equation (18) enables us to determine the trajectory of the dog in an explicit form, i.e. $y = y(x)$. To this end we eliminate time parameter from the equations. First we notice that one can write

$$\dot{y} = \frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} \equiv \dot{x}\,y'. \tag{24}$$

Substituting (20) into (18) we obtain

$$\frac{dy}{dx} \equiv y' = \frac{y - c\,t}{x} \quad \text{resp.} \quad x\,y' = y - c\,t,$$

and after differentiating this equation with respect to x,

$$x\,y'' = -c\,\frac{dt}{dx}.$$

Hence, under appropriate conditions,

$$\dot{x} = -\frac{c}{x\,y''}. \tag{25}$$

Since the motion takes place in the first quadrant, relations $x > 0$, $\dot{x} < 0$ hold, and subsequently $y'' > 0$. Substituting identity (24) to the first integral (23) we get

$$\dot{x}^2\left(1 + (y')^2\right) = v^2,$$

and after extracting the square root we can write

$$-\dot{x} = \frac{v}{\sqrt{1 + (y')^2}}.$$

Finally we compare the last equation with equation (25) and after separation of variables we gain the desired differential equation for the curve of pursuit

$$\frac{y''}{\sqrt{1+(y')^2}} = \frac{c}{v}\frac{1}{x}. \tag{26}$$

The fact that both sides of this equation can be written by means of total derivative with respect to x in the following way

$$\frac{d}{dx}\left[\ln\left(y' + \sqrt{1+(y')^2}\right)\right] = \frac{d}{dx}\left(\frac{c}{v}\ln x\right),$$

enables one a reduction of equation (26) to the following first order implicit differential equation

$$\ln\left(y' + \sqrt{1+(y')^2}\right) = \frac{c}{v}\ln x + \ln A, \tag{27}$$

where $\ln A$ is a constant which can be determined with help of initial conditions. Equation (27) can be written in a simpler form

$$y' + \sqrt{1+(y')^2} = A\,x^{\alpha},$$

where $\alpha = \frac{c}{v}$. Expressing y'

$$y' = \frac{1}{2}\left(A\,x^{\alpha} - \frac{1}{A\,x^{\alpha}}\right),$$

and after integration we obtain for $\alpha \neq 1$ a general solution described by the function

$$y = \frac{1}{2}\left[\frac{A}{1+\alpha}\,x^{1+\alpha} - \frac{1}{A(1-\alpha)}\,x^{1-\alpha}\right] + C,$$

where C is a constant to be determined with help of initial conditions. The final explicit form of the desired curve of pursuit is

$$y = y_0 + \frac{1}{2}\left[\frac{A}{1+\alpha}\,(x^{1+\alpha} - x_0^{1+\alpha}) - \frac{1}{A(1-\alpha)}\,(x^{1-\alpha} - x_0^{1-\alpha})\right],$$

where

$$A = \frac{y_0 + \sqrt{x_0^2 + y_0^2}}{x_0^{1+\alpha}},$$

and x_0, y_0 are coordinates of the initial position of the dog.

5.3 Pursuit of a general motion in a plane

Consider an object moving in a plane along an a-priori given curve described by parametric equations $x = \xi(t)$, $y = \eta(t)$, and consider a dog which starts from a point $[x_0, y_0]$, $x_0 \geq 0$, $y_0 \neq 0$, and pursues this object in the same way as above, i.e. that its velocity at each moment is given by the line connecting its instantaneous position and the instantaneous position of the object. We shall find equations of motion of the dog.

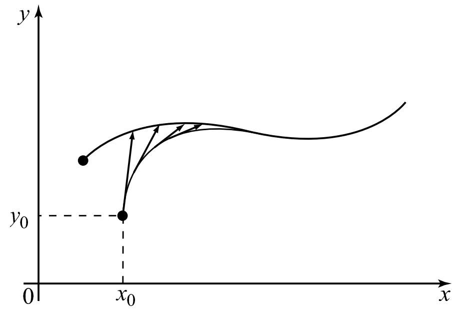

Figure 2

The configuration space Y, the Lagrangian λ and the mechanical system $[\alpha]$ are the same as above, however, restriction of the motion of the dog now is given by the corresponding generalization of the constraint (21) to

$$\dot{y} = G(t, x, y)\,\dot{x} = \frac{y - \eta(t)}{x - \xi(t)}\,\dot{x}. \tag{28}$$

This is again a rheonomic nonholonomic constraint affine in components of velocity, which defines a constraint submanifold Q in the phase space J^1Y. The canonical constraint 1-form (10) now reads

$$\varphi = -(y - \eta(t))\,dx + (x - \xi(t))\,dy.$$

The constrained system $[\alpha_Q]$ related to the mechanical system $[\alpha]$ (17) and the constraint Q given by (28) is again an equivalence class as follows,

$$\alpha_Q = A'_1\,\omega^1 \wedge dt + B'_{11}\,\omega^1 \wedge d\dot{x} + \bar{F} + \varphi_{(2)},$$

where

$$A'_1 = m\dot{x}\,\frac{\dot{\eta}\,(y-\eta)(x-\xi) - \dot{\xi}\,(y-\eta)^2}{(x-\xi)^3}, \quad B'_{11} = -\,m\left(1 + \frac{(y-\eta)^2}{(x-\xi)^2}\right),$$

and $\bar{F}$ is any 2-contact 2-form and $\varphi_{(2)}$ is any constraint 2-form on Q. Since

$$\det\,B'_{11} = -\,m\,\frac{(x-\xi)^2 + (y-\eta)^2}{(x-\xi)^2} \neq 0,$$

the constrained system $[\alpha_Q]$ is again regular.

The reduced equation of motion of the constrained system is

$$m\left[\dot{x}\,\frac{(y-\eta)}{(x-\xi)^2}\,\dot{\eta} - \dot{x}\,\frac{(y-\eta)^2}{(x-\xi)^3}\,\dot{\xi} - \ddot{x}\left(1 + \frac{(y-\eta)^2}{(x-\xi)^2}\right)\right] \circ J^2\bar{\gamma} = 0,$$

where $\bar{\gamma} = (t, x(t), y(t))$ is a Q-admissible section satisfying constraint equation (28). In particular, if we put $\xi(t) = 0$, $\eta(t) = c\,t$, i.e. we consider the motion along the y-axis with a constant speed c, we obtain motion equation (22).

In the same way as in the previous example we can write down Chetaev equations of motion, which have the same form as above,

$$\ddot{x} = \mu^* G(x, y, t)\,,$$
$$\ddot{y} = -\mu^*,$$

but now the function G is given by formula

$$G(x, y, t) = \frac{y - \eta(t)}{x - \xi(t)}.$$

Repeating the same procedure we obtain a first integral

$$\dot{x}^2 + \dot{y}^2 = v^2 = const.$$

However, now we cannot eliminate the time parameter from the equations because of the fact that the pursuing object moves along a curve determined by parametric equations $x = \xi(t)$, $y = \eta(t)$, which need not represent a straight motion with a constant velocity as in the previous example.

5.4 Motion of a particle in a homogeneous gravitational field with constant velocity

Consider a particle of mass m moving in a homogeneous gravitational field (the gravitational acceleration is denoted by G) from a point $(q^1(0), q^2(0), q^3(0))$, $q^3(0) > 0$, with the initial velocity given by a vector $(p^1(0), p^2(0), p^3(0))$, where all the components are non-zero and positive. The motion is restricted by the condition that the speed of the particle remains constant. (See [9], pp. 991, Example 4.2.)

This is a problem originally formulated by Leibnitz in 1689 as follows: find a curve along which a particle moves in a homogeneous gravitational field with a constant speed. A solution of the problem was found by Jacob Bernoulli in 1694 as a curve called the *paracentric isochrone*. However the problem was solved only from the kinematic point of view in the framework of differential geometry of curves. For a complete description of dynamics of the problem it is necessary to understand the requirement of the constant speed as a nonholonomic, so called *isotachystonic* constraint, which is nonlinear.

Our aim is to study the dynamics of the Leibnitz particle.

The configuration space is again $Y = \mathbb{R} \times \mathbb{R}^3$, (t, q^σ), $1 \leq \sigma \leq 3$, are fibered coordinates on Y. The Lagrangian has the form

$$\lambda = L\,dt = \left[\frac{1}{2} m \left((\dot{q}^1)^2 + (\dot{q}^2)^2 + (\dot{q}^3)^2\right) - mGq^3\right] dt.$$

The mechanical system $[\alpha]$ is represented by a Lepage 2-form

$$\alpha = \,-\,mG\,\omega^3 \wedge dt - m\left(\omega^1 \wedge d\dot{q}^1 + \omega^2 \wedge d\dot{q}^2 + \omega^3 \wedge d\dot{q}^3\right) + F, \tag{29}$$

where F is a 2-contact 2-form. The corresponding dynamical form is then

$$E = -mG\, dq^3 \wedge dt - \sum_{\sigma=1}^{3} m\ddot{q}^\sigma\, dq^\sigma \wedge dt\,.$$

The constraint on the motion is given by equation

$$f \equiv (\dot{q}^1)^2 + (\dot{q}^2)^2 + (\dot{q}^3)^2 - C = 0, \tag{30}$$

where $C = \big(p^1(0)\big)^2 + \big(p^2(0)\big)^2 + \big(p^3(0)\big)^2$ is the square of the initial speed of the particle. Equation (30) defines a constraint submanifold Q in J^1Y. It is a skleronomic nonholonomic constraint, affine of degree 2 in components of velocity. Let $U \subset J^1Y$ be the set of all points where $\dot{q}^3 > 0$ and consider on U the adapted coordinates $(t, q^1, q^2, q^3, \dot{q}^1, \dot{q}^2, \bar{f})$, where $\bar{f} = \dot{q}^3 - g$, $g = \sqrt{C - (\dot{q}^1)^2 - (\dot{q}^2)^2}$ is equation of the constraint (30) in normal form.

The constrained system $[\alpha_Q]$ related to the mechanical system $[\alpha]$ (29) and the constraint Q (30) is the equivalence class of 2-forms

$$\alpha_Q = \sum_{l=1,2} A'_l\, \omega^l \wedge dt + \sum_{l,s=1,2} B'_{ls}\, \omega^l \wedge d\dot{q}^s + \bar{F} + \varphi_{(2)} \tag{31}$$

on Q, where

$$A'_l = \left[-m\,G\,\frac{\dot{q}^l}{\dot{q}^3}\right]_\iota = -m\,G\,\frac{\dot{q}^l}{g}, \qquad 1 \le l \le 2,$$

$$B'_{ls} = \left[-\,m\left(\delta_{ls} + \frac{\dot{q}^l\dot{q}^s}{(\dot{q}^3)^2}\right)\right]_\iota = -\,m\left(\delta_{ls} + \frac{\dot{q}^l\dot{q}^s}{g^2}\right), \qquad 1 \le l, s \le 2,$$

and $\bar{F}$ is a 2-contact 2-form and $\varphi_{(2)}$ is a constraint 2-form defined on the constraint submanifold Q. The constrained system $[\alpha_Q]$ is regular since the matrix $(-B'_{ls})$ is the same in the second example above. The motion of this constrained system is described by two reduced equations

$$\left[m\,G\,\frac{\dot{q}^1}{g} + m\left(1 + \frac{(\dot{q}^1)^2}{g^2}\right)\ddot{q}^1 + m\frac{\dot{q}^1\dot{q}^2}{g^2}\,\ddot{q}^2\right] \circ J^2\bar{\gamma} = 0\,,$$

$$\left[m\,G\,\frac{\dot{q}^2}{g} + m\left(1 + \frac{(\dot{q}^2)^2}{g^2}\right)\ddot{q}^2 + m\frac{\dot{q}^1\dot{q}^2}{g^2}\,\ddot{q}^1\right] \circ J^2\bar{\gamma} = 0\,,$$

where $\bar{\gamma} = (t, x(t), y(t))$ is a Q-admissible section satisfying the constraint equation

$$\dot{q}^3 = \sqrt{C - (\dot{q}^1)^2 - (\dot{q}^2)^2}.$$

After simple computations equations of motion of the constrained system take the form

$$\ddot{q}^1(t) = \frac{G}{C}\,\dot{q}^1\,\sqrt{C - (\dot{q}^1)^2 - (\dot{q}^2)^2}\,,$$

$$\ddot{q}^2(t) = \frac{G}{C}\,\dot{q}^2\,\sqrt{C - (\dot{q}^1)^2 - (\dot{q}^2)^2}\,,$$

$$\dot{q}^3(t) = \sqrt{C - (\dot{q}^1)^2 - (\dot{q}^2)^2}\,.$$

The same equations were obtained in [9] by a different method.

The above system of differential equations can be reduced to the first order system

$$\dot p^1(t) = D\,p^1\,\sqrt{C-(p^1)^2-(p^2)^2}\,,$$
$$\dot p^2(t) = D\,p^2\,\sqrt{C-(p^1)^2-(p^2)^2}\,,$$
$$\dot q^3(t) = \sqrt{C-(p^1)^2-(p^2)^2}\,,$$

where we denoted $D = G/C$. Since $\dot p^1 p^2 - p^1 \dot p^2 = 0$, and if moreover $p^2 \neq 0$, then $p^1/p^2 = \kappa$ is a first integral of these equations, which has the positive value $\kappa = p^1(0)/p^2(0)$ determined by the given components of the initial velocity. If we suppose that in a certain interval of time the components p^1, p^2 of the instantaneous velocity are not zero, we can separate equations for p^1 and p^2 and integrate

$$\int \frac{dp^1}{p^1\sqrt{C-\left(1+\frac{1}{\kappa^2}\right)(p^1)^2}} = \int D\,dt$$
$$\int \frac{dp^2}{p^2\sqrt{C-(1+\kappa^2)\,(p^2)^2}} = \int D\,dt\,.$$

After integration we can write

$$\sqrt{C}\,\ln\left[\frac{\sqrt{\frac{C\,\kappa^2}{1+\kappa^2}}-\sqrt{\frac{C\,\kappa^2}{1+\kappa^2}-(p^1)^2}}{p^1}\right] = \frac{G}{C}\,t + b_1\,,$$
$$\sqrt{C}\,\ln\left[\frac{\sqrt{\frac{C}{1+\kappa^2}}-\sqrt{\frac{C}{1+\kappa^2}-(p^2)^2}}{p^2}\right] = \frac{G}{C}\,t + b_2\,,$$

where

$$\frac{\kappa^2}{1+\kappa^2} = \frac{\left(p^1(0)\right)^2}{\left(p^1(0)\right)^2+\left(p^2(0)\right)^2}, \qquad \frac{1}{1+\kappa^2} = \frac{\left(p^2(0)\right)^2}{\left(p^1(0)\right)^2+\left(p^2(0)\right)^2},$$

and b_1, b_2 are some integration constants. Expressing variables p^1, p^2 we obtain

$$p^1 = \frac{dq^1}{dt} = \sqrt{\frac{C\,\kappa^2}{1+\kappa^2}}\;\frac{2B_1 e^{\frac{G}{\sqrt{C}}t}}{B_1^2 e^{\frac{2G}{\sqrt{C}}t}+1}\,,$$
$$p^2 = \frac{dq^2}{dt} = \sqrt{\frac{C}{1+\kappa^2}}\;\frac{2B_2 e^{\frac{G}{\sqrt{C}}t}}{B_2^2 e^{\frac{2G}{\sqrt{C}}t}+1}\,, \tag{32}$$

where B_1, B_2 are constants determined by means of b_1, b_2 by the following relations $B_1 = e^{\sqrt{C}\,b_1}$, $B_2 = e^{\sqrt{C}\,b_2}$. If we take into account given components of the initial velocity $p^1(0), p^2(0), p^3(0)$ which are positive as we assumed, and with respect to the value of the first integral $\kappa = p^1(0)/p^2(0)$ we obtain that

$$B_1 = B_2 = B = \frac{\sqrt{C}-p^3(0)}{\sqrt{\left(p^1(0)\right)^2+\left(p^2(0)\right)^2}}.$$

We find the primitive function

$$\int \frac{e^{\alpha t}}{B^2 e^{2\alpha t} + 1} = \frac{1}{\alpha B} \arctan\left(B\, e^{\alpha t}\right),$$

where $\alpha = G/\sqrt{C}$. Hence the desired functions $q^1(t), q^2(t)$ are

$$q^1(t) = \frac{2C}{G} \sqrt{\frac{\kappa^2}{1+\kappa^2}} \arctan\left(B\, e^{\frac{G}{\sqrt{C}}\, t}\right) + A_1\,,$$

$$q^2(t) = \frac{2C}{G} \sqrt{\frac{1}{1+\kappa^2}} \arctan\left(B\, e^{\frac{G}{\sqrt{C}}\, t}\right) + A_2\,,$$

and A_1, A_2 are constants, which are determined by the initial position of the particle. After elimination of the parameter t from the last equations we can see, that the particle moves in the plane $q^1 - \kappa q^2 - A_1 + \kappa\, A_2 = 0$, which is parallel to the q^3-axis.

Now we can substitute the functions $p^1(t)$, $p^2(t)$ given by (32) into the constraint condition $\dot{q}^3 = \sqrt{C - (p^1)^2 - (p^2)^2}$:

$$\dot{q}^3 = \sqrt{C}\, \frac{|B^2\, e^{\frac{2G}{\sqrt{C}}\, t} - 1|}{B^2\, e^{\frac{2G}{\sqrt{C}}\, t} + 1}. \tag{33}$$

Indeed, for $t = 0$ we obtain $\dot{q}^3(0) = p^3(0)$.

We notice the fact that

$$B^2 = 1 - \frac{2\, p^3(0)\left(\sqrt{C} - p^3(0)\right)}{\left(p^1(0)\right)^2 + \left(p^2(0)\right)^2} < 1,$$

since all the components of the initial velocity are non-zero.

As a consequence of the above property and due to the physical reason that potential energy of a homogeneous gravitational field increases proportionally to q^3, it turns out that in some time $T = -\frac{\sqrt{C}}{G} \ln B$ the motion in the vertical direction stops, i.e. $\dot{q}^3(T) = 0$, and then it proceeds with $\dot{q}^3(t) < 0$. Hence for the time $t > T$ one has to consider the constraint condition in the form $\dot{q}^3 = -\sqrt{C - (p^1)^2 - (p^2)^2}$.

Integrating equation (33) we get that in the time interval $(0, T)$ the solution $q^3(t)$ is described by the function

$$q^3(t) = \frac{C}{2G} \ln \left[\frac{B^2\, e^{\frac{2G}{\sqrt{C}}\, t}}{\left(B^2\, e^{\frac{2G}{\sqrt{C}}\, t} + 1\right)^2} \right] + A_3 = -\frac{C}{G} \ln \left[2 \cosh \left(\frac{G\, t + b\, C}{\sqrt{C}} \right) \right] + A_3,$$

where the relationship between constants B and b is given by $b = 1/\sqrt{C}\, \ln B$, and A_3 is a constant, which can be determined by means of $q^3(0)$.

It is worth notice properties of the "nonholonomic fall" in a homogeneous gravitational field: One could expect that the motion will turn to the vertical direction and the particle will fall down with increasing acceleration. However, the constraint condition keeps the speed constant, therefore the components $\dot{q}^1(t)$, $\dot{q}^2(t)$ of the instantaneous velocity have to decrease proportionally, and after some time the motion will proceed in the vertical direction with a constant velocity determined by the vector $(0, 0, \sqrt{C})$.

5.5 Motion of a particle in a homogeneous gravitational field subject to a nonlinear constraint

Consider a particle of mass m in a homogeneous gravitational field (the same as in the previous example). The motion of the particle is now subjected to a nonholonomic condition $b^2\left((\dot q^1)^2+(\dot q^2)^2\right)-(\dot q^3)^2=0$, where b is a constant. (See [9], pp. 992, Example 4.3.)

This mechanical system is the same as above, i.e. it is represented by the Lepage form (29). However the constraint condition

$$f \equiv b^2\left((\dot q^1)^2+(\dot q^2)^2\right)-(\dot q^3)^2=0, \tag{34}$$

or equivalently in normal form

$$\dot q^3 = g = b\sqrt{(\dot q^1)^2+(\dot q^2)^2} \tag{35}$$

is different. The constraint (34) is again a skleronomic nonholonomic constraint, which is affine of degree 2 in components of velocity.

The corresponding constrained mechanical system is given by the equivalence class $[\alpha_Q]$ of 2-forms (31), where

$$A'_l = \left[-\,m\,G\,\frac{b^2\dot q^l}{\dot q^3}\right]_\iota = -\,m\,G\,\frac{b\,\dot q^l}{\sqrt{(\dot q^1)^2+(\dot q^2)^2}} \qquad 1\le l\le 2,$$
$$B'_{ls} = \left[-\,m\left(\delta_{ls}+b^4\frac{\dot q^l\dot q^s}{(\dot q^3)^2}\right)\right]_\iota = -\,m\left(\delta_{ls}+b^2\frac{\dot q^l\dot q^s}{(\dot q^1)^2+(\dot q^2)^2}\right) \quad 1\le l,s\le 2.$$

Reduced equations of motion become the following system of second order ODE's

$$\left[\frac{Gb\,\dot q^1}{\sqrt{(\dot q^1)^2+(\dot q^2)^2}}+\left(1+b^2\frac{(\dot q^1)^2}{(\dot q^1)^2+(\dot q^2)^2}\right)\ddot q^1+b^2\frac{\dot q^1\dot q^2}{(\dot q^1)^2+(\dot q^2)^2}\,\ddot q^2\right]\circ J^2\bar\gamma=0\,,$$
$$\left[\frac{Gb\,\dot q^2}{\sqrt{(\dot q^1)^2+(\dot q^2)^2}}+\left(1+b^2\frac{(\dot q^2)^2}{(\dot q^1)^2+(\dot q^2)^2}\right)\ddot q^2+b^2\frac{\dot q^1\dot q^2}{(\dot q^1)^2+(\dot q^2)^2}\,\ddot q^1\right]\circ J^2\bar\gamma=0\,,$$

where $\bar\gamma=(t,x(t),y(t))$ is a Q-admissible section satisfying constraint equation (35). Expressing the second derivatives we obtain

$$\begin{aligned}\ddot q^1(t) &= -\,b\,G\,\frac{\dot q^1}{(1+b^2)\sqrt{(\dot q^1)^2+(\dot q^2)^2}}\,,\\ \ddot q^2(t) &= -\,b\,G\,\frac{\dot q^2}{(1+b^2)\sqrt{(\dot q^1)^2+(\dot q^2)^2}}\,.\end{aligned} \tag{36}$$

The same equations are derived in [9] by a different method.

We shall solve the reduced equations. First we differentiate constraint equation (35)

$$\ddot q^3 = \frac{b^2}{\dot q^3}\,(\dot q^1\,\ddot q^1+\dot q^2\,\ddot q^2).$$

Substituting reduced equations (36) we obtain the equality

$$\ddot{q}^3 = -\frac{G\,b^2}{1+b^2},$$

which can be simply integrated

$$\dot{q}^3 \equiv b\sqrt{(\dot{q}^1)^2 + (\dot{q}^2)^2} = -\frac{G\,b^2}{1+b^2}\,t + K_1^3.$$

Finally we substitute the last equality back to (36), and we obtain simple differential equations, which can be reduced to first order equations with separable variables. A complete solution of the problem is obtained in the form

$$\begin{aligned} q^1(t) &= -\frac{1}{2}\,\frac{G\,b^2}{1+b^2}\,K_1^1\,t^2 + K_1^1K_1^3\,t + K_2^1\,, \\ q^2(t) &= -\frac{1}{2}\,\frac{G\,b^2}{1+b^2}\,K_1^2\,t^2 + K_1^2K_1^3\,t + K_2^2\,, \\ q^3(t) &= -\frac{1}{2}\,\frac{G\,b^2}{1+b^2}\,t^2 + K_1^3\,t + K_2^3\,, \end{aligned}$$

where K_j^i are constants, and the identity $(K_1^1)^2 + (K_1^2)^2 = 1/b^2$ holds true.

5.6 A rolling disc on a horizontal plane

Consider a disc of radius R rolling without sliding on a horizontal plane. Let $Oxyz$ be a fixed orthogonal system of coordinates with the x and y-axis in the horizontal plane and the z-axis directed vertically upwards. Then the position of the disc on the plane may be given by five generalized coordinates $x, y, \psi, \varphi, \vartheta$, where x and y are the coordinates of the point P of contact of the disc and the horizontal plane, ψ is the angle of proper rotation of the disc, φ is the angle between the tangent to the disc at the point P and the x-axis, and ϑ is the angle between the rotating axis of the disc and the parallel line to the z-axis which is going through the point P (i.e. $\pi/2 - \vartheta$ is the angle of inclination between the plane of the disc and the horizontal plane). (See [22], pp. 55.)

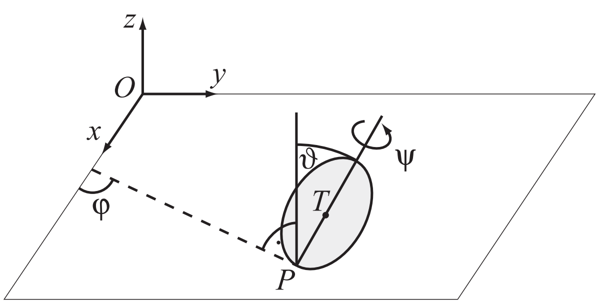

Figure 3

So the base space $X = \mathbb{R}$, the configuration space is $Y = \mathbb{R}\times\mathbb{R}^2\times S^1\times S^1\times S^1$ and phase space is $J^1Y = \mathbb{R}\times\mathbb{R}^2\times S^1\times S^1\times S^1\times\mathbb{R}^2\times S^1\times S^1\times S^1$. Hence fibered coordinates on Y are $(t, x, y, \psi, \varphi, \vartheta)$ and the associated coordinates on J^1Y are $(t, x, y, \psi, \varphi, \vartheta, \dot{x}, \dot{y}, \dot{\psi}, \dot{\varphi}, \dot{\vartheta})$.

The Lagrange function of this mechanical system is given by relation $L = T - V$. The kinetic energy T is given by the sum of the energy of translation and rotation of the disc:

$$\begin{aligned} T = {} & \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + R^2 \dot{\vartheta}^2 + R^2 \dot{\varphi}^2 \sin^2 \vartheta \right) - \\ & - mR \left(\dot{\vartheta} \cos \vartheta \left(\dot{x} \sin \varphi - \dot{y} \cos \varphi \right) + \dot{\varphi} \sin \vartheta \left(\dot{x} \cos \varphi + \dot{y} \sin \varphi \right) \right) + \\ & + \frac{1}{2} I_1 \left(\dot{\vartheta}^2 + \dot{\varphi}^2 \cos^2 \vartheta \right) + \frac{1}{2} I_2 \left(\dot{\psi} + \dot{\varphi} \sin \vartheta \right)^2 , \end{aligned} \tag{37}$$

where m is the mass, and I_1, I_2 are the principal moments of inertia of the disc. The potential energy of the disc is $V = mgR \cos \vartheta$. Formula (37) for kinetic energy of this problem is presented in [22] and is derived in detail in [27].

If we compute motion equation (5) of this Lagrangian system according to (2) and (3), where $1 \leq \sigma, \rho \leq 5$ and coordinates $(q^1, q^2, q^3, q^4, q^5)$ are substituted by corresponding coordinates $(x, y, \psi, \varphi, \vartheta)$, we obtain the following five Euler--Lagrange equations:

$$\begin{aligned} -m\ddot{x} + mR \left((\cos\varphi \sin\vartheta)\ddot{\varphi} + (\sin\varphi \cos\vartheta)\ddot{\vartheta} \right) - \\ - mR \left((\sin\varphi \sin\vartheta)(\dot{\varphi}^2 + \dot{\vartheta}^2) - (2\cos\varphi \cos\vartheta)\dot{\varphi}\dot{\vartheta} \right) = 0 \, , \\ -m\ddot{y} + mR \left((\sin\varphi \sin\vartheta)\ddot{\varphi} - (\cos\varphi \cos\vartheta)\ddot{\vartheta} \right) + \\ + mR \left((\cos\varphi \sin\vartheta)(\dot{\varphi}^2 + \dot{\vartheta}^2) + (2\sin\varphi \cos\vartheta)\dot{\varphi}\dot{\vartheta} \right) = 0 \, , \\ I_2(\ddot{\psi} + \sin\vartheta \ddot{\varphi}) + (I_2 \cos\vartheta)\dot{\varphi}\dot{\vartheta} = 0 \, , \\ mR \left((\cos\varphi \sin\vartheta)\ddot{x} + (\sin\varphi \sin\vartheta)\ddot{y} \right) - (I_2 \sin\vartheta)\ddot{\psi} - \\ - \left((mR^2 + I_2)\sin^2\vartheta + I_1 \cos^2\vartheta \right) \ddot{\varphi} - \\ - (I_2 \cos\vartheta)\dot{\psi}\dot{\vartheta} - 2(mR^2 - I_1 + I_2)(\sin\vartheta \cos\vartheta)\dot{\varphi}\dot{\vartheta} = 0 \, , \\ mR \left((\sin\varphi \cos\vartheta)\ddot{x} - (\cos\varphi \cos\vartheta)\ddot{y} \right) - (mR^2 + I_1)\ddot{\vartheta} + \\ + (mR^2 - I_1 + I_2)(\sin\vartheta \cos\vartheta)\dot{\varphi}^2 + (I_2 \cos\vartheta)\dot{\psi}\dot{\varphi} + mgR \sin\vartheta = 0 \, . \end{aligned}$$

The condition that the disc rolls without sliding on the horizontal plane means, that the instantaneous velocity of the point of contact of the disc is equal to zero at all times. This gives rise to the following nonholonomic constraints

$$f^1 \equiv \dot{x} - R \cos\varphi \dot{\psi} = 0, \qquad f^2 \equiv \dot{y} - R \sin\varphi \dot{\psi} = 0, \tag{38}$$

or in normal form

$$\dot{x} = g^1 \equiv R \cos\varphi \dot{\psi}, \qquad \dot{y} = g^2 \equiv R \sin\varphi \dot{\psi}.$$

One can see that constraints above are linear, or more precisely affine in components of velocities. Equations (38) define a constraint submanifold $Q \subset J^1Y$, since the condition (8) is satisfied, i.e.

$$\operatorname{rank}\left(\frac{\partial f^i}{\partial \dot{q}^\sigma}\right) = \operatorname{rank}\begin{pmatrix} 1 & 0 & -R\cos\varphi & 0 & 0 \\ 0 & 1 & -R\sin\varphi & 0 & 0 \end{pmatrix} = 2.$$

Thus $\dim Q = \dim J^1 Y - 2 = 9$. Constraint 1-forms (10) are in this case the following two forms

$$\varphi^1 = dx - R\cos\varphi d\psi, \qquad \varphi^2 = dy - R\sin\varphi d\psi.$$

Now one can construct the constrained system $[\alpha_Q]$ related to the mechanical system $[\alpha]$ and the constraint Q as the equivalence class of the 2-form

$$\alpha_Q = A'_1\,\omega^1 \wedge dt + A'_2\,\omega^2 \wedge dt + A'_3\,\omega^3 \wedge dt + \\ + \sum_{l=1}^{3} B'_{l1}\,\omega^l \wedge d\dot{\psi} + B'_{l2}\,\omega^l \wedge d\dot{\phi} + B'_{l3}\,\omega^l \wedge d\dot{\vartheta} + \bar{F} + \varphi_{(2)}$$

on Q, where $\omega^1 = d\psi - \dot{\psi}dt$, $\omega^2 = d\varphi - \dot{\varphi}dt$, $\omega^3 = d\vartheta - \dot{\vartheta}dt$ are the corresponding contact 1-forms, and where $\bar{F}$ is a 2-contact 2-form and $\varphi_{(2)}$ is a constraint 2-form defined on Q. Computing the coefficients A'_l according to (12) we obtain the following expressions:

$$\begin{aligned} A'_1 &= (2mR^2 - I_2)(\cos\vartheta)\dot{\varphi}\dot{\vartheta}\,, \\ A'_2 &= -I_2\cos\vartheta\dot{\psi}\dot{\vartheta} - 2(mR^2 - I_1 + I_2)(\sin\vartheta\cos\vartheta)\dot{\varphi}\dot{\vartheta}\,, \\ A'_3 &= (I_2 - mR^2)\cos\vartheta\dot{\psi}\dot{\varphi} + (mR^2 - I_1 + I_2)(\sin\vartheta\cos\vartheta)\dot{\varphi}^2 + mgR\sin\vartheta\,, \end{aligned}$$

and coefficients B'_{ls} according to (13) are

$$\begin{aligned} B'_{11} &= -(mR^2 + I_2)\,, & B'_{12} &= B'_{21} = (mR^2 - I_2)\sin\vartheta\,, \\ B'_{22} &= -(mR^2 + I_2)\sin^2\vartheta - I_1\cos^2\vartheta\,, & B'_{23} &= B'_{32} = 0\,, \\ B'_{33} &= -(mR^2 + I_1)\,, & B'_{31} &= B'_{13} = 0\,. \end{aligned}$$

Hence, reduced equations of motion (14) of the constrained system $[\alpha_Q]$ take the form (see also [26]):

$$\begin{gathered} (mR^2 + I_2)\ddot{\psi} + (I_2 - mR^2)(\sin\vartheta)\ddot{\varphi} + (I_2 - 2mR^2)(\cos\vartheta)\dot{\varphi}\dot{\vartheta} = 0\,, \\ (mR^2 - I_2)(\sin\vartheta)\ddot{\psi} - \left((mR^2 + I_2)\sin^2\vartheta + I_1\cos^2\vartheta\right)\ddot{\varphi} - \\ - I_2(\cos\vartheta)\dot{\psi}\dot{\vartheta} - 2(mR^2 - I_1 + I_2)(\sin\vartheta\cos\vartheta)\dot{\varphi}\dot{\vartheta} = 0\,, \\ -(mR^2 + I_1)\ddot{\vartheta} + (mR^2 - I_1 + I_2)(\sin\vartheta\cos\vartheta)\dot{\varphi}^2 + \\ + (I_2 - mR^2)(\cos\vartheta)\dot{\psi}\dot{\varphi} + mgR\sin\vartheta = 0\,. \end{gathered}$$

These equations can be solved numerically; it turns out that solutions are unstable with respect to a small change of initial conditions.

5.7 A homogeneous ball on a rotating table

Consider a homogeneous ball of radius R rolling without sliding on a horizontal plane which rotates with a nonconstant angular velocity $\Omega(t)$ around the vertical axis. We assume that except the constant gravitational force, no other external forces act on the ball. (See [22], pp. 131, Example 3.)

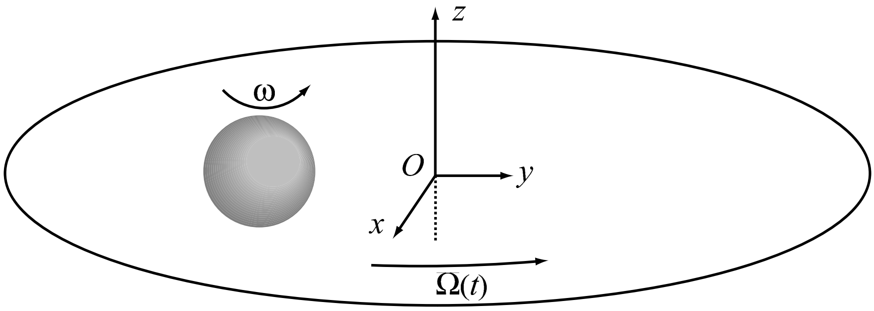

Figure 4

Let the z-axis of the fixed system of coordinates $Oxyz$ coincide with the axis of rotation. Let (x, y) denote the position of contact of the ball with the plane and ϑ, φ, ψ denote Euler angles of the rotating ball. The angle ϑ is the angle of inclination, the φ is the rotating angle and ψ is the angle of precession. Hence $(t, x, y, \vartheta, \varphi, \psi)$ are fibered coordinates on the configuration space $Y = \mathbb{R} \times \mathbb{R}^2 \times SO(3)$, where $SO(3)$ is the special orthogonal group parametrized by Euler angles, and $(t, x, y, \vartheta, \varphi, \psi, \dot{x}, \dot{y}, \dot{\vartheta}, \dot{\varphi}, \dot{\psi})$ are associated coordinates on $J^1Y = \mathbb{R} \times \mathbb{R}^2 \times SO(2) \times \mathbb{R}^2 \times SO(2)$.

The potential energy is constant, so without loss of generality we put $V = 0$. In addition, since we do not consider external forces, the Lagrange function is given by the kinetic energy of the rotating ball

$$L = T = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + k^2(\dot{\vartheta}^2 + \dot{\varphi}^2 + \dot{\psi}^2 + 2\dot{\varphi}\dot{\psi}\cos\vartheta)\right), \tag{39}$$

where k is the radius of gyration and the mass of the ball is $m = 1$.

The motion equations of this Lagrangian system in coordinates $(q^1, \ldots, q^5) = (x, y, \vartheta, \varphi, \psi)$ become:

$$\begin{aligned} \ddot{x} &= 0\,, \\ \ddot{y} &= 0\,, \\ k^2(\ddot{\vartheta} + \sin\vartheta\,\dot{\varphi}\dot{\psi}) &= 0\,, \\ k^2(\ddot{\varphi} + \cos\vartheta\,\ddot{\psi} - \sin\vartheta\,\dot{\vartheta}\dot{\psi}) &= 0\,, \\ k^2(\cos\vartheta\,\ddot{\varphi} + \ddot{\psi} - \sin\vartheta\,\dot{\vartheta}\dot{\varphi}) &= 0\,. \end{aligned}$$

Denoting by ω the instantaneous angular velocity of the ball, we write down the condition of rolling without sliding of the ball on the rotating plane

$$\dot{x} - R\omega_y + \Omega(t)\,y = 0, \qquad \dot{y} + R\omega_x - \Omega(t)\,x = 0, \tag{40}$$

or, using the Euler angles we obtain the following two equations

$$f^1 \equiv \dot{x} - R\sin\psi\,\dot{\vartheta} + R\sin\vartheta\cos\psi\,\dot{\varphi} + \Omega(t)\,y = 0\,,$$
$$f^2 \equiv \dot{y} + R\cos\psi\,\dot{\vartheta} + R\sin\vartheta\sin\psi\,\dot{\varphi} - \Omega(t)\,x = 0\,,$$

which represent two nonholonomic constraints affine in components of velocities. These equations evidently satisfy condition (8),

$$\operatorname{rank}\left(\frac{\partial f^i}{\partial \dot{q}^\sigma}\right) = \operatorname{rank}\begin{pmatrix} 1 & 0 & -R\sin\psi & R\sin\vartheta\cos\psi & 0 \\ 0 & 1 & -R\sin\varphi & R\sin\vartheta\cos\psi & 0 \end{pmatrix} = 2,$$

thus $\dim Q = \dim J^1Y - 2 = 9$. Constraint 1-forms (10) take the form

$$\varphi^1 = dx + \Omega(t)ydt - R\sin\psi d\vartheta + R\sin\vartheta\cos\psi d\varphi\,,$$
$$\varphi^2 = dy - \Omega(t)xdt + R\cos\psi d\vartheta + R\sin\vartheta\sin\psi d\varphi\,.$$

The constrained system $[\alpha_Q]$ is in this case represented by the equivalence class of a 2-form

$$\alpha_Q = A'_1\,\omega^1\wedge dt + A'_2\,\omega^2\wedge dt + A'_3\,\omega^3\wedge dt + \\ + \sum_{l=1}^{3} B'_{l1}\,\omega^l\wedge d\dot{\vartheta} + B'_{l2}\,\omega^l\wedge d\dot{\varphi} + B'_{l3}\,\omega^l\wedge d\dot{\psi} + \bar{F} + \varphi_{(2)}$$

on Q, where $\omega^1 = d\vartheta - \dot{\vartheta}dt$, $\omega^2 = d\varphi - \dot{\varphi}dt$, $\omega^3 = d\psi - \dot{\psi}dt$, and where for the coefficients A'_l we obtain

$$A'_1 = -(R^2+k^2)\dot{\varphi}\dot{\psi}\sin\vartheta + \\ + R\Omega(t)(\dot{x}\cos\psi + \dot{y}\sin\psi) + R\dot{\Omega}(t)(x\cos\psi + y\sin\psi)\,,$$
$$A'_2 = -R^2\dot{\vartheta}\dot{\varphi}\sin\vartheta\cos\vartheta + (R^2+k^2)\dot{\vartheta}\dot{\psi}\sin\vartheta + \\ + R\dot{\Omega}(t)\sin\vartheta(x\sin\psi - y\cos\psi) + R\Omega(t)\sin\vartheta(\dot{x}\sin\psi - \dot{y}\cos\psi)\,,$$
$$A'_3 = k^2\dot{\vartheta}\dot{\varphi}\sin\vartheta\,,$$

and for the coefficients B'_{ls} we have

$$B'_{11} = -(R^2+k^2)\,,\qquad B'_{12} = 0\,,\qquad B'_{13} = 0\,,$$
$$B'_{21} = 0\,,\qquad B'_{22} = -(R^2\sin^2\vartheta + k^2)\,,\qquad B'_{23} = -k^2\cos\vartheta\,,$$
$$B'_{31} = 0\,,\qquad B'_{32} = -k^2\cos\vartheta\,,\qquad B'_{33} = -k^2\,.$$

The motion of this constrained system is described by the following three reduced equations (see [26]):

$$(R^2+k^2)\,\ddot{\vartheta} + (R^2+k^2)\,\dot{\varphi}\,\dot{\psi}\sin\vartheta - \\ - R\,\Omega(t)(\dot{x}\cos\psi + \dot{y}\sin\psi) - R\,\dot{\Omega}(t)(x\cos\psi + y\sin\psi) = 0\,,$$
$$(R^2\sin^2\vartheta + k^2)\,\ddot{\varphi} + k^2\cos\vartheta\,\ddot{\psi} + \\ + R^2\dot{\vartheta}\,\dot{\varphi}\sin\vartheta\cos\vartheta - (R^2+k^2)\,\dot{\vartheta}\,\dot{\psi}\sin\vartheta - \\ - R\,\Omega(t)\sin\vartheta\,(\dot{x}\sin\psi - \dot{y}\cos\psi) - R\,\dot{\Omega}(t)\sin\vartheta\,(x\sin\psi - y\cos\psi) = 0\,,$$
$$k^2\cos\vartheta\ddot{\phi} + k^2\ddot{\psi} - k^2\dot{\vartheta}\,\dot{\varphi}\sin\vartheta = 0\,.$$

To simplify these equations we can use other coordinates, so called *quasicoordinates*. Recall that $\omega_x, \omega_y, \omega_z$ denote the components of the instantaneous angular velocity, which are determined by means of the Euler angles

$$\begin{aligned}\omega_x &= \dot\vartheta\cos\psi + \dot\varphi\sin\vartheta\sin\psi\,,\\ \omega_y &= \dot\vartheta\sin\psi - \dot\varphi\sin\vartheta\cos\psi\,,\\ \omega_z &= \dot\psi + \dot\varphi\cos\vartheta\,.\end{aligned} \tag{41}$$

Consider now "quasicoordinates" q^1, q^2, q^3 on the configuration space defined byt $\dot q^1 = \omega_x, \dot q^2 = \omega_y, \dot q^3 = \omega_z$. Denote by $(t, x, y, q^1, q^2, q^3, \dot x, \dot y, \omega_x, \omega_y, \omega_z)$ associated coordinates on J^1Y. Then the expression of Lagrangian (39) in quasicoordinates is as follows:

$$L = \frac{1}{2}\left(\dot x^2 + \dot y^2 + k^2(\omega_x^2 + \omega_y^2 + \omega_z^2)\right),$$

and equations of the constrained submanifold take the form (40). Reduced equations of motion of the constrained mechanical system in the quasicoordinates have the form

$$\begin{aligned}\left(R^2+k^2\right)\ddot q^1 - R^2\Omega(t)\,\dot q^2 - R\dot\Omega(t)\,x + R\Omega^2(t)\,y &= 0\,,\\ \left(R^2+k^2\right)\ddot q^2 + R^2\Omega(t)\,\dot q^1 - R\dot\Omega(t)\,y - R\Omega^2(t)\,x &= 0\,,\\ -\,k\,\ddot q^3 &= 0\,.\end{aligned} \tag{42}$$

Using the definition of the quasicoordinates q^1, q^2, q^3 we obtain that

$$\dot q^3 = \omega_z = C_3 = const,$$

and the first two equations of the system (42) can be reduced to a system of first order linear differential equations

$$\begin{aligned}\left(R^2+k^2\right)\dot\omega_x - R^2\Omega(t)\,\omega_y - R\,\dot\Omega(t)\,x + R\,\Omega^2(t)\,y &= 0\,,\\ \left(R^2+k^2\right)\dot\omega_y + R^2\Omega(t)\,\omega_x - R\,\dot\Omega(t)\,y - R\,\Omega^2(t)\,x &= 0\,.\end{aligned} \tag{43}$$

Substituting constraint equations (40) into equations (42) we get two first integrals:

$$\begin{aligned}\left(R^2+k^2\right)\omega_x - R\,\Omega(t)\,x &= D_1\left(R^2+k^2\right)\,,\\ \left(R^2+k^2\right)\omega_y - R\,\Omega(t)\,y &= D_2\left(R^2+k^2\right)\,,\end{aligned} \tag{44}$$

where D_1, D_2 are arbitrary constants. Comparing the expressions for ω_x, ω_y from the constraint equations (40) and from (44) we obtain

$$\dot x + \frac{k^2\Omega(t)}{R^2+k^2}\,y + R\,D_1 = 0, \qquad \dot y - \frac{k^2\Omega(t)}{R^2+k^2}\,x - R\,D_2 = 0. \tag{45}$$

Differentiating the last two equations we get the following system of second order differential equations

$$\ddot x + \frac{k^2\Omega(t)}{R^2+k^2}\,\dot y + \frac{k^2\dot\Omega(t)}{R^2+k^2}\,y = 0, \qquad \ddot y - \frac{k^2\Omega(t)}{R^2+k^2}\,\dot x + \frac{k^2\dot\Omega(t)}{R^2+k^2}\,x = 0 \tag{46}$$

for unknown functions $x(t), y(t)$, which describe the motion of the point of contact of the ball with the plane.

Let us suppose, that for a given function $\Omega(t)$ of the angular velocity of the rotating plane we have found a solution $x(t), y(t)$ of (46). If we put

$$A = \left(R^2 + k^2\right), \qquad b(t) = R^2\,\Omega(t),$$

and denote

$$\begin{aligned} F_1\left(t, x(t), y(t)\right) &= R\,\dot{\Omega}(t)\,x - R\,\Omega^2(t)\,y\,, \\ F_2\left(t, x(t), y(t)\right) &= R\,\dot{\Omega}(t)\,y + R\,\Omega^2(t)\,x\,, \end{aligned}$$

then the system (43) can be written in the form

$$\begin{aligned} A\,\dot{\omega}_x - b(t)\,\omega_y &= F_1\left(t, x(t), y(t)\right)\,, \\ A\,\dot{\omega}_y + b(t)\,\omega_x &= F_2\left(t, x(t), y(t)\right)\,. \end{aligned} \tag{47}$$

This is a system of two first order linear non-homogeneous differential equations with nonconstant coefficients. First, we solve the corresponding homogeneous system

$$\dot{\omega}_x = \frac{B(t)}{A}\,\omega_y, \qquad \dot{\omega}_y = -\frac{B(t)}{A}\,\omega_x$$

and obtain the following result

$$\begin{aligned} \omega_x^H(t) &= C_1 \sin\left(\frac{B(t)}{A}\right) + C_2 \cos\left(\frac{B(t)}{A}\right)\,, \\ \omega_y^H(t) &= -C_2 \sin\left(\frac{B(t)}{A}\right) + C_1 \cos\left(\frac{B(t)}{A}\right)\,, \end{aligned}$$

where $B(t) = \int b(t)\,dt$. Next we are looking for a particular solution by the standard procedure of variation of constants

$$\begin{aligned} \omega_x^P(t) &= C_1(t) \sin\left(\frac{B(t)}{A}\right) + C_2(t) \cos\left(\frac{B(t)}{A}\right)\,, \\ \omega_y^P(t) &= C_1(t) \cos\left(\frac{B(t)}{A}\right) - C_2(t) \sin\left(\frac{B(t)}{A}\right)\,, \end{aligned}$$

where $C_1(t), C_2(t)$ are obtained by integrating the following equations

$$\begin{aligned} \dot{C}_1(t) &= F_1\left(t, x(t), y(t)\right) \sin\left(\frac{B(t)}{A}\right) + F_2\left(t, x(t), y(t)\right) \cos\left(\frac{B(t)}{A}\right)\,, \\ \dot{C}_2(t) &= F_1\left(t, x(t), y(t)\right) \cos\left(\frac{B(t)}{A}\right) - F_2\left(t, x(t), y(t)\right) \sin\left(\frac{B(t)}{A}\right)\,. \end{aligned}$$

A general solution of equations (47) is then of the form

$$\begin{pmatrix} \omega_x(t) \\ \omega_y(t) \end{pmatrix} = \begin{pmatrix} \omega_x^H(t) \\ \omega_y^H(t) \end{pmatrix} + \begin{pmatrix} \omega_x^P(t) \\ \omega_y^P(t) \end{pmatrix}.$$

The solution in terms of quasicoordinates is then determined by elementary quadratures

$$q^1(t) = \int \omega_x(t)\,dt, \qquad q^2(t) = \int \omega_y(t)\,dt, \qquad q^3(t) = \int C_3\,dt,$$

and the solution in terms of Euler angles is described by differential equations (41).

In a particular case, when $\Omega(t) = \Omega_0 = const.$, (see [22]) the system (46) takes the form

$$\ddot{x} + \frac{k^2\Omega_0}{R^2+k^2}\,\dot{y} = 0, \qquad \ddot{y} - \frac{k^2\Omega_0}{R^2+k^2}\,\dot{x} = 0.$$

Using first integrals (45) we write:

$$\ddot{x} + \left(\frac{k^2\Omega_0}{R^2+k^2}\right)^2 x = -\frac{k^2 R\,\Omega_0}{R^2+k^2}\,D_2\,,$$
$$\ddot{y} + \left(\frac{k^2\Omega_0}{R^2+k^2}\right)^2 y = -\frac{k^2 R\,\Omega_0}{R^2+k^2}\,D_1\,.$$

A solution of the corresponding homogeneous system is:

$$x^H(t) = A_1 \sin\left[\left(\frac{k^2\Omega_0}{R^2+k^2}\right)^2 t\right] + A_2 \cos\left[\left(\frac{k^2\Omega_0}{R^2+k^2}\right)^2 t\right],$$
$$y^H(t) = A_3 \sin\left[\left(\frac{k^2\Omega_0}{R^2+k^2}\right)^2 t\right] + A_4 \cos\left[\left(\frac{k^2\Omega_0}{R^2+k^2}\right)^2 t\right],$$

where A_1, A_2, A_3, A_4 are arbitrary constants. Using the procedure of variation of constants we get a particular solution:

$$x^P(t) = -\,R\,D_2\,\frac{R^2+k^2}{k^2\Omega_0}, \qquad y^P(t) = -\,R\,D_1\,\frac{R^2+k^2}{k^2\Omega_0}.$$

Finally, the general solution takes the form

$$x(t) = A_1 \sin\left[\left(\frac{k^2\Omega_0}{R^2+k^2}\right)^2 t\right] + A_2 \cos\left[\left(\frac{k^2\Omega_0}{R^2+k^2}\right)^2 t\right] - R\,D_2\,\frac{R^2+k^2}{k^2\Omega_0}\,,$$
$$y(t) = A_3 \sin\left[\left(\frac{k^2\Omega_0}{R^2+k^2}\right)^2 t\right] + A_4 \cos\left[\left(\frac{k^2\Omega_0}{R^2+k^2}\right)^2 t\right] - R\,D_1\,\frac{R^2+k^2}{k^2\Omega_0}\,,$$

where D_1, D_2 are constants, which occur in the first integrals (44). Hence the ball on the rotating table moves along ellipses parameters of which depend on initial conditions.

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