Backward jump continuous-time random walk: An application to market trading

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The backward jump modification of the continuous-time random walk model or the version of the model driven by the negative feedback was herein derived for spatiotemporal continuum in the context of a share price evolution on a stock exchange. In the frame of the model, we described stochastic evolution of a typical share price on a stock exchange with a moderate liquidity within a high-frequency time scale. The model was validated by satisfactory agreement of the theoretical velocity autocorrelation function with its empirical counterpart obtained for the continuous quotation. This agreement is mainly a result of a *sharp* backward correlation found and considered in this article. This correlation is a reminiscence of such a bid-ask bounce phenomenon where backward price jump has the same or almost the same length as preceding jump. We suggested that this correlation dominated the dynamics of the stock market with moderate liquidity. Although assumptions of the model were inspired by the market high-frequency empirical data, its potential applications extend beyond the financial market, for instance, to the field covered by the Le Chatelier-Braun principle of contrariness.

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I. INTRODUCTION

The negative feedback is encountered both in nature and in socioeconomical systems as a counteraction against some exogenous factors, which aim at restoring of the initial conditions of these systems. This effect is well defined for systems in equilibrium or, approximately, in partial equilibrium by the commonly known Le Chatelier-Braun principle of contrariness. The most prominent example of this principle in finance could be the elimination of an arbitrage opportunity that appeared on a market. Moreover, the backward correlation[1] in consecutive jumps of a tagged particle subjecting a random walk within the fluctuating environment, observed even in systems far from equilibrium and on a short time scale, may be viewed as certain example of an extension of this principle.

Nearly three decades ago, the backward correlation was considered [1–4] (and references therein). This correlation leads to reduction of the tracer diffusion. For example, it is operative for metals for vanishing vacancy concentration [5] and in solid electrolytes [6] as well as it causes reduction of the hydrogen self-diffusion in transition metals where concentrations of vacancies can be arbitrary [2]. Furthermore, a phonon-associated tunneling forming a polaron could be another interesting example of this correlation [5,7]. Recently, the problem of backward correlation came back in a quite different, financial context [8–12] (and references therein).

The backward correlation occurs over two consecutive jumps of a tagged particle. This is because certainly, this particle leaves a vacancy behind when making a jump. Hence, there is an increased tendency for the tagged particle to make a backward jump. This tendency becomes weaker with lapse of time [2]. The backward correlation is time dependent because the vacancy can also be filled by other jumping particles of the neighborhood. In order to describe dynamics of this process in a lattice gas, the properly suited spatiotemporal waiting-time distributions (WTDs) were found [2] in the frame of the renewal theory. These WTDs enabled extension of formalism of the canonical continuous-time random walk (CTRW) in such a way that the time-dependent backward correlation became its dominant feature.

The canonical CTRW formalism was originally introduced by physicists, Montroll and Weiss, in 1965 [13] as a way to render time continuous in the classical random walk. Almost one decade later, Tunaley [14,15] extended this formalism by incorporating distinct WTD for the first jump. The CTRW model can be considered as an example of *renewal stochastic processes* [16] (and references therein), where time intervals between jumps or *holding time intervals* are random variables characterized by any (and not only Poisson) distribution. Then, also mathematicians developed a related theory of *Markov renewal processes* or *semi-Markov chains* [17] as well as *hidden semi-Markov models* [17]. Noticeably, the hidden semi-Markov models were applied in many fields ranging from biology through telecommunication to finance [18–29] including econometrics [30] and economics [31], and even to speech recognition [32].

The CTRW found innumerable applications in many other fields, still growing, such as the aging of glasses [33,34], a nearly constant dielectric loss response in structurally disordered ionic conductors [35] as well as in modeling of hydrological problems [36,37] and earthquakes [38]. Since CTRW was first successfully applied by Scher and Lax in 1973 [39–44] (and references therein) and independently by Moore one year later [45] to describe anomalous transient photocurrent in an amorphous glassy material manifesting the power-law relaxation, this formalism has achieved much more than its original goal.

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[1]The backward correlation is also called *anticorrelation* or *negative correlation*. In this article the names *correlation* and *autocorrelation* are used as synonyms.

In financial high-frequency or tick-by-tick time series of a single share price, the backward correlation between its consecutive jumps has been commonly observed for a long time [29] (and references therein) while correlations over three and more consecutive jumps have, in practice, been absent there. The strong mutual dependence between consecutive jumps of a share price has been observed on financial markets in contrast to its weak statistical dependence on time intervals between consecutive trades.

Importantly, this strong dependence originating in the market microstructure can deeper be understood by analysis of the order book. The order book [46,47] is a deterministic system developed to organize the *double auction* market [48,49]. This book contains different kinds of buy and sell orders [50]. The most prominent feature of this auction is the so-called *bid-ask spread* [29]. This spread is a positive and decisive difference between the lowest available sale offer (ask) price that sellers are willing to accept and the highest purchase (bid) price of an asset that buyers are willing to pay. The existence of the bid-ask spread and intraday dynamics of transaction prices in markets with a moderate liquidity, as is apparently the case of emerging markets (e.g., the Polish market), leads to the phenomenon called *bid-ask bounce* [29,51]. The presence of this bounce results in a strong anticorrelation of successive price changes.

The present work is inspired by the Montero-Masoliver [8] and our [11] recent versions of the CTRW model developed in the context of financial markets as well as by the Haus-Kehr [1] and Kehr-Kutner-Binder [2–4] papers. Furthermore, the present article remains under the influence of the canonical model of Roll [29,51] considering the time independent anticorrelation induced by the static bid-ask spread. Herein, we consider the fluctuating intertransaction time intervals in contrast to constant ones assumed in the Roll model. Although this fluctuation introduces a nonsynchronous trading [29] (and references therein), which can induce erroneous negative correlations between returns for a single stock proportional to square of the mean value of returns, this trading is in practice absent here as high-frequency empirical data give approximately a vanishing mean value of returns.

The principal aim of the present article is to describe stochastic evolution of a typical share price on a financial market with a moderate liquidity, on a high-frequency time scale. This evolution is a short-term anticorrelated stochastic process, which we describe in the frame of the backward jump CTRW model. The model was mainly validated by satisfactory agreement of our theoretical velocity autocorrelation function (VAF) with its empirical counterpart obtained for the continuous quotation or tick-by-tick data.

The paper is organized, as follows. In Sec. II, we discuss possible origin of the observed correlation and postulate an ansatz, which reflects the main feature of the empirical correlation. In Sec. III, our version of the CTRW model, based on the postulated ansatz or driven by the sharp anticorrelation, is developed. We obtained there a general analytical and closed formula in the Laplace domain for (i) different soft and sharp stochastic propagators and useful, related quantities (ii) the mean-square displacement, and (iii) the velocity autocorrelation function (VAF). The explicit time dependences of different VAFs were obtained in a closed form in Sec. IV by the inverse Laplace transform of the particular forms of the waiting-time distribution, namely, single exponential and double exponential functions. In this Section our theoretical VAF was also compared with empirical VAF. In Sec. V, our main results and conclusions were shortly summarized.

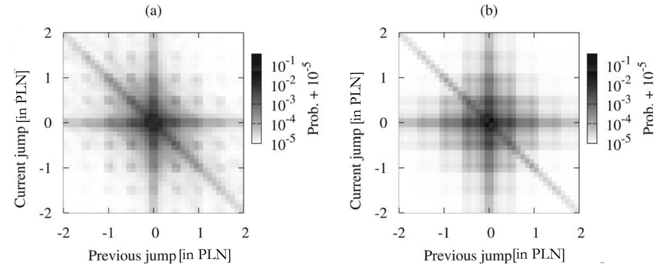

FIG. 1. Comparison of (a) empirical and (b) theoretical 2D shadow histograms, where larger joint probability is visualized by more intense grayness. The gray scale codes the decimal logarithms of probabilities. These probabilities are shifted by small number 10^{-5} to avoid singularity supplied by log0. The empirical histogram was obtained from empirical time series, for instance, for the PEKAO bank. The theoretical histogram is based on expression (1), where the single-variable distribution $h(r_n)$ is, however, empirical. The weight $\epsilon=0.198$ was obtained from the fit of expression (1) to the whole empirical histogram. Details of this fit were given in Sec. IV D.

II. MOTIVATION AND INITIAL HYPOTHESES

The basis for the present considerations arises from the public domain tick-by-tick empirical data for emerging, Polish market [52]. As an initial step, we consider an empirical histogram of the static part of the joint probability density, $h(r_n,r_{n-1})$, of two consecutive share price jumps, r_{n-1} and r_n. This histogram is presented in Fig. 1(a), for instance, for the PEKAO bank, which is the biggest private bank of the Polish origin operating on our domestic market. The accuracy of the empirical data as well as the histogram grid is $\eta=0.1$ PLN, which is larger than the tick size. The acronym PLN means the Polish Nominal or Polish New Złoty (Polish currency). This grid corresponds to the linear size of small squares seen in Fig. 1. This square size is not smaller than the share price currency unit.

Obviously, the empirical density $h(r_n,r_{n-1})$ mainly consists of the following components.

(i) The central cross defined by points $(r_{n-1},0)$ and $(0,r_n)$ confirming well established observation that at least one of the successive transactions can occur without share price change.

(ii) Points $(r_{n-1},\ r_n=-r_{n-1}\pm\eta)$ belonging to the antidiagonal, which approximately defines the term proportional to the Dirac delta $\delta(r_{n-1}+r_n)$. That is, direction of the jump of a share price is opposite to that of the preceding jump but length of both jumps is approximately the same because η is small. Such a dependence between two successive price jumps we call a *sharp* backward correlation. A similar histo-

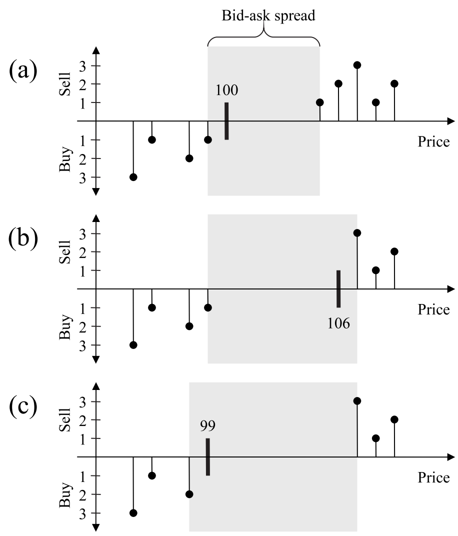

FIG. 2. The schematic view of two sides of the order book divided by the bid-ask spread, marked by the shadowed region, shown in three successive snapshot pictures (a), (b), and (c). By means of a typical example, these plots illustrate how the bid-ask bounce, visualized by the position of the tick, is working. Filled circles placed on bars above the abscissa denote the total volume of the sell offers at that price level. If this volume is larger then the bar is longer. The analogous offers below the abscissa denote the complementary buy offers.

gram for returns does not show such a sharp antidiagonal shape—it is much more diffused.

The first observation (i) is, indeed, a consequence of appearing of transactions without any share price change, i.e., the existence of the nonvanishing joint probability density of two successive price jumps where at least one of them is vanishing. This nonvanishing probability density would exist even if successive jumps were statistically independent.

The second observation (ii) can remind the Le Chatelier-Braun principle or, more precisely, the result of the bid-ask bounce. This phenomenon gives the bouncing of the transaction price from the lower to upper border of the bid-ask spread and back, repeating it many times; this phenomenon is also fluctuating with time. Such a behavior results in the increased, with respect to the case of independent successive jumps, the joint probability of two successive jumps of opposite signs but equal or almost equal length. This trading mechanism is hidden behind observation (ii) and considered in details in Sec. II A.

A. Assumptions of the bid-ask bounce mechanism

Let us assume that, initially, our order book of the financial market with moderate liquidity contains a set of different offers, shown by the snapshot picture in Fig. 2(a). Suppose that the last transaction price, marked by the tick, equals, e.g., 100 currency units. We consider, for example, that a *buy* market offer for three stocks was realized[2] and the price raised to 106 currency units. Hence, the upper border of the bid-ask spread increased to 107 currency units, as it was shown by the snapshot plot in Fig. 2(b), while the price of the highest buy offer still equals 99 charge units remaining unchanged. The next offer can be of the following type.

(a) A completed buy offer, that is either the market offer or the offer with the price limit, which is equal to the higher bid-ask spread border or placed above it. Note that sell offers with the limit higher than the last transaction price equal to 106 currency units [cf. Fig. 2(b)] were already present in the order book. Hence, we can approximately assume that the jump of the current transaction price (i.e., the forward jump) is independent of the preceding jump.

(b) An *internal* buy offer, which is that with the price limit within the bid-ask spread. This offer cannot be instantly realized; its possible realization is delayed. The offer shrinks bid-ask spread by shifting its lower border to the right, leaving the last transaction price, i.e., 106 currency units, inside the new bid-ask spread. This means that the next transaction price will be the result of completion either the internal buy offer or the one from sell offers waiting for realization and placed to the right of the last transaction price. Hence, next transaction price jump can occur to the left or to the right relative to this last transaction price, with approximately equal probability. Therefore, we can assume that, here, there is no correlation between the next and the previous transaction price jumps. Note that we deal here with a distribution of current jumps.

(c) A completed sell offer, that is either the market offer or the one with the price limit, which is equal to the lower bid-ask spread border or placed below it. Even if the sell offer is very small having, for example, the lowest nonvanishing volume that equals a single stock, the transition will take place and the price will locate in the vicinity of the lower bid-ask spread border, which is here 99 currency units [cf. Fig. 2(c)]. Longer price jumps, i.e., jumps corresponding to much lower share price, are much less probable as small market offers are more frequent. In this case we can assume that both preceding and current transaction price jumps have approximately the same length but opposite directions, i.e., they are sharply backward correlated.

(d) An *internal* sell offer, which is that with the price limit within the bid-ask spread. This offer also cannot be instantly realized, which leads to a fluctuating shrinkage of the bid-ask spread by shifting its upper border to the left. Right now, the last transaction price, i.e., 106 currency units, remains above the upper border of a new bid-ask spread. The next transaction price will result from completion of either the internal sell offer or that chosen from already present buy offers. For both cases, next price jump is directed to the left. In the former case, we can treat it as approximately independent of the preceding price jump, in analogy to the (a). In the latter case, the transaction price returns to the vicinity of the lower border of the bid-ask spread, as in (c). Moreover, we can

[2] By market offer it is simply understood an offer without any price limit, i.e., the sell offer with 0-limit and the buy offer with an ∞-limit.

postulate in this case that previous and next transaction price jumps have approximately the same lengths but opposite directions, which results in sharp backward correlation.

(e) Remaining offers can be ignored in this analysis because at a given moment they do not participate in the dynamics. If such an offer occurred, we would have to wait for another one.

The above sequence of steps is continuously repeated. This sequence constitutes the bid-ask bounce mechanism. In our model, we assume that situations leading to independent jumps of the transaction price appear with probability equal to $1-\epsilon$, where $0 \leq \epsilon \leq 1$, while situations leading to sharp backward correlation appear with complementary probability of ϵ.

We can add that assumptions introduced above are independent of how large is the initial jump of the transaction price, which in our example increases the price from 100 to 106 currency units. It is sufficient that this jump is greater than the smallest admissible share price change. An interesting observation is that the significant change of the transaction price can be restored or almost restored even by a transaction of a small volume. Such a transaction leads to the reverse price jump of the same or approximately the same length as the initial jump of the transaction price. The large volume transaction can be considered as a large fluctuation or the one exerted by an external force driving the system out of the equilibrium or partial equilibrium.

The analogous trading mechanism applies to the reverse sequence of the share price changes. In fact, the aim of the successive sections is a quantitative description of the bid-ask bounce mechanism. Thus, we selected events which constitute the basis for the analytical preparation of the conditional probability density $h(r_n|r_{n-1})$ in Sec. II B.

B. Basic relation

Being influenced by the explanation given in Sec. II A and inspired by empirical data shown in Fig. 1(a), we propose the conditional probability density $h(r_n|r_{n-1})$ of two consecutive share price jumps r_{n-1} and r_n in the form, which favors the sharp backward correlation

$$h(r_n|r_{n-1}) = (1-\epsilon)h(r_n) + \epsilon\delta(r_n + r_{n-1}). \quad (1)$$

Here, distribution $h(r_n)$ is an ϵ-independent, symmetric function, which means that no drift is considered in this work, and degree of correlation $0 \leq \epsilon \leq 1$ is a constant weight. The above defined conditional probability density consists of two terms. The first term appears with weight $1-\epsilon$. This term says that a new jump of the share price is drawn from the distribution $h(r_n)$, i.e., without any dependence on the previous jump. The second term, appearing with probability ϵ, describes the price returns to its previous value or sharply backward correlated successive price jumps.

The conditional probability density given by expression (1) seems to be better suites to describe our empirical data than the corresponding one

$$h(r_n|r_{n-1}) = h(r_n) - \epsilon h(r_n)\mathrm{sgn}(r_n)\mathrm{sgn}(r_{n-1}), \quad |\epsilon| \leq 1 \quad (2)$$

proposed by Montero-Masoliver [8]. This better suiting is due to the presence in it of the second, δ-Dirac term favoring a sharp backward correlation in consecutive jumps of a price, cf. the empirical histogram shown in Fig. 1(a) which clearly shows the antidiagonal line.

For weight $\epsilon \neq 0$ both expressions can coincide only within the two-state distribution defined by the probability density $h(x) = [\delta(x-c) + \delta(x+c)]/2$, where constant c means, e.g., a single-tick movement. This probability density is a particular case of Eq. (4) in [8] for weight $Q=0$. For $\epsilon=0$, both conditional probability densities given by expressions (1) and (2) become unconditional and identical, as it should be.

The conditional probability density [Eq. (1)] obeys self-consistency constraint

$$h(r_n) = \int_{-\infty}^{\infty} h(r_n|r_{n-1})h(r_{n-1})dr_{n-1},$$

$$h(r_{n-1}) = \int_{-\infty}^{\infty} h(r_n|r_{n-1})h(r_{n-1})dr_n, \quad (3)$$

cf [8]. for details. Indeed, expression (1) is implemented in Sec. III into the backward jump version of the CTRW model.

III. BACKWARD JUMP VERSION OF CTRW

Let us consider a single realization or trajectory of a jump process as an intraday high-frequency time series. We have to deal with the stochastic process of the share price where each step consists of the waiting time t_n prior to the jump of price r_n. Note that transactions with no price change are also permitted. We can quantify a single trajectory by using its turning points $(t_1, r_1; t_2, r_2; \ldots; t_n, r_n)$ and define the process by using the conditional probability density $\rho(r_n, t_n | r_{n-1}, t_{n-1}; r_{n-2}, t_{n-2}; \ldots; r_2, t_2; r_1, t_1)$. This probability density says that the price jump r_n occurring exactly at the end of the waiting time t_n is conditioned by the whole history $(t_1, r_1; t_2, r_2; \ldots; t_{n-1}, r_{n-1})$. Next, we make a self-restrain.

(1) In one, we introduce simplification, which reduces the memory only to one step back, i.e., approximation (4)

$$\rho(r_n, t_n | r_{n-1}, t_{n-1}; r_{n-2}, t_{n-2}; \ldots; r_2, t_2; r_1, t_1) \approx \rho(r_n, t_n | r_{n-1}) \approx h(r_n|r_{n-1})\psi(t_n), \quad (4)$$

is already sufficient for the analysis. Approximation (4) or the factorization was tacitly used in Sec. II. This factorization enables to consider two empirical variograms: (i) one consisting of the share price jumps and (ii) the other of intertransaction times, as mutually independent. Indeed, distribution $\psi(t_n)$ is the waiting-time distribution (WTD), which concerns only the temporal part of the overall spatiotemporal WTD, $\rho(r_n, t_n | r_{n-1})$.

(2) In the other, we assume that the stationary initial situation makes possible to neglect some daily pattern of investors' activity, e.g., the influence of the so-called *lunch effect*, remaining the crucial antidiagonal line present in histograms (shown in Fig. 1) still sufficiently sharp. Unfortunately, only crude methods of elimination of this effect are known up to now (see, for example [53]).

The aim of this section is to derive the conditional probability density, $P(X, t|\xi)$, to find share price value X at time t, at condition that the share price initial value was assumed as the origin reached by the share price preinitial jump ξ. Further in the text we call this probability the *soft* stochastic propagator, in contrast to the *sharp* one, which we define below. Note that t denotes here the clock or current time and *not* the waiting or pausing time. The derivation consists of few steps described in the following sections.

A. Stochastic propagators

The intermediate dynamic quantity describing the stochastic process is the stochastic, *sharp*, n-step propagator $Q_n(X,r_n;t|\xi)$, $n=1,2,\ldots$. This propagator is defined as the conditional probability density that the share price, which had initially (at $t=0$) the original value ($X=0$) reached by preinitial jump ξ, makes its nth jump by r_n from $X-r_n$ to X *exactly* at time t. The recursion relation between two successive sharp stochastic propagators can be written for any form of $h(r_n|r_{n-1})$, as follows:

$$Q_n(X,r_n;t|\xi) = \int_0^t dt' \psi(t') \int_{-\infty}^{\infty} dr_{n-1} h(r_n|r_{n-1}) \times Q_{n-1}(X-r_n,r_{n-1};t-t'|\xi), \quad (5)$$

where all spatial variables X, r_n, r_{n-1}, and ξ are continuous. Equation (5) relates successive sharp propagators by the spatiotempotral convolution. As the first jump must be treated differently, this equation is valid only for $n\geq 3$. Equation (5) is the fundamental relation used in the backward jump version of the CTRW model. As the space variables in Eq. (5) are continuous, this version is more general than the backward jump models developed only for regular lattices [1–4].

By substituting the concrete form of $h(r_n|r_{n-1})$, given by expression (1), into Eq. (5) and by performing the Fourier-Laplace transform, we obtain

$$\tilde{Q}_n(k,r_n;s|\xi) = \tilde{\psi}(s) e^{ikr_n} \int_{-\infty}^{\infty} dr_{n-1} [(1-\epsilon)h(r_n) + \epsilon\delta(r_n + r_{n-1})] \tilde{Q}_{n-1}(k,r_{n-1};s|\xi), \quad (6)$$

where $\tilde{O}$ means the Fourier, Laplace, or Fourier-Laplace transform of O.

Our practical aim is to obtain from Eq. (6) the summarized, indispensable for further considerations, stochastic, sharp n-step propagator

$$\tilde{Q}_n(k;s|\xi) = \int_{-\infty}^{\infty} dr_n \tilde{Q}_n(k,r_n;s|\xi). \quad (7)$$

Therefore, from Eq. (6) we derive the recursion equation in the algebraic form

$$\tilde{Q}_n(k,s|\xi) = (1-\epsilon)\tilde{h}(k)\tilde{\psi}(s)\tilde{Q}_{n-1}(k,s|\xi) + \epsilon\tilde{\psi}(s)^2 \tilde{Q}_{n-2}(k,s|\xi) \quad (8)$$

valid for $n\geq 3$. The summation of the recursion Eq. (8) over n from 1 to infinity yields, after simple algebraic manipulations,

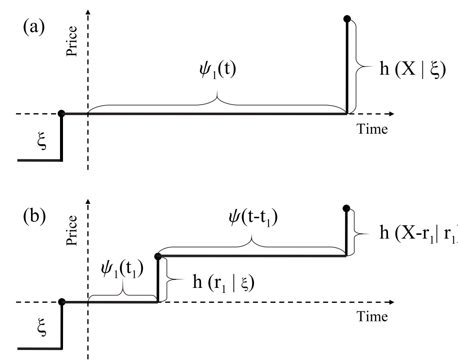

FIG. 3. The illustration of derivation of (a) the one-step stochastic sharp propagator and (b) the two-step propagator. For both cases, the characteristic sequences of basic probabilities (marked by braces) were visualized. Moreover, the preinitial jump ξ was marked (together with other jumps of prices) by vertical bars having small filled circles on the top.

$$\tilde{Q}(k,s|\xi) = \frac{\tilde{Q}_2(k,s|\xi) + \tilde{Q}_1(k,s|\xi)[1-\tilde{h}(k)(1-\epsilon)\tilde{\psi}(s)]}{1-(1-\epsilon)\tilde{h}(k)\tilde{\psi}(s) - \epsilon\tilde{\psi}(s)^2}, \quad (9)$$

depending on the unknown one- and two-step sharp propagator $\tilde{Q}_1(k,s|\xi)$ and $\tilde{Q}_2(k,s|\xi)$, respectively. Here, the summarized sharp propagator is defined as

$$\tilde{Q}(k,s|\xi) = \sum_{n=1}^{\infty} \tilde{Q}_n(k,s|\xi). \quad (10)$$

The illustration of the derivation of the one- and two-step stochastic sharp propagators is shown in Figs. 3(a) and 3(b). Note that we cannot use the same waiting-time distribution for the first jump as for other jumps. This is because the previous (preinitial) jump might occur at any time before $t=0$. In the stationary state, which we consider here, the time origin can be chosen arbitrarily. Otherwise, the time homogeneity of the process would be destroyed. Therefore, we can average over all possible time intervals of the preinitial jump, i.e., over all time differences t' between time origin $t=0$ and the time of the last transaction. Hence and following Eqs. (3.3) and (3.4) shown in [4], we assume that

$$\psi_1(t) = \frac{\int_0^{\infty} dt' \psi(t+t')}{\int_0^{\infty} dt'' \int_0^{\infty} dt' \psi(t'+t'')} \Leftrightarrow \tilde{\psi}_1(s) = \frac{1}{\langle t\rangle} \frac{1-\tilde{\psi}(s)}{s}, \quad (11)$$

where expected (mean) waiting-time $\langle t\rangle = \int_0^{\infty} t\psi(t)dt < \infty$. The denominator in the first equation in Eq. (11) is required for normalization. The only case when $\psi_1(t)=\psi(t)$ is an exponential waiting-time distribution of a Poisson process. Generally, the choice of $\psi_1(t)$ decisively depends on the type of the considered problem [4,10,53,54], namely, on whether a random walk has stationary or nonstationary character. Besides, $\psi_1(t)$ can be arbitrary chosen.

For arbitrary conditional probability distribution $h(r_n|r_{n-1})$ we can write

$$Q_1(X,t|\xi) = \psi_1(t)h(X|\xi)$$

and

$$Q_2(X,t|\xi) = \int_0^t dt_1 \psi_1(t_1)\psi(t-t_1)\int_{-\infty}^{\infty} dr_1 h(r_1|\xi)h(X-r_1|r_1). \tag{12}$$

By writing above equations in the Fourier-Laplace domain and substituting an explicit form [Eq. (1)] of distribution $h(r_1|\xi)$, we obtain the searched one- and two-step propagators. Substituting these propagators into Eq. (9) we have

$$\tilde{Q}(k,s|\xi) = \tilde{\psi}_1(s)\frac{(1-\epsilon)\tilde{h}(k) + \epsilon[e^{ik(-\xi)} + \tilde{\psi}(s)]}{1-(1-\epsilon)\tilde{h}(k)\tilde{\psi}(s) - \epsilon\tilde{\psi}(s)^2}, \tag{13}$$

as an important successive intermediate step.

B. Soft stochastic propagator, its variance, and the velocity autocorrelation function

In order to find an unconditional stochastic propagator $\tilde{Q}(k;s)$ we take the following average over the pre-initial jump vector ξ,

$$\tilde{Q}(k;s) = \int_{-\infty}^{\infty} d\xi\, \tilde{Q}(k,s|\xi)h(\xi)$$

$$= \tilde{\psi}_1(s)\frac{\tilde{h}(k) + \epsilon\tilde{\psi}(s)}{1-(1-\epsilon)\tilde{h}(k)\tilde{\psi}(s) - \epsilon\tilde{\psi}(s)^2}. \tag{14}$$

Equation (14) only slightly differs from the corresponding expression (4.18) derived in [4] by different way, as the variable ξ is a continuous one. The present approach uses more fundamental parent Eq. (5) than the corresponding Eq. (4.15) in [4].

Finally, we obtain the *soft* stochastic propagator in the form of the superposition of two essentially different terms

$$\tilde{P}(k,s) = \tilde{\Psi}_1(s) + \tilde{\Psi}(s)\tilde{Q}(k;s), \tag{15}$$

where sojourn probabilities (in time and Laplace domains) are defined by the corresponding waiting-time distributions

$$\Psi(t) = \int_t^{\infty}\psi(\tau)d\tau \Leftrightarrow \tilde{\Psi}(s) = \frac{1-\tilde{\psi}(s)}{s}$$

and

$$\Psi_1(t) = \int_t^{\infty}\psi_1(\tau)d\tau \Leftrightarrow \tilde{\Psi}_1(s) = \frac{1-\tilde{\psi}_1(s)}{s}. \tag{16}$$

The first term in expression (15), which we can call the *passive* one, describes no jumps of the share price for $t>0$, including only the information concerning the initial state of the process. The second term describes any nonvanishing number of jumps; we can call it the *active* term.

Substituting expressions (14) and (16) into Eq. (15) we obtain

$$\tilde{P}(k,s) = \frac{1}{s} - \frac{1}{\langle t\rangle s^2}\frac{[1-\tilde{\psi}(s)][1-\epsilon\tilde{\psi}(s)][1-\tilde{h}(k)]}{1-(1-\epsilon)\tilde{h}(k)\tilde{\psi}(s)-\epsilon\tilde{\psi}(s)^2}. \tag{17}$$

For $\epsilon\to 0$, Eq. (17) corresponds to Eq. (3.18) in [4] where both backward and forward correlations are absent; this latter equation concerns the nonseparable CTRW.

Now, we are ready to calculate the variance of the soft propagator in the Laplace domain

$$\langle\tilde{X}^2\rangle(s) = -\left.\frac{\partial^2\tilde{P}(k,s)}{\partial k^2}\right|_{k=0} = \frac{\mu_2}{s^2\langle t\rangle}\frac{1-\epsilon\tilde{\psi}(s)}{1+\epsilon\tilde{\psi}(s)},$$

where

$$\mu_2 = \int_{-\infty}^{\infty} dx x^2 h(x). \tag{18}$$

The Laplace transform of the VAF and VAF itself are given by

$$\tilde{C}(s) = \frac{s^2}{2}\langle\tilde{X}^2\rangle(s) = \frac{\mu_2}{2\langle t\rangle}\frac{1-\epsilon\tilde{\psi}(s)}{1+\epsilon\tilde{\psi}(s)} \Leftrightarrow C(t)$$

$$= \frac{\mu_2}{2\langle t\rangle}\delta(t) - \epsilon\frac{\mu_2}{\langle t\rangle}\mathcal{L}_t^{-1}\left\{\frac{\tilde{\psi}(s)}{1+\epsilon\tilde{\psi}(s)}\right\}, \tag{19}$$

where $\mathcal{L}_t^{-1}\{\ldots\}$ is an inverse Laplace transform to the time domain. The second equation in Eq. (19) is the main formula of the present work similar to the corresponding one derived in our early paper [2]. However, in this paper we restricted our approach only to the random walk of a tagged particle on a regular lattice. Moreover, it is straightforward to obtain from VAF the related useful quantities, such as the *power spectrum* and the *frequency-dependent diffusion coefficient*. Hence, Eq. (19) can particularly be useful to study a wide spectrum of random walks. Obviously, in order to derive explicit form of VAF, the explicit form of WTD is necessary.

IV. COMPARISON WITH TYPICAL FINANCIAL DATA

In this section we consider VAF normalized in the time domain

$$C^n(t) = \delta(t) - 2\epsilon\mathcal{L}_t^{-1}\left\{\frac{\tilde{\psi}(s)}{1+\epsilon\tilde{\psi}(s)}\right\}. \tag{20}$$

The upper index n means that the quantity is normalized to the Dirac δ value at $t=0$. Particularly useful here is expression (20) for the normalized VAF, although more popular in literature is an approximate approach where, instead of VAF, autocorrelations of price changes are calculated at a fixed small time step. Fortunately, both quantities are equal, with good approximation, after the normalization.

The expressions corresponding to Eq. (20) were derived in [11] by assuming the Montero-Masoliver form of the conditional distribution $h(r_n|r_{n-1})$ defined by expression (2),

$$C^n_{MM}(t) = \delta(t) - 2\epsilon\frac{\mu_1^2}{\mu_2}\mathcal{L}_t^{-1}\left\{\frac{\tilde{\psi}(s)}{1+\epsilon\tilde{\psi}(s)}\right\},$$

where

$$\mu_1 = \int_{-\infty}^{\infty} dx|x|h(x). \tag{21}$$

The generic useful property of expression (20) is that normalized VAF does not depend on the single-variable distribution of jump lengths. This VAF is only scaled by factor 2ϵ, which for $t>0$ measures the strength of the correlation of two successive price jumps. Hence, the formal difference between our expression (20) and that of Montero-Masoliver [Eq. (21)] is given by the scaling factor μ_1^2/μ_2, which depends on the ratio of the corresponding moments of the single-variable distribution. In both cases, however, the time dependence of VAF is fully determined only by the waiting-time distribution and ϵ. In case of two-state model mentioned in Sec. II B, we have $\mu_1^2=\mu_2=c^2$ and both formulas (20) and (21) become identical, as expected.

It is illustrative to consider two simple cases of VAF by using two different forms of the waiting-time distribution, which makes possible to perform the inverse Laplace transformation $\mathcal{L}_t^{-1}$ to analytic, closed expressions.

A. Single exponential waiting-time distribution

As a simple reference example, we provide WTD of a Poisson process,

$$\psi(t) = \frac{1}{\langle t\rangle}e^{-t/\langle t\rangle} \Leftrightarrow \tilde{\psi}(s) = \frac{1}{\langle t\rangle s+1}. \tag{22}$$

With the use of expression (22), the normalized autocorrelation function corresponding to expression (20) takes the following form:

$$C^n(t) = \delta(t) - \frac{2\epsilon}{\langle t\rangle}e^{-(1+\epsilon)t/\langle t\rangle}. \tag{23}$$

To perform numerical calculations, the value of the mean waiting time $\langle t\rangle$ is required as the weight ϵ was calculated separately. However, the Montero-Masoliver model additionally requires, according to expression (21), the knowledge of the ratio of the moments μ_1^2/μ_2.

B. Waiting-time distribution given by sum of two exponentials

A more realistic form of the waiting-time distribution seems to be a superposition (weighted sum) of two exponentials

$$\psi(t) = \frac{w}{\tau_1}e^{-t/\tau_1} + \frac{1-w}{\tau_2}e^{-t/\tau_2} \Leftrightarrow \tilde{\psi}(s) = \frac{w}{1+s\tau_1} + \frac{1-w}{1+s\tau_2}, \tag{24}$$

where $0\le w\le 1$ is the weight while τ_1 and τ_2 are the corresponding (partial) relaxation times. This form of WTD leads to

$$C^n(t) = \delta(t) - 2\epsilon[A_1e^{-\nu_1 t} + A_2e^{-\nu_2 t}], \tag{25}$$

where

$$\nu_{1,2} = \frac{1}{2}[\omega_1+\omega_2+\epsilon v \pm \sqrt{(\omega_1+\omega_2+\epsilon v)^2 - 4\omega_1\omega_2(1+\epsilon)}],$$

$$A_j = (-1)^j\omega_1\omega_2\frac{1-\nu_1 v}{\nu_1-\nu_2},$$

$$\omega_j = 1/\tau_j, \quad j=1,2, \tag{26}$$

where $v=w\omega_1+(1-w)\omega_2$ while coefficients A_j play the role of the un-normalized weights. As it is seen, the formulas for partial relaxation times $1/\nu_1$ and $1/\nu_2$ essentially differ from the corresponding times τ_1 and τ_2.

C. Numerical algorithm for calculation of VAF

Direct calculation of the velocity autocorrelation function for the tick-by-tick data is a bit more complicated than analogous calculation for the discretized (by a fixed time-step) empirical time series. We are not using any method of detrending the empirical data or removing daily activity pattern to protect the observed negative, sharp correlations expressed by Eq. (1). Despite the observed nonstationarity in the stock price empirical data, we consider the stock price change as a stationary process. Otherwise, we could not calculate velocity autocorrelation function by using the moving average. In other words, we permit at most slowly varying influence of the nonstationarity pattern on studied normalized quantities.

Our procedure of calculation a VAF is straightforward. The empirical data [52] contain d_{max} days. In each day $d=1,2,\ldots,d_{max}$, the number of transactions n_d is fluctuating. Let us denote by $r_{d,i}$ the price change (jump) at ith transaction at day d and by $t_{d,i}$ the time interval directly preceding the ith transaction at day d. The time accuracy or grid of the data is one second. So, with such accuracy we calculate the velocity autocorrelation function using the following moving average

$$C(t) \approx \frac{1}{T(t)}\sum_{d=1}^{d_{max}}\sum_{i=1}^{n_d}\sum_{j=1}^{n_d} r_{d,i}r_{d,j}\delta\left(\sum_{k=i+1}^{j} t_{d,k} - t\right) - \langle v\rangle^2,$$

where

$$T(t) = \sum_{d=1}^{d_{max}}\sum_{i=1}^{n_d} t_{d,i} - td_{max}$$

and

$$\langle v\rangle = \frac{1}{T(0)}\sum_{d=1}^{d_{max}}\sum_{i=1}^{n_d} r_{d,i}. \tag{27}$$

Here, all time intervals are fluctuating, as it is supplied by the continuous quotation on a stock market. Hence, there is here no possibility to create artifacts (e.g., artificial autocorrelation) in contrast to the traditional approach assuming a fixed (small) discretization time step. Obviously, it is sufficient to divide $C(t)$ (which is, of course, a histogram) by the factor $C(0)$ in order to calculate the normalized VAF, $C^n(t)$.

D. Results and discussion

Our data analysis consists of three stages. Within first two stages the parameters were calculated, which are mutually independent. This is because they were obtained from independent data sets. They are needed for the third, final stage. The obtained results are systematically presented and discussed in this section.

1. Initial stage

This stage was realized in Sec. II, where

(i) the functional form of the conditional probability distribution, $h(r_n|r_{n-1})$, was established together with

(ii) numerical calculation of weight ϵ,

both supported by the same set of empirical data.

In fact, our empirical analysis is based on the public domain database [52] containing tick-by-tick market data since 2000–11–17 till 2009–1-31. We focused on the 20 largest companies composing the WIG20 index of the Warsaw Stock Exchange. This database contains sufficiently large data sets to calculate the required estimators with acceptable accuracy not exceeded 10%. For instance, we disposed the data set containing 938 264 records for PEKAO.

To determine our basic parameter ϵ, we constructed the empirical histogram shown in Fig. 1(a) for the price jumps counted with accuracy of $\eta=0.1$ PLN and confined to a sufficiently wide range [−3 PLN, +3 PLN]. That is, the joint distribution $h(r_n,r_{n-1})$ is represented here by 61×61 matrix. Next, we fitted to this empirical histogram our theoretical histogram based on approximate distribution [Eq. (1)], where

(iii) the single-jump distribution, $h(r_n)$, was determined from the corresponding empirical single-variable histogram, prepared also in this stage.

For completeness, we calculated here

(iv) the factor $\mu_1^2/\mu_2=0.228$, which is required by the Montero-Masoliver approach for their normalized VAF [Eq. (21)].

We applied two different fitting routines to estimate the value of ϵ. The first one was the method of least-squares while the second one was the Maximum Likelihood Method. We found parameter ϵ varying no more than about 10% for each company, when we changed the method from one to the other. For 20 companies, we obtained weight ϵ between 0.169 and 0.413. For the PEKAO bank, as for the most other companies, both methods gave to a good approximation the same weight, here $\epsilon=0.198$.

We repeated the whole procedure by dividing the set of all empirical share price changes r_{n-1} and r_n into the positive, negative, and vanishing value groups, still keeping the grid accuracy at $\eta=0.1$ PLN. Thus, we prepared an empirical distribution in the form of 3×3 sign matrix, used e.g., by Tsay [29] and in Montero-Masoliver paper [9], classifying the price movements into "down," "unchanged," and "up" ranges. By applying the above mentioned fitting routines we found again the set of values of ϵ for 20 companies which differ only by a few percentage from the above given more refined values. For example, for the PEKAO bank this difference was merely 4%. Therefore, we can state that the estimation of ϵ is stable.

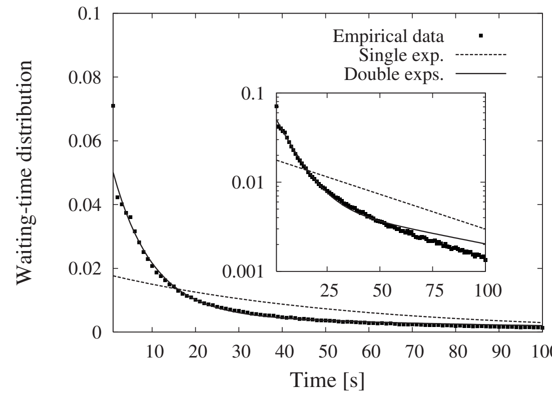

FIG. 4. Comparison of the empirical waiting-time distribution (small black squares) and that theoretical (solid curve) fitted to it. The latter is defined by two exponentials given by the first expression in Eq. (24); the former WTD concerns, for example, the PEKAO bank. For completeness, the dashed curve shows the prediction of WTD given by the single exponential [cf. the first expression in Eq. (22)]. Remarkably, its slope (in the semilogarithmic coordinates) is (to a good approximation) the same for longer time as that of the empirical curve; i.e., it is approximately correct only for the long-term dynamics.

2. Intermediate stage

The unknown parameters [in case of double exponential WTD given by the first expression in Eq. (24)], namely,

(i) weight w as well as

(ii) partial relaxation time τ_1 and τ_2,

were obtained by the least-square fit of the first relation in Eq. (24) to the empirical histogram of the intertransaction time intervals for $t\in[2,100]$, cf. solid curve and small black squares in Fig. 4, respectively. Notably, the data set used here is complementary to that used in the previous stage.

However, we used the reduced number of fit parameters for the fit. That is, from those mentioned above we reduced three (w,τ_1,τ_2) to two (τ_1,τ_2) by using relation $\langle t\rangle=w\tau_1+(1-w)\tau_2$. In this relation the mean waiting time, $\langle t\rangle$, was calculated in this stage directly from empirical data as an arithmetic average over all intertransation times, separately for each company. For example, we obtained $\langle t\rangle=55.647$ s for the PEKAO bank considered here. Noticeably, very similar values of mean waiting-time intervals were found by Montero *et al.* [9] for several companies quoted on NYSE for the period of 1995–1998. From the fit, we obtained also partial relaxation times, $\tau_1=9.017$ s and $\tau_2=93.148$ s, and derived weight, $w=0.446$.

As it is seen in Fig. 4, WTD given by the first expression in Eq. (24) (solid curve) much better fits to the empirical data (small squares) than that given by the first expression in Eq. (22) (dashed curve). WTD given by this latter expression has no free parameters as $\langle t\rangle$ was already calculated independently and it is common for all WTD considered here.

However, the double exponential form of WTD is still the first approximation as for extremely short time (the order of 1 s) and the long one, i.e., longer than $\langle t\rangle$ (equals here $=55.647$ s), some deviation is observed in the semilogarithmic scale (cf. insert to Fig. 4). Moreover, as systematic deviations for longer time are placed on the exponential tail,

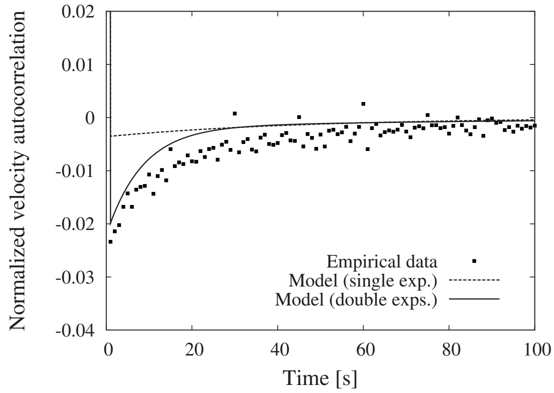

FIG. 5. The normalized velocity autocorrelation functions: empirical (small black squares) and theoretical ones given by Eq. (25) (solid curve) and Eq. (23) (dashed curve), respectively, for example, for the PEKAO bank. Although anticorrelation is small (being at most of the order of few percentages) as a result of a small mean value of the bid-as spread, it is sufficiently distinct and significant. Obviously, at $t=0$ the values of all normalized velocity autocorrelation functions appear above the top of the plot.

they seem to have less meaning. Nevertheless, this result suggests that WTD consisting of more exponentials could better approximate the overall empirical curve. However, it would require extensive calculations involving too many parameters.

3. Final stage

Finally, we were able to calculate values of the parameters $1/\nu_1=8.272$ s and $1/\nu_2=84.619$ s as well as $A_1=0.050$ and $A_2=0.005$ defining normalized VAF [Eq. (25)]. It was expected that the components of each pair of parameters τ_1, $1/\nu_1$ and τ_2, $1/\nu_2$ are of the same order of magnitude and one pair of parameters differs from the other by one order of magnitude. However, a quite surprising result could be that the coefficient A_2 is by one order of magnitude smaller than the coefficient A_1.

In Fig. 5, we compared predictions of our model with empirical VAF (cf. curves and small black squares, respectively). Noticeably, as theoretical VAFs [here given by expressions (22) and (24)] have no free parameters, curves shown in Fig. 5 are not a fit. This theoretical VAF refer to the empirical data set, which is independent from that used within the second stage to construct waiting-time distribution. Although the agreement between prediction of formula (25) (solid curve) and empirical data (small black squares) is quite satisfactory, a systematic deviation is observed. Nevertheless, this result much better reproduces the main relaxation phenomenon than that for the Poisson VAF (dashed curve). Hence, we can state that observed correlation effect is mainly driven by the *sharp* backward correlation where the current backward share price jump has the same or almost the same length as the preceding jump.

V. SUMMARY AND CONCLUDING REMARKS

The understanding, including the quantitative description, of the financial markets' evolution is a long standing challenge strongly depending on the time scale considered. Therefore, so useful principle of the time scales separation was tacitly used here. In the present work, in complementary to the Montero-Masoliver one [8], we realized an ambitious goal to study intraday, tick-by-tick or high-frequency empirical data, i.e., to come down in our study to the resolution of single offers.

Despite the potential importance of the high-frequency time scale as the most significant microscopic time scale, no model systematically accounts up to now the microstructure of so richly fluctuating stock market with a moderate liquidity, even in crude approximation. In fact, this microstructure is defined by all offers contained in the stock market order book, on the time scale yet as short as it is possible. By presenting the high-frequency trading, concerning a single, typical company on the level of its order book, we argued that statistical dependence of two successive jumps of a given share price is mainly of the sharp backward type. Moreover, we argued that this is a dominating property caused by the fluctuating bid-ask spread leading to bid-ask bouncing.

Although the mean value of the bid-ask spread is small, the temporary bid-ask spread can be large, constituting the frame for the significant sharp backward correlation studied in this article. We suggest that this backward correlation is an universal property mainly dominating order books of stock markets of small and even intermediate sizes.

We described the static part of the backward correlation by the conditional probability density [Eq. (1)], which was directly inspired by corresponding empirical data, cf. histogram (a) in Fig. 1. The main dynamical part was approximately described by WTD consisting of two exponentials [Eq. (24)] again inspired by the corresponding empirical data (cf. Fig. 4). Due to the use of so simple forms of expressions (1) and (24), we were able to apply the generic formula, Eq. (20), of the backward jump CTRW model by comparing their predictions with the corresponding empirical data (cf. Fig. 5).

Note that in the frame of the backward jump CTRW model we derived useful formulas for

(i) sharp propagators given by expressions (13) and (14), and

(ii) the soft propagator given by expression (17); this latter expression easily gives

(iii) the mean-square displacement [Eq. (18)] and hence the velocity autocorrelation function (19).

All tools given above can describe in detail not only the dynamics of the share price changes but are potentially able to describe an evolution based on more complicated negative feedback; in this sense the dynamics has a generic character.

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