Doctoral School of Exact and Natural Sciences – Physical Sciences Examination

In the solutions, present the reasoning leading to your results. Write down the final results of the calculations with accuracy of 3 or 2 significant digits, after appropriate rounding, for example $1.23456 \cdot 10^{-19} \approx 1.23 \cdot 10^{-19}$ or $1.2 \cdot 10^{-19}$.

Values of selected constants

speed of light in vacuum	$c \approx 3.00 \cdot 10^{8}$ m/s
elementary charge	$e \approx 1,60 \cdot 10^{-19}$ C
Coulomb constant	$k_e \approx 8.99 \cdot 10^{9}$ N·m^2/C^2
Planck constant	$h \approx 6.63 \cdot 10^{-34}$ Js $\approx 4.14 \cdot 10^{-15}$ eVs
reduced Planck constant	$\hbar = \frac{h}{2\pi} \approx 1.05 \cdot 10^{-34}$ Js $\approx 6.58 \cdot 10^{-16}$ eVs
gravitational constant	$G \approx 6.67 \cdot 10^{-11}$ Nm^2/kg^2
Avogadro constant	$N_A \approx 6.02 \cdot 10^{23}$ mol^{-1}
gas constant	$R \approx 8.31$ J / (mol·K)
Boltzmann constant	$k_B \approx 1.38 \cdot 10^{-23}$ J/K$\approx 8.62 \cdot 10^{-5}$ eV/K
Rydberg constant	$R_\infty \approx 1.10 \cdot 10^{7}$ m^{-1}
Rydberg unit of energy	$Ry \approx 13.6$ eV
electron mass	$m_e \approx 9.11 \cdot 10^{-31}$ kg ≈ 511 keV/c^2
proton mass	$m_p \approx 1.67 \cdot 10^{-27}$ kg ≈ 938 MeV/c^2
unified atomic mass unit	$u \approx 931$ MeV/c^2

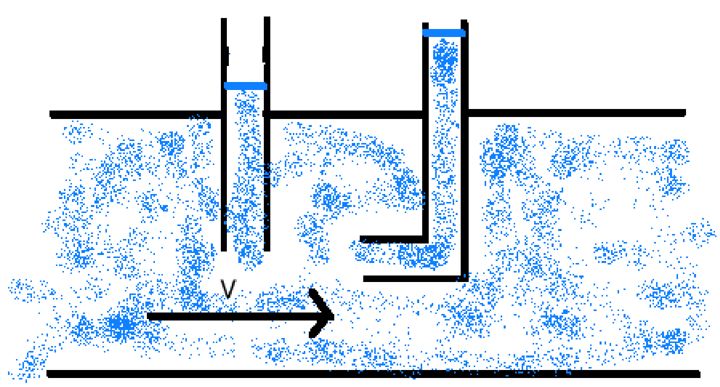
Problems 1—7 are easier. Submit solutions of only four of them.
For each of them you can get up to 6 points.
Problems 8-12 are more difficult. Submit solutions of only two of them.
For each of these solutions you can get up to 8 points.

Easier problems

Problem 1. Tunnel through the center of the Earth.
If it were possible to drill a tunnel through the center of the Earth (along its rotation axis) and throw a stone into one end of this tunnel, how long would it take for the stone to travel to the other end? Assume that the Earth is a perfect ball with a constant density, mass $M_Z = 5.97 \cdot 10^{24}$ kg and radius $R = 6370$ km. The gravitational constant
$G = 6.67 \cdot 10^{-11}$ m^3/(kg·s^2).

Problem 2. Particle in fields $\vec{E}$ and $\vec{B}$.
A particle of mass m and charge q moves in constant and uniform electric and magnetic fields given by $\vec{E} = -E\hat{e}_y$ and $\vec{B} = B\hat{e}_z$. Initially, the particle is at the origin of the coordinate system and its initial velocity is $\vec{v}_0 = v_0\hat{e}_x$. Find how the position of the particle depends on time and sketch the trajectory of the particle. The velocity of the particle is always much smaller than the speed of light in vacuum, c.

Problem 3. Water flowing.
An ideal fluid, i.e. non-viscous and non-compressible, is flowing through a horizontally oriented pipe. The flow of the fluid is uniform, i.e. the velocity is the same everywhere. There are two smaller, thin pipes placed perpendicularly to the wall and with ends at the same level and oriented as in the figure. The difference of the fluid levels inside them is $\Delta h = 0.2$ m. Find the velocity of the fluid in the horizontal pipe.



Problem 3. Water flowing.

Problem 4. Isothermal expansion.
Nitrogen gas of mass $m = 1.5$ kg expanding isothermally at the temperature $T = 300$ K used $Q = 2.5 \cdot 10^5$ J of heat . How many times have the pressure and the volume changed? Molar mass of nitrogen is $\mu = 28$ kg/kmol and the gas constant $R = 8.31 \cdot 10^3$ J/(kmol·K). Treat nitrogen as an ideal gas.

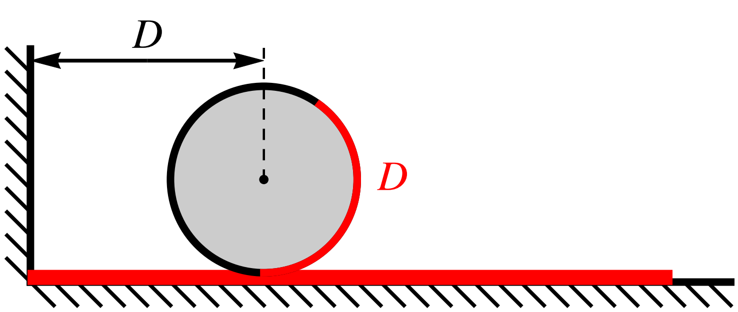
Problem 5. Diffraction on a crystal.
X-ray diffraction experiment on the protein crystal was performed using radiation with a wavelength of $\lambda_1 = 3.0$ Å. Calculate the maximum resolution, d_{max1}, with which the protein structure can be derived from these data. What should be the wavelength λ_2 of the radiation used to enable the experiment to record data with the two-fold better resolution d_{max2}? Hint: The maximum resolution is equivalent to the minimum distance of lattice planes from which reflection can be observed.

Problem 6. Compression of a PVC tube
A tube made of PVC is $L = 10$ m long when put horizontally. What would be the length of this tube standing vertically?
Mass density of PVC, $\rho = 1300$ kg/m^3, its Young's modulus, $Y = 3.4$ GPa, and the standard acceleration due to gravity, $g = 9.81$ m/s^2.

Problem 7. Soap bubbles.
Two spherical soap bubbles of radii r_1 and r_2, respectively, merge to form a spherical soap bubble of a radius r_3. Show that the volume of the final bubble is larger than the sum of the volumes of the initial bubbles and that the surface area of the final bubble is smaller than the sum of the surface areas of the initial bubbles.
Hint: For a spherical soap bubble of a radius r, the relation between the inner pressure p_{in} and the outer pressure p_{out} reads $p_{\mathrm{in}} = p_{\mathrm{out}} + 2\sigma/r$, where σ is the surface tension.

More difficult problems

Problem 8. Elastic rubber band.
An elastic rubber band lies on a smooth table and uniformly extends when a force is applied to one of the ends of the rubber band, while the other end remains fixed. A cylindrical roll is rotated without sliding by the rubber band moving beneath it. The axis of the roll is fixed, perpendicular to the direction in which the rubber band is extended and parallel to the surface of the table. The contact point between the roll and the rubber band remains at a distance D from the fixed end of the rubber band. The band is stretched from an initial length L_1 to the final length L_2 in such a way that a distance travelled by a given point of the surface of the roll was equal to D and the rubber band was always stretched. Find the ratio L_2/L_1.



Problem 8. Elastic rubber band (drawing).

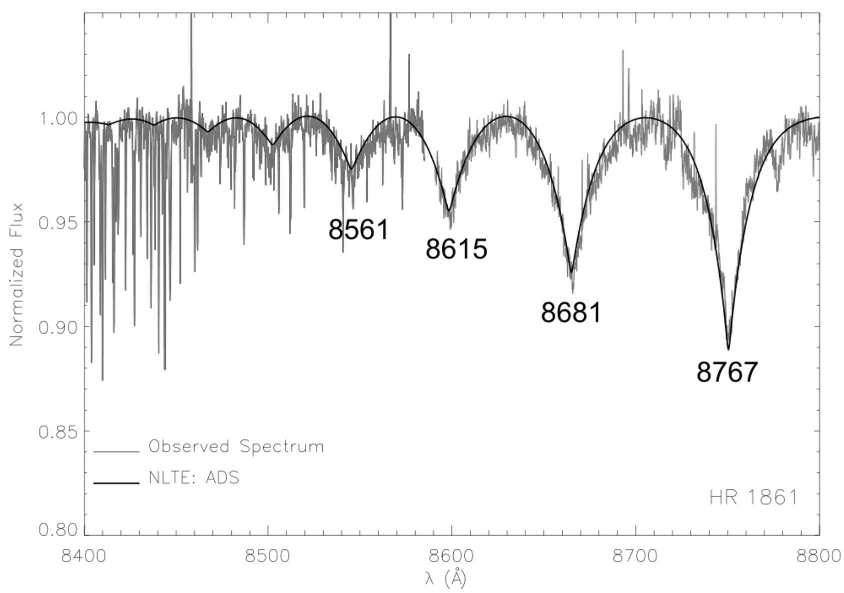
Problem 8. Elastic rubber band (photo). Constructed by J. Grabarczyk.

Problem 9. Reaction threshold.
Consider endothermic nuclear reaction

$$^{2}\mathrm{H}+^{14}\mathrm{N}\rightarrow^{6}\mathrm{Li}+^{10}\mathrm{B}$$

with reaction energy $Q = -10.1$ MeV. If in the laboratory the deuteron ^{2}H is incident on stationary ^{14}N, what minimum kinetic energy must it have for the reaction to occur?

Problem 10. Hydrogen spectrum.
The graph below presents a part of the spectrum of light emitted by the star HR1861 [1], in which several absorption lines belonging to one of the so-called hydrogen spectral series can be seen. Subsequent lines correspond to transitions from the same lower energy level to the consecutive upper levels.



Problem 10. Hydrogen spectrum

Perform the necessary calculations and answer the following questions:

a. What is the principal quantum number n of the lower energy level of transitions belonging to this series?

b. What is the short-wavelength limit of this series, i.e. below which wavelength one cannot observe lines belonging to this series?

c. What is the longest wavelength of lines belonging to this series?

Wavelengths in the graph are given in angstroms, 1Å$= 10^{-10}$ m.
The Rydberg constant $R = 13.6$ eV,
the Planck constant $h = 6.63 \cdot 10^{-34}$ J·s $= 4.14 \cdot 10^{-15}$ eV·s, speed of light $c = 3 \cdot 10^8$ m/s.

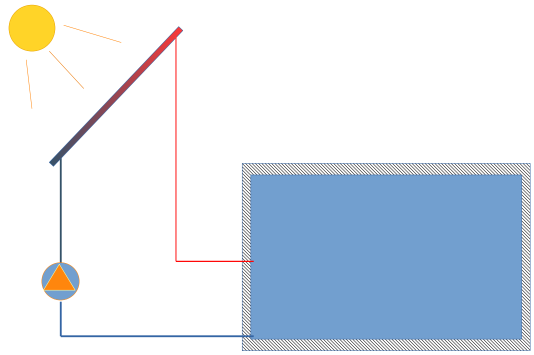
Problem 11. Determining the concentration of a solution on the basis of absorbance measurement.
The UV absorption spectrum of a certain substance was measured by placing its solution in a quartz cuvette with an optical path length of $l = 0.5$ cm. The spectrum shows two narrow bands with a maximum at 280 nm and 220 nm, with absorbance $A_{280} = 0.50$ and $A_{220} = 3.05$, respectively, while the absorbance recorded for wavelengths greater than 320 nm is constant and equals to $A_{>320} = 0.10$. Then the solution was twofold diluted (with the same solvent in which the initial solution was prepared) and the spectrum of the diluted solution was recorded using the same cuvette. This time, at the maxima for the wavelength of 280 nm and 220 nm, the absorbance was $A'_{280} = 0.30$ and $A'_{220} = 1.75$, respectively, while the absorbance for wavelengths greater than 320 nm did not change. Determine the concentration of the initial solution of the tested substance, knowing that under the experimental conditions used in this experiment its decimal molar absorption (also called extinction) coefficients for the wavelengths of 280 nm and 220 nm are respectively $\varepsilon_{280} = 10\ 000$ M^{-1}cm^{-1} and $\varepsilon_{220} = 150\ 000$ M^{-1}cm^{-1}. Justify your answer. Remarks: Absorbance is the decimal logarithm of the ratio of the intensity of the radiation beam incident on the test sample to the intensity of the radiation beam after it has passed through the test sample. The decimal molar absorption coefficient is the absorbance value of a 1 M solution tested in a cuvette with an optical path length of 1 cm.

Problem 12. Heating the pool.
Consider a system consisting of a pool with water (covered and isolated from the environment), a solar heater and a pump. The pump moves water from the pool through the solar heater and back, with the mass flow rate $\frac{\mathrm{d}m}{\mathrm{d}t} = 0.04$ kg/s. How much time is needed to warm up the water in the pool by $\Delta T = 1$°C? What is the heating time if the pump capacity is doubled, that is, the mass flow rate through the solar heater $\frac{\mathrm{d}m}{\mathrm{d}t}$ doubles)? Neglect heat losses and assume that the efficiency of the heater is constant $\eta = \eta_0 = 50\%$. Assume the mass of water in the pool to be $M = 5000$ kg, the specific heat of the water $c_w = 4200$ J/(kg K) and assume that the temperature of the water in the pool is homogeneous throughout the volume and initially equals $T_0 = 15$°C. The intensity of solar radiation is $I = 800$ W/m^2, and the surface area of the heater $A = 10$ m^2.

Hint: Efficiency of the solar heater reads:

$$\eta = \frac{P}{AI},$$

where $P = \frac{\mathrm{d}m}{\mathrm{d}t} c_w (T_{\mathrm{out}} - T_{\mathrm{in}})$ is the power of the solar heater, T_{in} and T_{out} are temperatures at the inlet and outlet of the solar heater, respectively.



Problem 12. Heating the pool.