Doctoral School, physical sciences, exam

In the solutions, present the reasoning leading to the result.

Write down the final results of the calculations with an accuracy of 3 or 2 significant digits, after appropriate rounding, for example $1.234\,56 \cdot 10^{-19} \approx 1.23 \cdot 10^{-19}$ or $1.2 \cdot 10^{-19}$.

Values of selected constants

speed of light in vacuum	$c \approx 3.00 \cdot 10^{8}$ m/s
elementary charge	$e \approx 1.60 \cdot 10^{-19}$ C
Coulomb constant	$k_e \approx 8.99 \cdot 10^{9}\ \mathrm{N\,m^2/C^2}$
Planck constant	$h \approx 6.63 \cdot 10^{-34}\ \mathrm{J\,s} \approx 4.14 \cdot 10^{-15}\ \mathrm{eV\,s}$
reduced Planck constant	$\hbar = \frac{h}{2\pi} \approx 1.05 \cdot 10^{-34}\ \mathrm{J\,s} \approx 6.58 \cdot 10^{-16}\ \mathrm{eV\,s}$
gravitational constant	$G \approx 6.67 \cdot 10^{-11}\ \mathrm{N\,m^2/kg^2}$
Avogadro constant	$N_A \approx 6.02 \cdot 10^{23}\ \mathrm{mol^{-1}}$
gas constant	$R \approx 8.31\ \mathrm{J/(K\,mol)}$
Boltzmann constant	$k_B \approx 1.38 \cdot 10^{-23}\ \mathrm{J/K} \approx 8.62 \cdot 10^{-5}\ \mathrm{eV/K}$
Rydberg constant	$R_\infty \approx 1.10 \cdot 10^{7}\ \mathrm{m^{-1}}$
Rydberg unit of energy	$\mathrm{Ry} \approx 13.6\ \mathrm{eV}$
electron mass	$m_e \approx 9.11 \cdot 10^{-31}\ \mathrm{kg} \approx 511\ \mathrm{keV}/c^2$
proton mass	$m_p \approx 1.67 \cdot 10^{-27}\ \mathrm{kg} \approx 938\ \mathrm{MeV}/c^2$

Problems 1–5 are easier.

Send solutions of only <u>four</u> of them!

For each of them you can get up to 6 points.

Problems 6–9 are more difficult.

Send solutions of only <u>two</u> of them!

For each of these solutions you can get up to 8 points.

1 Problem – Electrons and diamond

In the crystal lattice of diamond, the nearest-neighbour atoms are at the distance $d = 1.545 \cdot 10^{-10}$ m. Electrons accelerated by the voltage U fall on the crystal. The length of matter wave (de Broglie wave) of the electron is d. The electrons were initially at rest. Acceleration takes place in a vacuum. The energy losses due to the electromagnetic radiation of the accelerated charge should be ignored.

Calculate the voltage U.

2 Problem – Laser and evaporation

A surgical laser emits infrared radiation of wavelength 10.6 µm. This radiation falls on a tissue of volume 0.1 mm^3. The tissue is mainly composed of water. Over time 1.00 ms, the temperature of the tissue illuminated by the laser rises from 36 °C to 100 °C and then the tissue evaporates. It should be assumed that the specific heat, the heat of evaporation, and the tissue density are constant and respectively equal $c = 4200$ J/(kg K), $l = 2257$ kJ/kg, $\rho = 997$ kg/m^3.

Calculate the number of photons absorbed by the tissue.

3 Problem – Harmonic oscillator

A particle of mass m can move along the X axis. The potential energy of the particle reads:

$$V(x) = \frac{1}{2} m\omega^2 x^2$$

where ω is a constant. In the framework of quantum mechanics, the ground state of this system, i.e. the state with the lowest possible energy, is described by the following wave function

$$\psi_0\left(x\right) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \exp\left(-\frac{m\omega}{2\hbar}x^2\right)$$

On the other hand, when this system with the same energy is considered in the framework of classical mechanics, the range of possible values of x turns out to be limited.

Calculate the probability of finding the particle outside the classically accessible x region for this quantum system in the ground state.

Hint.

$$\int_1^\infty e^{-t^2}\,\mathrm{d}t \approx 0.139$$

4 Problem – Process $e^+e^- \to \gamma\gamma$

A positron moving along a straight line collides with an electron at rest, so that the particles annihilate and two photons (gamma particles) are created. The direction of motion of each of the photons forms an angle 30° with the initial direction of motion of the positron.

Calculate the positron energy before the collision.

5 Problem – Two atoms

The half-life of a certain isotope is equal to $T_{1/2} = 4$ min. Two atoms of this isotope were placed in a trap. It should be assumed that atoms are distinguishable and that their decays are independent.

Calculate the probability that in the time $t = 2\,T_{1/2} = 8\,\mathrm{min}$ after placing the atoms in the trap **only one** of them will decay.

Problems 6–9 are more difficult.

Send solutions of only <u>two</u> of them!

For each of these solutions you can get up to 8 points.

6 Problem – Dumping coal

A cart of mass m is initially filled with coal of mass M; the cart is initially at rest. The cart is subject to a constant force of value F, parallel to a straight track, on which the cart can move. The cart has a hole in the floor; the coal escapes through that hole at a constant rate, such that after time T the cart is empty. Friction and air resistance should be ignored.

a) Determine the speed of the cart at the moment when the cart becomes empty.

b) Discuss the case $M \ll m$.

7 Problem – Flowing-out liquid

The interior of a container is a cylinder with a height of $h = 10$ m and a base radius $R = 1$ m. The axis of rotational symmetry of the container is vertical. The container is filled with an ideal fluid, i.e. nonviscous and incompressible fluid. In the bottom of the container there is a circular hole of area $S_0 = 0.1\ \mathrm{m}^2$. The density of the liquid is equal to $\rho = 1000\ \mathrm{kg/m}^{\,3}$. Gravitational acceleration $g = 10\ \mathrm{m/s}^2$ should be assumed. The change in atmospheric pressure between the base and the top of the cylinder should be ignored.

Calculate the time in which the liquid will flow out of the container.

8 Problem – Quantum cylinder

A carbon nanotube is a single layer of graphite (graphene) rolled into a cylinder. In this two-dimensional surface, electrons can move. In order to build a model of the carbon nanotube the side surface of the straight cylinder with the height L and the circumference D could be considered with the following assumptions. The probability density of finding an electron should approach 0 at the edge of the surface. The potential energy of an electron at the surface is 0.

a) Determine the position dependence of the wave function for one electron of a given energy, and the corresponding energy together with the values of quantum numbers (normalization of the function is not required).

b) For $D = 2L$, determine the ground state energy E_0 for one electron. Then write down: the energy values of the next four eigenstates as multiples of E_0, the corresponding quantum numbers and degeneration levels.

c) For $D = 2L$, determine the ground state energy of 10 electrons placed in this system, as a multiple of E_0.

Hint. The Laplace operator in the cylindrical coordinate system (ϕ, z) on the surface of a cylinder with radius R has the form:

$$\Delta = \frac{1}{R^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}$$

9 Problem – Heat flow

One mole of a monatomic perfect gas underwent the following transformation: with increasing volume, the pressure decreased linearly from the value $3p_0$ for the volume of V_0 to the value p_0 for the volume of $5V_0$, where p_0 and V_0 are constants. During this transformation the gas initially absorbed heat and then released it.

Determine the volume of the gas at which the direction of the heat flow changed.