Habilitation Thesis

“Applications of geometric structures to Lie systems”

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Personal information

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Degrees

1. **Profesor contratado doctor:** I obtained my accreditation as a '*Profesor contratado doctor*'[1] from the ANECA (Agencia Nacional de Evaluación de la Calidad y Acreditación, National Agency for the Evaluation of Excellence and Accreditation) in 2012.
2. **PhD in Physics:** I defended my PhD thesis '*Lie systems and applications to Quantum Mechanics*', written under the supervision of prof. J.F. Cariñena Marzo, in the Faculty of Sciences of the University of Zaragoza (Spain) on October 23rd, 2009. My PhD dissertation obtained the highest distinction (SOBRESALIENTE CUM LAUDE), and I was awarded the *'Premio Extraordinario de Doctorado 2009/2010'* (Special award for Doctoral Theses 2009/2010) of the Faculty of Physics of the University of Zaragoza in 2011[2].
3. **MSc in Physics**. Faculty of Physics, University of Salamanca, 2004. I was awarded a grant '*Beca de colaboración*' for the best students of the last year of the degree on Physics in 2004[3].

Scientific Career

1999–2004	MSc studies, Degree in Physics, University of Salamanca, Salamanca (Spain).
2004–2006	PhD studies, Department of Mathematics, University of Salamanca, Salamanca (Spain).
2006–2009	PhD studies, Department of Theoretical Physics, University of Zaragoza, Zaragoza (Spain).
2007–2009	Assistant professor, Department of Theoretical Physics, University of Zaragoza, Zaragoza (Spain).
2009–2012	Postdoc fellowship, Assistant professor, IMPAN, Warsaw (Poland).
2012–2013	Assistant professor, Faculty of Mathematics and Natural Sciences, School of Exact Sciences, Cardinal Stefan Wyszyński University, Warsaw (Poland).
2013–2018	Assistant professor, Department of Mathematical Methods in Physics, University of Warsaw (Poland).

[1]This term refers to researchers who have passed an official tenure examination that is approximately equivalent to the Polish habilitation. It allows researchers to supervise PhD students and to acquire a permanent position in academia. Indeed, I have already been the assistant supervisor of the PhD student, Cristina Sardón Muñoz, from 2013 until 2015. She defended her PhD thesis in the University of Salamanca in May 2015. She obtained the '*Special Prize for Doctoral Theses of the University of Salamanca*' for outstanding PhD theses in 2016. About the 70% of her PhD thesis was carried out exclusively under my supervision (the main supervisor was P. García Estévez) as illustrated by her works and my habilitation essay.

[2]The University of Saragossa offers one such an award for each of its faculties yearly

[3]The obligation of writing a master thesis appeared in the Spanish university system only after the introduction of the Bologna process.

Contents

Summary

A *Lie system* is a non-autonomous system of first-order ordinary differential equations whose general solution can be described as an autonomous function, the *superposition rule*, of a generic family of particular solutions and a set of constants [**CL.PH12, PW, CGM00, CGM07**]. Relevant types of Lie systems are non-autonomous linear systems of first-order ordinary differential equations and most types of Riccati equations, e.g. matrix and conformal Riccati equations [**CL.PH12, PW, AW**].

The interest of Lie systems is twofold. On the one hand, their study involves the analysis of relevant properties of geometric structures such as k-symplectic structures [**LV.H1**], Dirac structures [**CGL.H6**], momentum maps [**CGL.H6, CGL.H8**], or the local classification of finite-dimensional Lie algebras of vector fields on manifolds [**BBHL.H2, GKO92**]. On the other hand, Lie systems occur in physics, biology, medicine, etcetera [**BHL.H3**]. As Lie systems are not very well known [**CL.PH12**], their applications to old and modern problems allow for approaches that have passed unadvertised so far (cf. [**LV.H1**]).

This habilitation thesis describes my postdoctoral results on the application of modern geometric and algebraic structures to Lie systems, superposition rules, and their applications in physics, biology, and mathematics. My findings were published in a series of nine articles:

- LV.H1. J. de Lucas and S. Vilariño, k-Symplectic Lie systems: theory and applications, *J. Differential Equations* **258**, 2221–2255 (2015). (Ranking JCR[4]: 14/312 in Mathematics) My contribution was around the 85%.
- BBHL.H2. A. Ballesteros, A. Blasco, F.J. Herranz, J. de Lucas, and C. Sardón, Lie–Hamilton systems on the plane: Properties, classification and applications, *J. Differential Equations* **258**, 2873–2907 (2015). (Ranking JCR: 14/312 in Mathematics) My contribution was around the 60%.
- BHL.H3. A. Blasco, F.J. Herranz, J. de Lucas, and C. Sardón, Lie–Hamilton systems on the plane: applications and superposition rules, *J. Phys. A* **48**, 345202 (2015). (Ranking JCR: 11/53 in Physics, Mathematical) My contribution was around the 65%.
- LTV.H4. J. de Lucas, M. Tobolski, and S. Vilariño, A new application of k-symplectic Lie systems, *Int. J. Geom. Methods Mod. Phys.* **12**, 1550071 (2015). (Ranking JCR: 41/53 in Physics) My contribution was around the 55%.
- HL.H5. F.J. Herranz, J. de Lucas, and C. Sardón, Jacobi–Lie systems: theory and low dimensional classification in: *The 10th AIMS Conference on Dynamical Systems, Differential Equations and Applications*, AIMS Proceedings 2015, p. 605–614. My contribution was around the 70%.
- CGL.H6. J.F. Cariñena, J. Grabowski, J. de Lucas, and C. Sardón, Dirac–Lie systems and Schwarzian equations, *J. Differential Equations* **257**, 2303–2340 (2014). (Ranking JCR: 16/312 in Mathematics) My contribution was around the 55%.
- BCHL.H7. A. Ballesteros, J.F. Cariñena, F.J. Herranz, J. de Lucas, and C. Sardón, From constants of motion to superposition rules for Lie–Hamilton systems, *J. Phys. A* **46**, 285203 (2013). (Ranking JCR: 26/78 in Physics, Multidisciplinary) My contribution was around the 55%.
- CGL.H8. J.F. Cariñena, J. de Lucas, and C. Sardón, Lie–Hamilton systems: theory and applications, *Int. J. Geom. Methods Mod. Phys.* **10**, 1350047 (2013). (Ranking JCR: 45/55 in Physics) My contribution was around the 65%.

[4]Ranking JCR stands for the ranking in the Journal Citation Reports in the year of publication of the paper. Additional information is detailed in the curriculum of this habilitation.

CLS.H9. J.F. Cariñena, J. de Lucas, and C. Sardón, A new Lie systems approach to second-order Riccati equations, *Int. J. Geom. Methods Mod. Phys.* **9**, 1260007 (2012). (Ranking JCR: 34/55 in Physics, Mathematical) My contribution was around the 70%.

There were just a very few and simple applications of symplectic structures and Drinfel'd doubles to Lie systems when I finished my PhD in 2009 (see e.g. [**CGM00, CR05**]). After analysing several new Lie systems of Hamiltonian type [**CGL.H8, CLS.H9**], I noticed that geometric structures can be of the utmost relevance in this research field. Subsequently, I applied Poisson, Dirac, Jacobi and k-symplectic structures to construct superposition rules and other invariants of Lie systems appearing in physical, mathematical, and biological related problems, e.g. Ermakov-Ince invariants [**CGL.H8**], viral models [**BHL.H3**] or *Casimir tensor fields* [**BBHL.H2**]. As a first consequence, this allowed me to understand known and new invariants related to Lie systems geometrically [**BCHL.H7, LTV.H4**]. Second, that gave rise to techniques simplifying previous methods to obtain superposition rules, constants of motion, etcetera [**BCHL.H7**]. Third, my research led to derive new results and applications of modern geometric theories [**LV.H1**]. As a byproduct, I obtained many findings on the existence, properties, and applications of Lie algebras of vector fields on the plane [**BBHL.H2, HL.H5**]. This has inspired further research by other authors [**LS16, CS16, CS16II**].

In particular, the well-known superposition rule for Riccati equations [**LS**] was derived through a symplectic invariant constructed by means of a Casimir element of $\mathfrak{sl}(2)$ [**BCHL.H7**]. This led me to devise methods to simplify the calculation of superposition rules for Lie-Hamilton systems via Poisson coalgebras [**BBHL.H2, BHL.H3, BCHL.H7**]. I proved that k-symplectic structures can be related to Poisson algebras of functions. This was previously considered as impossible and/or useless [**LV.H1**]. Nevertheless, I proved that these Poisson algebras enable us to simplify the calculation of superposition rules and I applied this to study systems of first-order ordinary differential equations, control systems, and physical models [**LV.H1**]. This opened a new field of applications for k-symplectic structures, which are mainly applied nowadays to systems of partial differential equations appearing in field theories [**LSV16**].

The line of research described in this habilitation is far from being exhausted since other geometric structures can be applied to Lie systems as noticed in [**LS16**], where Lie systems were related to *Nambu-Poisson structures*. I think that Lie systems can also be studied through multisymplectic structures, twisted-Dirac manifolds, and Lie algebroids [**CGL.H6**]. I expect to develop these ideas in a near future.

Previously to my postdoctoral research, Lie systems were mainly applied to types of matrix Riccati equations, Ermakov-like systems, and a few Lie systems appearing in control theory and quantum mechanics by Cariñena, Marmo, Winternitz, and their coworkers [**CL.PH12, PW, CGM00**]. Meanwhile, this habilitation thesis details a much larger family of applications as illustrated by my analysis of Lie systems occurring in viral systems, Kummer–Schwarz equations, diffusion models, equations of the Riccati hierarchy, Buchdahl equations, and many more (see e.g. [**BBHL.H2, BHL.H3, CLS.H9**] and Table 2).

The analysis of the above-mentioned examples and their physical meaning was carried out in collaboration with my previous PhD student, dr. C. Sardón, and they form part of her PhD dissertation [**CS15**]. In our common articles, I was mostly concerned with the theoretical part of the work, and I suggested models for their application. C. Sardón applied my results to physical systems, e.g. in Winternitz–Smorodinski oscillators, and she also took part in the redaction of our common works. Her contribution was modest in some of our first common works, but the last paper of her PhD dissertation, namely [**EHL.PH1**], was almost entirely performed by her.

This habilitation thesis consists of two chapters. The first one details the main results of this habilitation. Since Lie systems are rather unknown to the general public, I describe quite in detail my findings and I illustrate them with recent examples from my research. The second chapter details other scientific activities accomplished during my postdoctoral research. In particular, it very briefly describes fourteen papers, named as [**EHL.PH1**]–[**CL.PH14**], that I wrote as a continuation of my research line as a PhD. Additionally, I also describe other works, namely [**E1**] and [**E2**], I wrote on the geometric properties and applications of differential equations, e.g. on the application of infinite-dimensional jet bundles to the description of symmetries of nonlinear oscillators. The second chapter also describes my attendance at conferences, seminars, my stays in research centers, and the talks I gave in international conferences, universities, and other research centers. A more detailed information can be found in the curriculum accompanying this dissertation.

CHAPTER 1

Lie systems and geometric structures

1. Introduction to Lie systems

This section surveys the most fundamental concepts and techniques of the theory of Lie systems (see [**CL.PH12, LS, CGM07, CGM00**] for details). It also provides a brief state-of-the-art of the subject prior to the works of my habilitation to help referees with the assessment of the impact of my findings. Although most results of this section are standard in the literature, their presentation has been devised mainly by my colleagues and me over the recent years.

Structures are hereafter considered to be real, smooth, and globally defined. This simplifies the presentation and highlights its main points. More precise results can easily be obtained by adding appropriate technical assumptions and details. Differential equations are always assumed to be non-autonomous systems of ordinary differential equations.

Let $(V, [\cdot, \cdot])$ be a Lie algebra with a Lie bracket $[\cdot\,, \cdot] : V \times V \to V$. We define $\mathrm{Lie}(\mathcal{B}, V, [\cdot, \cdot])$ to be the smallest Lie subalgebra of $(V, [\cdot, \cdot])$ containing a subset $\mathcal{B} \subset V$. When its meaning is clear from context, we denote $(V, [\cdot, \cdot])$ by V and $\mathrm{Lie}(\mathcal{B}, V, [\cdot, \cdot])$ by $\mathrm{Lie}(\mathcal{B}, [\cdot, \cdot])$ or simply $\mathrm{Lie}(\mathcal{B})$.

Definition 1.1. A *t-dependent vector field* X on N is a mapping $X : \mathbb{R} \times N \to TN$ such that $\tau \circ X = \pi$, where $\pi : (t, x) \in \mathbb{R} \times N \mapsto x \in N$ and $\tau : TN \to N$ is the canonical tangent bundle projection onto N.

Every t-dependent vector field X on N amounts to a family $\{X_t\}_{t\in\mathbb{R}}$ of standard vector fields $X_t : x \in N \mapsto X(t, x) \in TN$ on N.

Definition 1.2. ([**CL.PH12**], Definition 2.1) The *minimal Lie algebra* of a t-dependent vector field X on N is the smallest real Lie algebra of vector fields, V^X, containing $\{X_t\}_{t\in\mathbb{R}}$, i.e. $V^X = \mathrm{Lie}(\{X_t\}_{t\in\mathbb{R}}, [\cdot, \cdot])$.

Definition 1.3. An *integral curve* of a t-dependent vector field X on N is an integral curve $\gamma : \mathbb{R} \to \mathbb{R} \times N$ of the vector field $\bar{X} := \partial_t + X(t, x)$ on $\mathbb{R} \times N$ that is also a section of the fiber bundle $\mathrm{pr} : (t, x) \in \mathbb{R} \times N \mapsto t \in \mathbb{R}$, i.e. $\mathrm{pr} \circ \gamma = \mathrm{Id}_N$, where Id_N is the identity map on N.

In other words, the integral curves of a t-dependent vector field X on N are the sections $\gamma : \mathbb{R} \to \mathbb{R} \times N$ of the fiber bundle $\mathrm{pr} : (t, x) \in \mathbb{R} \times N \mapsto t \in \mathbb{R}$ that satisfy

$$\frac{\mathrm{d}\pi \circ \gamma}{\mathrm{d}t} = X(t, \pi \circ \gamma). \tag{1.1}$$

System (1.1) is the so-called *associated system* of X. Conversely, a first-order system of ordinary differential equations in normal form on N determines a unique t-dependent vector field X on N whose integral curves, γ, are such that $\pi \circ \gamma$ are particular solutions to (1.1). This justifies denoting by X both a t-dependent vector field and its associated system [**CL.PH12**].

Definition 1.4. ([**CGL.H8**], Definition 4) Let X be a t-dependent vector field on N. Its *associated distribution*, $\mathcal{D}^X$, is the generalised distribution on N given by

$$\mathcal{D}^X_x := \{Y_x \mid Y \in V^X\} \subset T_xN, \qquad \forall x \in N,$$

and its *associated co-distribution*, $\mathcal{V}^X$, is the generalised co-distribution on N of the form

$$\mathcal{V}^X_x := \{\vartheta \in T^*_xN \mid \vartheta(Z_x) = 0, \forall\ Z_x \in \mathcal{D}^X_x\} = (\mathcal{D}^X_x)^\circ \subset T^*_xN, \qquad \forall x \in N,$$

where $(\mathcal{D}^X_x)^\circ$ is the *annihilator* of $\mathcal{D}^X_x$ for each $x \in N$.

The generalised distribution $\mathcal{D}^X$ is involutive and regular on each connected component of an open dense subset U^X of N. Hence, $\mathcal{V}^X$ becomes a regular co-distribution on each connected component of U^X (see [**CGL.H8**, p. 5] for details). The most relevant case for us is when $\mathcal{D}^X$ is spanned by a finite-dimensional V^X. In that case $\mathcal{D}^X$ becomes integrable on N (cf. [**JP**, p. 63]). These structures are interesting to obtain constants of motion and superposition rules for Lie systems.

Definition 1.5. A *Lie system* is a system X whose V^X is finite-dimensional. A finite-dimensional Lie algebra V containing V^X is called a *Vessiot–Guldberg Lie algebra* of X [**BBHL.H2, CL.PH12, PW**].

The Lie algebra V^X of a Lie system X contains relevant information about it, e.g. a solvable V^X allows us to integrate X by quadratures [**CRG**].

Example 1.6. Every Lie system on $\mathbb{R}$ is locally diffeomorphic to a Riccati equation [**Eg07, PW, LS**], namely a differential equation of the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = a_1(t) + a_2(t)x + a_3(t)x^2, \qquad x \in \mathbb{R}, \tag{1.2}$$

where $a_1(t), a_2(t), a_3(t)$ are arbitrary t-dependent real functions. There is a wealth of applications of the Riccati equation in physics [**NR02, Ra71**].

The Riccati equation (1.2) has an associated t-dependent vector field $X = \sum_{\alpha=1}^{3} a_\alpha(t)X_\alpha$, where

$$X_1 := \partial_x, \qquad X_2 := x\partial_x, \qquad X_3 := x^2\partial_x. \tag{1.3}$$

Since

$$[X_1, X_2] = X_1, \qquad [X_1, X_3] = 2X_2, \qquad [X_2, X_3] = X_3,$$

these vector fields span a Lie algebra $V \simeq \mathfrak{sl}(2)$. As X takes values in V, i.e. $X_t \in V$ for every $t \in \mathbb{R}$, it follows that $V^X \subset V$ and V^X is finite-dimensional. Hence, X is a Lie system and V is a Vessiot–Guldberg Lie algebra for X. The form of V^X depends on the t-dependent coefficients of X, e.g. $V^X = 0$ for $a_\alpha(t) = 0$ with $\alpha = 1, 2, 3$.

Example 1.7. Let us give another relevant Lie system analyzed in [**BBHL.H2**, Example 2.1]. Consider the system of differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = a_1(t) + a_2(t)x + a_3(t)(x^2 - y^2), \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = a_2(t)y + a_3(t)2xy, \qquad (x, y) \in \mathbb{R}^2, \tag{1.4}$$

where $a_1(t), a_2(t), a_3(t)$ are arbitrary t-dependent real functions. The system (1.4) is a particular type of planar Riccati equation briefly studied in [**Eg07**]. Writing $z := x + \mathrm{i}y$, system (1.4) amounts to

$$\frac{\mathrm{d}z}{\mathrm{d}t} = a_1(t) + a_2(t)z + a_3(t)z^2, \qquad z \in \mathbb{C}, \tag{1.5}$$

which is a particular type of complex Riccati equations, which plays a relevant role in several physical problems, like in the study of non-autonomous Schrödinger equations [**Sc12**] and others [**BHL.H3**]. Particular solutions of periodic equations of this type have been investigated in [**Ca97, Or12**] and other special types of complex Riccati equations appear in [**FMR10**]. The differential equations (1.5) also appear in the application of Wei–Norman method in t-dependent quantum harmonic oscillators [**CK13**].

The system (1.4) is related to the t-dependent vector field $X := \sum_{\alpha=1}^{3} a_\alpha(t)X_\alpha$, where

$$X_1 := \partial_x, \qquad X_2 := x\partial_x + y\partial_y, \qquad X_3 := (x^2 - y^2)\partial_x + 2xy\partial_y \tag{1.6}$$

span a Vessiot–Guldberg Lie algebra $V \simeq \mathfrak{sl}(2)$. Hence, $\{X_t\}_{t\in\mathbb{R}} \subset V^X \subset V$ and V^X is finite-dimensional, which turns X to be a Lie system. The Lie algebra V is a Vessiot–Guldberg Lie algebra for X, and the systems of the form (1.4) are Lie systems.

Real and complex Riccati equations are types of Riccati equations over a normed division algebra with unity which are in turn particular cases of conformal Riccati equations [**LT.PH2**]. Example 1.7 is, additionally, a particular type of Cayley–Klein Riccati equation, defined and studied in [**BHL.H3**]. I proved that all these differential equations can be described through Lie systems. Details were analyzed together with my collaborators and students in [**BBHL.H2, BHL.H3, LT.PH2**].

Riccati equations are differential equations determined by a t-dependent second-order polynomial. Despite this apparent simplicity, there is no method to integrate a Riccati equation with arbitrary t-dependent coefficients [**Ince**]. Nonetheless, the general solution, $x(t)$, to a Riccati equation (1.2) can be written as

$$x(t) := \frac{x_{(1)}(t)(x_{(3)}(t) - x_{(2)}(t)) + kx_{(2)}(t)(x_{(3)}(t) - x_{(1)}(t))}{x_{(3)}(t) - x_{(2)}(t) + k(x_{(3)}(t) - x_{(1)}(t))}, \qquad k \in \mathbb{R},$$

in terms of three different particular solutions $x_{(1)}(t), x_{(2)}(t), x_{(3)}(t)$ [**Ince**]. This reduces the integration of the whole Riccati equation to determine a set of three particular solutions. This simplifies the integration by means of numerical methods of Riccati equations [**PW**], and it also provides techniques to study the behaviour of their solutions [**LT.PH2**]. It turns out that (1.4) admits a similar property and, in general, Lie proved that every Lie system does also [**LS**]. This motivated the following definition (see [**CGM07, LCO09**] for further details).

Definition 1.8. A *superposition rule* depending on m particular solutions for a system X on N is a function $\Phi : N^m \times N \to N$, $x = \Phi(x_{(1)}, \ldots, x_{(m)}; \lambda)$, such that the general solution $x(t)$ of X can be brought into the form $x(t) = \Phi(x_{(1)}(t), \ldots, x_{(m)}(t); \lambda)$, where $x_{(1)}(t), \ldots, x_{(m)}(t)$ is any generic family of particular solutions and λ is a point of N to be related to initial conditions.

Example 1.9. Riccati equations admit a superposition rule $\Phi : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ given by

$$x := \frac{x_{(1)}(x_{(3)} - x_{(2)}) + kx_{(2)}(x_{(3)} - x_{(1)})}{x_{(3)} - x_{(2)} + k(x_{(3)} - x_{(1)})}, \qquad k \in \mathbb{R}.$$

The *Lie–Scheffers Theorem* establishes the necessary and sufficient conditions for a system X to have a superposition rule [**LS**, Theorem 44]. A modern statement of this relevant result is described next (for a modern geometric description see [**CGM07**, Theorem 1]) .

Theorem 1.10. *(Lie–Scheffers Theorem* [**LS, CGM07**]*) A system X admits a superposition rule if and only if $X = \sum_{\alpha=1}^{r} b_\alpha(t) X_\alpha$ for a certain family $b_1(t), \ldots, b_r(t)$ of t-dependent functions and a collection $X_1, \ldots, X_r$ of vector fields spanning an r-dimensional real Lie algebra. In other words, a system X admits a superposition rule if and only if V^X is finite-dimensional.*

Although the Lie–Scheffers Theorem characterize Lie systems, it can be difficult to determine whether a particular system X is a Lie system, namely whether V^X is finite-dimensional or not. To solve this, it is helpful to classify all possible finite-dimensional Lie algebras V^X on a fixed manifold. Lie proved that every Vessiot–Guldberg Lie algebra on $\mathbb{R}$ is locally diffeomorphic around a *generic point*, namely a point around which the elements of the Lie algebra generate a locally regular distribution, to $\langle \partial_x, x\partial_x, x^2\partial_x \rangle \simeq \mathfrak{sl}(2)$ [**LS, Lie1880**]. The corresponding result on $\mathbb{R}^2$ was solved by Lie [**Lie1880III**], but his proof led to many misunderstandings, which were finally fixed by González, Kamran and Olver [**GKO92**]. They proved that every finite-dimensional Lie algebra of vector fields on $\mathbb{R}^2$ is locally diffeomorphic around a generic point to one of the classes in Table 1. That is why we call this classification *GKO classification*. Additionally, Winternitz and coworkers classified almost all complex primitive intransitive Lie algebras of complex vector fields on $\mathbb{C}^n$, and they applied this result to obtain superposition rules for a very general class of complex Lie systems in certain canonical forms [**SW84, SW84II**]. Since Winternitz's classification is based on canonical forms, it can be easier in relevant cases to obtain the superposition rule by means of other methods.

The geometrical description of superposition rules as well as one of the techniques for their determination is based upon the notion of *diagonal prolongation* [**CGM07**].

Definition 1.11. Given a t-dependent vector field $X(t,x) := \sum_{i=1}^{n} X^i(t,x)\partial_{x^i}$ on N, the t-dependent vector field $\widetilde{X}^{[m+1]}$ on N^{m+1} of the form $\widetilde{X}^{[m+1]} := \sum_{a=0}^{m}\sum_{i=1}^{n} X^i(t, x_{(a)})\partial_{x^i_{(a)}}$, is called the *diagonal prolongation* of X to $N^{(m+1)}$.

TABLE 1. The GKO (González, Kamran, Olver) classification [**GKO92**] of classes of finite-dimensional real Lie algebras of vector fields on $\mathbb{R}^2$ and their most relevant characteristics. My contribution to the following table relies on the calculation of the domains and modular generating systems for every Lie algebra (the first one or two vector fields which are written between brackets form a modular generating system). These structures were found in [**BBHL.H2**, Definitions 3.1 and 4.3] to classify finite-dimensional Lie algebras of Hamiltonian vector fields on $\mathbb{R}^2$. Finally, $\mathfrak{g} \simeq \mathfrak{g}_1 \ltimes \mathfrak{g}_2$ means that $\mathfrak{g}$ is the direct sum (as linear subspaces) of $\mathfrak{g}_1$ and $\mathfrak{g}_2$, where $\mathfrak{g}_2$ is an ideal of $\mathfrak{g}$.

#	Primitive	Basis of vector fields X_i	Domain
P_1	$A_\alpha \simeq \mathbb{R} \ltimes \mathbb{R}^2$	$\{\partial_x, \partial_y\}, \alpha(x\partial_x + y\partial_y) + y\partial_x - x\partial_y, \quad \alpha \geq 0$	$\mathbb{R}^2$
P_2	$\mathfrak{sl}(2)$	$\{\partial_x, x\partial_x + y\partial_y\}, (x^2 - y^2)\partial_x + 2xy\partial_y$	$\mathbb{R}^2_{y\neq 0}$
P_3	$\mathfrak{so}(3)$	$\{y\partial_x - x\partial_y, (1 + x^2 - y^2)\partial_x + 2xy\partial_y\}, 2xy\partial_x + (1 + y^2 - x^2)\partial_y$	$\mathbb{R}^2$
P_4	$\mathbb{R}^2 \ltimes \mathbb{R}^2$	$\{\partial_x, \partial_y\}, x\partial_x + y\partial_y, y\partial_x - x\partial_y$	$\mathbb{R}^2$
P_5	$\mathfrak{sl}(2) \ltimes \mathbb{R}^2$	$\{\partial_x, \partial_y\}, x\partial_x - y\partial_y, y\partial_x, x\partial_y$	$\mathbb{R}^2$
P_6	$\mathfrak{gl}(2) \ltimes \mathbb{R}^2$	$\{\partial_x, \partial_y\}, x\partial_x, y\partial_x, x\partial_y, y\partial_y$	$\mathbb{R}^2$
P_7	$\mathfrak{so}(3,1)$	$\{\partial_x, \partial_y\}, x\partial_x + y\partial_y, y\partial_x - x\partial_y, (x^2 - y^2)\partial_x + 2xy\partial_y, 2xy\partial_x + (y^2 - x^2)\partial_y$	$\mathbb{R}^2$
P_8	$\mathfrak{sl}(3)$	$\{\partial_x, \partial_y\}, x\partial_x, y\partial_x, x\partial_y, y\partial_y, x^2\partial_x + xy\partial_y, xy\partial_x + y^2\partial_y$	$\mathbb{R}^2$

#	Imprimitive	Basis of vector fields X_i	Domain
I_1	$\mathbb{R}$	$\{\partial_x\}$	$\mathbb{R}^2$
I_2	$\mathfrak{h}_2$	$\{\partial_x\}, x\partial_x$	$\mathbb{R}^2$
I_3	$\mathfrak{sl}(2)$ (type I)	$\{\partial_x\}, x\partial_x, x^2\partial_x$	$\mathbb{R}^2$
I_4	$\mathfrak{sl}(2)$ (type II)	$\{\partial_x + \partial_y, x\partial_x + y\partial_y\}, x^2\partial_x + y^2\partial_y$	$\mathbb{R}^2_{x\neq y}$
I_5	$\mathfrak{sl}(2)$ (type III)	$\{\partial_x, 2x\partial_x + y\partial_y\}, x^2\partial_x + xy\partial_y$	$\mathbb{R}^2_{y\neq 0}$
I_6	$\mathfrak{gl}(2)$ (type I)	$\{\partial_x, \partial_y\}, x\partial_x, x^2\partial_x$	$\mathbb{R}^2$
I_7	$\mathfrak{gl}(2)$ (type II)	$\{\partial_x, y\partial_y\}, x\partial_x, x^2\partial_x + xy\partial_y$	$\mathbb{R}^2_{y\neq 0}$
I_8	$B_\alpha \simeq \mathbb{R} \ltimes \mathbb{R}^2$	$\{\partial_x, \partial_y\}, x\partial_x + \alpha y\partial_y, \quad 0 < \lvert\alpha\rvert \leq 1$	$\mathbb{R}^2$
I_9	$\mathfrak{h}_2 \oplus \mathfrak{h}_2$	$\{\partial_x, \partial_y\}, x\partial_x, y\partial_y$	$\mathbb{R}^2$
I_{10}	$\mathfrak{sl}(2) \oplus \mathfrak{h}_2$	$\{\partial_x, \partial_y\}, x\partial_x, y\partial_y, x^2\partial_x$	$\mathbb{R}^2$
I_{11}	$\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$	$\{\partial_x, \partial_y\}, x\partial_x, y\partial_y, x^2\partial_x, y^2\partial_y$	$\mathbb{R}^2$
I_{12}	$\mathbb{R}^{r+1}$	$\{\partial_y\}, \xi_1(x)\partial_y, \ldots, \xi_r(x)\partial_y, \quad r \geq 1$	$\mathbb{R}^2$
I_{13}	$\mathbb{R} \ltimes \mathbb{R}^{r+1}$	$\{\partial_y\}, y\partial_y, \xi_1(x)\partial_y, \ldots, \xi_r(x)\partial_y, \quad r \geq 1$	$\mathbb{R}^2$
I_{14}	$\mathbb{R} \ltimes \mathbb{R}^r$	$\{\partial_x, \eta_1(x)\partial_y\}, \eta_2(x)\partial_y, \ldots, \eta_r(x)\partial_y, \quad r \geq 1$	$\mathbb{R}^2$
I_{15}	$\mathbb{R}^2 \ltimes \mathbb{R}^r$	$\{\partial_x, y\partial_y\}, \eta_1(x)\partial_y, \ldots, \eta_r(x)\partial_y, \quad r \geq 1$	$\mathbb{R}^2$
I_{16}	$C^r_\alpha \simeq \mathfrak{h}_2 \ltimes \mathbb{R}^{r+1}$	$\{\partial_x, \partial_y\}, x\partial_x + \alpha y\partial y, x\partial_y, \ldots, x^r\partial_y, \quad r \geq 1, \qquad \alpha \in \mathbb{R}$	$\mathbb{R}^2$
I_{17}	$\mathbb{R} \ltimes (\mathbb{R} \ltimes \mathbb{R}^r)$	$\{\partial_x, \partial_y\}, x\partial_x + (ry + x^r)\partial_y, x\partial_y, \ldots, x^{r-1}\partial_y, \quad r \geq 1$	$\mathbb{R}^2$
I_{18}	$(\mathfrak{h}_2 \oplus \mathbb{R}) \ltimes \mathbb{R}^{r+1}$	$\{\partial_x, \partial_y\}, x\partial_x, x\partial_y, y\partial_y, x^2\partial_y, \ldots, x^r\partial_y, \quad r \geq 1$	$\mathbb{R}^2$
I_{19}	$\mathfrak{sl}(2) \ltimes \mathbb{R}^{r+1}$	$\{\partial_x, \partial_y\}, x\partial_y, 2x\partial_x + ry\partial_y, x^2\partial_x + rxy\partial_y, x^2\partial_y, \ldots, x^r\partial_y, \quad r \geq 1$	$\mathbb{R}^2$
I_{20}	$\mathfrak{gl}(2) \ltimes \mathbb{R}^{r+1}$	$\{\partial_x, \partial_y\}, x\partial_x, x\partial_y, y\partial_y, x^2\partial_x + rxy\partial_y, x^2\partial_y, \ldots, x^r\partial_y, \quad r \geq 1$	$\mathbb{R}^2$

Since every vector field can be naturally considered as a t-dependent vector field, the above definition also applies to vector fields. Their diagonal prolongations are also vector fields [**CL.PH12**].

A method for determining superposition rules obtained by Cariñena, Grabowski, and Marmo is briefly described as follows (see [**CGM07, CL.PH7**] for details and examples). It is worth noting that this method is a theoretical improvement of *the method of invariants* previously applied by Winternitz [**PW**]. Indeed, this procedure is implicitly described in the proof of the Lie–Scheffers Theorem [**LS**].

(1) Take a basis $X_1, \ldots, X_r$ of a Vessiot–Guldberg Lie algebra V associated with the Lie system under study.
(2) Choose the smallest positive integer m so that the diagonal prolongations of $X_1, \ldots, X_r$ to N^m are linearly independent at a generic point.
(3) Take coordinates $x^1, \ldots, x^n$ on N. By defining this coordinate system on each copy of N within N^{m+1}, we get a coordinate system $\{x^i_{(a)} \mid i = 1, \ldots, n,\ a = 0, \ldots, m\}$ on N^{m+1}. Obtain n first-integrals $F_1, \ldots, F_n$ common to the diagonal prolongations $\widetilde{X}_1^{[m+1]}, \ldots, \widetilde{X}_r^{[m+1]}$ such that $\partial(F_1, \ldots, F_n)/\partial(x^1_{(0)}, \ldots, x^n_{(0)}) \neq 0$.
(4) Assume the above first-integrals to take certain real constant values, i.e. $F_i = k_i$ for $i = 1, \ldots, n$. By means of these equations, calculate the expressions of the variables $x^1_{(0)} \ldots, x^n_{(0)}$ in terms of $x^1_{(a)}, \ldots, x^n_{(a)}$, with $a = 1, \ldots, m$, and $k_1, \ldots, k_n$.
(5) The obtained expressions give rise to a superposition rule in terms of any generic family of m particular solutions and the constants $k_1, \ldots, k_n$.

Since the above method involves solving a system of PDEs to determine $F_1, \ldots, F_n$, the procedure becomes difficult (or impossible) to be applied. Other techniques to derive superposition rules, e.g. those described by Winternitz in [**PW, AHW81**], also demand the integration of the vector fields $X_1, \ldots, X_r$, which is frequently complicated and long [**PW, CL.PH7**]. The canonical forms derived by Winternitz [**SW84, SW84II**] are useful when it is simple to map our Lie system onto one of the canonical forms, e.g. that is the case of octonionic Riccati equations [**LT.PH2**] or the equations of the Riccati chain hierarchy [**GL16**]. This has applications in many physical systems and in quaternionic quantum mechanics [**LT.PH2, GL16**].

Previously to this habilitation thesis, applications of Lie systems were mostly restricted to matrix Riccati equations, thoroughly analyzed by Winternitz and his collaborators [**HWA83, LW96**], several Milne–Pinney equations, and a few control systems [**CR03, CCR03**]. Mathematically, geometric structures, like symplectic and/or Poisson structures, were very rarely used in the analysis of Lie systems. A few applications were done in integrable systems [**CGM00**] and during the study of classical and quantum systems [**CR03**].

2. Lie–Hamilton systems

2.1. On the relevance of Lie–Hamilton systems. I surprisingly found that there are more applications for Lie systems admitting a Vessiot–Guldberg Lie algebra of Hamiltonian vector fields with respect to a *Poisson bivector* [**IV**] than for Lie systems not related to any geometric structure [**CGL.H8**]. This is illustrated in Table 2, which summarizes most Lie systems of this type I found in [**LV.H1, BBHL.H2, BHL.H3, LTV.H4, HL.H5, CGL.H6, BCHL.H7, CGL.H8, CLS.H9**]. This led me to define the following notion.

Definition 2.1. [**CGL.H8**, Definition 10] A system X on N is a *Lie–Hamilton system* if V^X is a finite-dimensional Lie algebra of Hamiltonian vector fields relative to a Poisson structure on N.

Example 2.2. ([**BCHL.H7**, Section 7.4] and [**CGL.H8**, Section 4]) Let us analyze the Hamilton equations for an n-dimensional Winternitz–Smorodinsky oscillator [**WSUF67**] on $\mathrm{T}^*\mathbb{R}_0^n$, with $\mathbb{R}_0 := \mathbb{R}\backslash\{0\}$, i.e.

$$\begin{cases} \dfrac{dx_i}{dt} = p_i, \\ \dfrac{dp_i}{dt} = -\omega^2(t)x_i + \dfrac{k}{x_i^3}, \end{cases} \qquad i = 1, \ldots, n, \tag{2.1}$$

where $\omega(t)$ is an arbitrary t-dependent function and $k \in \mathbb{R}$. These oscillators have attracted quite much attention in classical and quantum mechanics for their special properties [**GPS06, HBS05**]. Note that (2.1) reduce to t-dependent isotropic harmonic oscillators when $k = 0$.

System (2.1) describes the integral curves of the t-dependent vector field on $\mathrm{T}^*\mathbb{R}^n_0$ given by

$$X = \sum_{i=1}^{n} \left[p_i \partial_{x_i} + \left(-\omega^2(t) x_i + \frac{k}{x_i^3} \right) \partial_{p_i} \right]. \tag{2.2}$$

This allows us to write $X_t = X_3 + \omega^2(t) X_1$ by defining

$$X_1 := -\sum_{i=1}^{n} x_i \partial_{p_i}, \qquad X_2 := \sum_{i=1}^{n} \frac{1}{2} \left(p_i \partial_{p_i} - x_i \partial_{x_i} \right), \qquad X_3 := \sum_{i=1}^{n} \left(p_i \partial_{x_i} + \frac{k}{x_i^3} \partial_{p_i} \right). \tag{2.3}$$

Since

$$[X_1, X_3] = 2X_2, \qquad [X_1, X_2] = X_1, \qquad [X_2, X_3] = X_3, \tag{2.4}$$

it follows that (2.1) is a Lie system related to a Vessiot–Guldberg Lie algebra isomorphic to $\mathfrak{sl}(2)$. This Lie algebra also consists of Hamiltonian vector fields relative to the natural Poisson bivector $\Lambda := \sum_{i=1}^{n} \partial_{x_i} \wedge \partial_{p_i}$. Indeed, let $\widehat{\Lambda}$ be the vector bundle morphism $\widehat{\Lambda} : \mathrm{T}^*\mathbb{R}^n_0 \to \mathrm{T}\mathbb{R}^n_0$ given by $[\widehat{\Lambda}(\theta)](\theta') := \Lambda(\theta, \theta')$ for arbitrary $\theta, \theta' \in \mathrm{T}^*\mathbb{R}^n_0$. Then, $X_\alpha = -\widehat{\Lambda}(dh_\alpha)$, with $\alpha = 1, 2, 3$ and

$$h_1 = \frac{1}{2} \sum_{i=1}^{n} x_i^2, \qquad h_2 = -\frac{1}{2} \sum_{i=1}^{n} x_i p_i, \qquad h_3 = \frac{1}{2} \sum_{i=1}^{n} \left(p_i^2 + \frac{k}{x_i^2} \right). \tag{2.5}$$

Hence, X becomes a Lie–Hamilton system.

Each vector field X_t, with $t \in \mathbb{R}$, admits a Hamiltonian function $h_t = h_3 + \omega^2(t) h_1$ and

$$\{h_1, h_2\}_\Lambda = -h_1, \qquad \{h_1, h_3\}_\Lambda = 2h_2, \qquad \{h_2, h_3\}_\Lambda = h_3,$$

where $\{\cdot, \cdot\}_\Lambda$ is the Poisson bracket related to Λ [**IV**]. Hence, X can be associated with a t-dependent Hamiltonian function $h := h_3 + \omega^2(t) h_1$ taking values in a finite-dimensional Lie algebra of functions (over the reals). I found a similar structure while studying Kummer–Schwarz equations of second-order [**BCHL.H7, CGL.H8**] and certain integrable systems with trigonometric non-linearities [**BCHL.H7, ADR12**]. This led me to give a definition for the above structure and to study its properties.

Definition 2.3. ([**CGL.H8**, Definition 11]) A *Lie–Hamiltonian structure* is a triple (N, Λ, h), where (N, Λ) stands for a Poisson manifold and h represents a t-parametrised family of functions $h_t : N \to \mathbb{R}$ such that $\mathfrak{W} := \mathrm{Lie}(\{h_t\}_{t\in\mathbb{R}}, \{\cdot, \cdot\}_\Lambda)$ is finite-dimensional. We call $\mathfrak{W}$ the *Lie–Hamilton algebra* of (N, Λ, h). A t-dependent vector field X *admits, or possesses, a Lie–Hamiltonian structure* (N, Λ, h) if $X_t = -\widehat{\Lambda}(dh_t)$ for all $t \in \mathbb{R}$.

Roughly speaking, a Lie–Hamiltonian structure is a curve within a certain Lie algebra of functions relative to a Poisson structure, and a system X admits a Lie–Hamiltonian structure if it can be determined by means of one. The following example illustrates the above.

Example 2.4. ([**CGL.H8**, p. 10–12]) The system X given by (2.2) admits a Lie–Hamiltonian structure given by $(\mathrm{T}^*\mathbb{R}^n_0, \Lambda, h = h_3 + \omega^2(t) h_1)$. Indeed, $X_t = -\widehat{\Lambda}(dh_t)$ for each $t \in \mathbb{R}$. The corresponding Lie–Hamilton algebra reads $\mathrm{Lie}(\{h_t\}_{t\in\mathbb{R}})$. If $\{X_t\}_{t\in\mathbb{R}} = \langle X_1, X_3 \rangle$, then $\{h_t\}_{t\in\mathbb{R}} = \{h_1, h_3\}$ and $\mathfrak{W} = \{h_1, h_2, h_3\}$.

The main theorem of the theory of Lie–Hamilton systems is the following one. Roughly speaking, it states that every Lie–Hamilton system comes from a curve in a finite-dimensional Lie algebra of functions relative to a Poisson structure.

Theorem 2.5. *(*[**CGL.H8**, Theorem 16]*) A system X admits a Lie–Hamiltonian structure if and only if it is a Lie–Hamilton system.*

TABLE 2. Specific Lie–Hamilton systems on the plane according to their class given in Table 1. All of these systems have t-dependent real coefficients except for P_1. The systems marked with '$*$' ($\mathrm{I}_{14A}^{r=2}$ and $\mathrm{I}_{14B}^{r=2}$) have been studied in [**BBHL.H2**], while the one marked with '†' in P_3 can be found in [**BCHL.H7, ADR12**]. All these Lie systems were discovered in the papers [**LV.H1**]–[**CLS.H9**]. Further information on this table can be found in [**BHL.H3**].

#	LH systems
P_1	Complex Bernoulli equation $\dot{z} = \mathrm{i}a(t)z + b(t)z^n$ for real $a(t)$ and complex $b(t)$
P_2	Complex Riccati equation
	Milne–Pinney and Kummer–Schwarz equations with $c > 0$
P_3	Projective Schrödinger equations on $\mathbb{CP}^1$
	Planar system with trigonometric nonlinearities†
P_5	Dissipative harmonic oscillator
	Second-order Riccati equation in Hamiltonian form
I_4	Split-complex Riccati equation
	Coupled Riccati equations
	Milne–Pinney and Kummer–Schwarz equations with $c < 0$
	Planar diffusion Riccati system for $c_0 = 1$
I_5	Dual-Study Riccati equation
	Milne–Pinney and Kummer–Schwarz equations with $c = 0$
	Harmonic oscillator
	Planar diffusion Riccati system for $c_0 = 0$
$\mathrm{I}_{14A}^{r=1}$	Complex Bernoulli equation $\dot{z} = a_1(t)z + a_2(t)z^n$
	Generalised Buchdahl equations
	Lotka–Volterra systems
$\mathrm{I}_{14A}^{r=2}$	Quadratic polynomial systems $\dot{x} = bx + c(t)y + f(t)y^2$, $\dot{y} = y$ with $b \notin \{1, 2\}^*$
$\mathrm{I}_{14B}^{r=2}$	Quadratic polynomial systems $\dot{x} = bx + c(t)y + f(t)y^2$, $\dot{y} = y$ with $b \in \{1, 2\}^*$
	A primitive model of viral infection*

2.2. Existence and classification of Lie–Hamilton systems on low-dimensional manifolds. After defining Lie–Hamilton systems, I tried to determine conditions characterizing when a Lie system can be considered as a Lie–Hamilton system relative to a certain Poisson structure. The only Lie–Hamilton system on the real line is $X = 0$ [**CGL.H6**]. In general, there exists no easy criterium to check whether a Lie system on a fixed manifold is a Lie–Hamilton system. This section summarizes my main results on this topic.

The following criterium is useful to determine when a Lie system cannot be considered as a Lie–Hamilton system.

Proposition 2.6. [**CGL.H6**, Proposition 5.1] *If X is a Lie system on an odd-dimensional manifold N and $\mathcal{D}^X_{x_0} = T_{x_0}N$ for a point x_0 in N, then X is not a Lie–Hamilton system on N.*

In virtue of the previous proposition certain Lie systems, e.g. third-order Kummer–Schwarz equations [**CGL.H6**], cannot be studied as Lie–Hamilton systems. This motivated to study Lie systems with Vessiot–Guldberg Lie algebras of Hamiltonian vector fields relative to more general structures, e.g. k-symplectic structures (see [**LV.H1, CGL.H6**] for details).

The GKO classification allowed me to classify in [**BBHL.H2**] all Vessiot–Guldberg Lie algebras of Hamiltonian vector fields on the plane around a generic point of the Lie algebra. The *domain of a Lie algebra of vector fields* is the set of generic points of the Lie algebra. My classification along with the calculation of domains of Lie algebra of vector fields on the plane and other related notions is shown in Tables 1, 3 and 4 [**BBHL.H2, BHL.H3**]. This gave rise to a local classification of Lie–Hamilton systems on the plane. I hereafter resume my main results on the topic. To simplify the notation, U will hereafter stand for a contractible open subset of $\mathbb{R}^2$.

A *volume form* Ω on an n-dimensional manifold N is a non-vanishing n-form on N. The *divergence* of a vector field X on N with respect to Ω is the unique function $\mathrm{div}X : N \to \mathbb{R}$ satisfying $\mathcal{L}_X\Omega = (\mathrm{div}\, X)\,\Omega$, where $\mathcal{L}_X$ denotes the Lie derivative in terms of X. An *integrating factor* for X on $U \subset N$ is a function $f : U \to \mathbb{R}$ such that $\mathcal{L}_{fX}\Omega = 0$ on U.

The key notion to accomplish the classification of Vessiot–Guldberg Lie algebras of Hamiltonian vector fields on the plane is given next.

Definition 2.7. ([**BBHL.H2**, Definition 4.3]) Let V be a vector space of vector fields on U, we say that V admits a *modular generating system* $(U_1, X_1, \ldots, X_p)$ if U_1 is a dense open subset of U such that every $X \in V|_{U_1}$ can be brought into the form $X|_{U_1} = \sum_{i=1}^{p} g_i X_i|_{U_1}$ for certain functions $g_1, \ldots, g_p \in C^\infty(U_1)$ and $X_1, \ldots, X_p \in V$.

Example 2.8. ([**BBHL.H2**, Example 4.1]) Given the Lie algebra $\mathrm{P}_3 \simeq \mathfrak{so}(3)$ on $\mathbb{R}^2$ of Table 1, the vector fields

$$X_1 = y\partial_x - x\partial_y, \qquad X_2 = (1 + x^2 - y^2)\partial_x + 2xy\partial_y$$

of P_3 satisfy that $X_3 = g_1X_1 + g_2X_2$ on $U_1 := \{(x, y) \in \mathbb{R}^2 \mid x \neq 0\}$ for the functions $g_1, g_2 \in C^\infty(U_1)$ given by $g_1 := (x^2 + y^2 - 1)/x$ and $g_2 := y/x$. Obviously, U_1 is an open dense subset of $\mathbb{R}^2$. As every element of V is a linear combination of X_1, X_2 and $X_3 = g_1X_1 + g_2X_2$, then every $X \in V|_{U_1}$ can be written as a linear combination with smooth functions on U_1 of X_1 and X_2. So, (U_1, X_1, X_2) forms a generating modular system for P_3.

I derived a modular generating system for every Lie algebra of the GKO classification ([**BCHL.H7**, Table 1] and Table 1 in this dissertation). My results appear in Table 1. This notion gave rise to the following theorem and corollary.

Theorem 2.9. *(*[**BCHL.H7**, Theorem 4.4]*) Let V be a Lie algebra of vector fields on $U \subset \mathbb{R}^2$ admitting a modular generating system $(U_1, X_1, \ldots, X_p)$. We have that:*
1) The space V consists of Hamiltonian vector fields relative to a symplectic form on U if and only if:
i) Let $g_1, \ldots, g_p$ be certain smooth functions on $U_1 \subset U$. Then,

$$X|_{U_1} = \sum_{i=1}^{p} g_i X_i|_{U_1} \in V|_{U_1} \Longrightarrow \mathrm{div}X|_{U_1} = \sum_{i=1}^{p} g_i \mathrm{div}X_i|_{U_1}. \tag{2.6}$$

ii) The elements $X_1, \ldots, X_p$ admit a common non-vanishing integrating factor on U.
2) If the rank of $\mathcal{D}^V$ is two on U, then the symplectic form is unique up to a multiplicative non-zero constant.

Corollary 2.10. *(*[**BCHL.H7**, Corollary 4.5]*) If a Lie algebra of vector fields V on a $U \subset \mathbb{R}^2$ consists of Hamiltonian vector fields with respect to a symplectic form and admits a modular generating system whose elements are divergence free, then every element of V is divergence free.*

The application of Theorem 2.9 and Corollary 2.10 to the GKO classification allows for the local classification of Vessiot–Guldberg Lie algebras of Hamiltonian vector fields on $\mathbb{R}^2$. My result is detailed in Table 3.

Table 3 enables us to determine if X is a Lie-Hamilton system and it details all important structures related to this fact provided it is possible to determine to which class of the GKO classification V^X is diffeomorphic to. If so, a change of variables mapping V^X into the Lie algebra given in Table 3 enables us to derive a symplectic structure turning X into a Lie–Hamilton system. If V^X is isomorphic only to one class of the GKO classification, then it is simple to establish whether X is a Lie-Hamilton system or not.

TABLE 3. The classification of the $4+8$ classes of finite-dimensional real Lie algebras of Hamiltonian vector fields on $\mathbb{R}^2$ found in [**BBHL.H2**]. For I_{12}, I_{14A} and I_{16}, we have $j = 1, \ldots, r$ and $r \geq 1$; in I_{14B} the index $j = 2, \ldots, r$.

#	Primitive	Hamiltonian functions h_i	ω	Lie–Hamilton algebra
P_1	$A_0 \simeq \mathfrak{iso}(2)$	$y,\ -x,\ \frac{1}{2}(x^2+y^2),\ 1$	$\mathrm{d}x \wedge \mathrm{d}y$	$\overline{\mathfrak{iso}}(2)$
P_2	$\mathfrak{sl}(2)$	$-\frac{1}{y},\ -\frac{x}{y},\ -\frac{x^2+y^2}{y}$	$\frac{\mathrm{d}x \wedge \mathrm{d}y}{y^2}$	$\mathfrak{sl}(2)$ or $\mathfrak{sl}(2) \oplus \mathbb{R}$
P_3	$\mathfrak{so}(3)$	$\frac{-1}{2(1+x^2+y^2)},\ \frac{y}{1+x^2+y^2},\ -\frac{x}{1+x^2+y^2},\ 1$	$\frac{\mathrm{d}x \wedge \mathrm{d}y}{(1+x^2+y^2)^2}$	$\mathfrak{so}(3)$ or $\mathfrak{so}(3) \oplus \mathbb{R}$
P_5	$\mathfrak{sl}(2) \ltimes \mathbb{R}^2$	$y,\ -x,\ xy,\ \frac{1}{2}y^2,\ -\frac{1}{2}x^2,\ 1$	$\mathrm{d}x \wedge \mathrm{d}y$	$\overline{\mathfrak{sl}(2) \ltimes \mathbb{R}^2} \simeq \mathfrak{h}_6$
#	**Imprimitive**	**Hamiltonian functions h_i**	**ω**	**Lie–Hamilton algebra**
I_1	$\mathbb{R}$	$\int^y f(y')\mathrm{d}y'$	$f(y)\mathrm{d}x \wedge \mathrm{d}y$	$\mathbb{R}$ or $\mathbb{R}^2$
I_4	$\mathfrak{sl}(2)$ (type II)	$\frac{1}{x-y},\ \frac{x+y}{2(x-y)},\ \frac{xy}{x-y}$	$\frac{\mathrm{d}x \wedge \mathrm{d}y}{(x-y)^2}$	$\mathfrak{sl}(2)$ or $\mathfrak{sl}(2) \oplus \mathbb{R}$
I_5	$\mathfrak{sl}(2)$ (type III)	$-\frac{1}{2y^2},\ -\frac{x}{y^2},\ -\frac{x^2}{2y^2}$	$\frac{\mathrm{d}x \wedge \mathrm{d}y}{y^3}$	$\mathfrak{sl}(2)$ or $\mathfrak{sl}(2) \oplus \mathbb{R}$
I_8	$B_{-1} \simeq \mathfrak{iso}(1,1)$	$y,\ -x,\ xy,\ 1$	$\mathrm{d}x \wedge \mathrm{d}y$	$\overline{\mathfrak{iso}}(1,1) \simeq \mathfrak{h}_4$
I_{12}	$\mathbb{R}^{r+1}$	$-\int^x f(x')\mathrm{d}x', -\int^x f(x')\xi_j(x')\mathrm{d}x'$	$f(x)\mathrm{d}x \wedge \mathrm{d}y$	$\mathbb{R}^{r+1}$ or $\mathbb{R}^{r+2}$
I_{14A}	$\mathbb{R} \ltimes \mathbb{R}^r$ (type I)	$y,\ -\int^x \eta_j(x')\mathrm{d}x',\quad 1 \notin \langle \eta_j \rangle$	$\mathrm{d}x \wedge \mathrm{d}y$	$\mathbb{R} \ltimes \mathbb{R}^r$ or $(\mathbb{R} \ltimes \mathbb{R}^r) \oplus \mathbb{R}$
I_{14B}	$\mathbb{R} \ltimes \mathbb{R}^r$ (type II)	$y,\ -x,\ -\int^x \eta_j(x')\mathrm{d}x',\ 1$	$\mathrm{d}x \wedge \mathrm{d}y$	$\overline{(\mathbb{R} \ltimes \mathbb{R}^r)}$
I_{16}	$C_{-1}^r \simeq \mathfrak{h}_2 \ltimes \mathbb{R}^{r+1}$	$y,\ -x,\ xy,\ -\frac{x^{j+1}}{j+1},\ 1$	$\mathrm{d}x \wedge \mathrm{d}y$	$\overline{\mathfrak{h}_2 \ltimes \mathbb{R}^{r+1}}$

There are several isomorphic classes of Vessiot–Guldberg Lie algebras in the GKO classification that are not diffeomorphic, e.g. I_5, P_2, and I_4. The following results permit us to determine to which of them a Vessiot–Guldberg Lie algebra is diffeomorphic to.

Definition 2.11. ([**BHL.H3**, Definition 4.3]) Let V be a finite-dimensional real Lie algebra of vector fields and let $S_2(V)$ be the space of 2-contravariant symmetric tensors of elements on V. We call *Casimir tensor field* of V an element $R \in S_2(V)$ such that $\mathcal{L}_X R = 0$ for every $X \in V$.

Theorem 2.12. *(*[**BHL.H3**, Theorem 4.4]*) Let V be a Vessiot–Guldberg Lie algebra diffeomorphic to either* P_2, I_4 *or* I_5. *Let R be a non-zero Casimir tensor field for V. Writing $R = \sum_{\alpha,\beta=1}^2 R^{\alpha\beta}\partial_\alpha \otimes \partial_\beta$ with $\partial_1 = \partial_x$ and $\partial_2 = \partial_y$, we define*

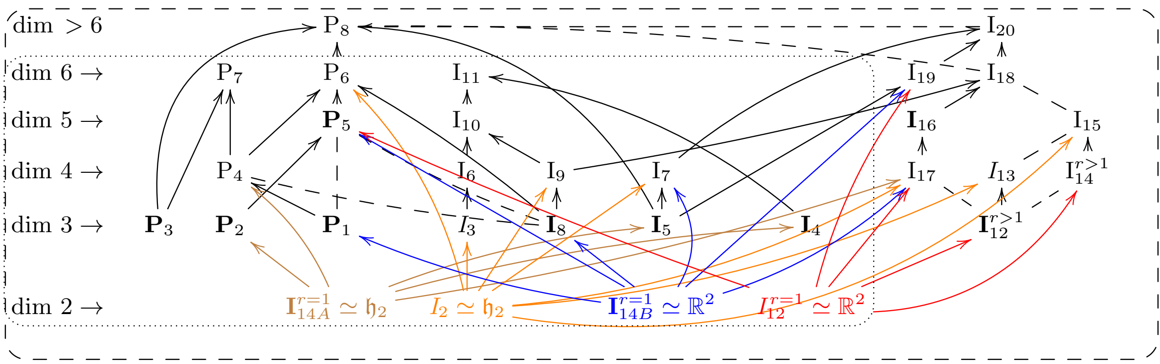
$$\mathcal{I}(V) := \operatorname{sign}\left(\det(R^{\alpha\beta}(x))\right), \qquad \forall x \in \operatorname{dom} V,$$

where $\operatorname{dom} V$ is the domain of V. If $\mathcal{I}(V) > 0$, then V is locally diffeomorphic to P_2*; when $\mathcal{I}(V) < 0$, then V is locally diffeomorphic to* I_4*; if $\mathcal{I}(V) = 0$, then V is locally diffeomorphic to* I_5.

Above result was applied by my colleagues and my PhD student to determine to which class of the GKO classification belong Smorodinsky–Winternitz oscillators, harmonic oscillators, Kummer–Schwarz equations, and other known and new types of Lie systems on the plane [**BHL.H3**].

Instead of using a change of variables to map V^X into a class of Table 3, I found that the following propositions allow us to determine a symplectic structure turning the elements of V^X into Hamiltonian vector fields without using changes of variables.

TABLE 4. Nonexhastive tree of inclusion relations between classes of the GKO classification [**BBHL.H2**]. This diagram significantly completes the inclusion relations between the classes of the GKO classification given in [**GKO92**]. The Lie algebras appearing out of the dot line are considered to have dimension bigger than 6. We write $A \to B$ when a subclass of A is diffeomorphic to a Lie subalgebra of B. Every Lie algebra includes I_1. In bold and italics are classes with Hamiltonian Lie algebras and rank one associated distribution, respectively. Colors help distinguishing the arrows.



Proposition 2.13. *(*[**BHL.H3**, Proposition 3.1]*) Let V be a Vessiot–Guldberg Lie algebra of planar vector fields. The vector fields of V are Hamiltonian with respect to a bivector field $\Lambda \in V \wedge V\backslash\{0\}$ if and only if V admits a one-dimensional trivial Lie algebra representation within $V \wedge V$.*

Theorem 2.14. *(*[**BHL.H3**, Theorem 3.6]*) If V is a planar Vessiot–Guldberg Lie algebra admitting a two-dimensional ideal I such that $I \wedge I \neq \{0\}$ and the elements of V act on I by traceless operators, namely the mappings $\vartheta_X : Y \in I \mapsto [X, Y] \in I$ are traceless for each $X \in V$, then V becomes a Lie algebra of Hamiltonian vector fields with respect to every element of $I \wedge I\backslash\{0\}$.*

The application of previous results gave rise to the determination of symplectic structures for most classes of Lie–Hamilton systems on the plane (see Examples 3.4, 3.5, 3.8–3.11 in [**BHL.H3**]).

2.3. Lie–Hamilton algebras. The other ingredient appearing in the study of Lie–Hamilton systems are Lie–Hamilton algebras. Their analysis is crucial to derive superposition rules [**BCHL.H7**] and constants motion of Lie–Hamilton systems [**CGL.H8**]. I will now detail my main results on the topic.

Lie–Hamilton algebras are not uniquely defined in general [**BBHL.H2**, Example 5.1]. Moreover, the existence of different types of Lie–Hamilton algebras for the same Lie–Hamilton system is important for their linearization and the use of certain methods [**CGL.H8**]. For instance, if a Lie–Hamilton system X on N admits a strong comomentum map, Lie–Hamilton algebra isomorphic to V^X and $\dim V^X = \dim N$, then X can be linearized along with its associated Poisson structure [**CGL.H8**, Proposition 23].

The following propositions were found and employed in [**BBHL.H2**] in order to obtain all Lie–Hamilton algebras for Lie–Hamilton systems on the plane. Their classification is detailed in Table 3.

Proposition 2.15. *(*[**BBHL.H2**, Proposition 5.1] *A Lie–Hamilton system X on a symplectic connected manifold (N, ω) possesses an associated Lie–Hamilton algebra $(\mathcal{H}_\Lambda, \{\cdot,\cdot\}_\omega)$ isomorphic to V^X if and only if every Lie–Hamilton algebra non-isomorphic to V^X is isomorphic to $V^X \oplus \mathbb{R}$.*

Proposition 2.16. *(*[**BBHL.H2**, Proposition 5.2] *If a Lie–Hamilton system X on a symplectic connected manifold (N, ω) admits an associated Lie–Hamilton algebra $(\mathcal{H}_\Lambda, \{\cdot,\cdot\}_\omega)$ isomorphic to V^X, then it admits a Lie–Hamilton algebra isomorphic to $V^X \oplus \mathbb{R}$.*

Corollary 2.17. *(*[**BBHL.H2**, Corollary 5.3] *If X is a Lie–Hamilton system with respect to a symplectic connected manifold (N, ω) admitting a Lie–Hamilton algebra $(\mathcal{H}_\Lambda, \{\cdot,\cdot\}_\omega)$ satisfying that $1 \in \{\mathcal{H}_\Lambda, \mathcal{H}_\Lambda\}_\omega$, then X does not possess any Lie–Hamilton algebra isomorphic to V^X.*

Proposition 2.18. *(*[**BBHL.H2**, Proposition 5.4] *If X is a Lie–Hamilton system on a connected manifold N admitting a V^X of Hamiltonian vector fields with respect to a symplectic structure ω that does not possess any Lie–Hamilton algebra $(\mathcal{H}_\Lambda, \{\cdot,\cdot\}_\omega)$ isomorphic to V^X, then all its Lie–Hamilton algebras (with respect to the Lie bracket $\{\cdot,\cdot\}_\omega$) are isomorphic.*

2.4. Constants of motion for Lie–Hamilton systems. Lie–Hamiltonian structures allows one to study and to derive of constants of motion, Lie symmetries, and other properties of Lie–Hamilton system. Let us start by the following proposition, constituting an extension of the celebrated result for constants of motion for autonomous Hamiltonian systems [**FM**].

Proposition 2.19. [**BCHL.H7**, Proposition 7] *Given a Lie–Hamilton system X on N admitting a Lie–Hamiltonian structure (N, Λ, h), a t-independent function is a constant of motion for X if and only if it Poisson commutes with the elements of $\mathcal{H}_\Lambda$. The family $\mathcal{I}^X$ of t-independent constants of motion of X form a Poisson algebra $(\mathcal{I}^X, \cdot, \{\cdot,\cdot\}_\Lambda)$.*

The t-dependent extension of the previous proposition reads as follows.

Proposition 2.20. *(*[**BCHL.H7**, Lemma 9 and Proposition 12]*) A Poisson manifold (N, Λ) induces a Poisson manifold $(\mathbb{R}\times N, \bar{\Lambda})$ with Poisson structure*

$$\{f, g\}_{\bar{\Lambda}}(t, x) := \{f_t, g_t\}_\Lambda(x), \qquad (t, x) \in \mathbb{R}\times N.$$

If X is a Lie–Hamilton system on N possessing a Lie–Hamiltonian structure (N, Λ, h), then $(\overline{\mathcal{I}}^X, \cdot, \{\cdot,\cdot\}_{\bar{\Lambda}})$, where $\overline{\mathcal{I}^X}$ is the space of t-dependent constants of motion of the system X, is a Poisson algebra.

The Poisson structure accompanying Lie–Hamilton systems allows for powerful methods to obtain particular types of constants of motion: the *Lie integrals* and the *polynomial Lie integrals* defined in [**BCHL.H7**] and appearing in physical problems [**Ma95**]. They will play a relevant role in the description of superposition rules for Lie–Hamilton systems. Additionally, they are useful in the study of physical systems as illustrated in coming examples.

Definition 2.21. ([**BCHL.H7**, Definitions 13 and 16]) Given a Lie–Hamilton system X on N possessing a Lie–Hamiltonian structure (N, Λ, h), a *polynomial Lie integral* for X with respect to (N, Λ, h) is a t-dependent constant of motion f for X of the form $f_t := \sum_{I\in M} \lambda_I(t) h^I$, where the I's are r-multi-indexes: sets $(i_1, \ldots, i_r)$ of nonnegative integers with $r \in \mathbb{N}$, the set M is a finite family of multi-indexes, the $\lambda_I(t)$ are certain t-dependent functions, and $h^I := h_1^{i_1} \cdot \ldots \cdot h_r^{i_r}$ for a fixed basis $\{h_1, \ldots, h_r\}$ for the Lie Hamilton algebra $\mathcal{H}_\Lambda$. A *Lie integral* is a polynomial Lie integral such that $\lambda_J = 0$ for every J with $|J| := \sum_{\alpha=1}^r i_\alpha \neq 1$.

Proposition 2.22. *(*[**BCHL.H7**, Propositions 14 i 15]*) Given a Lie–Hamilton system X with a Lie–Hamiltonian structure (N, Λ, h), the space $\mathfrak{L}_h^\Lambda$ of Lie integrals relative to (N, Λ, h) gives rise to a Lie algebra $(\mathfrak{L}_h^\Lambda, \{\cdot,\cdot\}_{\bar{\Lambda}})$ isomorphic to $(\mathcal{H}_\Lambda, \{\cdot,\cdot\}_\Lambda)$. The Lie algebra $\mathfrak{L}_h^\Lambda$ consists of t-independent constants of motion if and only if $\mathcal{H}_\Lambda$ is Abelian.*

We consider $S_\mathfrak{g}$ and $U_\mathfrak{g}$ to be the symmetric and the enveloping algebras of $\mathfrak{g}$ [**Va84**]. Both can be considered as Poisson Lie algebras, the second begin a non-commutative one, with respect to their natural associative products and the Lie brackets $\{\cdot,\cdot\}_{S_\mathfrak{g}}$ and $[\cdot,\cdot]_{U_\mathfrak{g}}$ (see [**Va84, CL99, BCHL.H7**]). The so-called *symmetrizer map* $\lambda : S_\mathfrak{g} \to U_\mathfrak{g}$ is a isomorphism of $\mathfrak{g}$-spaces, i.e. $\lambda(\{v, P\}_{S_\mathfrak{g}}) = [v, \lambda(P)]_{U_\mathfrak{g}}$ for every $P \in S_\mathfrak{g}$ and $v \in \mathfrak{g}$ [**Va84**].

Proposition 2.23. *(*[**BCHL.H7**, Proposition 19]*) A function f is a polynomial Lie integral for a Lie–Hamilton system X with respect to the Lie–Hamiltonian structure (N, Λ, h) if and only if $f_t = D(P_t)$ for every $t \in \mathbb{R}$, where $D : (S_\mathfrak{g}, \cdot, \{\cdot,\cdot\}_{S_\mathfrak{g}}) \to (C^\infty(N), \cdot, \{\cdot,\cdot\}_\Lambda)$ is the Poisson algebra morphism satisfying that its restriction to $\mathfrak{g}$ is Lie algebra injective mapping with $D(\mathfrak{g}) = \mathcal{H}_\Lambda$, and the curve P_t is a* *linear combination with linear t-dependent coefficients of a family of polynomials satisfying the differential equation*

$$\frac{\mathrm{d}P}{\mathrm{d}t} + \{P, w_t\}_{S_\mathfrak{g}} = 0, \qquad P \in S_\mathfrak{g}, \tag{2.7}$$

where w_t stands for a curve in $\mathfrak{g}$ such that $D(w_t) = h_t$ for every $t \in \mathbb{R}$.

Corollary 2.24. *(*[**BCHL.H7**, Corollary 21]*) Let X be a Lie–Hamilton system that possesses a Lie–Hamiltonian structure (N, Λ, h) inducing a Poisson algebra morphism $D : S_\mathfrak{g} \to C^\infty(N)$ as in Proposition 2.23. The function $F := D(C)$, where C is a Casimir element of $S_\mathfrak{g}$, is a t-independent constant of motion of X. If $\mathcal{C}$ is a Casimir element of $U_\mathfrak{g}$, then $F = D(\lambda^{-1}(\mathcal{C}))$ is t-independent constant of motion for X.*

Example 2.25. ([**BCHL.H7**, Section 7.1]) Let us consider the classical Ermakov system [**CL.PH12**]:

$$\begin{cases} \frac{\mathrm{d}^2x}{\mathrm{d}t^2} &= -\omega^2(t)x + \frac{b}{x^3}, \\ \frac{\mathrm{d}^2y}{\mathrm{d}t^2} &= -\omega^2(t)y, \end{cases}$$

where $\omega(t)$ is a non-constant t-dependent frequency and $b \in \mathbb{R}$. This system appears in a number of applications related to problems in quantum and classical mechanics [**LA08**]. By writing this system as a first-order one

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} &= v_x, \qquad & \frac{\mathrm{d}v_x}{\mathrm{d}t} &= -\omega^2(t)x + \frac{b}{x^3}, \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= v_y, \qquad & \frac{\mathrm{d}v_y}{\mathrm{d}t} &= -\omega^2(t)y, \end{cases} \tag{2.8}$$

it becomes a Lie system related to a Vessiot–Guldberg Lie algebra V isomorphic to $\mathfrak{sl}(2)$ [**CL.PH12**]. In fact, system (2.8) describes the integral curves of the t-dependent vector field $X = X_3 + \omega^2(t)X_1$, where the vector fields

$$X_1 := -x\partial_{v_x} - y\partial_{v_y}, \quad X_2 := \frac{1}{2}\left(v_x\partial_{v_x} + v_y\partial_{v_y} - x\partial_x - y\partial_y\right), \quad X_3 := v_x\partial_x + v_y\partial_y + \frac{b}{x^3}\partial_{v_x},$$

satisfy the commutation relations

$$[X_1, X_2] = X_1, \qquad [X_1, X_3] = 2X_2, \qquad [X_2, X_3] = X_3. \tag{2.9}$$

This is a Lie–Hamilton system. Indeed, the vector fields X_1, X_2, X_3 are Hamiltonian with respect to the Poisson bivector $\Lambda = \partial_x \wedge \partial_{v_x} + \partial_y \wedge \partial_{v_y}$. Indeed, their Hamiltonian functions:

$$h_1 = \frac{1}{2}(x^2 + y^2), \qquad h_2 = -\frac{1}{2}(xv_x + yv_y), \qquad h_3 = \frac{1}{2}\left(v_x^2 + v_y^2 + \frac{b}{x^2}\right),$$

form a basis for $(\mathcal{H}_\Lambda, \{\cdot,\cdot\}_\Lambda) \simeq (\mathfrak{sl}(2), [\cdot,\cdot])$ as they fullfil

$$\{h_1, h_2\} = -h_1, \qquad \{h_1, h_3\} = -2h_2, \qquad \{h_2, h_3\} = -h_3. \tag{2.10}$$

Since $X = X_3 + \omega^2(t)X_1$ and $\omega(t)$ is not a constant, every t-independent constant of motion f for X is a common first-integral for X_1, X_2, X_3. Instead of searching an f by solving the system of PDEs given by $X_1f = X_2f = X_3f = 0$, we use Corollary 2.24. This easily provides such a first integral through the Casimir element of the symmetric algebra of $\mathfrak{sl}(2)$. Explicitly, given a basis $\{v_1, v_2, v_3\}$ for $\mathfrak{sl}(2)$ satisfying

$$[v_1, v_2] = -v_1, \qquad [v_1, v_3] = -2v_2, \qquad [v_2, v_3] = -v_3, \tag{2.11}$$

the Casimir element of $\mathfrak{sl}(2)$ reads $\mathcal{C} = \frac{1}{2}(v_1\widetilde{\otimes}v_3 + v_3\widetilde{\otimes}v_1) - v_2\widetilde{\otimes}v_2 \in U_{\mathfrak{sl}(2)}$. Then, the inverse of symmetrizer morphism [**Va84**], $\lambda^{-1} : U_{\mathfrak{sl}(2)} \to S_{\mathfrak{sl}(2)}$, gives rise to the Casimir element of $S_{\mathfrak{sl}(2)}$:

$$C = \lambda^{-1}(\mathcal{C}) = v_1v_3 - v_2^2. \tag{2.12}$$

The isomorphism $\phi : \mathfrak{sl}(2) \to \mathcal{H}_\Lambda$ defined by $\phi(v_\alpha) = h_\alpha$ for $\alpha = 1, 2, 3$ induces the Poisson algebra morphism D in Proposition 2.23. Subsequently, via Corollary 2.24, we obtain

$$F = D(C) = \phi(v_1)\phi(v_3) - \phi^2(v_2) = h_1h_3 - h_2^2 = (v_yx - v_xy)^2 + b\left(1 + \frac{y^2}{x^2}\right).$$

In this way, we recover, up to an additive and non-zero multiplicative constant, the well-known Lewis–Riesenfeld invariant [**LA08**]. If $\omega(t)$ is a constant, then $V^X \subset V$ and F is also a constant of motion for X.

2.5. Superposition rules for Lie–Hamilton systems. One of the reasons to use Lie–Hamilton systems is that their associated Lie–Hamiltonian structures can be used to derive their superposition rules providing several advantages with respect to previous methods:

- It avoids the integration for PDEs and ODEs to derive them: this differs from the methods followed by Winternitz, Cariñena, and their collaborators [**CGM07, CGM00**],
- There is no need to map the system into a canonical form as needed, for instance, to use previous Winternitz's results on superposition rules [**PW, SW84, SW84II**].
- It naturally provides a geometric understanding of the superposition rule, while other methods focus only on the expression for the superposition rule.

My method can be applied to Lie–Hamilton systems and its generalizations, like Dirac-Lie systems [**CGL.H6**]. As most Lie systems of interest fall into this class, my method is worth of analysis. This approach requires the use of Poisson coalgebras. Let us briefly introduce this notion.

If $(A, \star_A, \{\cdot,\cdot\}_A)$ and $(B, \star_B, \{\cdot,\cdot\}_B)$ are Poisson algebras and $\star_A$, $\star_B$ are commutative, then $A\otimes B$ becomes a Poisson algebra $(A\otimes B, \star_{A\otimes B}, \{\cdot,\cdot\}_{A\otimes B})$ by defining

$$(a\otimes b)\star_{A\otimes B}(c\otimes d) := (a\star_A c)\otimes(b\star_B d),$$
$$\{a\otimes b, c\otimes d\}_{A\otimes B} := \{a,c\}_A\otimes b\star_B d + a\star_A c\otimes\{b,d\}_B$$

for all $a, c\in A,\quad \forall b,d\in B$. Similarly, a Poisson structure on $A^{(m)} := \overbrace{A\otimes\ldots\otimes A}^{m-\text{times}}$ can be constructed by induction.

A *Poisson coalgebra* is a triple $(A, \star_A, \{\cdot,\cdot\}_A, \Delta)$ such that $(A, \star_A, \{\cdot,\cdot\}_A)$ is a Poisson algebra and $\Delta : (A, \star_A, \{\cdot,\cdot\}_A)\to(A\otimes A, \star_{A\otimes A}, \{\cdot,\cdot\}_{A\otimes A})$, the so-called *coproduct*, is a Poisson algebra homomorphism which is *coassociative* [**CP95**], i.e. $(\Delta\otimes\mathrm{Id})\circ\Delta = (\mathrm{Id}\otimes\Delta)\circ\Delta$.

Every Lie–Hamilton system X on N can be endowed with a Lie–Hamiltonian structure and an associated Lie–Hamilton algebra $\mathcal{H}_\Lambda\simeq\mathfrak{g}$. Hence, there exists a natural injective Lie algebra morphism $\phi:\mathfrak{g}\to C^\infty(N)$ mapping every element of $\mathfrak{g}$ with its corresponding element of $\mathcal{H}_\Lambda$. Additionally, $S_\mathfrak{g}$ becomes a Poisson coalgebra relative to the unique coproduct $\Delta: S_\mathfrak{g}\to S_\mathfrak{g}\otimes S_\mathfrak{g}$ satisfying $\Delta(v) = v\otimes 1 + 1\otimes v$ (cf. [**BCHL.H7**]). Moreover,

Lemma 2.26. *(*[**BCHL.H7**, Lemma 23]*) The map* $\Delta^{(m)} : (S_\mathfrak{g}, \cdot, \{\cdot,\cdot\}_{S_\mathfrak{g}})\to(S_\mathfrak{g}^{(m)}, \cdot_{S_\mathfrak{g}^{(m)}}, \{\cdot,\cdot\}_{S_\mathfrak{g}^{(m)}})$ *, with* $m>1$*, defined by recursion*

$$\Delta^{(m)} := (\overbrace{\mathrm{Id}\otimes\ldots\otimes\mathrm{Id}}^{(m-2)-\text{times}}\otimes\Delta^{(2)})\circ\Delta^{(m-1)},\qquad m>2, \tag{2.13}$$

where $\Delta^{(2)} := \Delta$ *is the natural coproduct on* $S_\mathfrak{g}$*, is a Poisson algebra morphism.*

Lemma 2.27. *(*[**BCHL.H7**, Lemma 24]*) The Lie algebra morphism* $\phi:\mathfrak{g}\hookrightarrow C^\infty(N)$ *gives rise to a family of Poisson algebra morphisms* $D^{(m)} : S_\mathfrak{g}^{(m)}\hookrightarrow C^\infty(N)^{(m)}\subset C^\infty(N^m)$ *satisfying, for all* $v_1,\ldots,v_m\in\mathfrak{g}\subset S_\mathfrak{g}$*, that*

$$\Big[D^{(m)}(v_1\otimes\ldots\otimes v_m)\Big](x_{(1)},\ldots,x_{(m)}) = [D(v_1)](x_{(1)})\cdot\ldots\cdot[D(v_m)](x_{(m)}), \tag{2.14}$$

where $x_{(i)}$ *is a point of the manifold* N *placed in the* i*-position within the product* $N\times\ldots\times N := N^m$ *and* D *is the Lie algebra morphism induced by* ϕ *given in Proposition 2.23.*

The above results allow us to prove Theorem 2.29, providing a method to obtain t-independent constants of motion for the diagonal prolongations of a Lie–Hamilton system. From this result one may obtain superposition rules for Lie–Hamilton systems in an algebraic way. Additionally, we remark that such a theorem is a generalization, only valid in the case of primitive coproduct maps, of the integrability theorem for coalgebra symmetric systems given in [**BR**].

Proposition 2.28. *(*[**BCHL.H7**, Proposition 25]*) If X is a Lie–Hamilton system on N with a Lie–Hamiltonian structure (N, Λ, h), then the diagonal prolongation $\widetilde{X}^{[m+1]}$ to each N^{m+1} is also a Lie–Hamilton system endowed with a Lie–Hamiltonian structure $(N^{m+1}, \Lambda^{m+1}, \widetilde{h})$ given by*

$$\Lambda^{m+1}(x_{(0)}, \ldots, x_{(m)}) := \sum_{a=0}^{m} \Lambda(x_{(a)}),$$

where we made use of the vector bundle isomorphism $\mathrm{T}N^{m+1} \simeq \mathrm{T}N \oplus \cdots \oplus \mathrm{T}N$ (m+1 copies*), and $\widetilde{h}_t := D^{(m+1)}(\Delta^{(m+1)}(h_t))$, where $D^{(m+1)}$ is the Poisson algebra morphism (2.14) induced by the Lie algebra morphism $\mathfrak{g} \hookrightarrow \mathcal{H}_\Lambda \subset C^\infty(N)$.*

Theorem 2.29. *(*[**BCHL.H7**, Theorem 26]*) If X is a Lie–Hamilton system with a Lie–Hamiltonian structure (N, Λ, h) and C is a Casimir element of the Poisson algebra $(S_\mathfrak{g}, \cdot, \{,\}_{S_\mathfrak{g}})$, then:*
(i) The functions defined as

$$F^{(k)} = D^{(k)}(\Delta^{(k)}(C)), \qquad k = 2, \ldots, m, \tag{2.15}$$

are t-independent constants of motion for the diagonal prolongation $\widetilde{X}$ of X to N^m. Furthermore, if all the $F^{(k)}$ are non-constant functions, they form a set of $(m-1)$ functionally independent functions in involution.
(ii) The functions given by

$$F_{ij}^{(k)} = S_{ij}(F^{(k)}), \qquad 1 \le i < j \le k, \qquad k = 2, \ldots, m, \tag{2.16}$$

where S_{ij} is the permutation of variables $x_{(i)} \leftrightarrow x_{(j)}$, are t-independent constants of motion for the diagonal prolongation $\widetilde{X}$ to N^m.

3. Dirac–Lie systems

The no-go theorem for Lie–Hamilton systems, namely Theorem 2.6, allows us to prove that Schwarz equations cannot be considered as Lie–Hamilton systems [**CGL.H6**]. Consider a Schwarzian equation [**Be07, OT09**]

$$\{x, t\} = \frac{\mathrm{d}^3 x}{\mathrm{d}t^3}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^{-1} - \frac{3}{2}\left(\frac{\mathrm{d}^2 x}{\mathrm{d}t^2}\right)\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^{-2} = 2b_1(t), \tag{3.1}$$

where $\{x, t\}$ is the refereed to as *Schwarzian derivative* of the function $x(t)$ in terms of the variable t and $b_1(t)$ is an arbitrary t-dependent function [**NM13**]. This equation is a particular case of a third-order Kummer–Schwarz equation [**CGL.PH6**] and it appears in the study of iterative differential Riccati and second-order Kummer–Schwarz equations [**NM13**]. For simplicity, we hereafter assume $b_1(t)$ to be non-constant.

The first-order system of differential equations obtained by adding the variables $v := \mathrm{d}x/\mathrm{d}t$ and $a := \mathrm{d}^2x/\mathrm{d}t^2$ to (3.1), i.e.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = v, \qquad \frac{\mathrm{d}v}{\mathrm{d}t} = a, \qquad \frac{\mathrm{d}a}{\mathrm{d}t} = \frac{3}{2}\frac{a^2}{v} + 2b_1(t)v, \tag{3.2}$$

is a Lie system. Indeed, it is the associated system to the t-dependent vector field

$$X_t^{3KS} = v\partial_x + a\partial_v + \left(\frac{3}{2}\frac{a^2}{v} + 2b_1(t)v\right)\partial_a = Y_3 + b_1(t)Y_1,$$

where the vector fields on $\mathcal{O}_2 := \{(x, v, a) \in \mathrm{T}^2\mathbb{R} \,|\, v \neq 0\}$, with $\mathrm{T}^2\mathbb{R}$ being the second tangent bundle to $\mathbb{R}$ [**LM87**], given by

$$Y_1 := 2v\partial_a, \qquad Y_2 := v\partial_v + 2a\partial_a, \qquad Y_3 := v\partial_x + a\partial_v + \frac{3}{2}\frac{a^2}{v}\partial_a, \tag{3.3}$$

span a Lie algebra of vector fields V^{3KS} isomorphic to $\mathfrak{sl}(2)$ and X^{3KS} becomes a t-dependent vector field taking values in V^{3KS}, i.e. X^{3KS} is a Lie system. Since $\mathcal{D}^{3KS} = T\mathcal{O}_2$ and $\mathcal{O}_2$ is a three-dimensional manifold, the no-go theorem for Lie-Hamilton systems, Theorem 2.6, states that this is not a Lie–Hamilton system and another approach is required. In spite of this, V^{3KS} consists of Hamiltonian vector fields with respect to a presymplectic form

$$\omega_1 := \frac{\mathrm{d}v \wedge \mathrm{d}a}{v^3}.$$

Indeed,

$$\iota_{Y_1}\omega_1 = \mathrm{d}\left(\frac{2}{v}\right), \qquad \iota_{Y_2}\omega_1 = \mathrm{d}\left(\frac{a}{v^2}\right), \qquad \iota_{Y_3}\omega_1 = \mathrm{d}\left(\frac{a^2}{2v^3}\right),$$

The above seems to justify the definition and study of Lie systems with a Vessiot–Guldberg Lie algebra of Hamiltonian vector fields relative to a presymplectic structure. Nevertheless, it is more appropriate to consider presymplectic structures as particular cases of Dirac structures [**IV**] and to study Lie systems with a Vessiot–Guldberg Lie algebra of Hamiltonian vector fields relative to a Dirac structure. Let us introduce these notions.

The *Pontryagin* or *generalized tangent bundle* over a manifold N is the vector bundle $\mathcal{P}N := T^*N \oplus_N TN$ over N, where $\oplus_N$ stands for the Whitney sum of vector bundles. The Pontryagin bundle can be endowed with the natural pairing between forms and vectors. A *Dirac manifold* is a pair (N, L), where N is a manifold and L is a subbundle of $TN \oplus_N T^*N$ such that: a) L is maximally isotropic relative to the pairing between forms and vectors, b) its space of sections, $\Gamma(L)$, is integrable relative to the so-called *Courant bracket* $[\cdot,\cdot]_C$ on $\Gamma(\mathcal{P}N)$ (see [**Co90**] for details on Dirac structures).

Previous example on third-order Kummer-Schwarz equations along with many others developed in [**LV.H1, LTV.H4, HL.H5, CGL.H6**] motivates the study of Lie systems admitting a Vessiot–Guldberg Lie algebra of vector fields of Hamiltonian vector fields relative to a Dirac structure. Recall that a Hamiltonian vector field relative to a Dirac manifold (N, L), an *L-Hamiltonian vector field*, is a vector field X on N such that $X + \mathrm{d}h \in \Gamma(L)$ for a certain function $h \in C^\infty(N)$ called L-Hamiltonian. We write $\mathrm{Adm}(N, L)$ for the space of L-Hamiltonian functions. If $X \in \Gamma(L)$, then it is sais that X is a *gauge vector field*. The Dirac structure allows us to define a Poisson bracket $\{\cdot,\cdot\}_L$ on the space of L-hamiltonian functions. Additionally, every Dirac manifold induces a Lie algebroid structure $(\mathcal{P}N, [\cdot,\cdot]_C, \rho : \mathcal{P}N \to TN)$, where ρ is the natural projection of $\mathcal{P}N$ onto TN (see [**Ma08**] for details on Lie algebroids).

The above example shows the existence of Lie systems that are not of Lie–Hamilton type and admit a Vessiot–Guldberg Lie algebra of Hamiltonian vector fields relative to a Dirac structure. The previous and other similar examples can be found in [**CGL.H6**]. In view of the above, the natural generalization of Lie–Hamilton systems to the realm of Dirac structures reads as follows.

Definition 3.1. ([**CGL.H6**, Definition 5.2]) A *Dirac–Lie system* is a triple (N, L, X), where (N, L) stands for a Dirac manifold and X is a Lie system admitting a Vessiot–Guldberg Lie algebra of L-Hamiltonian vector fields.

Definition 3.2. ([**CGL.H6**, Definition 6.1]) A *Dirac–Lie Hamiltonian* is a triple (N, L, h), where (N, L) stands for a Dirac manifold and h represents a t-parametrized family of admissible functions $h_t : N \to \mathbb{R}$ such that Lie$(\{h_t\}_{t\in\mathbb{R}}, \{\cdot,\cdot\}_L)$ is a finite-dimensional real Lie algebra. A t-dependent vector field X is said to admit (or to possess) a Dirac–Lie Hamiltonian (N, L, h) if $X_t + dh_t \in \Gamma(L)$ for all $t \in \mathbb{R}$.

Theorem 3.3. *(*[**CGL.H6**, Theorem 6.4]*) Every Dirac–Lie system* (N, L, X) *admits a Dirac–Lie Hamiltonian* (N, L, h).

3.1. Prolongations of Dirac–Lie systems and superposition rules. Let us study the properties of diagonal prolongations of Dirac–Lie systems. This allows us to apply these structures to obtain superposition rules and to introduce some new concepts of interest generalizing the diagonal prolongation of t-dependent vector fields.

Let $\tau : E \to N$ be a vector bundle. Its *diagonal prolongation* to N^m is the Cartesian product bundle $E^{[m]} := E \times \cdots \times E$ of m copies of E, viewed as a vector bundle over N^m in a natural way:

$$E^{[m]}_{(x_{(1)},\dots,x_{(m)})} \simeq E_{x_{(1)}} \oplus \cdots \oplus E_{x_{(m)}} .$$

Every section $X : N \to E$ of E has a natural *diagonal prolongation* to a section $X^{[m]}$ of $E^{[m]}$:

$$X^{[m]}(x_{(1)}, \dots, x_{(m)}) := X(x_{(1)}) + \cdots + X(x_{(m)}) .$$

Given a function $f : N \to \mathbb{R}$, we call *diagonal prolongation* of f to N^m the function $\widetilde{f}^{[m]}(x_{(1)}, \dots, x_{(m)}) = f(x_{(1)}) + \ldots + f(x_{(m)})$.

We can consider also sections $X^{(j)}$ of $E^{[m]}$ given by

$$X^{(j)}(x_{(1)}, \dots, x_{(m)}) := 0 + \cdots + X(x_{(j)}) + \cdots + 0 . \tag{3.4}$$

It is clear that, if (X_i) is a basis of local sections of E, then $(X_i^{(j)})$ is a basis of local sections of $E^{[m]}$.

Since there are obvious canonical isomorphisms

$$(TN)^{[m]} \simeq TN^m \quad \text{and} \quad (T^*N)^{[m]} \simeq T^*N^m ,$$

we can interpret the diagonal prolongation $X^{[m]}$ of a vector field on N as a vector field $\widetilde{X}^{[m]}$ on N^m, and the diagonal prolongation $\alpha^{[m]}$ of a 1-form on N as a 1-form $\widetilde{\alpha}^{[m]}$ on N^m. In the case when m is fixed, we will simply write $\widetilde{X}$ and $\widetilde{\alpha}$.

Proposition 3.4. *(*[**CGL.H6**, Proposition 7.1]*) The diagonal prolongation to N^m of a vector field X on N is the unique vector field $\widetilde{X}^{[m]}$ on N^m projectable with respect to the map $\pi : (x_{(1)}, \dots, x_{(m)}) \in N^m \mapsto x_{(1)} \in N$ onto X and invariant relative to the permutation of variables $x_{(i)} \leftrightarrow x_{(j)}$, with $i, j = 1, \dots, m$. The diagonal prolongation to N^m of a 1-form α on N is the unique 1-form $\widetilde{\alpha}^{[m]}$ on N^m such that $\widetilde{\alpha}^{[m]}(\widetilde{X}^{[m]}) = \widetilde{\alpha(X)}^{[m]}$ for every vector field $X \in \Gamma(TN)$. We have $d\widetilde{\alpha} = \widetilde{d\alpha}$ and $\mathcal{L}_{\widetilde{X}^{[m]}}\widetilde{\alpha}^{[m]} = \widetilde{\mathcal{L}_X\alpha}$. In particular, if α is closed (exact), so is its diagonal prolongation $\widetilde{\alpha}^{[m]}$ to N^m.*

Let (x^a) be a local coordinate system on N and let $(x^a_{(i)})$ be the induced coordinate system on N^m. If $X = \sum_a X^a(x)\partial_{x^a}$ and $\alpha = \sum_a \alpha_a(x)dx^a$, then

$$\widetilde{X}^{[m]} = \sum_{a,i} X^a(x_{(i)})\partial_{x^a_{(i)}} \quad \text{and} \quad \widetilde{\alpha}^{[m]} = \sum_{a,i} \alpha_a(x_{(i)})dx^a_{(i)} . \tag{3.5}$$

Let us fix m. If X_1, X_2 are two vector fields on N, then $\widetilde{[X_1, X_2]}^{[m]} = [\widetilde{X_1}^{[m]}, \widetilde{X_2}^{[m]}]$. In consequence, the prolongations to N^m of the elements of a finite-dimensional real Lie algebra V of vector fields on N form a real Lie algebra $\widetilde{V}^{[m]}$ isomorphic to V. Similarly to standard vector fields, we can define the diagonal prolongation of a t-dependent vector field X on N to N^m as the only t-dependent vector field $\widetilde{X}^{[m]}$ on N^m satisfying that $\widetilde{X_t}^{[m]}$ is the diagonal prolongation of X_t to N^m for each $t \in \mathbb{R}$.

When X is a Lie–Hamilton system, its diagonal prolongations are also Lie–Hamilton systems in a natural way [**BCHL.H7**]. Let us now focus on proving an analogue of this result for Dirac–Lie systems.

Definition 3.5. ([**CGL.H6**, Definition 7.2]) Given two Dirac manifolds (N, L_N) and (M, L_M), we say that $\varphi : N \to M$ is a *forward Dirac map* between them if $(L_M)_{\varphi(x)} = \mathfrak{P}_\varphi(L_N)_x$, where

$$\mathfrak{P}_\varphi(L_N)_x := \{\varphi_{*x}X_x + \omega_{\varphi(x)} \in T_{\varphi(x)}M \oplus T^*_{\varphi(x)}M \,|\, X_x + (\varphi^*\omega_{\varphi(x)})_x \in (L_N)_x\},$$

for all $x \in N$.

Proposition 3.6. *(*[**CGL.H6**, Proposition 7.3]*) Given a Dirac structure (N, L) and the natural isomorphism*

$$(TN^m \oplus_{N^m} T^*N^m)_{(x_{(1)},\dots,x_{(m)})} \simeq (T_{x_{(1)}}N \oplus T^*_{x_{(1)}}N) \oplus \cdots \oplus (T_{x_{(m)}}N \oplus T^*_{x_{(m)}}N),$$

the diagonal prolongation $L^{[m]}$, viewed as a vector subbundle in $TN^m \oplus_{N^m} T^*N^m = \mathcal{P}N^{[m]}$, is a Dirac structure on N^m.

The forward image of $L^{[m]}$ through each $\pi_i : (x_{(1)}, \dots, x_{(m)}) \mapsto N^m \to x_{(i)} \in N$, with $i = 1, \dots, m$, equals L. Additionally, $L^{[m]}$ is invariant under the permutations $x_{(i)} \leftrightarrow x_{(j)}$, with $i, j = 1, \dots, m$.

Corollary 3.7. *(*[**CGL.H6**, Corollary 7.4]*) Given a Dirac manifold (N, L), we have $\rho_m(L^{[m]}) = \rho(L)^{[m]}$, where ρ_m is the projection $\rho_m : \mathcal{P}N^m \to TN^m$ of the Dirac manifold $(N^m, L^{[m]})$. Then, if X is an L-Hamiltonian vector field with respect to L, then its diagonal prolongation $\widetilde{X}^{[m]}$ to N^m is an L-Hamiltonian vector field with respect to $L^{[m]}$. Moreover, $\rho_m^*(L^{[m]}) = \rho^*(L)^{[m]}$, where ρ_m^* is the canonical projection $\rho_m^* : \mathcal{P}N^m \to T^*N^m$.*

Corollary 3.8. *(*[**CGL.H6**, Corollary 7.4]*) If (N, L, X) is a Dirac–Lie system, then $(N^m, L^{[m]}, \widetilde{X}^{[m]})$ is also a Dirac–Lie system.*

Proposition 3.9. *(*[**CGL.H6**, Proposition 7.6]*) Let X be a vector field and f be a function on N. Then:*

(a) *If f is an L-Hamiltonian function for X, its diagonal prolongation $\widetilde{f}$ to N^m is an $L^{[m]}$-Hamiltonian function of the diagonal prolongation $\widetilde{X}^{[m]}$ on N^m.*
(b) *If $f \in \mathrm{Cas}(N, L)$, then $\widetilde{f}^{[m]} \in \mathrm{Cas}(N^m, L^{[m]})$.*
(c) *The map $\lambda : (\mathrm{Adm}(N, L), \{\cdot, \cdot\}_L) \ni f \mapsto \widetilde{f}^{[m]} \in (\mathrm{Adm}(N^m, L^{[m]}), \{\cdot, \cdot\}_{L^{[m]}})$ is an injective Lie algebra morphism.*

Note, however, that in the above we cannot ensure that λ is a Poisson algebra morphism, as in general $\widetilde{fg}^{[m]} \neq \widetilde{f}^{[m]}\widetilde{g}^{[m]}$.

Using the above proposition, we can easily prove the following corollaries.

Corollary 3.10. *(*[**CGL.H6**, Corollary 7.7]*) If $h_1, \dots, h_r : N \to \mathbb{R}$ is a family of functions on a Dirac manifold (N, L) spanning a finite-dimensional real Lie algebra of functions with respect to the Lie bracket $\{\cdot, \cdot\}_L$, then their diagonal prolongations $\widetilde{h}_1^{[m]}, \dots, \widetilde{h}_r^{[m]}$ to N^m close an isomorphic Lie algebra of functions with respect to the Lie bracket $\{\cdot, \cdot\}_{L^{[m]}}$ induced by the Dirac structure $(N^m, L^{[m]})$.*

Corollary 3.11. *(*[**CGL.H6**, Corollary 7.8]*) If (N, L, X) is a Dirac–Lie system admitting a Dirac–Lie Hamiltonian (N, L, h), then $(N^m, L^{[m]}, \widetilde{X}^{[m]})$ is a Dirac–Lie system with a Dirac–Lie Hamiltonian $(N^m, L^{[m]}, h^{[m]})$, where $h_t^{[m]} := \widetilde{h}_t^{[m]}$ is the diagonal prolongation of h_t to N^m.*

Previous results can be employed to study superposition rules for Dirac–Lie systems as illustrated by the examples in [**CGL.H6**]. The main idea I had is to extend the Poisson coalgebra method for Poisson manifolds to Dirac manifolds, which increases the field of application of Poisson coalgebras.

4. k-Symplectic Lie systems

The study of k-symplectic structures was pioneered by M. de León [**LMS88, LMS93, LSV16**], Awane [**Aw92**], and Gunther [**Gu87**] among others. They appeared as a generalization of symplectic geometry to study field theories [**Gu87**]. Formally, a k-symplectic structure can be defined as follows.

Definition 4.1. Let N be an $n(k+1)$-dimensional manifold and $\omega_1, \dots, \omega_k$ a set of k closed two-forms on N. We say that $(\omega_1, \dots, \omega_k)$ is a *k–symplectic structure* if $\bigcap_{i=1}^k \ker \omega_i(x) = \{0\}$, for all $x \in N$. We call $(N, \omega_1, \dots, \omega_k)$ a *k–symplectic manifold*.

A *k-symplectic structure* $(\omega_1, \dots, \omega_k)$ on an $n(k+1)$-dimensional manifold N amounts to a closed nondegenerate form $\Omega := \sum_{n=1}^k \omega_k \otimes e^k$ on N taking values in $\mathbb{R}^k$. The Ω is called a *polysymplectic form* and it allows us to simplify the formalism on k-symplectic structures. The k-symplectic structures present several problems: they are not naturally associated with a Poisson algebra of functions and cannot be applied to study all partial differential equations [**LSV16, LV.H1**].

Instead of following the standard approach, I proved that k-symplectic structures are useful to analyze systems of first-order ordinary differential equations and natural tools from k-symplectic geometry can be used to study them. In particular, a generalization of the Poisson algebras of functions from symplectic geometry appears, and it gives rise to methods to obtain superposition rules. I also found numerous Lie systems possessing Vessiot–Guldberg Lie algebras of Hamiltonian vector fields relative to a k-symplectic structure, and I applied my methods to them. Let us detail my main finding on Lie systems and k-symplectic structures.

Consider again the Schwarzian equation (3.1). Let us prove that V^{3KS} consists of Hamiltonian vector fields with respect to the presymplectic forms of a two-symplectic manifold $(\mathcal{O}_2, \omega_1, \omega_2)$. To do so, we look for presymplectic forms ω satisfying that Y_1, Y_2 and Y_3 are Hamiltonian vector fields relative to it, i.e. $\mathcal{L}_{Y_\alpha}\omega = 0$ for $\alpha = 1, 2, 3$ and $\mathrm{d}\omega = 0$. By solving the latter system of partial differential equations for ω, we find the presymplectic forms

$$\omega_1 := \frac{\mathrm{d}v \wedge \mathrm{d}a}{v^3}, \qquad \omega_2 := -\frac{2}{v^3}(x\,\mathrm{d}v \wedge \mathrm{d}a + v\,\mathrm{d}a \wedge \mathrm{d}x + a\,\mathrm{d}x \wedge \mathrm{d}v). \tag{4.1}$$

Since $\ker\omega_1 = \langle\partial_x\rangle, \ker\omega_2 = \langle x\partial_x + v\partial_v + a\partial_a\rangle$ and $v \neq 0$ on $\mathcal{O}_2$, it turns out that ω_1 and ω_2 have constant rank equal to two and $\ker\omega_1 \cap \ker\omega_2 = \{0\}$ on $\mathcal{O}_2$. Therefore, (ω_1, ω_2) forms a two-symplectic structure.

More interestingly, Y_1, Y_2 and Y_3 are Hamiltonian vector fields relative to ω_1, ω_2:

$$\begin{gathered}\iota_{Y_1}\omega_1 = \mathrm{d}\left(\frac{2}{v}\right), \qquad \iota_{Y_2}\omega_1 = \mathrm{d}\left(\frac{a}{v^2}\right), \qquad \iota_{Y_3}\omega_1 = \mathrm{d}\left(\frac{a^2}{2v^3}\right),\\ \iota_{Y_1}\omega_2 = -\mathrm{d}\left(\frac{4x}{v}\right), \quad \iota_{Y_2}\omega_2 = \mathrm{d}\left(2 - \frac{2ax}{v^2}\right), \quad \iota_{Y_3}\omega_2 = \mathrm{d}\left(\frac{2a}{v} - \frac{a^2x}{v^3}\right).\end{gathered} \tag{4.2}$$

Although system (3.2) cannot be studied through a Lie–Hamilton system, the use of the above presymplectic structures will allow us to study such systems through similar techniques to those developed for Lie–Hamilton and Dirac–Lie systems [**CGL.H6, CGL.H8**]. Indeed, the above system possesses a Lie algebra of Hamiltonian vector fields relative to all the presymplectic forms of a k–symplectic structure. Although we already know that (3.1) can be studied through a Dirac–Lie system, this approach only takes into account the existence of a presymplectic compatible form. Nevertheless, the k-symplectic structure is much richer and allows for more powerful tools.

Table 5 resumes many Lie systems admitting a Vessiot–Guldberg Lie algebra of k-Hamiltonian vector fields [**LV.H1**]. This table could be enlarged with other systems such as quaternionic Riccati equations. These findings suggests us the following definitions.

Definition 4.2. Given a k–symplectic structure $(\omega_1, \ldots, \omega_k)$ on an $n(k+1)$ dimensional manifold N, we say that a vector field Y on N is *k–Hamiltonian* if it is a Hamiltonian vector field with respect to the presymplectic forms $\omega_1, \ldots, \omega_k$.

It also makes sense to say that X is Ω–Hamiltonian for a polysymplectic form Ω if X is k–Hamiltonian for a k–symplectic manifold possessing Ω as an associated polysymplectic form. From now on, we will talk about *k–Hamiltonian* and/or *Ω–Hamiltonian vector fields* indistinctly. We write $\mathrm{Ham}(\Omega)$, where Ω is a polysymplectic form induced by $(\omega_1, \ldots, \omega_k)$, for the space of k–Hamiltonian vector fields.

In view of previous comments, it is justified to define k–symplectic Lie systems as follows.

Definition 4.3. ([**LV.H1**, Definition 3.2]) We say that a system X is a *k–symplectic Lie system* if V^X is a finite-dimensional real Lie algebra of k–Hamiltonian vector fields with respect to a k–symplectic structure $(\omega_1, \ldots, \omega_k)$. We call $(\omega_1, \ldots, \omega_k)$ a *compatible* k–symplectic structure.

The above can be restated by saying that a system X on a manifold N is a k–symplectic Lie system if and only if it admits a Vessiot–Guldberg Lie algebra of k–Hamiltonian vector fields with respect to a certain k–symplectic structure on N. Lie–Hamilton systems are a particular type of k–symplectic Lie systems.

Nevertheless, not every k–symplectic Lie system is a Lie–Hamilton system as shown above by means of Schwarz equations and the no-go theorem for Lie–Hamilton systems given in [**CGL.H6**, Theorem 4.4].

Every k–symplectic Lie system can be considered as a Dirac–Lie system [**LV.H1**]. More specifically, if X is a k–symplectic Lie system relative to the k–symplectic structure $(\omega_1, \ldots, \omega_k)$, then V^X is a family of Hamiltonian vector fields with respect to each one of the presymplectic forms $\omega_1, \ldots, \omega_k$. So, V^X is a Lie algebra of Hamiltonian vector fields relative to each Dirac structure L^{ω_r} induced by the presymplectic form ω_r (see [**CGL.H6, Co90**] for details). Following our previous notation, we say that (N, L^{ω_r}, X) is a Dirac–Lie system. Meanwhile, not every Dirac–Lie system can be considered as a k–symplectic Lie system, e.g. a Lie system given by an autonomous vector field $X \neq 0$ on the real line gives rise to a Dirac–Lie system $(\mathbb{R}, T\mathbb{R}, X)$, but it is not a k-symplectic one. Nevertheless, the main advantage of k–symplectic Lie systems is that they can be considered as Dirac–Lie systems in different ways. This suggests us to find a natural approach to the study of these systems, which is given by k–symplectic structures.

Determining whether a Lie system is a k–symplectic Lie system generally requires solving a system of PDEs to find a compatible k–symplectic structure. It is in general difficult to establish whether this system of PDEs has enough solutions to construct a compatible k–symplectic structure. It is difficult to establish whether such a system of PDEs admits enough solutions to obtain a compatible k-symplectic structure. This motivates to find simple necessary and/or sufficient conditions to ensure or to discard that a Lie system is a k–symplectic Lie system. I now describe the no-go theorem for k-symplectic structures, which gives conditions ensuring that a Lie system is not a k–symplectic Lie system (see [**LV.H1**, Theorem 4.4] for details). The main idea is that the minimal Lie algebra of the Lie system under study must leave stable, in the sense given next, the kernels of the presymplectic forms of any k–symplectic structure compatible with the Lie system. This condition is easier to verify than showing straightforwardly that the system of PDEs describing the presymplectic forms compatible with a Lie system does not contain enough solutions to construct a compatible k–symplectic structure.

Definition 4.4. ([**LV.H1**, Definition 4.1]) A distribution $\mathcal{D}$ on N is *stable* relative to the action of a Lie algebra V of vector fields on N when $[X, Y] \in \mathcal{D}$ for every $Y \in \mathcal{D}$ and $X \in V$.

Definition 4.5. ([**LV.H1**, Definition 4.2]) Given a finite-dimensional real Lie algebra V of vector fields on N, we say that V is *s–primitive* when there exists no distribution $\mathcal{D}$ of rank s stable with respect to the action of V. We call V *odd–primitive* when V is s–primitive for every odd value of $s < \dim N$.

Remark 4.6. The above definition is a generalisation of the notion of a primitive Lie algebra of vector fields on the plane given in [**GKO92**].

Theorem 4.7. (No-go k–symplectic Lie systems theorem [**LV.H1**, Theorem 4.4]**)** *If X is a Lie system on an odd dimensional manifold N and V^X is odd-primitive, then X is not a k–symplectic Lie system.*

The above theorem was applied in [**LV.H1**, Example 1] to show that certain Lie systems are not k-symplectic Lie systems. For instance, consider the Lie system

$$\frac{\mathrm{d}g}{\mathrm{d}t} = X^G(t, g) := \sum_{\alpha=1}^{r} b_\alpha^R(t) X_\alpha^R(g) + \sum_{\alpha=1}^{r} b_\alpha^L(t) X_\alpha^L(g), \qquad g \in G, \tag{4.3}$$

where G is a Lie group, $X_1^R, \ldots, X_r^R$ and $X_1^L, \ldots, X_r^L$ form basis of right- and left-invariant vector fields on G respectively, and $b_1^L(t), \ldots, b_r^L(t), b_1^R(t), \ldots, b_r^R(t)$ are arbitrary t-dependent functions. Additionally, we assume G to be connected. Systems of the type (4.3) appear when searching for transformations mapping a Lie system into a new one, e.g. in a reduction process [**CL.PH14**]. Additionally, each Lie system on a manifold can be solved by means of a particular solution of systems like (4.3) where only right-invariant or left-invariant vector fields appear. Moreover, such systems appear in control theory and Darboux integrable systems [**CCR03, IV**]. It was proved in [**LV.H1**] that the Lie system (4.3) possesses an odd-primitive Vessiot–Guldberg Lie algebra of vector fields when $\dim G$ is odd and the Lie algebra of G is simple. In virtue of Theorem 4.7, it is not a k-symplectic Lie system.

4.1. On Ω–Hamiltonian functions. Every k–Hamiltonian vector field can be associated with a family $h_1,\ldots,h_k$ of Hamiltonian functions (each one relative to a different presymplectic form of a k–symplectic structure). It is therefore convenient to introduce some generalisation of the Hamiltonian function notion for presymplectic forms to deal simultaneously with all $h_1,\ldots,h_k$. Let us resume the main properties of such a generalisation I described in [**LV.H1**]. My findings extend to our k–symplectic structures several theorems devised by Awane in [**Aw92**] for a more particular type of k–symplectic structures.

Definition 4.8. ([**LV.H1**, Definition 5.1]) Given a polysymplectic structure $\Omega := \sum_{i=1}^k \omega_i\otimes e^i$ on N, we say that $h := \sum_{i=1}^k h_i \otimes e^i$ is an Ω*–Hamiltonian function* if there exists a vector field X_h on N such that $\iota_{X_h}\omega_i = dh_i$ for $i = 1,\ldots,k$. In this case, we call h an Ω*–Hamiltonian function* for X_h. We write $C^\infty(\Omega)$ for the space of Ω–Hamiltonian functions.

Example 4.9. ([**LV.H1**, Section 7]) In view of the relations (4.2), the vector fields $Y_1 = 2v\partial/\partial a$, $Y_2 = v\partial_v + 2a\partial_a$ and $X_3 = v\partial_x + a\partial_v + 3a^2/(2v)\partial_a$ given in (3.3) have Ω–Hamiltonian functions

$$f = \tfrac{2}{v}\otimes e^1 - \tfrac{4x}{v}\otimes e^2$$
$$g = \tfrac{a}{v^2}\otimes e^1 + \left(2 - \tfrac{2ax}{v^2}\right)\otimes e^2, \qquad h = \tfrac{a^2}{2v^3}\otimes e^1 + \left(\tfrac{2a}{v} - \tfrac{a^2x}{v^3}\right)\otimes e^2,$$

relative to the polysymplectic structure $\Omega := \omega_1\otimes e^1 + \omega_2 \otimes e^2$ obtained from the two–symplectic structure (ω_1,ω_1) constructed from the presymplectic forms (4.1).

Proposition 4.10. *(*[**LV.H1**, Proposition 5.2]*) If $\Omega := \sum_{i=1}^k \omega_i\otimes e^i$ is a polysymplectic structure, then every Ω–Hamiltonian vector field is associated, at least, to an Ω–Hamiltonian function. Conversely, every Ω–Hamiltonian function induces a unique Ω–Hamiltonian vector field.*

Proposition 4.11. *(*[**LV.H1**, Proposition 5.3]*) The space $C^\infty(\Omega)$ is a linear space over $\mathbb{R}$ with the natural operations:*

$$h+g := \sum_{i=1}^k (h_i+g_i)\otimes e^i, \qquad \lambda\cdot h := \sum_{i=1}^k \lambda h_i\otimes e^i$$

where $h = \sum_{i=1}^k h_i\otimes e^i$, $g = \sum_{i=1}^k g_i \otimes e^i \in C^\infty(\Omega)$ and $\lambda\in\mathbb{R}$.

Proposition 4.12. *(*[**LV.H1**, Proposition 5.4]*) The space $C^\infty(\Omega)$ becomes a Lie algebra when endowed with the bracket $\{\cdot,\cdot\}_\Omega : C^\infty(\Omega)\times C^\infty(\Omega)\to C^\infty(\Omega)$ of the form*

$$\{h_1\otimes e^1+\ldots+h_k\otimes e^k, h'_1\otimes e^1+\ldots+h'_k\otimes e^k\}_\Omega = \{h_1,h'_1\}_{\omega_1}\otimes e^1+\ldots+\{h_k,h'_k\}_{\omega_k}\otimes e^k, \quad (4.4)$$

where $\{\cdot,\cdot\}_{\omega_i}$ is the Poisson bracket naturally induced by the presymplectic form ω_i, with $i=1,\ldots,k$.

We cannot ensure $C^\infty(\Omega)$ to be a Poisson algebra in general. If $h := \sum_{i=1}^k h_i\otimes e^i$ and $g := \sum_{i=1}^k g_i\otimes e^i \in C^\infty(\Omega)$, then the function

$$h\cdot g := (h_1g_1)\otimes e^1 + \ldots + (h_kg_k)\otimes e^k \quad (4.5)$$

is not in general a $C^\infty(\Omega)$–function (see [**LV.H1**] for a particular example). Since we cannot ensure that $(C^\infty(\Omega),\cdot,\{\cdot,\cdot\}_\Omega)$ is a Poisson algebra, we cannot neither say that $\{\cdot,h\}_\Omega : g\in C^\infty(\Omega)\mapsto \{g,h\}_\Omega \in C^\infty(\Omega)$, with $h\in C^\infty(\Omega)$, is a derivation with respect to the product (4.5) of Ω–Hamiltonian functions. This shows that k–symplectic geometry becomes quite different from Poisson and presymplectic geometry, where previous analysed properties hold. Nevertheless, we can still ensure that $\{h,g\}_\Omega = 0$ for every locally constant function g and, moreover, we can still prove other properties of this Lie algebra. For instance, we have the following results.

Proposition 4.13. *(*[**LV.H1**, Proposition 5.5]*) Consider a polysymplectic manifold (N,Ω). Every Ω–Hamiltonian vector field X acts as a derivation on the Lie algebra $(C^\infty(\Omega),\{\cdot,\cdot\}_\Omega)$ in the form*

$$Xf = \{f,h\}_\Omega, \qquad \forall f\in C^\infty(\Omega),$$

where h is an Ω–Hamiltonian function for X.

Theorem 4.14. *(*[**LV.H1**, Theorem 5.6]*) Given a polysymplectic form $\Omega := \sum_{i=1}^{k} \omega_i \otimes e^i$ on a manifold N, we can define an exact sequence of Lie algebras:*

$$0 \hookrightarrow \overbrace{\mathrm{H}^0_{\mathrm{dH}}(N) \oplus \ldots \oplus \mathrm{H}^0_{\mathrm{dH}}(N)}^{k} \hookrightarrow C^\infty(\Omega) \xrightarrow{B_\Omega} \mathrm{Ham}(\Omega) \to 0, \tag{4.6}$$

where $B_\Omega(f) := -X_f$ is the Ω–Hamiltonian vector field corresponding to f and $\mathrm{H}^0_{\mathrm{dH}}(N)$ is the first De Rham cohomology group of N.

4.2. Derived Poisson algebras. Starting from a k–symplectic manifold $(N, \omega_1, \ldots, \omega_k)$, I constructed several Poisson algebras on certain subsets of $C^\infty(N)$, the hereafter called *derived Poisson algebras*. This is very important to study the geometric properties of superposition rules for k–symplectic Lie systems. This structure passed so far unadvised since it becomes relevant only to study systems of ordinary differential equations, which were not very much analyzed through k-symplectic structures previously to my works.

The k–symplectic structure $(\omega_1, \ldots, \omega_k)$ along with a basis $e^1, \ldots, e^k$ for $\mathbb{R}^k$, and an element $\theta \in (\mathbb{R}^k)^*$ allow us to define a polysymplectic form $\Omega = \sum_{i=1}^{k} \omega_i \otimes e^i$. The contraction $\Omega_\theta := \langle \Omega, \theta \rangle = \sum_{i=1}^{k} \theta(e^i)\omega_i$ is a presymplectic form on N. We call $\mathrm{Adm}(\Omega_\theta)$ the set of admissible functions with respect to (N, Ω_θ). We hereafter denote by X_f, where f is an admissible function on N relative to Ω_θ, a Hamiltonian vector field of f relative to the presymplectic form Ω_θ. Recall that when f is a k–Hamiltonian function, X_f denotes the k–Hamiltonian vector field associated to f.

Proposition 4.15. *(*[**LV.H1**, Proposition 6.1]*) Let $\Omega := \sum_{i=1}^{k} \omega_i \otimes e^i$ be a polysymplectic structure and $\theta \in (\mathbb{R}^k)^*$. Every Ω–Hamiltonian function h gives rise to an admissible function $h_\theta := \langle h, \theta \rangle$ with respect to (N, Ω_θ).*

Proposition 4.16. *(*[**LV.H1**, Proposition 6.2]*) Let $(\omega_1, \ldots, \omega_k)$ be a k–symplectic structure and let $\{e^1, \ldots, e^k\}$ be a basis of $\mathbb{R}^k$. The k–symplectic structure induces a k-polysymplectic form $\Omega := \sum_{i=1}^{k} \omega_i \otimes e^i$ and a family of Poisson algebras $(\mathrm{Adm}(\Omega_\theta), \cdot, \{\cdot, \cdot\}_\theta)$, where $\{\cdot, \cdot\}_\theta$ is the Poisson bracket induced by the presymplectic form Ω_θ, with $\theta \in (\mathbb{R}^k)^*$, on its space of admissible functions.*

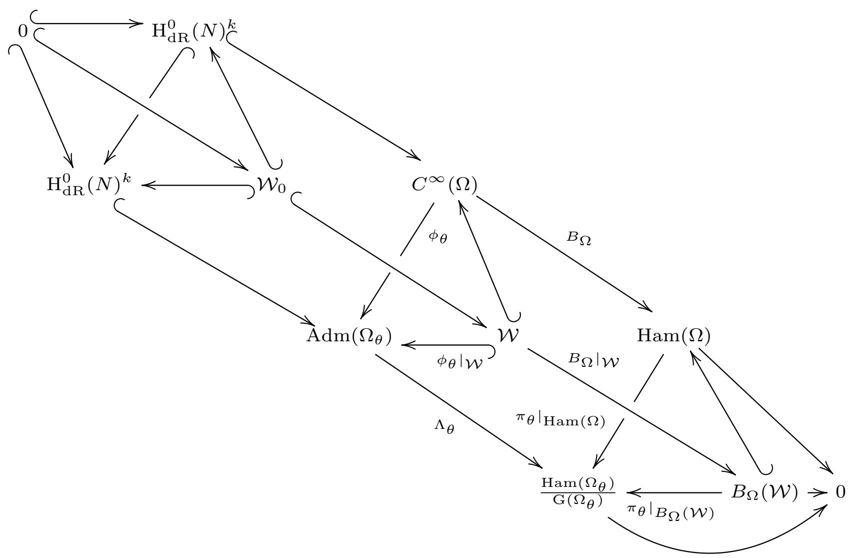
Proposition 4.17. *(*[**LV.H1**, Proposition 6.3]*) Given a polysymplectic form $\Omega := \sum_{i=1}^{k} \omega_i \otimes e^i$, every Ω–Hamiltonian vector field X_h is a derivation on all the Lie algebras $(\mathrm{Adm}(\Omega_\theta), \{\cdot, \cdot\}_\theta)$ with $\theta \in (\mathbb{R}^k)^*$ of the form $X_h f = \{f, h_\theta\}_\theta, \forall f \in \mathrm{Adm}(\Omega_\theta)$. Additionally,*

$$\begin{array}{rccc} \phi_\theta & : (C^\infty(\Omega), \{\cdot, \cdot\}_\Omega) & \to & (\mathrm{Adm}(\Omega_\theta), \{\cdot, \cdot\}_\theta) \\ & h & \mapsto & h_\theta = \langle h, \theta \rangle \end{array}$$

is a Lie algebra morphism. Hence, every finite-dimensional Lie algebra $(\mathcal{W} \subset C^\infty(\Omega), \{\cdot, \cdot\}_\Omega)$ is a Lie algebra extension of the Lie algebra $(\phi_\theta(\mathcal{W}), \{\cdot, \cdot\}_\theta)$[1].

[1]The following diagram can be found in [**LV.H1**] and summarises different structures, which I found while analyzing k-symplectic Lie systems. The arrows of the form $A \hookrightarrow B$ mean the inclusion of A in B.

Finally, we have the commutative exact diagram aside, where $\mathcal{W}_0 = \mathrm{H}^0(N)^k \cap \mathcal{W}$ and



we recall that $\mathrm{G}(\Omega_\theta)$ is the space of gauge vector fields of Ω_θ, we call $\pi_\theta : X \in \mathrm{Ham}(\Omega_\theta) \mapsto [X] \in \mathrm{Ham}(\Omega_\theta)/G(\Omega_\theta)$ the quotient map onto $\mathrm{Ham}(\Omega_\theta)/G(\Omega_\theta)$, and $\Lambda_\theta : \mathrm{Adm}(\Omega_\theta) \to \mathrm{Ham}(\Omega_\theta)/G(\Omega_\theta)$ is the Lie algebra morphism mapping each $f \in \mathrm{Adm}(\Omega_\theta)$ to the class $[-X_f]$.

4.3. ***k*-symplectic Lie–Hamiltonian structures.** A k–symplectic Lie system is associated with many different Lie algebras of functions which can be employed to study the properties of the system.

Consider again the Schwarzian equations in first-order form (3.2). Remind that Y_1, Y_2 and Y_3 are Hamiltonian vector fields (3.3) with respect to the presymplectic structures ω_1 and ω_2. In particular, from the relations (4.2), the vector fields Y_1, Y_2 and Y_3 have Hamiltonian functions

$$h_1^1 = \frac{2}{v}, \qquad h_1^2 = \frac{a}{v^2}, \qquad h_1^3 = \frac{a^2}{2v^3}, \tag{4.7}$$

and

$$h_2^1 = -\frac{4x}{v}, \quad h_2^2 = 2 - \frac{2ax}{v^2}, \quad h_2^3 = \left(\frac{2a}{v} - \frac{a^2x}{v^3}\right), \tag{4.8}$$

with respect to the presymplectic forms ω_1 and ω_2 given by (4.1), correspondingly. Moreover,

$$\left\{h_i^1, h_i^2\right\}_{\omega_i} = -h_i^1, \qquad \left\{h_i^1, h_i^3\right\}_{\omega_i} = -2h_i^2, \qquad \left\{h_i^2, h_i^3\right\}_{\omega_i} = -h_i^3, \qquad i = 1, 2.$$

Consequently, the functions h_i^α, with $\alpha = 1, 2, 3$ and a fixed i, span a finite-dimensional real Lie algebra of functions isomorphic to $\mathfrak{sl}(2)$. The same applies to $h_1^\alpha + h_2^\alpha$, with $\alpha = 1, 2, 3$, and in general for any linear combination $\mu_1 h_1^\alpha + \mu_2 h_2^\alpha$, with fixed $(\mu_1, \mu_2) \in \mathbb{R}^2 \backslash \{(0, 0)\}$.

Now, we consider the space $C^\infty(\Omega)$ of Ω–Hamiltonian functions related to the two-symplectic structure (ω_1, ω_2). From the relations (4.2), the functions

$$h^\alpha := h_1^\alpha \otimes e^1 + h_2^\alpha \otimes e^2,$$

with $\alpha = 1, 2, 3$, span a finite-dimensional Lie algebra when endowed with the Lie bracket (4.4).

Thus, every X_t^{3KS} is an Ω–Hamiltonian vector field with Ω–Hamiltonian function

$$h_t^{3KS} = (h_1^3 + b_1(t) h_1^1) \otimes e^1 + (h_2^3 + b_1(t) h_2^1) \otimes e^2.$$

Since we assume $b_1(t)$ to be non-constant, the space $\mathrm{Lie}(\{h_t^{3KS}\}_{t\in\mathbb{R}}, \{\cdot, \cdot\}_\Omega)$ becomes a real Lie algebra isomorphic to $\mathfrak{sl}(2)$. Similarly, every system in Table 5 can be associated with a t-dependent Ω-Hamiltonian function of the form $\sum_{\alpha=1}^k a_\alpha(t) h_\alpha$. It makes therefore sense to provide the following definitions and the important Theorem 4.20, which associates every k-symplectic Lie system with a t-dependent Ω-Hamiltonian function as previously.

Definition 4.18. ([**LV.H1**, Definition 7.1]) A *k–symplectic Lie–Hamiltonian structure* is a triple (N, Ω, h) where (N, Ω) is a polysymplectic manifold and h represents a t-parametrised family of Ω–Hamiltonian functions $h_t \colon N \to \mathbb{R}^k$ such that $\mathrm{Lie}(\{h_t\}_{t\in\mathbb{R}}, \{\cdot, \cdot\}_\Omega)$ is a finite-dimensional real Lie algebra.

TABLE 5. Lie systems admitting a Lie algebra of Hamiltonian vector fields relative to a k-symplectic form (for further details see [**LV.H1**]). For simplicity, we define $\omega_{ij} := dx_i \wedge dx_j$, $\partial_{x_i} = \partial_i$.

Application	Basis of vector fields X_i	Ω-Hamiltonian functions h_i	k-symplectic structure ω_i
Superposition rules	$\sum_{i=1}^4 \partial_i$	$\left(\frac{1}{x_1-x_2}+\frac{1}{x_3-x_4}\right)\otimes e_1+\left(\sum_{i<j=1}^4 \frac{1}{x_i-x_j}\right)\otimes e_2$	$\frac{\omega_{12}}{(x_1-x_2)^2}+\frac{\omega_{34}}{(x_3-x_4)^2}$
for Riccati equations	$\sum_{i=1}^4 x_i\partial_i$	$\frac{1}{2}\left(\frac{x_1+x_2}{x_1-x_2}+\frac{x_3+x_4}{x_3-x_4}\right)\otimes e_1+\frac{1}{2}\left(\sum_{i<j=1}^4 \frac{x_i+x_j}{x_i-x_j}\right)\otimes e_2$	$\sum_{i<j=1}^4 \frac{\omega_{ij}}{(x_i-x_j)^2}$
$\sum_{\alpha=1}^3 a_\alpha(t)X_\alpha$	$\sum_{i=1}^4 x_i^2\partial_i$	$\left(\frac{x_1x_2}{x_1-x_2}+\frac{x_3x_4}{x_3-x_4}\right)\otimes e_1+\left(\sum_{i<j=1}^4 \frac{x_ix_j}{x_i-x_j}\right)\otimes e_2$	
Control system	∂_1,	$x_2\otimes e_1+x_3\otimes e_2+x_4\otimes e_3+\frac{1}{3}x_2^3\otimes e_4$	ω_{12}
$\sum_{\alpha=1}^2 a_\alpha(t)X_\alpha$	$\partial_2+x_1(\partial_3+x_1\partial_4+2x_2\partial_5)$	$-x_1\otimes e_1-\frac{1}{2}x_1^2\otimes e_2-\frac{1}{3}x_1^3\otimes e_3+(x_5-x_1x_2^2)\otimes e_4.$	ω_{13}
	$\partial_3+2x_1\partial_4+2x_2\partial_5$	$x_1\otimes e_2-x_1^2\otimes e_3-x_2^2\otimes e_4$	ω_{14}
	∂_4	$-x_1\otimes e_3$	$\omega_{25}+x_2^2\,\omega_{12}$
	∂_5	$-x_2\otimes e_4$	
Control system	$\partial_1-x_2\partial_3+x_2^2\partial_5$,	$x_2\otimes e_1-\frac{1}{3}x_2^3\otimes e_2+x_4\otimes e_3+(x_1x_2+x_3)\otimes e_4$	ω_{12}
$\sum_{\alpha=1}^2 a_\alpha(t)X_\alpha$	∂_{x_4}	$x_1\otimes e_1+x_5\otimes e_2-\frac{1}{3}x_1^3\otimes e_3-x_1^2\otimes e_4$	ω_{25}
	∂_5	$\frac{1}{2}x_2^2\otimes e_2-\frac{1}{2}x_1^2\otimes e_3-x_1\otimes e_4$	ω_{13}
	$\partial_2+x_1\partial_3+x_1^2\partial_{x_4}$	$-x_1\otimes e_3$	$\omega_{13}+x_1\omega_{12}$
	$\partial_3+x_1\partial_{x_4}-x_2\partial_5$	$-x_2\otimes e_2$	
Diffusion equations	$4x_1^2\partial_1+4x_1x_2\partial_2+x_2^2\partial_3$,	$(4x_1x_3-8\frac{x_1^2x_3^2}{x_2^2}-\frac{x_2^2}{2})\otimes e_1+(x_1-4\frac{x_1^2x_3}{x_2^2})\otimes e_2$	$\frac{\omega_{23}}{x_2}+\frac{4x_3^2\omega_{12}}{x_2^3}-\frac{4x_3\omega_{13}}{x_2^2}$
$\sum_{\alpha=1}^3 a_\alpha(t)X_\alpha$,	$2x_1\partial_1+x_2\partial_2$	$-2\frac{x_3^2}{x_2^2}\otimes e_1-4\frac{x_3}{x_2^2}\otimes e_2$	$-\frac{4\omega_{13}}{x_2^2}+\frac{8x_3\omega_{12}}{x_2^3}.$
	∂_1	$(x_3-4\frac{x_1x_3^2}{x_2^2})\otimes e_1-8\frac{x_1x_3}{x_2^2}\otimes e_2$	
Lotka-Volterra system $a(t)X_1+b(t)X_2$,	$\sum_{i=1}^5 x_i\partial_i$	$(\frac{x_1+x_2}{x_1-x_2}+\frac{x_3+x_4}{x_3-x_4})\otimes e_1+(\frac{x_1+x_2}{x_1-x_2}+\frac{x_3+x_5}{x_3-x_5})\otimes e_2$ $(\frac{x_1+x_2}{x_1-x_2}+\frac{x_4+x_5}{x_4-x_5})\otimes e_3+(\frac{x_1+x_3}{x_1-x_3}+\frac{x_4+x_5}{x_4-x_5})\otimes e_4$	$\frac{\omega_{12}}{(x_1-x_2)^2}+\frac{\omega_{34}}{(x_3-x_4)^2}$ $\frac{\omega_{12}}{(x_1-x_2)^2}+\frac{\omega_{35}}{(x_3-x_5)^2}$
	$\sum_{i=1}^5 x_i^2\partial_i$	$\left(\frac{x_1x_2}{x_1-x_2}+\frac{x_3x_4}{x_3-x_4}\right)\otimes e_1+\left(\frac{x_1x_2}{x_1-x_2}+\frac{x_3x_5}{x_3-x_5}\right)\otimes e_2$ $\left(\frac{x_1x_2}{x_1-x_2}+\frac{x_4x_5}{x_4-x_5}\right)\otimes e_3+\left(\frac{x_1x_3}{x_1-x_3}+\frac{x_4x_5}{x_4-x_5}\right)\otimes e_4$	$\frac{\omega_{12}}{(x_1-x_2)^2}+\frac{\omega_{45}}{(x_4-x_5)^2}$ $\frac{\omega_{13}}{(x_1-x_3)^2}+\frac{\omega_{45}}{(x_4-x_5)^2}$

Definition 4.19. ([**LV.H1**, Definition 7.2]) A t-dependent vector field X is said to admit a k–symplectic Lie–Hamiltonian structure (N, Ω, h) if $B_\Omega(h_t) = -X_t$, for all $t \in \mathbb{R}$.

Theorem 4.20. *(*[**LV.H1**, Theorem 7.3]*)A system X admits a k–symplectic Lie–Hamiltonian structure if and only if it is a k–symplectic Lie system.*

4.4. On general properties of k–symplectic Lie systems. Let us describe the analogue for k–symplectic Lie systems of the basic properties of general Lie systems found in [**LV.H1**]. Additionally, we show how the derived algebras enable us to investigate their t-independent constants of motion.

Recall that, as for every Lie system, the general solution $x(t)$ of a k–symplectic Lie system X on N can be brought into the form $x(t) = \varphi(g(t), x_0)$, where $x_0 \in N$ and $\varphi \colon G \times N \to N$ is a Lie group action. If G is additionally connected, every curve $\bar{g}(t)$ in G induces a t-dependent change of variables mapping a Lie system X taking values in a Lie algebra V^X into another Lie system Y, with general solution $y(t) = \varphi(\bar{g}(t), x(t))$, taking values in the same Lie algebra V^X [**CGL09, CRG**]. If X is a k–symplectic Lie system, then V^X consists of k–Hamiltonian vector fields with respect to some k–symplectic structure. Since the vector fields $\{Y_t\}_{t\in\mathbb{R}}$ belong to V^X also, they are k–Hamiltonian vector fields and Y is again a k–symplectic Lie system.

Each particular solution of a Lie system X is contained within an orbit S of φ. Then, it makes sense to define the restriction $X|_S$ of X to each orbit S. Therefore, the integration of a Lie system X reduces to integrating its restrictions to each orbit of φ, which are Lie systems also. If X is a k-symplectic Lie system, then it is interesting to know whether $X|_S$ is again a k–symplectic Lie system. This requires to study the notion of *l–symplectic submanifold* ($l \leq k$) of a k–symplectic manifold $(N, \omega_1, \ldots, \omega_k)$, which was developed by S. Vilariño and M. de León in [**LV13**].

Definition 4.21. Given a k–symplectic manifold $(N, \omega_1, \ldots, \omega_k)$, a submanifold $S \subset N$ is said to be an *l–symplectic submanifold* with respect to $(N, \omega_1, \ldots, \omega_k)$, ($l \leq k$) if $\dim S = n_l(l+1)$ for an integer n_l and

$$(T_pS)^{\perp,l} \cap T_pS = \{0\}, \qquad \forall p \in S, \tag{4.9}$$

where $(T_pS)^{\perp,l}$ is the l–th orthogonal complement of T_pS relative to the k–symplectic structure $(N, \omega_1, \ldots, \omega_k)$, i.e. $T_pS^{\perp,l} := \{v \in T_pN : \omega_1(v, w) = \ldots = \omega_l(v, w) = 0, \forall w \in T_pS\}$.

Condition (4.9) is equivalent to $\bigcap_{i=1}^l (T_pS)^{\perp_i} \cap T_pS = \{0\}, \forall p \in S$,where $(T_pS)^{\perp_i}$ is the presymplectic annihilator of T_pS, i.e. $T_pS^{\perp_i} = \{v \in T_pN : \omega_i(v, w) = 0, \forall w \in T_pS\}$. If a submanifold $S \subset M$ is endowed with an l–symplectic structure $(\iota^*\omega_1, \ldots, \iota^*\omega_l)$ with $l < k$, then for all l' such that $l \leq l' \leq k$ (it is necessary that there exists $n_{l'}$ such that $\dim S = n_{l'}(l'+1)$), $(\iota^*\omega_1, \ldots, \iota^*\omega_{l'})$ is an l'–symplectic structure on S. Taking into account the above, the following results follow.

Proposition 4.22. *(*[**LV.H1**, Proposition 8.4]*) Let $(\omega_1, \ldots, \omega_k)$ be a k–symplectic structure on N and let X be a k–symplectic Lie system relative to it. Given an l–symplectic submanifold S such that $\mathcal{D}^X \subset TS$, the restriction of X to S is an l–symplectic Lie system.*

Proposition 4.23. *(*[**LV.H1**, Proposition 8.5]*) Let X be a k–symplectic Lie system on N with k–symplectic Lie–Hamiltonian structure (N, Ω, h). For each $\theta \in (\mathbb{R}^k)^*$, the space $\mathcal{I}^X_\theta$ of t-independent constants of motion of X admissible relative to Ω_θ is a Poisson algebra with respect to each Poisson bracket $\{\cdot, \cdot\}_\theta$ induced by Ω_θ.*

Proposition 4.24. *(*[**LV.H1**, Proposition 8.6]*) Let X be a k–symplectic Lie system on a manifold N with k–symplectic Lie Hamiltonian structure (N, Ω, h). For each $\theta \in (\mathbb{R}^k)^*$, the function $f : N \to \mathbb{R}$ is a constant of motion for X admissible relative to Ω_θ if and only if f Poisson commutes with all elements of each $\phi_\theta(\mathrm{Lie}(\{h_t\}_{t\in\mathbb{R}}, \{\cdot, \cdot\}_\Omega))$.*

Every autonomous Hamiltonian system is a k–symplectic Lie system with respect a symplectic form ω. It also possesses a k–Hamiltonian structure (N, Ω, h), where h is a t-independent Hamiltonian. In consequence, the above proposition shows that the t-independent constants of motion for a Hamiltonian system are those functions that Poisson commute with its Hamiltonian, recovering as a particular case this well-known result.

4.5. Diagonal prolongations of k–symplectic Lie systems. This section surveys my results on superposition rules for k–symplectic Lie systems. In short, the k-symplectic structure related to these systems provides us with methods to work out these superposition rules far more efficiently than standard methods or by using Dirac or symplectic structures. The fundamental result on our study is the following one:

Proposition 4.25. *(*[**LV.H1**, Proposition 9.1]*) If X is a k–symplectic Lie system relative to $(\omega_1,\ldots,\omega_k)$, then $\widetilde{X}^{[m]}$ is a k–symplectic Lie system relative to $(\omega_1^{[m]},\ldots,\omega_k^{[m]})$.*

Let us illustrate the above notion through a remarkable example I developed along with one of my collaborators in [**LV.H1**]. Consider again the Schwarzian equation (3.1) as a first-order system. The works [**CGL.H6, LS13**] derived a superposition rule for such equations by solving a system of PDEs to obtain three functionally independent t-independent constants of motion for the diagonal prolongation of (3.2) to $\mathcal{O}_2^{[2]}$. Let us derive such constants of motion through k-symplectic structures in order to show their advantages.

Schwarzian equations admit a Vessiot–Guldberg Lie algebra of Hamiltonian vector fields related to a two-symplectic structure (ω_1,ω_2) on $\mathcal{O}_2$ given by (4.1). Proposition 4.25 ensures that their diagonal prolongations to $\mathcal{O}_2^{[2]}$, i.e. the presymplectic forms

$$\omega_1^{[2]}=\sum_{i=1}^{2}\frac{\mathrm{d}v_{(i)}\wedge\mathrm{d}a_{(i)}}{v_{(i)}},\qquad \omega_2^{[2]}=-\sum_{i=1}^{2}\frac{2}{v_{(i)}^3}(x_{(i)}\mathrm{d}v_{(i)}\wedge\mathrm{d}a_{(i)}+v_{(i)}\mathrm{d}a_{(i)}\wedge\mathrm{d}x_{(i)}+a_{(i)}\mathrm{d}x_{(i)}\wedge\mathrm{d}v_{(i)}),$$

give rise to a two-symplectic structure on $\mathcal{O}_2^{[2]}$. Their kernels are given by

$$\ker\omega_1^{[2]}=\left\langle\frac{\partial}{\partial x_{(1)}},\frac{\partial}{\partial x_{(2)}}\right\rangle,\qquad \ker\omega_2^{[2]}=\bigoplus_{i=1}^{2}\left\langle x_{(i)}\frac{\partial}{\partial x_{(i)}}+v_{(i)}\frac{\partial}{\partial v_{(i)}}+a_{(i)}\frac{\partial}{\partial a_{(i)}}\right\rangle.$$

Both kernels have zero intersection as expected.

Recall that given a polysymplectic form $\Omega:=\sum_{i=1}^k\omega_i\otimes e^i$ on N, its diagonal prolongation to N^m is the polysymplectic form $\Omega^{[m]}=\sum_{i=1}^k\omega_i^{[m]}\otimes e^i$. Using (4.2), we obtain that the k–Hamiltonian functions for the diagonal prolongations of the vector fields (3.3) to $(\mathcal{O}_2)^2$ read

$$h^{1,[2]}=\sum_{i=1}^{2}\left(\frac{2}{v_{(i)}}\otimes e^1-\frac{4x_{(i)}}{v_{(i)}}\otimes e^2\right),\qquad h^{2,[2]}=\sum_{i=1}^{2}\left[\frac{a_{(i)}}{v_{(i)}^2}\otimes e^1+\left(2-\frac{2a_{(i)}x_{(i)}}{v_{(i)}^2}\right)\otimes e^2\right]$$

and

$$h^{3,[2]}=\sum_{i=1}^{2}\left[\frac{a_{(i)}^2}{2v_{(i)}^3}\otimes e^1+\left(\frac{2a_{(i)}}{v_{(i)}}-\frac{a_{(i)}^2x_{(i)}}{v_{(i)}^3}\right)\otimes e^2\right].$$

It follows that

$$\left\{h^{1,[2]},h^{2,[2]}\right\}_{\Omega^{[2]}}=h^{1,[2]},\qquad \left\{h^{1,[2]},h^{3,[2]}\right\}_{\Omega^{[2]}}=2h^{2,[2]},\qquad \left\{h^{2,[2]},h^{3,[2]}\right\}_{\Omega^{[2]}}=h^{3,[2]}.$$

So, these functions close a Lie algebra isomorphic to $\mathfrak{sl}(2)$. Next, we will use the derived algebras to obtain several t-independent constants of motion for these systems.

We can induce from $\Omega^{[2]}$ several presymplectic structures $\Omega_\xi^{[2]}:=\langle\Omega^{[2]},\xi\rangle$, for an arbitrary $\xi\in(\mathbb{R}^2)^*$. For instance, let $\{\theta_1,\theta_2\}$ be the dual basis to $\{e^1,e^2\}$. We therefore have

$$\Omega_{\xi_1}\equiv\langle\Omega^{[2]},\theta_1\rangle=\omega_1^{[2]},\qquad \Omega_{\xi_2}\equiv\langle\Omega^{[2]},\theta_2\rangle=\omega_2^{[2]}.$$

From Proposition 4.17, the Hamiltonian functions $(h^{1,[2]})_\xi,(h^{2,[2]})_\xi,(h^{3,[2]})_\xi$, for every $\xi\in(\mathbb{R}^2)^*$, span a real Lie algebra $\mathfrak{W}$ such that $\mathfrak{sl}(2)$ is a Lie algebra extension. Since $\mathfrak{sl}(2)$ is simple, $\mathfrak{W}$ is isomorphic to $\mathfrak{sl}(2)$ or zero.

If $\mathfrak{W}$ is isomorphic to $\mathfrak{sl}(2)$, it was proved in [**CGL.H6, BCHL.H7**] that $\{C_\xi, (h_i)_\xi\}_\xi = 0$, where $i = 1, 2, 3$, the bracket $\{\cdot, \cdot\}_\xi$ is the Poisson bracket on the space of admissible functions of $\Omega^{[2]}_\xi$, and

$$C_\xi = (h^{1,[2]})_\xi (h^{3,[2]})_\xi - (h^{2,[2]})^2_\xi.$$

It is relevant that C_ξ can be obtained from a Casimir element of a Lie algebra isomorphic to $\mathfrak{sl}(2)$ constructed induced by $h^{1,[2]}, h^{2,[2]}, h^{3,[2]}$. Observe that C_ξ is a t-independent constant of motion for the prolongated system $\widetilde{X}^{[2]}_{3KS}$. More generally, a similar procedure can be developed for other Lie algebras of functions associated with k–symplectic Lie systems. If we write $\xi = \lambda_1\theta_1 + \lambda_2\theta_2$, with $\lambda_1, \lambda_2 \in \mathbb{R}$, then we have $C_\xi = \lambda_1^2 C_{\xi_1} + \lambda_2^2 C_{\xi_2} + \lambda_1\lambda_2 F_{\xi_1\xi_2}$, where C_{ξ_1}, C_{ξ_2}, and $F_{\xi_1\xi_2}$ are three constants of motion given by

$$C_{\xi_1} = (h^{1,[2]})_{\xi_1}(h^{3,[2]})_{\xi_1} - (h^{2,[2]})^2_{\xi_1} = \frac{(a_2v_1 - a_1v_2)^2}{v_1^3 v_2^3},$$

$$C_{\xi_2} = (h^{1,[2]})_{\xi_2}(h^{3,[2]})_{\xi_2} - (h^{2,[2]})^2_{\xi_2} = -4\left(-x_1x_2 + \frac{2v_1v_2(v_1x_2 - v_2x_1)}{a_1v_2 - v_1a_2}\right)\frac{(a_2v_1 - a_1v_2)^2}{v_1^3v_2^3} - 4^2,$$

$$\begin{aligned} F_{\xi_1\xi_2} = (h^{1,[2]})_{\xi_1}(h^{3,[2]})_{\xi_2} + (h^{3,[2]})_{\xi_1}(h^{1,[2]})_{\xi_2} - 2(h^{2,[2]})_{\xi_2}(h^{2,[2]})_{\xi_1} \\ = -\frac{2(a_2v_1 - v_2a_1)^2}{v_1^3v_2^3}\left(x_1 + x_2 - \frac{2v_1v_2(v_1 - v_2)}{a_1v_2 - v_1a_2}\right). \end{aligned}$$

Using that C_{ξ_1} is a t-independent constant of motion, we obtain that $C_{\xi_2}, F_{\xi_1\xi_2}$ allow us to define three simpler t-independent constants of motion F_1, F_3, F_4:

$$F_1 = x_1x_2 - \frac{2v_1v_2(v_1x_2 - v_2x_1)}{a_1v_2 - v_1a_2}, \qquad F_3 = x_1 + x_2 - \frac{2v_1v_2(v_1 - v_2)}{a_1v_2 - v_1a_2},$$

$$F_4 = \sqrt{F_3^2 - 4F_1 + \frac{16}{C_{\xi_1}}} = x_1 - x_2 - \frac{2v_1v_2(v_1 + v_2)}{a_1v_2 - v_1a_2}.$$

The t-independent constants of motion C_{ξ_2}, F_3 and F_4 are the first-integrals employed in [**CGL.H6, LS13**] to obtain the superposition rule for Schwarzian equations in first-order form. In those works, C_{ξ_2}, F_3, F_4 were obtained by means of several geometric methods. In [**LS13**] they were derived by means of the method of characteristics, which is quite long and tedious. In [**CGL.H6**], the techniques for Dirac–Lie systems enabled us to obtain F_1 and C_{ξ_2}. Meanwhile, F_4 had to be obtained through a Lie symmetry. Meanwhile, C_{ξ_2}, F_3, F_4 appear simultaneously from the k–symplectic structure of Schwarzian equations. This is the key point of the usefulness of this approach to obtain superposition rules. The k–symplectic structure provides a framework to exploit the geometric properties of k–symplectic Lie system better than Dirac–Lie systems. My techniques were employed further in [**LTV.H4**] for obtaining superposition rules to study diffusion equations through k-symplectic structures.

5. Jacobi–Lie systems

Let us finally introduce Jacobi–Lie systems as Lie systems admitting a Vessiot–Guldberg Lie algebra of Hamiltonian vector fields relative to a Jacobi manifold (for a detailed account on Jacobi manifolds see [**Ki76, IV, Li77**]). A *Jacobi manifold* is a triple (N, Λ, R), where Λ is a bivector field on N and R is a vector field on N, the so-called *Reeb vector field*, satisfying $[\Lambda, \Lambda]_{SN} = 2R \wedge \Lambda$ and $[R, \Lambda]_{SN} = 0$. A vector field X on N is *Hamiltonian* relative to the Jacobi manifold (N, Λ, R) if there exists $f \in C^\infty(N)$ such that

$$X = [\Lambda, f]_{SN} + fR = \widehat{\Lambda}(df) + fR.$$

It is therefore said that f is a *Hamiltonian function* of X, and we write $X = X_f$. I defined f to be a *good Hamiltonian function* and X_f a *good Hamiltonian vector field* if f is a first-integral of the Reeb vector field [**HL.H5**, Definition 3.4].

The space $\mathrm{Ham}(N,\Lambda,R)$ of Hamiltonian vector fields relative to (N,Λ,R) is a Lie algebra with respect to the standard Lie bracket of vector fields. Additionally, a Jacobi manifold allows us to define a Lie bracket on $C^\infty(N)$ given by

$$\{f,g\}_{\Lambda,R} = \Lambda(\mathrm{d}f,\mathrm{d}g) + fRg - gRf.$$

This Lie bracket becomes a Poisson bracket if and only if $R=0$. Moreover, the morphism $\phi_{\Lambda,R} : f \in C^\infty(N) \mapsto X_f \in \mathrm{Ham}(\Lambda,R)$ is a Lie algebra morphism. It is important to emphasize that it may not be injective.

Similarly to previous sections, I found that it makes sense to propose the following definition.

Definition 5.1. ([**HL.H5**, Definition 4.1]) A *Jacobi–Lie system* (N,Λ,R,X) consists of a Jacobi manifold (N,Λ,R) and a Lie system X satisfying that $V^X \subset \mathrm{Ham}(N,\Lambda,R)$.

Example 5.2. ([**HL.H5**, Example 4.3]) Consider the Lie group $\mathbb{G} := SL(2)$ of matrices 2×2 with real entries $\alpha,\beta,\gamma,\delta$ satisfying $\alpha\delta-\beta\gamma=1$. Close to its neutral element, $\{\alpha,\beta,\gamma\}$ form a local coordinate system for $\mathbb{G}$. A short calculation shows that

$$X_1^R = \alpha\partial_\alpha + \beta\partial_\beta - \gamma\partial_\gamma, \qquad X_2^R = \gamma\partial_\alpha + \frac{1+\beta\gamma}{\alpha}\partial_\beta, \qquad X_3^R = \alpha\partial_\gamma$$

is a basis of the space of right-invariant vector fields on $\mathbb{G}$. If we define

$$\Lambda_{\mathbb{G}} := \alpha\beta\partial_\alpha\wedge\partial_\beta - (1+\beta\gamma)\partial_\beta\wedge\partial_\gamma, \qquad R_{\mathbb{G}} := \alpha\partial_\alpha - \beta\partial_\beta + \gamma\partial_\gamma, \tag{5.1}$$

we obtain that $[\Lambda_{\mathbb{G}},\Lambda_{\mathbb{G}}]_{SN} = -2\alpha\partial_\alpha\wedge\partial_\beta\wedge\partial_\gamma = 2R_{\mathbb{G}}\wedge\Lambda_{\mathbb{G}}$ and $[R_{\mathbb{G}},\Lambda_{\mathbb{G}}]_{SN} = 0$. So, $(\mathbb{G},\Lambda_{\mathbb{G}},R_{\mathbb{G}})$ is a Jacobi manifold. Consider now the system on $\mathbb{G}$ given by $\frac{\mathrm{d}\mathcal{G}}{\mathrm{d}t} = \sum_{i=1}^3 b_i(t)X_i^R(\mathcal{G})$, $\mathcal{G}\in\mathbb{G}$, for any t-dependent functions $b_i(t)$. Since $X^{\mathbb{G}} = \sum_{i=1}^3 b_i(t)X_i^R$ takes values in the Lie algebra $V^{\mathbb{G}} = \langle X_1^R, X_2^R, X_3^R\rangle$, the system $X^{\mathbb{G}}$ is a Lie system. System $X^{\mathbb{G}}$ occurs in the study of Briosche–Darboux–Halphen equations, Kummer–Schwarz equations, Milne–Pinney equations, etcetera [**CGL.H6, EHL.PH1, CL.PH12**].

We now prove that $(\mathbb{G},\Lambda_{\mathbb{G}},R_{\mathbb{G}},X^{\mathbb{G}})$ is a Jacobi–Lie system. In fact, X_1^R,X_2^R,X_3^R are Hamiltonian relative to $(\mathbb{G},\Lambda_{\mathbb{G}},R_{\mathbb{G}})$ with good Hamiltonian functions

$$h_1 = 1+2\beta\gamma, \qquad h_2 = \frac{\gamma}{\alpha}(1+\beta\gamma), \qquad h_3 = -\beta\alpha. \tag{5.2}$$

These functions are first-integrals of X_1^R,X_2^R,X_3^R, respectively, and $R_{\mathbb{G}}$. This allows us to use $X_i^R + \mathrm{d}h_i$ with $i=1,2,3$, and $R_{\mathbb{G}}$ to span a sub-bundle $L_{\mathbb{G}}$ of $T\mathbb{G}\oplus_{\mathbb{G}} T^*\mathbb{G}$ originating a Dirac structure on $\mathbb{G}$ [**Co90**]. Vector fields X_1^R,X_2^R,X_3^R are Hamiltonian relative to $L_{\mathbb{G}}$ giving rise to a Dirac–Lie system $(\mathbb{G},L_{\mathbb{G}},X^{\mathbb{G}})$ [**CGL.H6**].

5.1. Jacobi–Lie Hamiltonian systems. Similarly to Lie systems admitting a Vessiot–Guldberg Lie algebra of Hamiltonian vector fields relative to other structures, Jacobi–Lie systems can be related to a t-dependent Hamiltonian function relative to a Jacobi manifold.

Definition 5.3. ([**HL.H5**, Definition 5.1]) We call *Jacobi–Lie Hamiltonian system* a quadruple (N,Λ,R,h), where (N,Λ,R) is a Jacobi manifold and $h:(t,x)\in\mathbb{R}\times N\mapsto h_t(x)\in N$ is a t-dependent function such that $\mathrm{Lie}(\{h_t\}_{t\in\mathbb{R}},\{\cdot,\cdot\}_{\Lambda,R})$ is finite-dimensional. Given a system X on N, we say that X admits a *Jacobi–Lie Hamiltonian system* (N,Λ,R,h) if X_t is a Hamiltonian vector field with Hamiltonian function h_t (with respect to (N,Λ,R)) for each $t\in\mathbb{R}$.

Example 5.4. ([**HL.H5**, Example 5.3]) Relative to the Lie bracket induced by $(G,\Lambda_{\mathbb{G}},R_{\mathbb{G}})$ given in (5.1), the functions (5.2) satisfy that

$$\{h_1,h_2\}_{\Lambda_{\mathbb{G}},R_{\mathbb{G}}} = -2h_2, \qquad \{h_1,h_3\}_{\Lambda_{\mathbb{G}},R_{\mathbb{G}}} = 2h_3, \qquad \{h_2,h_3\}_{\Lambda_{\mathbb{G}},R_{\mathbb{G}}} = -h_1.$$

So, $(\mathbb{G},\Lambda_{\mathbb{G}},R_{\mathbb{G}},h := \sum_{i=1}^3 b_i(t)h_i)$ is a Jacobi–Lie Hamiltonian system for $X^{\mathbb{G}}$.

The analogues for Jacobi–Lie systems of Theorems 2.5, 3.3, 4.20 were devised in [**HL.H5**] and read as follows.

Theorem 5.5. *(*[**HL.H5**, Theorem 5.4]*) If (N, Λ, R, h) is a Jacobi–Lie Hamiltonian system, then the system X of the form $X_t := X_{h_t}$, $\forall t \in \mathbb{R}$, gives rise to a Jacobi–Lie system (N, Λ, R, X). If X is a Lie system and the $\{X_t\}_{t\in\mathbb{R}}$ are good Hamiltonian vector fields, then X admits a Jacobi–Lie Hamiltonian.*

Jacobi–Lie Hamiltonian systems can be employed for the study of Jacobi–Lie systems.

Proposition 5.6. *(*[**HL.H5**, Proposition 1]*) Let (N, Λ, R, X) be a Jacobi–Lie system admitting a Jacobi–Lie Hamiltonian (N, Λ, R, h) of good Hamiltonian functions $\{h_t\}_{t\in\mathbb{R}}$. Then, $f \in C^\infty(N)$ is a t-independent constant of motion for X if and only if f commutes with all the elements of* $\mathrm{Lie}(\{h_t\}_{t\in\mathbb{R}}, \{\cdot,\cdot\}_{\Lambda,R})$ *relative to $\{\cdot,\cdot\}_{\Lambda,R}$.*

Example 5.7. ([**HL.H5**, Example 5.5]) Consider again the functions h_1, h_2, h_3 given in (5.2) and the Jacobi manifold $(\mathbb{G}, \Lambda_\mathbb{G}, R_\mathbb{G})$, with $\Lambda_\mathbb{G}$ and $R_\mathbb{G}$ given by (5.1). Then, $\{h_1^2 + 4h_2h_3, h_i\}_{\Lambda_\mathbb{G},R_\mathbb{G}} = 0$ for $i = 1, 2, 3$. So, $C = h_1^2 + 4h_2h_3$ is a constant of motion for $X^\mathbb{G}$.

5.2. Jacobi–Lie systems on low dimensional manifolds. This section summarises my findings concerning the classification of Lie algebras of Hamiltonian vector fields relative to Jacobi structures on $\mathbb{R}$ and $\mathbb{R}^2$ obtained in [**HL.H5**]. My main results are detailed in Table 6.

I proved in [**HL.H5**] that a Riccati equation (1.2) can be associated with a Jacobi–Lie system $(\mathbb{R}, \Lambda = 0, R = \partial_{x_1})$. Indeed, the elements of the basis $X_1, X_2, X_3 \in V$ of the Vessiot–Guldberg Lie algebra of (1.2) admit Hamiltonian functions $h_1 = 1, h_2 = x_1, h_3 = x_1^2$. Hence, $(\mathbb{R}, \Lambda = 0, R = \partial_{x_1}, a_0(t)X_1 + a_1(t)X_2 + a_2(t)X_3)$ is a Jacobi–Lie system. Since every Lie system on $\mathbb{R}$ can be brought into this form through a local diffeomorphism on $\mathbb{R}$ [**GKO92, Lie1880, LS**], every Lie system on the real line can be considered as a Jacobi–Lie system.

We now classify Jacobi–Lie systems $(\mathbb{R}^2, \Lambda, R, X)$, where we may assume Λ and R to be locally equal or different from zero. There exists just one Jacobi–Lie system with $\Lambda = 0$ and $R = 0$: $(\mathbb{R}^2, \Lambda = 0, R = 0, X = 0)$.

Jacobi–Lie systems of the form $(\mathbb{R}^2, \Lambda \neq 0, R = 0)$ are Lie–Hamilton systems, whose Vessiot–Guldberg Lie algebras were obtained in [**BHL.H3**]. In Table 4 we indicate these cases by writing P (*Poisson*). A Jacobi–Lie system $(\mathbb{R}^2, \Lambda = 0, R \neq 0, X)$ is such that if $Y \in V^X$, then $Y = fR$ for certain $f \in C^\infty(\mathbb{R}^2)$. All cases of this type can easily be obtained out of the bases given in Table 6. We describe them by writing $(0, R)$ at the last column.

Propositions 5.8 and 5.9 below show that the Vessiot–Gulbderg Lie algebras of Table 6 that do not fall into the mentioned categories are not Vessiot–Guldberg Lie algebras of Hamiltonian vector fields with respect to Jacobi manifolds $(\mathbb{R}^2, \Lambda \neq 0, R \neq 0)$. So, every $(\mathbb{R}^2, \Lambda, R, X)$ admits a Vessiot–Guldberg Lie algebra belonging to one of the previously mentioned classes[2].

Proposition 5.8. *(*[**HL.H5**, Proposition 2]*) Let V be a Vessiot–Guldberg Lie algebra on $\mathbb{R}^2$ containing $X_1, X_2 \in V\backslash\{0\}$ such that $[X_1, X_2] = X_1$ and $X_1 \wedge X_2 = 0$. Then V does not consist of Hamiltonian vector fields relative to any Jacobi manifold with $R \neq 0$ and $\Lambda \neq 0$.*

Proposition 5.9. *(*[**HL.H5**, Proposition 3]*) There exists no Jacobi manifold on the plane with $\Lambda \neq 0$ and $R \neq 0$ turning the elements of a Lie algebra diffeomorphic to $V := \langle \partial_x, \partial_y, x\partial_x + \alpha y\partial_y \rangle$ with $\alpha \notin \{0, -1\}$ into Hamiltonian vector fields.*

6. Outlook

The results of this habilitation can be extended in many ways. Indeed, I have now three papers under review developing different aspects of this dissertation [**GL17, CCJL17, LHT17**]. My previous PhD student continues studying the Lie systems I proposed her during her PhD [**LS16**]. I am continuing my research and I found even more interesting results. Additionally, my research concerns now a larger set of topics: Lie bialgebras, quantum groups, infinite-dimensional jets, differential equations on supermanifolds, stochastic differential equations, BRST symmetries, and their applications in mathematics and physics.

[2]To exclude P_1 with $\alpha \neq 0$ and I_{17}, we need a trivial modification of Proposition 5.9 using exactly the same line of thought.

TABLE 6. Vessiot–Guldberg Lie algebra of Hamiltonian vector fields on $\mathbb{R}^2$ relative to a Jacobi manifold (see [**HL.H5**] for details). P means Poisson. Functions $1, \xi_1(x), \ldots, \xi_r(x)$ are linearly independent, and $\eta_1(x), \ldots, \eta_r(x)$ form a basis of solutions for a $d^r f/dx^r = \sum_{\alpha=0}^{r-1} c_\alpha d^\alpha f/dx^\alpha, c_\alpha \in \mathbb{R}$.

#	Lie algebra	Basis of vector fields X_i	Jacobi
P_1	$A_\alpha \simeq \mathbb{R} \ltimes \mathbb{R}^2$	$\partial_x, \partial_y, \alpha(x\partial_x + y\partial_y) + y\partial_x - x\partial_y, \quad \alpha \geq 0$	$(\alpha = 0)$ P
P_2	$\mathfrak{sl}(2)$	$\partial_x, x\partial_x + y\partial_y, (x^2 - y^2)\partial_x + 2xy\partial_y$	P
P_3	$\mathfrak{so}(3)$	$y\partial_x - x\partial_y, (1 + x^2 - y^2)\partial_x + 2xy\partial_y,$ $2xy\partial_x + (1 + y^2 - x^2)\partial_y$	P
P_4	$\mathbb{R}^2 \ltimes \mathbb{R}^2$	$\partial_x, \partial_y, x\partial_x + y\partial_y, y\partial_x - x\partial_y$	No
P_5	$\mathfrak{sl}(2) \ltimes \mathbb{R}^2$	$\partial_x, \partial_y, x\partial_x - y\partial_y, y\partial_x, x\partial_y$	P
P_6	$\mathfrak{gl}(2) \ltimes \mathbb{R}^2$	$\partial_x, \partial_y, x\partial_x, y\partial_x, x\partial_y, y\partial_y$	No
P_7	$\mathfrak{so}(3,1)$	$\partial_x, \partial_y, x\partial_x + y\partial_y, y\partial_x - x\partial_y, (x^2 - y^2)\partial_x + 2xy\partial_y,$ $2xy\partial_x + (y^2 - x^2)\partial_y$	No
P_8	$\mathfrak{sl}(3)$	$\partial_x, \partial_y, x\partial_x, y\partial_x, x\partial_y, y\partial_y, x^2\partial_x + xy\partial_y, xy\partial_x + y^2\partial_y$	No
I_1	$\mathbb{R}$	∂_x	P, $(0, \partial_x)$
I_2	$\mathfrak{h}_2$	$\partial_x, x\partial_x$	P, $(0, \partial_x)$
I_3	$\mathfrak{sl}(2)$ (type I)	$\partial_x, x\partial_x, x^2\partial_x$	P, $(0, \partial_x)$
I_4	$\mathfrak{sl}(2)$ (type II)	$\partial_x + \partial_y, x\partial_x + y\partial_y, x^2\partial_x + y^2\partial_y$	P
I_5	$\mathfrak{sl}(2)$ (type III)	$\partial_x, 2x\partial_x + y\partial_y, x^2\partial_x + xy\partial_y$	P
I_6	$\mathfrak{gl}(2)$ (type I)	$\partial_x, \partial_y, x\partial_x, x^2\partial_x$	No
I_7	$\mathfrak{gl}(2)$ (type II)	$\partial_x, y\partial_y, x\partial_x, x^2\partial_x + xy\partial_y$	No
I_8	$B_\alpha \simeq \mathbb{R} \ltimes \mathbb{R}^2$	$\partial_x, \partial_y, x\partial_x + \alpha y\partial_y, \quad 0 < \lvert\alpha\rvert \leq 1$	$(\alpha = -1)$ P
I_9	$\mathfrak{h}_2 \oplus \mathfrak{h}_2$	$\partial_x, \partial_y, x\partial_x, y\partial_y$	No
I_{10}	$\mathfrak{sl}(2) \oplus \mathfrak{h}_2$	$\partial_x, \partial_y, x\partial_x, y\partial_y, x^2\partial_x$	No
I_{11}	$\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$	$\partial_x, \partial_y, x\partial_x, y\partial_y, x^2\partial_x, y^2\partial_y$	No
I_{12}	$\mathbb{R}^{r+1}$	$\partial_y, \xi_1(x)\partial_y, \ldots, \xi_r(x)\partial_y$	P, $(0, \partial_y)$
I_{13}	$\mathbb{R} \ltimes \mathbb{R}^{r+1}$	$\partial_y, y\partial_y, \xi_1(x)\partial_y, \ldots, \xi_r(x)\partial_y$	P, $(0, \partial_y)$
I_{14}	$\mathbb{R} \ltimes \mathbb{R}^r$	$\partial_x, \eta_1(x)\partial_y, \eta_2(x)\partial_y, \ldots, \eta_r(x)\partial_y$	P
I_{15}	$\mathbb{R}^2 \ltimes \mathbb{R}^r$	$\partial_x, y\partial_y, \eta_1(x)\partial_y, \ldots, \eta_r(x)\partial_y$	No
I_{16}	$C_\alpha^r \simeq \mathfrak{h}_2 \ltimes \mathbb{R}^{r+1}$	$\partial_x, \partial_y, x\partial_x + \alpha y\partial y, x\partial_y, \ldots, x^r\partial_y, \quad \alpha \in \mathbb{R}$	$(\alpha = -1)$ P
I_{17}	$\mathbb{R} \ltimes (\mathbb{R} \ltimes \mathbb{R}^r)$	$\partial_x, \partial_y, x\partial_x + (ry + x^r)\partial_y, x\partial_y, \ldots, x^{r-1}\partial_y$	No
I_{18}	$(\mathfrak{h}_2 \oplus \mathbb{R}) \ltimes \mathbb{R}^{r+1}$	$\partial_x, \partial_y, x\partial_x, x\partial_y, y\partial_y, x^2\partial_y, \ldots, x^r\partial_y$	No
I_{19}	$\mathfrak{sl}(2) \ltimes \mathbb{R}^{r+1}$	$\partial_x, \partial_y, x\partial_y, 2x\partial_x + ry\partial_y, x^2\partial_x + rxy\partial_y, x^2\partial_y, \ldots, x^r\partial_y$	No
I_{20}	$\mathfrak{gl}(2) \ltimes \mathbb{R}^{r+1}$	$\partial_x, \partial_y, x\partial_x, x\partial_y, y\partial_y, x^2\partial_x + rxy\partial_y, x^2\partial_y, \ldots, x^r\partial_y$	No

The first paper [**CCJL17**] concerns the applications of Lie-Hamilton systems and Dirac-Lie systems to non-autonomous Schrödinger equations. This works establishes that many quantum systems admit a non-linear superposition rule depending on fewer solutions than the standard linear one. This works also concerns the study of Lie systems possessing a Vessiot–Guldberg Lie algebra of Kähler vector fields.

In my second work under review [**GL17**], I proved that all differential equations of the Riccati hierarchy, which frequently appear in the study of integrable systems, can be studied through Lie systems admitting a Lie algebra of conformal vector fields relative to a Riemannian metric. This means that one can apply Winternitz's methods to obtain their superposition rules.

The last paper under review, i.e. [**LHT17**], shows how to obtain superposition rules for Lie-Hamilton systems admitting a Vessiot–Guldberg Lia algebra of Killing vector fields relative to a certain metric of constant curvature.

Nowadays, I am working on obtaining an analogue of the Marsden-Weinstein reduction for Lie systems possessing a compatible Vessiot–Guldberg Lie algebra of Hamiltonian vector fields relative to a multisymplectic structure. Additionally, I am using Lie bialgebras and quantum groups to construct integrable systems, which can be understood as a quantum deformation of a Lie-Hamilton systems. More specifically, I am working on a quantum deformation of a Milne–Pinney equation.

CHAPTER 2

Other scientific achievements and activities

1. Research forming a continuation of the PhD thesis

During my postdoctoral research I also studied the basic properties and applications of Lie systems without compatible geometric structures. Additionally, I also searched for generalizations of Lie systems that could be led to the study of more general differential equations. Next, I provide a list of my works as a postdoc concerning the above-mentioned topics.

GHL.PH1. P. Garcia-Estevez, F.J. Herranz, J. de Lucas and C. Sardón, Lie symmetries for Lie systems: Applications to systems of ODEs and PDEs, *Appl. Math. Comp.* **273**, 435–452 (2016). IF = 1.014 (2015), (Q1 - 56/312 w Mathematics), citations = 1(0). My contribution is about the 15% of the work.

LT.PH2. J. de Lucas, M. Tobolski and S. Vilariño, Geometry of Riccati equations over normed division algebras, *J. Math. Anal. Appl.* **440**, 394–414 (2016). IF = 1.014 (2015), (Q1 - 56/312 w Mathematics), citations = 1(1). My contribution is about the 40% of the work.

CL.PH3. J.F. Cariñena and J. de Lucas, Quasi–Lie families, schemes, invariants and their applications to Abel equations, *J. Math. Anal. Appl.* **430**, 648–671 (2015). IF = 1.014 (2015), (Q1 - 25/53 w Physics, Mathematical), citations = 0. My contribution is about the 80% of the work.

CL.PH4. J.F. Cariñena, J. de Lucas and P. Guha, A quasi-Lie schemes approach to the Gambier equation, *SIGMA* **9**, 026 (2013). IF = 1.299 (2013), (Q2 - 25/55 w Physics, Mathematical), citations = 5(4). My contribution is about the 75% of the work.

GL.PH5. J. Grabowski i J. de Lucas, Mixed superposition rules and the Riccati hierarchy, *J. Differential Equations* **254**, 179–198 (2013). IF =1.570 (2013), (Q1 - 13/302 w Mathematics), citations = 6(2). My contribution is about the 70% of the work.

CL.PH6. J.F. Cariñena, J. de Lucas and J. Grabowski, Superposition rules for higher-order systems and their applications, *J. Phys. A: Math. Theor.* **45**, 185202 (2012).IF = 1.766 (2012), (Q2 - 13/55 w Physics, Mathematical), citations = 14(4). My contribution is about the 60% of the work.

CL.PH7. J.F. Cariñena and J. de Lucas, Superposition rules and second-order Riccati equations, *J. Geom. Mech.* **3**, 1–22 (2011). IF = 0.812, (Q2 - 101/245 w Physics, Mathematical), citations = 24(13). My contribution is about the 75% of the work.

CL.PH8. J.F. Cariñena and J. de Lucas, Integrability of Lie systems through Riccati equations, J. Nonl. Math. Phys. **18**, 29–54 (2011). IF = 0.543 (2011), (Q4 - 47/55 w Physics, Mathematical), Cytowania = 5(3). My contribution is about the 70% of the work.

CL.PH9. J.F. Cariñena, J. de Lucas and M.F. Rañada, A geometric approach to integrability of Abel differential equations, *Int. J. Theor. Phys.* **50**, 2114–2124 (2011). IF = 0.845 (2011), (Q3 - 48/84 w Physics, Multidisciplinary), Cytowania = 8(3). My contribution is about the 30% of the work.

CGL.PH10. J.F. Cariñena, J. Grabowski and J. de Lucas, Lie families: theory and applications, *J. Phys. A: Math. Theor.* **43**, 305201 (2010). IF =1.641 (2010), (Q2 - 17/54 w Physics, Mathematical), citations = 6(0). My contribution is about the 60% of the work.

FL.PH11. R. Flores, J. de Lucas and Y. Vorobiev, Phase splitting for periodic Lie systems, J. Phys A. **43**, 205208 (2010). IF = 1.641 (2010), (Q2 - 17/54 w Physics, Mathematical), citations = 7(1). My contribution is about the 20% of the work.

CL.PH12. J.F. Cariñena and J. de Lucas, Lie systems: theory, generalizations, and applications, Diss Math. **479**, 1–169 (2011). IF = 0.214, (Q4 - 279/289 w Mathematics), citations = 28(13). My contribution is about the 90% of the work.

AC.PH13. F. Avram, J.F. Cariñena and J. de Lucas, A Lie systems approach for the first passage-time of piecewise deterministic processes, w: *Modern Trends of Controlled Stochastic Processes: Theory and Applications*, Luniver Press, 2010, pp. 144–160. Swój układ oceniam na 50% of the work.

CL.PH14. J.F. Cariñena, J. de Lucas and M.F. Rañada, *Lie systems and integrability conditions for t-dependent frequency harmonics oscillators,* Int. J. Geom. Methods Mod. Phys. **7**, 289–310 (2010). IF = 1.612, (Q2 - 18/47 w PHYSICS, Mathematical), citations = 5(2) . My contribution is about the 60% of the work.

2. Other postdoctoral research

I have studied the general properties of differential equations, e.g. the infinite-dimensional jet formalism, Jacobi multipliers, non-local symmetries, etc. I applied my results to systems of mathematical of physical relevance, like types of an-harmonic oscillators [**CL.PH14**].

E1. J.F. Cariñena, J. de Lucas and M.F. Rañada, Jacobi multipliers, non-local symmetries, and nonlinear oscillators, *J. Math. Phys.* **56**, 063505 (2015). My contribution is about the 80% of the work.

E2. P.G. Estevez, M.L. Gandarias and J. de Lucas, Classical Lie symmetries and reductions of a nonisospectral Lax pair , J. Nonlinear Math. Phys. **18**, 51–60 (2011). My contribution is about the 30% of the work.

3. Editor in books

I was editor of the proceedings book: *Geometry of Jets and Fields - in honour of Professor Janusz Grabowski* (eds. K. Grabowska, M. Jóźwikowski, J. De Lucas and M. Rotkiewicz), Banach Center Publications **18**, Vol. 110, Warsaw, 2016.

4. Prizes, scholarships and research grants

- 2016 - Award in recognition of achievements affecting the development and prestige of the University of Warsaw, University of Warsaw.
- 2015 - Individual prize of third degree, Faculty of Physics, University of Warsaw.
- 2014 - Best teacher of the Faculty of Physics, University of Warsaw (UW Student council).
- 2013 - Didactic Award for outstanding classes and lectures, Summer term, University of Warsaw.
- 2011 - Postdoc fellowship for young researchers, IMPAN.
- 2010 - Special Award for Doctoral Theses, University of Zaragoza, year 2009/2010.
- 2010 - Postdoc fellowship for young researchers, IMPAN.
- 2009 - Postdoc fellowship for young researchers, IMPAN.

5. Other research activity

5.1. Longer research visits.

- August 6-September 6, 2016: Centre Recherches Mathématiques, CRM, University of Montreal, Canada.
- August 9-September 6, 2015: Centre Recherches Mathématiques, CRM, University of Montreal, Canada.
- August 28-September 29, 2012: University of Burgos, Burgos, Spain.
- October 1-December 31, 2011: University of Zaragoza, Zaragoza, Spain.

5.2. Chosen short (up to 3 weeks) research visits.

- École Normale Supérieure, Paris, France, February, 2017.
- University of Burgos, Burgos, Spain, December, 2016.
- University of Saragossa, Spain, June, 2015.
- Polytechnic University of Catalonia, Barcelona, Spain, December, 2015.
- University of Salamanca, Salamanca, Spain, May, 2010.
- University of Salamanca, Salamanca, Spain, September, 2010.

5.3. Conference and seminar lectures.

(1) Talk: *Control Lie systems and applications*, **Geometry of constraints and control**, IMPAN, Warsaw, Poland, October 25–31, 2009.
(2) Talk: *Lie families: theory and applications*, **IV International Summer School on Control, Geometry and Mechanics**, University of Santiago de Compostela, Santiago de Compostela, Spain, July 5–9, 2010.
(3) Poster:*Lie systems:theory, generalizations, and applications.*, **IV International Summer School on Control, Geometry and Mechanics**, University of Santiago de Compostela, Santiago de Compostela, Spain, July 5–9, 2010.
(4) Poster: *Superposition rules and second-order Riccati equations*, University of Porto, Porto, Portugal, September 6–9, 2010.
(5) Invited talk: *Teoria y aplicaciones de los sistemas de Lie y los esquemas de quasi-Lie*, Faculty of Mathematics, University of Salamanca, Salamanca, Spain, September 22, 2010.
(6) Talk: *Geometric structures and superposition rules*, **Centennial congress of the Spanish Royal Mathematical Society R.S.M.E. 2011**, Ávila, Spain, February 1–5, 2011.
(7) Talk: *Lie–Hamilton systems: theory and applications*, **5th Summer School on Geometry, Mechanics and Control**, La Cristalera, Miraflores de la Sierra, Spain, July 4–8, 2011.
(8) Invited talk: *Lie–Hamilton systems*, **Congreso de la Sociedad Matematica Mexicana**, University of San Luís de Potosí, San Luís de Potosí, Mexico, October 9–14, 2011.
(9) Invited talk: *Superposition rules and Lie systems*, University of Sonora, Hermosillo, Mexico, October 16, 2011.
(10) Invited talk: *Superposition rules and Lie systems*, University of Salamanca, Salamanca, Spain, May 15, 2012.
(11) Talk: *Mixed Superposition rules: theory and some applications.*, **XXI Fall workshop on Geometry and Physics**, University of Burgos, Burgos, Spain, August 30–September 1, 2012.
(12) Invited talk: *Lie-Hamilton Systems: theory and applications*, Faculty of Physics, University of Burgos, Burgos, Spain, September 2, 2012.
(13) Invited talk: *Mixed Superposition rules: theory and applications*, University of Burgos, Burgos, Spain, October 16, 2012.
(14) Talk: *Dirac–Lie systems: theory and applications*, **Thematic day on Dirac Structures and Applications**, University of Zaragoza, Zaragoza, Spain, February 2, 2013.
(15) Talk: *Dirac–Lie systems: theory and applications*, **I Meeting on Lie systems: theory, generalisations, and applications**, IMPAN, Warsaw, May 20–24, 2013.
(16) Invited Talk: *Dirac–Lie systems: theory and applications*, **XXIII Meeting on Differential Equations and Applications CEDYA**, University Jaume I, Castellon, Spain, September 9–13, 2013.
(17) Invited Talk: *Geometric structures and Lie systems: Theory and applications*, University of Burgos, Burgos, Spain, December 20, 2013.
(18) Talk: *New trends on Lie systems*, **II Meeting on Lie systems: theory, generalisations, and applications**, IMPAN, Poland, September 22–27, 2014.
(19) Invited talk: *Lie–Hamilton systems: theory and applications*, University of Łódz, Łódz, Poland, May 24, 2015.

(20) Talk: *Geometry and applications of Lie–Hamilton systems on the plane*, **III Meeting on Lie systems: theory, generalisations, and applications**, IMPAN, Warsaw, September 21–26, 2015.
(21) Invited talk: *k-symplectic Lie systems: theory and applications*, **III Young researchers conference of the RSME**, University of Murcia, Murcia, Spain, September 7–11, 2015.
(22) Talk: *A Lie systems approach to the Riccati hierarchy and PDEs*, **50th Sophus Lie Seminar**, Research and Conference Centre IMPAN, Będlewo, Poland, September 26–October 1, 2016.
(23) Invited talk: *Applications of Lie systems to Bernoulli-type equations*, University of Burgos, Burgos, Spain, December 16, 2016 r.

6. Attendance to conferences, courses, congresses.

(1) **School on Combinatorics and Control**, Benasque, Spain, from April 11–17, 2010.
(2) **XIII Winter Meeting on Geometry, Mechanics and Control Theory**, Saragossa, Spain, January 26–27, 2011.
(3) **XIII Thematic day on: Classic Field Theory**, Saragossa, Spain, January 28, 2011 r.
(4) **Geometry of Manifolds and Mathematical Physics**, Crakow, Poland, June 27–July 1, 2011.
(5) **III Iberoamerican meeting on Geometry, Mechanics and Control**, Salamanca, Spain, September 3–7, 2012.
(6) **XV Winter meeting on Mechanics, Geometry and Control**, Saragossa, Spain, January 30–31, 2013.
(7) **8th Symposium on Integrable Systems**, Department of Physics and Applied Mathematics, University of Łódź, Łódź, Poland, July 3–4, 2015.
(8) **Quantum Spacetime '16**, Zakopane, Poland, February 6–12, 2016.
(9) **Geometry of Jets and Fields**, Będlewo Conference Center, Będlewo, Poland, May 10–16, 2016.

7. Organization of conferences

- **I Meeting on Lie systems: theory, generalisations, and applications**, IMPAN, Warsaw, Poland, May 20–24, 2013.
- **II Meeting on Lie systems: theory, generalisations, and applications**, IMPAN, Warsaw, Poland, September 22–27, 2014.
- **III Meeting on Lie systems: theory, generalisations, and applications**, IMPAN, Warsaw, Poland, September 21–26, 2015.
- **Geometry of Fields and Jets**, Będlewo Conference Center, Będlewo, Poland, May 10–16, 2016.
- **50th Sophus Lie Seminar**, Research and conference centre IMPAN, Będlewo, Poland, September 26–October 1, 2016.

8. Activity as a referee, memberships, etc.

- Referee for projects of the Portuguese Foundation for Science and Technology.
- Referee for J. Phys. A, Adv. Math. Phys., Rep. Math. Phys., J. Dyn. Contr. Systems, Annals of Physics, Proc. Royal Soc. A, Int. J. Geom. Methods Mod. Physics, Advances in Mathematical Physics, Symmetry, EPJP and others.

9. International collaborations

I work with researchers from the Universities of Saragossa and Burgos (Spain), the Centre de Recherches Mathématiques of the University of Montreal (Canada), the Politechnic of Catalunya (Spain), IMPAN (Poland), ICMAT (Spain), Universidad Complutense de Madrid (Spain), etc. I previously worked with researchers from the S.N. Bose National Centre for Basic Sciences (India), the University of Hermosillo (Mexico) and the University of Pau (France).

I took part as a member of a HARMONIA project of international collaboration with title: '*Lie systems: theory, generalizations and applications*'. Among the duties of this collaboration, I accomplished the organization along with Prof. Janusz Grabowski of three conferences in Warsaw conglomerating researches from Poland and Spain.

10. Languages

- Spanish: mother tongue
- English: advanced level.
- Polish: advanced level.
- German: basic level.
- Russian: basic level.
- French: basic level.

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