Self-presentation

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4. Indication of achievement under Art. 16.2 of the Act of 14 March 2003 on Academic Degrees and Title and Degrees and Title in Art (Journal of Laws No. 65, item 595, as amended.): series of 5 publications

   a) title:

      Integrable equations in the theory of elasticity

   b) Publications of the series:

      [H1] W. K. Schief, **A. Szereszewski**, C. Rogers, *The Lamé equation in shell membrane theory*, J. Math. Phys. **48**, No. 7, 073510 (2007).

      [H2] W. K. Schief, **A. Szereszewski**, C. Rogers, *On shell membranes of Enneper type: generalized Dupin cyclides*, J. Phys. A, Math. Theor. **42**, No. 40, 404016 (2009).

      [H3] C. Rogers, **A. Szereszewski**, *A Bäcklund transformation for L-isothermic surfaces*, J. Phys. A, Math. Theor. **42**, No. 40, 404015 (2009).

      [H4] **A. Szereszewski**, *L-isothermic and L-minimal surfaces*, J. Phys. A, Math. Theor. **42**, No. 11, 115203 (2009).

      [H5] C. Rogers, W. K. Schief, **A. Szereszewski**, *Loop soliton interaction in an integrable nonlinear telegraphy model: reciprocal and Bäcklund transformations*, J. Phys. A, Math. Theor. **43**, No. 38, 385210 (2010).

   c) Discussion of the scientific goal of the above works and achieved results together with discussion of their applications.

1 Introduction

The governing equations in many of the physically important areas are intrinsically nonlinear. The distinguished amongst these are integrable equations characterised by the very specific mathematical properties. In addition, solitonic phenomena described by the equations of that kind have been observed in widely diverse areas in nature. In mathematical physics solitonic equations appear in theory of relativity, field theory, nonlinear optics, elasticity, hydrodynamics, magnetohydrodynamics, plasma physics and many others.

There is no widely accepted precise definition of complete integrability. Hitchin in [1] proposed that integrability of a system of differential equations should manifest itself through the following features: (i) the existence of many conserved quantities, (ii) the presence of algebraic geometry, (iii) the ability to give explicit solutions. In practice, the last feature is of great importance. Indeed, Bäcklund transformation has proved to be important tool in the generation of solutions to solitonic equations. Moreover, a nonlinear superposition principles provide purely algebraic algorithms for the generation of new solutions. The next property of integrable equations is the presence of Lax pair, which constitutes a linear system with compatibility conditions given by nonlinear equations under consideration.

At the present time almost all solitonic equations appear in physics. Moreover, the same equation occurs sometimes in completely different branches of physics. An adequate example is provided by the sin-Gordon equation [1]

$$\omega_{uv} = \frac{1}{\rho^2}\sin\omega \tag{sG}$$

examined by distinguished mathematicians such as Bour [2], Bianchi [3],[4] and Bäcklund [5] who analysed the pseudospherical surfaces. In the physical context sin-Gordon equation appears in crystal dislocation theory [6], study of the tunnelling in superconductors [7], propagation of ultrashort light pulses [8] and nonlinear theory of particle interaction [9]. One of the most important facts is that a nonlinear superposition principle, now termed a permutability theorem is not only a mathematical curiosity but allows to generate solutions of considerable physical importance. Lamb [10] applied the permutability theorem for the above mentioned sin-Gordon equation in the theory of ultrashort optical propagation. In the context of solitonic solutions of (sG) he analysed the decomposition of $2N\pi$ light pulses into N stable 2π pulses. This decomposition phenomena was observed experimentally in rubidium vapour by Gibbs and Slusher [11].

Until recently the nonlinear equations which describe solitons have typically been derived by approximation and expansion methods. Obvious example is the famous Korteweg-de Vries [12] equation

$$u_t + 6uu_x + u_{xxx} = 0, \tag{KdV}$$

admitting solution $u(x,t) = \frac{v}{2}\cosh^{-2}\left[\frac{\sqrt{v}}{2}(x - vt)\right]$ which models a solitary long wave travelling with constant speed v in a rectangular channel. The (KdV) equation arises from the hydrodynamic equations in the case of irrotational 2-dimensional motion of an incompressible inviscid fluid, bounded above by a free surface and below by a rigid horizontal plane. To derive (KdV) it is necessary to make many approximations, e.g. the relevant length scale in the direction of movement is much longer than the fluid depth, the wave amplitudes are small, etc. Other solitonic equations are obtained in a similar manner. The prime exception is the integrable Ernst equation

$$(\mathcal{E} + \bar{\mathcal{E}})\left(\mathcal{E}_{\rho\rho} + \frac{1}{\rho}\mathcal{E}_\rho + \mathcal{E}_{zz}\right) = 2\left(\mathcal{E}_\rho^2 + \mathcal{E}_z^2\right), \qquad \mathcal{E} = \mathcal{E}(\rho, z) \tag{Er}$$

which arises out of Einstein's vacuum equations $R_{\mu\nu} = 0$ in general relativity in the case when the spacetime admits two commuting Killing vectors which are hypersurface-orthogonal. The integrable properties of (Er) were discovered by Maison in late seventies of the XX century [14],[15]. It is not

[1]For notational convenience, the partial derivtives are denoted by the proper indices $u, v, x, y, t, \alpha, \beta \ldots$, e.g. $\omega_{uv} = \frac{\partial^2\omega}{\partial u \partial v}$

surprising, because Ernst equation (Er) nowadays may be viewed as a so-called Lelieuvre system [16],[17] for 2-dimensional surfaces embedded in 3-dimensional Minkowski space [18],[19]. Nevertheless, it is an open question whether the full Einstein's equations are integrable. We still do not know the answer but the linear system which compatibility conditions reduce to vacuum Einstein's equations with cosmological constant $R_{\mu\nu} = \Lambda g_{\mu\nu}$ is known. It is provided by linear Rarita-Schwinger equation $\nabla_{A\dot{B}}\Psi^{AC\dot{D}} = 0$ governing the spinor field $\Psi^{AB\dot{C}}$ which seems to describe spin-3/2 particles.

The solitonic equations which we are concerned in series of articles [H1]-[H5] were all discovered by the reduction techniques in the similar manner as the Ernst equation.

1.1 Geometry of 2-dimensional surfaces embedded in 3-dimensional Euclidean space

There exists a strong connection between integrable equations and differential geometry of 2-dimensional surfaces embedded in 3-dimensional flat space. Soliton equations appear in this context as the reductions of compatibility conditions of system describing embedding, i.e. Gauss-Mainardi-Codazzi equations.

Below we give a short description of sin-Gordon equation (sG) which is the most prominent example among soliton equations. This also allows us to introduce notation and geometric quantities which will be used later.

The sin-Gordon equation (sG) appears naturally in analysis of 2-dimensional surfaces with constant Gauss curvature embedded in 3-dimensional Euclidean space $\mathbb{E}^3$. The geometry of 2-dimensional surfaces plays an important role in discussed articles.

Let $\Sigma \subset \mathbb{E}^3$ be a 2-dimensional surface,

$$\Sigma : (u, v) \mapsto \boldsymbol{r}(u, v) \in \mathbb{E}^3 \tag{1}$$

parametrized in terms of local coordinates $u, v \in \mathbb{R}$. In a generic point p, the vectors $\boldsymbol{r}_u$ oraz $\boldsymbol{r}_v$ are tangential to Σ at p and, at such points,

$$\boldsymbol{N} = \frac{\boldsymbol{r}_u \times \boldsymbol{r}_v}{|\boldsymbol{r}_u \times \boldsymbol{r}_v|} \tag{2}$$

determines the unit normal to Σ. According to Bonnet theorem, the surface Σ is defined up to its position in space $\mathbb{E}^3$ by the first fundamental form

$$g_I = d\boldsymbol{r} \cdot d\boldsymbol{r} = E\, du^2 + 2F\, dudv + G\, dv^2 \tag{3}$$

and second fundamental form

$$g_{II} = -d\boldsymbol{r} \cdot d\boldsymbol{N} = e\, du^2 + 2f\, dudv + g\, dv^2, \tag{4}$$

where $\cdot$ denotes the usual scalar product in $\mathbb{E}^3$. It is known [17] that there always exist local coordinates (α, β) (called curvature coordinates) in which both fundamental forms are diagonal, namely

$$g_I = A_1^2\, d\alpha^2 + A_2^2\, d\beta^2, \qquad g_{II} = \kappa_1 A_1^2\, d\alpha^2 + \kappa_2 A_2^2\, d\beta^2. \tag{5}$$

Functions A_1, A_2, κ_1, κ_2 are not arbitrary, but satisfy the compatibility conditions (given in (9)). The third fundamental form

$$g_{III} = d\boldsymbol{N} \cdot d\boldsymbol{N} \tag{6}$$

defines the metric on a sphere given by Gauss map which assigns to every point $p \in \Sigma$ a normal vector $\boldsymbol{N}$ in this point. It is noted that the third fundamental form g_{III} is not an auxiliary geometric quantity necessary to describe the embedding of $\Sigma \subset \mathbb{E}^3$. It can be easily checked that $g_{III} = 2\mathcal{H}g_{II} - \mathcal{K}g_I$, where $\mathcal{H} = \frac{1}{2}(\kappa_1 + \kappa_2)$ is mean curvature and $\mathcal{K} = \kappa_1\kappa_2$ is Gauss curvature. Nevertheless, g_{III} will play the fundamental role in succeeding sections.

As it was mentioned before, the sin-Gordon equation (sG) appears as a compatibility condition of a system associated with psedosphere which is a surface with negative Gauss curvature $\mathcal{K} = -1/\rho^2 = const.$ If now Σ is pseudospherical surface parametrized in terms of (u, v) such that both fundamental forms read $g_I = du^2 + 2\cos\omega\, dudv + dv^2$, $g_{II} = \frac{2}{\rho}\sin\omega\, dudv$, then function ω necessarily fulfils sin-Gordon equation (sG). Moreover, Gauss and Weingarten equations are linear systems which are compatible modulo (sG). The Bäcklund transformation and permutability theorem for sin-Gordon equation can now be obtained by means of the geometric methods.

2 Integrable equations in the theory of elasticity

The series of articles [H1]-[H5] constitutes a contribution to the project *Hidden Geometric Structure in Nonlinear Physical Systems*, which aims to searching, recognition and description of integrable systems obtaining from mathematical physics. It is emphasized that all integrable systems considered as part of this project are dervied without using approximation or expansion methods. They are obtained by the suitable reductions, usually by imposing the additional geometric constraints. The equations analysed in series of articles have been derived exactly in this way. In [H1] the system of equilibrium equations for membranes under the assumption that the internal stress distributions are not uniquely determined by their shape has been considered. A detailed study of this case leads to the algebraic and geometric characterization of these surfaces [H1]. It turns out that the surfaces which correspond to such membranes are necessarily L-isothermic, more precisely they are L-isothermic surfaces subject to additional constraint. The algebraic description introduced in [H1] allowed to distinguish and examine the special type of L-isothermic surfaces on which the lines of curvatures are planar (they are subclass of so called Enneper surfaces). The prominent examples of the latter are generalized Dupin cyclides [H2]. The method of description of L-isothermic surfaces has proved to be very successful and it was subsequently developed in [H3]-[H4]. The main advantage of this method being that the problem is governed by complex nonhomogeneous linear equation. To the author knowledge this approach does not appear to be widely known. In [H3] this method has been used to analyse Bäcklund transformation for L-isothermic surfaces. It turns out that this transformation corresponds to Darboux transformation of linear equation associated with a surface. The main result of [H4] is a construction of Weierstrass representation for special subclass of L-isothermic surfaces, which are called L-minimal. The developed approach proved to be very fruitful in this case. In [H5] the telegraphy equation has been analysed. It is shown to be related to the equation describing a propagation of stress in a class of model ideally hard inhomogeneous elastic materials with the special equation of state. The construction of Bäcklund transformation shows the integrable character of the equation in question. The main result of [H5] was the discovery of a novel 'exchange particle' phenomenon. This was made by performing the detailed analysis of a two-loop soliton solution.

It is worth mentioning that there are no general algorithms for obtaining integrable systems. Therefore, any new solitonic equation is welcome.

The remaining part of this section is concerned with a detailed consideration and description of results obtained in series of articles [H1]-[H5]. These results (with suitable references to formulae) are summarized in the last section.

We start with the short description of classical shell and membrane equilibrium equations [21]. It will be subsequently shown that the appropriate reduction of these equations leads to complete integrable system.

In the theory of elasticity, the term shell is applied to bodies bounded by two curved surfaces Σ_-, Σ_+ (see figure 1). It is also assumed that the distance between these surfaces is small in comparison with the other dimensions and that all properties of shell can be described by a 2-dimensional middle surface placed between Σ_- and Σ_+.

Figure 1: A model of shell.

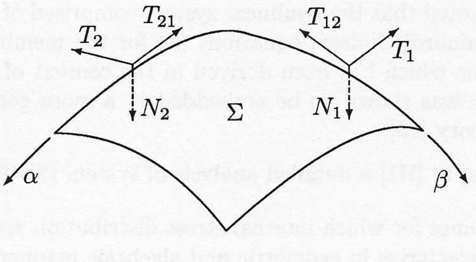

Figure 2: Forces acting on the surface Σ.

In the study of the theory of elasticity, the investigation of the geometrical properties of the deformations of shells are followed by that of the corresponding stresses. It is then convenient to introduce instead of these stresses, statically equivalent forces and bending moments. The corresponding forces[2](per unit length) T_1, T_{12}, N_1, T_2, T_{21}, N_2 acting on Σ are depicted in figure 2. In addition there exist four momenta which, together with the above mentioned forces, constitute the system of six classical shell equilibrium equations. We are going to consider the shell membrane theory which allows us to reject the momenta from the latter system. This reduction leads to $N_1 = N_2 = 0$, $T_{12} = T_{21} := S$ and the equilibrium equations read

$$
\begin{aligned}
(A_2T_1)_\alpha + (A_1S)_\beta + A_{1\beta}S - A_{2\alpha}T_2 + A_1A_2p_1 &= 0,\\
(A_1T_2)_\beta + (A_2S)_\alpha + A_{2\alpha}S - A_{1\beta}T_1 + A_1A_2p_2 &= 0,\\
\kappa_1T_1 + \kappa_2T_2 + p_3 &= 0,
\end{aligned} \tag{7}
$$

where (p_1, p_2, p_3) are coefficients of surface loading, i.e. external pressure acting on the membrane. For any given membrane geometry, which means that fundamental forms g_I, g_{II} are known, and prescribed external surface loading, the membrane equilibrium equations (7) constitute a well-determined linear system for stresses T_1, T_2 and S. It is recalled that the functions A_1, A_2, κ_1 and κ_2 have to obey Gauss-Mainardi-Codazzi equations (9).

In articles [H1]-[H2] the equilibrium system (7) in the absence of shear ($S = 0$) with constant purely normal loading ($p_3 = Z = const.$) has been considered. The system reduces in this case to

$$
\begin{aligned}
T_{1\alpha} + (\log A_2)_\alpha(T_1 - T_2) &= 0,\\
T_{2\beta} + (\log A_1)_\beta(T_2 - T_1) &= 0,\\
\kappa_1T_1 + \kappa_2T_2 + Z &= 0,
\end{aligned} \tag{8}
$$

where functions A_1, A_2, κ_1 i κ_2 satisfy the Gauss-Mainardi-Codazzi equations (GMC)

$$
\begin{aligned}
\kappa_{2\alpha} + (\log A_2)_\alpha(\kappa_2 - \kappa_1) &= 0,\\
\kappa_{1\beta} + (\log A_1)_\beta(\kappa_1 - \kappa_2) &= 0,\\
\left(\frac{A_{2\alpha}}{A_1}\right)_\alpha + \left(\frac{A_{1\beta}}{A_2}\right)_\beta + \kappa_1\kappa_2A_1A_2 &= 0.
\end{aligned} \tag{9}
$$

The equations (8) and (9) form a well-determined coupled nonlinear system which indicates that the shape of a membrane in equilibrium is restricted. In the simplest case of a constant stresses $T_1 = T_2 = c = const.$ the equilibrium equations (8) reduce to Young-Laplace equation

$$
\mathcal{H} = \kappa_1 + \kappa_2 = -c^{-1}Z = const.,
$$

which clearly shows that the membrane is of constant mean curvature while when $p_3 = 0$ then the membrane is minimal.

[2]Index 1 corresponds to coordinate α, index 2 to β, while 3 corresponds to the direction normal to the surface

It should be noted that the nonlinear system comprised of the equilibrium equations (8) augmented by the Gauss-Mainardi-Codazzi equations (9) for the membranes may be located in a large class of integrable systems which has been derived in the context of so-called O surfaces [22]. Moreover, the nonlinear system was shown to be embedded in a more general elastic shell system of revelance to liquid crystal theory [23].

By performing in [H1] a detailed analysis of system (8)-(9) (for $Z \neq 0$) it has been shown that

i) the membranes for which internal stress distribution are not uniquely determined by their shape may be characterize in geometric and algebraic manner,

ii) many examples of the membranes from i) can be explicitly constructed

In generic case the stress resultans T_1 and T_2 can be found by the following procedure. By differentiation $(8)_3$ with respect to α and β we obtain two additional equations from which we get $T_{1\beta}$ and $T_{2\alpha}$. Having all differentials of T_i (remaning differentials are given in $(8)_{1,2}$) we can calculate the following compatibility conditions

$$\mu T_1 + \nu T_2 = 0, \tag{10}$$

where functions μ, ν are expressed in terms of A_1, A_2, κ_1, κ_2. Unless both μ and ν are nonvanishing the pair $(8)_3$ i (10) may be solved for T_1 and T_2. If $\mu = \nu = 0$ there exists a one-parameter family of stress distribution. This case is described by the following conditions

$$\begin{gathered}\left[\log\left(\frac{A_1\kappa_1}{A_2\kappa_2}\right)\right]_{\alpha\beta} = 0, \\ \log(\kappa_1\kappa_2)_{\alpha\beta} + (\log A_1)_\beta(\log\kappa_1)_\alpha + (\log A_2)_\alpha(\log\kappa_2)_\beta - (\log\kappa_1)_\alpha(\log\kappa_2)_\beta = 0.\end{gathered} \tag{11}$$

A detailed analysis shows in fact that the condition $(11)_2$ is a consequence of $(11)_1$, $(8)_3$ and (10). This means that both constraints (11) may be replaced by $(11)_1$ which in turn indicates that the third fundamental form g_{III} is conformally flat when written in appriopriate rescaled curvature coordinates (α, β), namely

$$g_{III} = e^{2\theta}\left(d\alpha^2 + d\beta^2\right). \tag{12}$$

Hence we obtain te following [H1]

Fact 1. *The stress distribution of a membrane is uniquely determined by its geometry unless the third fundamental form of the membrane is conformally flat with respect to suitably scaled curvature coordinates.*

There is also a possibility of geometric characterization in terms of Combescure transformation of the membrane to a minimal surface.

It turns out that the membranes under consideration can be also characterized algebraically. The following fact holds:

Fact 2. *Membrane geometries for which there exists a one-parameter family of stress resultants T_1 and T_2 are determined by common solutions of the Liouville and the Moutard equations*

$$\theta_{\alpha\alpha} + \theta_{\beta\beta} = -e^{2\theta}, \qquad \left(e^{-\theta}\right)_{\alpha\beta} = -\frac{f'g'}{4(f+g)^2}e^{-\theta}, \tag{13}$$

where $f = f(\alpha)$, $g = g(\beta)$.

It is noted that functions f and g are not arbitrary. The combatibility of both equations (13) leads to

$$\left(\partial_\alpha^2 + \partial_\beta^2\right)\frac{f'g'}{(f+g)^2} = 0, \tag{14}$$

which means that in generic case f and g are elliptic functions. The Fact 2 can be reformulated in such a way that the nonlinear equations (13) are replaced by a linear one and explicit expression for position vector $\boldsymbol{r}$ is available. The first step to achieve that is to integrate the Gauss-Weingarten system

$$\begin{pmatrix}\boldsymbol{X}\\ \boldsymbol{Y}\\ \boldsymbol{N}\end{pmatrix}_\alpha = \begin{pmatrix}0 & -\theta_\beta & -e^\theta\\ \theta_\beta & 0 & 0\\ e^\theta & 0 & 0\end{pmatrix}\begin{pmatrix}\boldsymbol{X}\\ \boldsymbol{Y}\\ \boldsymbol{N}\end{pmatrix}, \qquad \begin{pmatrix}\boldsymbol{X}\\ \boldsymbol{Y}\\ \boldsymbol{N}\end{pmatrix}_\beta = \begin{pmatrix}0 & \theta_\alpha & 0\\ -\theta_\alpha & 0 & -e^\theta\\ 0 & e^\theta & 0\end{pmatrix}\begin{pmatrix}\boldsymbol{X}\\ \boldsymbol{Y}\\ \boldsymbol{N}\end{pmatrix}, \tag{15}$$

where $\{\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{N}\}$ is an orthonormal frame with $\boldsymbol{N} = \boldsymbol{X} \times \boldsymbol{Y}$. The latter can be solved and solution may be expressed in terms of holomorphic function $\rho(z)$, such that

$$e^\theta = \frac{2|\rho'|}{1+\rho\bar{\rho}}, \qquad \rho = \rho(z), \quad z = \alpha + i\beta \tag{16}$$

fulfils Liouvile equation $(13)_1$. Now, the direct method of finding the position vector r requires to integrate the following system

$$r_\alpha = A_1 \boldsymbol{X}, \qquad r_\beta = A_2 \boldsymbol{Y}, \tag{17}$$

which are quite difficult to solve (e.g. we need to calculate A_1 and A_2 first). Surprisingly, integration of the latter may be replaced by differentiation. To show that let us introduce the new quantity

$$b = \boldsymbol{r} \cdot \boldsymbol{N} \tag{18}$$

which measures the distance from the origin in $\mathbb{E}^3$ to the tangent plane to the membrane Σ at the point $\boldsymbol{r}$. On use of b, the equations (9), (15), (17) may be rewritten which allows us to proof the following [H1]

Fact 3. *The position vector of the membrane with undetermined stresses reads*

$$\boldsymbol{r} = e^{-\theta} b_{0\alpha} \boldsymbol{X} + e^{-\theta} b_{0\beta} \boldsymbol{Y} + (b_0 + \mathfrak{b}) \boldsymbol{N}, \tag{19}$$

where $\mathfrak{b} = const.$, *function* b_0 *is given by*

$$b_0 = \frac{2T_0}{|\Phi_1|^2 + |\Phi_2|^2} \tag{20}$$

and T_0 *is any particular real solution of the inhomogeneous Lamé equation*

$$T_{zz} + \left(\frac{1}{4}\wp(z) + C\right)T = \frac{P}{4}, \qquad C = const. \tag{21}$$

Here, function $P(\alpha, \beta)$ *is in the form* $P^2 = f(\alpha) + g(\beta)$, *functions* $\Phi_1(z)$, $\Phi_2(z)$ *fulfil the homogeneous version of (21) and orthonormal frame* $(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{N})$ *is defined by*

$$\begin{aligned} \boldsymbol{X} + i\boldsymbol{Y} &= \frac{1}{|\Phi_1|^2 + |\Phi_2|^2}\begin{pmatrix}\Phi_2^2 - \Phi_1^2\\ i(\Phi_1^2 + \Phi_2^2)\\ 2\Phi_1\Phi_2\end{pmatrix}, \\ \boldsymbol{N} &= -\frac{1}{|\Phi_1|^2 + |\Phi_2|^2}\begin{pmatrix}\Phi_1\bar{\Phi}_2 + \bar{\Phi}_1\Phi_2\\ i(\bar{\Phi}_1\Phi_2 - \Phi_1\bar{\Phi}_2)\\ |\Phi_1|^2 - |\Phi_2|^2\end{pmatrix}. \end{aligned} \tag{22}$$

The above fact is one of the main results of [H1].

The function $\wp$ which appears in potential $\mathsf{U} = \frac{1}{4}\wp + C$ of equation (21) is a Weierstrass elliptic function. It is worth mentioning that according to many mathematicians the presence of elliptic functions suggests the integrable character of equations discussed here (the elliptic funcion $\wp$ occurs sometimes in algebraic geometry, e.g. in problems associated with an algebraic curve of third order $y^2 = 4x^3 - g_2 x - g_3$).

The Fact 3 is of considerable importance in the sequel. It will be shown that it allows to construct many interesting examples including so-called generalized Dupin cyclides [H2]. Moreover, the position vector (19) does not describe a single surface but the set of parallel surfaces. This clearly indicates that if $\boldsymbol{r}$ is a position vector of the surface Σ then the parallel surface $\Sigma^{||}$ with a position vector $\boldsymbol{r}^{||} = \boldsymbol{r} + \mathfrak{b}\boldsymbol{N}$ is also a solution of the problem. To summarise, the solution appears always as a set of parallel surfaces. It is noted that this result could be also obtained at the beginning of analysis of equations (8)-(9).

The mathematical approach contained in Fact 3 can be used to examine more general surfaces. The dependence of potential U on Weierstrass function $\wp$ is a consequence of condition $(11)_2$. If we consider the surfaces with conformally flat third fundamental form then it is necessary to relax the conditions and assume that only $(11)_1$ is satisfied. In this case we obtain the linear equation of type (21) with arbitrary complex potential U. The surfaces of that kind were analysed in [H3]-[H4].

The case when the discriminant $g_2^3 - 27g_3^2 = 0$ and the Weierstrass $\wp(z; g_2, g_3)$ functions reduce to elementary functions was considered in detail. Accordingly, we have

$$\wp = \frac{1}{z^2}, \qquad \wp = c^2\left(\frac{1}{\sin^2(cz)} - \frac{1}{3}\right), \qquad \wp = c^2\left(\frac{1}{\sinh^2(cz)} + \frac{1}{3}\right), \tag{23}$$

where $c = const.$ Explicit examples of surfaces related to the above degenerate Weierstrass $\wp$ functions were constructed in [H1]. The solutions T_0 and Φ_1, Φ_2 of equation (21) were found for all functions in (23). Two surfaces of this kind are depicted in figure 3).

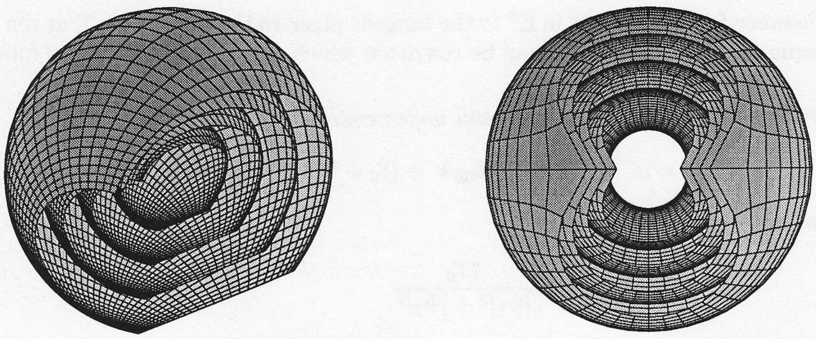

Figure 3: Sets of parallel surfaces associated with functions $(23)_1$ and $(23)_2$.

The important class of surfaces for which equation $(13)_2$ reduces to

$$(e^{-\theta})_{\alpha\beta} = 0 \tag{24}$$

has been analyzed in [H2]. The condition (24) arises by setting at lest one of the functions f or g equal to constant. On the other hand, this condition is equivalent to assuming that all lines of curvature on a surface are planar. Such surfaces are special cases of Enneper surfaces [24] on which there exists one family of planar lines of curvature. The physicial meaning is that the stress resultants tangent to the given curvature line $\alpha = const.$ or $\beta = const.$ lie on the same plane. The mathematical consequence of (24) is that potential U is constant and function P depends only on a single variable. Hence, this case is governed by the following inhomogeneous linear equation

$$T_{zz} + CT = \frac{P(\alpha)}{4}, \qquad C = const. \tag{25}$$

Geometrically, we are concerned with subclass of canal surfaces which are the envelopes of a one-parameter family of spheres of radius r with centers located on a curve γ. In our case (for $C > 0$) the curve is given by

$$\gamma = (-a_0 F_1(\alpha), F_2(\alpha), 0)^{\mathsf{T}}, \tag{26}$$

where

$$F_1(\alpha) = \int P(\alpha)\sin\alpha\, d\alpha, \quad F_2(\alpha) = \int P(\alpha)\cos\alpha\, d\alpha, \quad r = |c_0 F_1(\alpha) + \mu|, \quad \mu = const. \tag{27}$$

Constants a_0, c_0 obey the relation $a_0^2 - c_0^2 = 1$.

By introducing a new local coordinate $u = u(\beta)$ the position vector of the surface yields

$$r(\alpha, u) = -\psi e^{\varphi} \begin{pmatrix} \dfrac{1}{c_0} \\ \sin\alpha\cos u \\ \sin u \end{pmatrix} + \begin{pmatrix} \dfrac{a_0}{c_0}\mu \\ F_2 \\ 0 \end{pmatrix}, \tag{28}$$

where $\psi(\alpha) = c_0 F_1 + \mu$, $e^{-\varphi} = a_0 - c_0 \cos\alpha\cos u$. The surfaces (28) are called the generalised Dupin cyclides [H1]-[H2]. Dupin cyclides are special subcases of (28) and can be obtained if $P = const.$ Usually they are defined by the property that all lines of curvature thereon are circles. The Dupin cyclides were introduced in 1882 by the mathematician and naval architect Dupin [25]. They were also extensively investigated in the nineteenth century by Maxwell [26] and Cayley [27]. In recent years, it was demonstrated that Dupin cyclides could be even used in computer-aided design. They also arise in connection with integrable Hamiltonian systems of hydrodynamic type [28]. One of the most important property is that they constitute the isothermic surfaces which frequently appear in the context of solitonic systems [29]. It is recalled that on any isothermic surface there always exists a coordinate system in which the first fundamental form is conformally flat while the second one is diagonal.

Any planar curve may be (locally) obtained from (26) by a suitable choice of the parameter α and the function $P(\alpha)$ in (27). For example, the function $P(\alpha) = -\frac{3}{2}\sin 2\alpha$ generates (scaled) astroid $\gamma = (a_0 \sin^3\alpha, \cos^3\alpha)^{\mathsf{T}}$, $\alpha \in [0, 2\pi)$). The sets of parallel surfaces associated with an ellipse and Talbot's curve are displayed in figure 4.

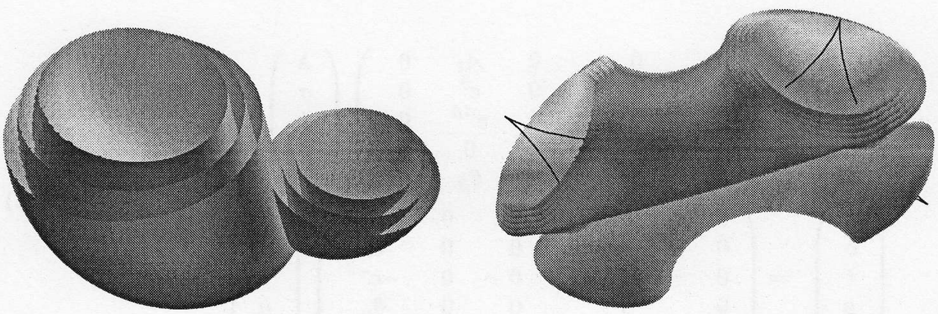

Figure 4: The families of parallel surfaces generated by an ellipse and Talbot's curve.

It is worth mentioning that the class of generalized Dupin cyclides which correspond to negative values of C are also of great interest. It consists of minimal surfaces with planar lines of curvatures. Although not all minimal surfaces of that kind are generalized Dupin cyclides, the following alternative characterization of them (for any C) can be made [H2]:

Fact 4. *Membranes on which the lines of curvature (and therefore the lines of principal stress) are planar are the membranes which may be mapped via a Combescure transformation to minimal surfaces with planar lines of curvature. All membranes of the latter class of minimal surfaces may be generated in this way.*

The class of minimal surfaces which may be mapped to the generalized Dupin cyclides (28) is given by

$$r_{\min} = \begin{pmatrix} \sqrt{1 - d_0^2}\cos\alpha\cosh\beta \\ \sin\alpha\cosh\beta - d_0\alpha \\ d_0\cos\alpha\sinh\beta - \beta \end{pmatrix}, \tag{29}$$

where $d_0 = c_0/a_o$ and $u = \arccos(\mathrm{sech}\beta)$.

The analysis of the nonlinear system of equilibrium equations developed so far allows to calculate the stress resultants

$$T_1 = -\frac{Z}{2\kappa_2}\Big(1 - \frac{g(\beta)}{A_2^2}\Big) + \frac{\varepsilon}{\kappa_2 A_2^2}, \qquad T_2 = -\frac{Z}{2\kappa_1}\Big(1 - \frac{f(\alpha)}{A_1^2}\Big) - \frac{\varepsilon}{\kappa_1 A_1^2}, \tag{30}$$

where f and g are elliptic functions in general and $\varepsilon = const.$ All intrinsic geometric quantities required to calculate T_i can be derived analytically.

The next evidence which convince us of solitonic behaviour of the equations is the existence of so-called Lax pair, that is, the linear system with compatibility conditions encoded in (8)-(9). However, the system (8)-(9) possesses another important property. The equations $(8)_{1,2}$ are identical with the Gauss-Mainardi-Codazzi equations $(9)_{1,2}$ if the correspondence $(T_1, T_2) \leftrightarrow (\kappa_2, \kappa_1)$ is made. This is the principal feature of the system which makes it integrable. Moreover, this similarity allows to rewrite the Lax pair system of equation which includes Combescure transforms of Σ to minimal surface in a compact form.

All surfaces considered so far are L-isothermic which means that they possess conformally flat third fundamental form g_{III}. There exists a Bäcklund transformation for such surfaces:

Proposition (A Bäcklund transformation for L-isothermic surfaces). Let $\boldsymbol{r}$ be the position vector of an L-isothermic surface Σ. Then, a second L-isothermic surface $\tilde{\Sigma}$ is given by

$$\tilde{\boldsymbol{r}} = \boldsymbol{r} - \frac{\lambda}{m\sigma t}\Big(\mu \boldsymbol{X} + \nu \boldsymbol{Y} + \sigma \boldsymbol{N}\Big), \tag{31}$$

where m is a real "Bäcklund parameter" and λ, σ, t, μ, ν are "eigenfunctions" of the compatible linear system

$$\begin{aligned}
\begin{pmatrix} \lambda \\ \sigma \\ t \\ \mu \\ \nu \end{pmatrix}_{\alpha} &= \begin{pmatrix} 0 & 0 & 0 & A_1 & 0 \\ 0 & 0 & 0 & e^{\theta} & 0 \\ 0 & 0 & 0 & e^{-\theta} & 0 \\ 0 & me^{-\theta} - e^{\theta} & me^{\theta} & 0 & -\theta_\beta \\ 0 & 0 & 0 & \theta_\beta & 0 \end{pmatrix} \begin{pmatrix} \lambda \\ \sigma \\ t \\ \mu \\ \nu \end{pmatrix}, \\
\begin{pmatrix} \lambda \\ \sigma \\ t \\ \mu \\ \nu \end{pmatrix}_{\beta} &= \begin{pmatrix} 0 & 0 & 0 & 0 & A_2 \\ 0 & 0 & 0 & 0 & e^{\theta} \\ 0 & 0 & 0 & 0 & -e^{-\theta} \\ 0 & 0 & 0 & 0 & \theta_\alpha \\ 0 & -me^{-\theta} - e^{\theta} & me^{\theta} & -\theta_\alpha & 0 \end{pmatrix} \begin{pmatrix} \lambda \\ \sigma \\ t \\ \mu \\ \nu \end{pmatrix},
\end{aligned} \tag{32}$$

which satisfy the admissible constraint $\mu^2 + \nu^2 + \sigma^2 = 2m\sigma t$.

The above transformation was examined by Bianchi [30] and Eisenhart [31]. It is recalled that Fact 3 allows to construct L-isothermic surfaces by the solutions of nonhomogeneous linear equation

$$T_{zz} + \mathsf{U}T = \frac{1}{4}P, \qquad z = \alpha + i\beta, \tag{33}$$

where $\mathsf{U}(z)$ is a complex potential and a real function $P(\alpha, \beta)$ satisfies Moutard equation[3] $P_{\alpha\beta} = 2(\mathrm{Im}\mathsf{U})P$. The following problem arises

> Let U be a potential associated with L-isothemic surface Σ. What potential $\tilde{\mathsf{U}}$ is related to a surface $\tilde{\Sigma}$ which is obtained from Σ via the Bäcklund transformation (31)-(32)?

It turns out [H3] that the Bäcklund transformation (31)-(32) corresponds to the well-known Darboux transformation [32] of the potential U, namely

$$\tilde{\mathsf{U}} = \mathsf{U} + 2\big(\log \hat{\sigma}\big)_{zz}, \tag{34}$$

[3]Previously, in Fact 3, Moutard equation was fulfilled due to the condition $P^2 = f + g$.

where a real function $\hat{\sigma} = 2e^{-\theta}\sigma$ satisfies modified linear equation with shifted potential

$$\hat{\sigma}_{zz} + \mathsf{U}\hat{\sigma} = \frac{m}{2}\hat{\sigma}. \tag{35}$$

The presence of the Darboux transformation frequently appearing in the context of integrable systems is natural. It allows to express the functions σ, t, μ, ν satisfying (32) in terms of solutions of homogeneous linear equations with potentials U and $\tilde{\mathsf{U}}$. Moreover, the Bäcklund transformation from Proposition leads to permutability theorem for the L-isothermic surfaces. We will see here that the powerful solitonic method provides the new solutions of nonlinear equations. Accordingly, let $\boldsymbol{r}$ be a position vector of an L-isothermic surface and $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ be two Bäcklund transforms of $\boldsymbol{r}$ with parameters m_1 and m_2. The above mentioned permutability theorem allows construction of a new L-isothermic surface Σ_{12} from $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ in purely algebraic manner (without any integration). Its position vector reads [H3]

$$\mathbf{R} = \begin{vmatrix} \boldsymbol{r} & \lambda_1 & \lambda_2 \\ \boldsymbol{r}_1 & j_1 & j_3 \\ \boldsymbol{r}_2 & j_4 & j_2 \end{vmatrix} \Bigg/ \begin{vmatrix} j_1 & j_3 \\ j_4 & j_2 \end{vmatrix}, \tag{36}$$

where functions j_i $(i = 1, \ldots, 4)$ may be expressed in terms of solutions of (32) with constants m_1 oraz m_2. In article [H3] the Bäcklund transformation applied to generalized Dupin cyclides has been analysed extensively and the explicit solutions to (32) have been constructed. The method outlined so far suggests that there should potential U_{12} associated to the surface Σ_{12}. Indeed, the calculation shows that $\mathsf{U}_{12} = \mathsf{U} + 2\partial_{zz}\left(S_1 S_{2z} - S_{1z} S_2\right)$, where S_1 and S_2 satisfy (35) with m_1 and m_2 respecively.

The fact that the linear equation (33) is connected with geometry of L-isothermic surfaces leads to another interesting question. Let the solution T_0 of nonhomogeneous equation (33) together with functons Φ_1, Φ_2 satisfying homogeneous version of this equation be given.

> What are the geometric transformations which correspond to a linear combination of solutions Φ_1, Φ_2 and addition the homogeneous solution to nonhomoheneous T_0?

The transformations mentioned above:

$$\begin{gathered} \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} \mapsto \begin{pmatrix} \Phi_1' \\ \Phi_2' \end{pmatrix} = \mathbf{S} \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}, \qquad \mathbf{S} = \begin{pmatrix} s_1 & s_2 \\ s_3 & s_4 \end{pmatrix} \in SL(2, \mathbb{C}), \\ T_0 \mapsto T_0' = T_0 + a_1|\Phi_1|^2 + a_2\Phi_1\bar{\Phi}_2 + \bar{a}_2\bar{\Phi}_1\Phi_2 + a_3|\Phi_2|^2, \quad a_1, a_3 \in \mathbb{R},\ a_2 \in \mathbb{C} \end{gathered} \tag{37}$$

depend on ten real parameters.

It is not difficult to answer the question. It is well-known that L-isothermic surfaces appear naturally in Laguerre geometry [33] where spheres and planes in $\mathbb{E}^3$ are fundamental objects. The group which leaves the contact between these objects invariant is Laguerre group and it is isomorphic to 10-dimensional Poincaré group $\mathbb{R}^4 \rtimes SO(1,3)$. The Laguerre transformations can be described by considering 6-dimensional space $\mathbb{R}^6$ eqiupped with the scalar product h of signature $(-++++-)$. Now, a 2-dimensional surface in $\mathbb{E}^3$ corresponds to the pair of 2-dimensional submanifolds in null quadric defined by h [33],[34],[35],[H4]. A detailed investigation shows that the transformation $(37)_2$ corresponds to translation of Σ in $\mathbb{E}^3$ and (for $a_2 = 0$ and $a_1 = a_3$) to the transition of Σ to its parallel surface $\Sigma^{||}$ with position vector given by $\boldsymbol{r}^{||} = \boldsymbol{r} + 2a_1\boldsymbol{N}$. The transformations $(37)_1$ in which $S \in SU(2) \subset SL(2, \mathbb{C})$ generate rotations of Σ. However, if matrices S in $(37)_1$ do not belong to $SU(2)$ group, then the transformation of a surface is much more complicated. Using the terminology of special theory of relativity this transformation corresponds to boost and it induces the new surface with the following position vector

$$\boldsymbol{r} \mapsto \boldsymbol{r}' = \boldsymbol{r} - \frac{\sinh(n)\boldsymbol{N} + \left(\cosh(n) - 1\right)\mathbf{n}}{\cosh(n) + \sinh(n)\boldsymbol{N}\cdot\mathbf{n}}\,\boldsymbol{r}\cdot\mathbf{n}, \tag{38}$$

where $n \in \mathbb{R}$ and $\mathbf{n} = (n_1, n_2, n_3)$ is unit constant vector.

It turns out that the description of L-isothermic surfaces in terms of solutions of linear nonhomogeneous equation (33) is very powerful when applied to so-called L-minimal surfaces. The latter are analog of minimal surfaces in Euclidean space and can be defined as a critical points of the funcional $\mathcal{W} = \int(\mathcal{H}^2/\mathcal{K} - 1)dA$, where $\mathcal{H}$ and $\mathcal{K}$ are mean and Gaussa curvature respectively. Locally, the curvatures of L-minimal surfaces satisfy the following fourth order differential equation

$$\Delta_{III}\Big(\frac{\mathcal{H}}{\mathcal{K}}\Big) = 0, \tag{39}$$

where Δ_{III} is a Laplace operator with respect to the third fundamental form (12). In case of the L-minimal surface which is also L-isothermic the equation (39) reduces to $P_{zz} + \mathsf{U}P = 0$ which may be interpreted as a additional constraint on inhomogeneous part of (33). Hence, function P can be put in the form $P = n_1|\Phi_1|^2 + n_2\Phi_1\bar{\Phi}_2 + \bar{n}_2\bar{\Phi}_1\Phi_2 + n_3|\Phi_2|^2$, where $n_1, n_3 \in \mathbb{R}$ and $n_2 \in \mathbb{C}$. Using Laguerre transformation the function P can be reduced to the quadratic form: $P = |\Phi_1|^2 + \varepsilon|\Phi_2|^2$, where $\varepsilon = -1, 0, 1$. This leads to the Weierstrass representation of L-minimal surface which is also L-isothermic:

$$r = \mathrm{Re}\begin{pmatrix} -\int(1-\varepsilon\rho^2)F(\rho)d\rho \\ i\int(1+\varepsilon\rho^2)F(\rho)d\rho \\ (1+\varepsilon)\int\rho F(\rho)d\rho \end{pmatrix} + \frac{\mathcal{H}}{\mathcal{K}}\frac{1}{1+\rho\bar{\rho}}\begin{pmatrix} \rho+\bar{\rho} \\ i(\rho-\bar{\rho}) \\ 1-\rho\bar{\rho} \end{pmatrix}, \tag{40}$$

where

$$\frac{\mathcal{H}}{\mathcal{K}} = (1-\varepsilon)\mathrm{Re}\int\rho F(\rho)d\rho - \mathfrak{b}, \qquad \mathfrak{b} = const. \tag{41}$$

The L-minimal surface (40)-(41) is defined by a holomorphic function $F(\rho)$, where ρ is a local complex coordinate. The case $\varepsilon = 1$, $\mathfrak{b} = 0$ corresponds to well-known Weierstrass representation of minimal surface. Geometric characterization of (40)-(41) depends on value of parameter ε. Accordingly, we have

$\varepsilon = -1$: Surfaces whose central sphere congruence has centers lying in the plane $z = 0$ in $\mathbb{E}^3$,

$\varepsilon = 0$: Surfaces whose central sphere congruence is tangential to a fixed plane $z = \mathfrak{b}$ in $\mathbb{E}^3$,

$\varepsilon = 1$: Surfaces parallel to minimal surfaces.

The above classification was known to Blaschke [33].

By choosing different functions $F(\rho)$ one can construct new L-minimal surfaces, especially the ones which are related to known minimal surfaces. The examples for $F(\rho) = \frac{1}{\rho^2}$ and $F(\rho) = 1 - \frac{1}{\rho^4}$ were depicted in figure 5. They are L-minimal analog of helicoid and surface of Henneberg. The figure 6 shows closed L-minimal generalized cyclide of Dupin (left) [H2] and an example of L-minimal surface together with its Bäcklund transformation (right) [H3].

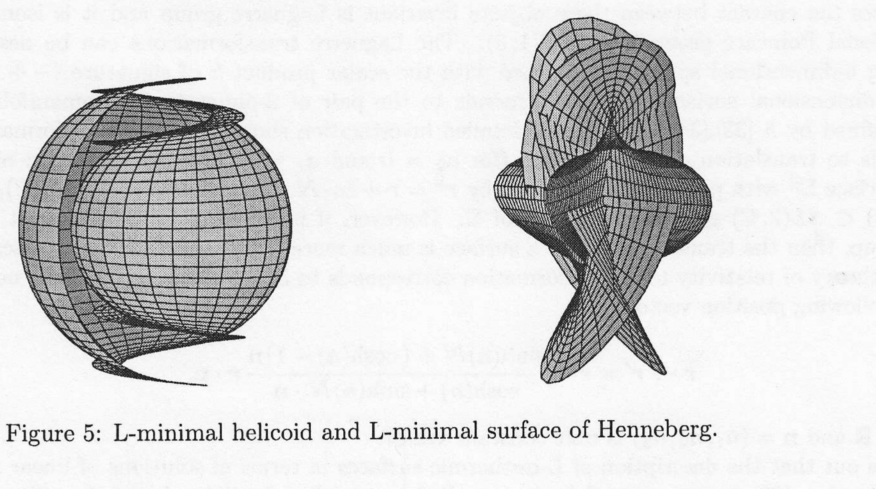

Figure 5: L-minimal helicoid and L-minimal surface of Henneberg.

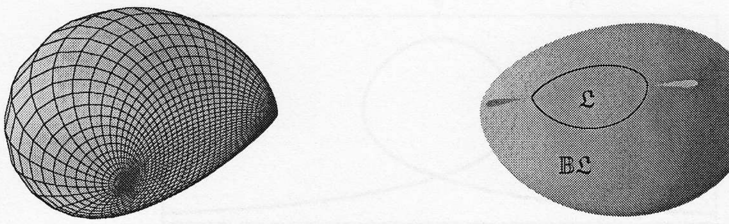

Figure 6: Closed L-minimal generalized cyclide of Dupin and L-minimal surface $\mathfrak{L}$ together with its Bäcklund trnasformation $\mathbb{B}\mathfrak{L}$.

In the subsequent part of the presentation we focus upon another integrable equation which arises out from the theory of elasticity, namely the following non-linear telegraphy equation

$$\Phi_{\rho\rho} - \left(\frac{\Phi_t}{(1+\Phi^2)^2}\right)_t + \Phi = 0, \qquad \Phi = \Phi(\rho, t). \tag{42}$$

The transformation $T = \sqrt{1+X^2}\,\Phi, \quad X = \tan\rho$ reduces (42) to non-linear equation

$$T_{XX} = \left[\frac{T_t}{(1+X^2+T^2)^2}\right]_t \tag{43}$$

descriptive of stress propagation T in a class of model inhomogeneous ideally hard elastic materials with the constitutive law $T = T(\epsilon, X)$ given parametrically by the relations

$$T = \sqrt{1+X^2}\,\tan\theta, \quad \epsilon = \frac{1}{2(1+X^2)^{3/2}}(\theta + \sin\theta\cos\theta). \tag{44}$$

Variable X is a Lagrangian material coordinate, t is the time, while ϵ is the strain. It turns out that equation (42) is integrable and its solutions exhibits a novel 'exchange particle' phenomenon, which probably was not observed before. It is known [36], that any pseudospherical surface in $\mathbb{E}^3$ can be related to solution to (43). This suggests a relation between (43) and sin-Gordon equation (sG). Both equations are indeed associated with two different parametrizations of the same 2-dimensional integral manifolds of a cc ideal [H5]. The reciprocal transformation allows to construct linear system which is compatible modulo sin-Gordon equation (sG). This implies, in turn, the Bäcklund transformation, a tool to produce many solutions to (42). The starting point is a trivial solution $\Phi = 0$. Applying the Bäcklund transformation to it we generate the one-soliton solution

$$\tilde{\Phi} = \left[\left(\frac{1+\mu^2}{2\mu}\right)^2 \cosh^2\eta - 1\right]^{-1/2}, \qquad \eta = \mu\alpha - \frac{\beta}{\mu}, \tag{45}$$

where α, β are non-linear functions of new coordinates $\tilde{\rho}$ i $\tilde{t}$. Constant parameter μ controls the shape and velocity of soliton. The solution (45) is a loop-shaped localized wave travelling at a constant speed keeping its shape unchanged. Figure 7 shows the solution (45) for a fixed time $\tilde{t}$ and parameter $\mu = 1/2$. Geometrically, this corresponds to a Dini surface [37] representing the solution of sin-Gordon equation (sG). A second application of the Bäcklund transformation with parameters μ and ν leads to very interesting solution $\hat{\Phi}$, which is given in explicite form by

$$\hat{\Phi} = \tan\left(\arcsin\frac{2(\mu^2-\nu^2)}{(1+\mu^2)(1+\nu^2)}\frac{\nu(1+\mu^2)\cosh\eta_1 - \mu(1+\nu^2)\cosh\eta_2}{(\mu^2+\nu^2)\cosh\eta_1\cosh\eta_2 - 2\mu\nu\sinh\eta_1\sinh\eta_2 - 2\mu\nu}\right), \tag{46}$$

where $\eta_1 = \mu\alpha - \beta/\mu, \quad \eta_2 = \nu\alpha - \beta/\nu$, α and β are functions of $\hat{\rho}$ and time $\hat{t}$. A detailed analysis of solution (46) not only confirms a standard behaviour of soliton type solutions but also reveals some

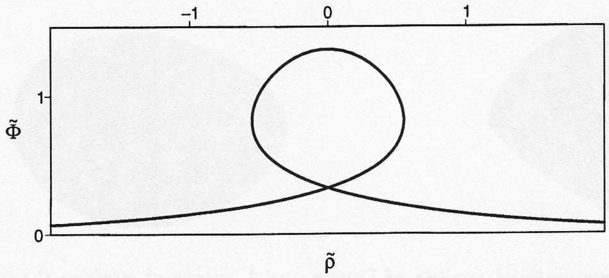

Figure 7: A loop soliton $\tilde{\Phi}$ for $\mu = 1/2$ at a fixed time.

novel features. The latter solution describes two traveling loop solitons (Figure 8). At the beginning, when solitons are far apart, they travel at a constant speed, although the larger is faster than the smaller. It is noted that the two-loop solitons do not seem to pass through but rather travel past each other.

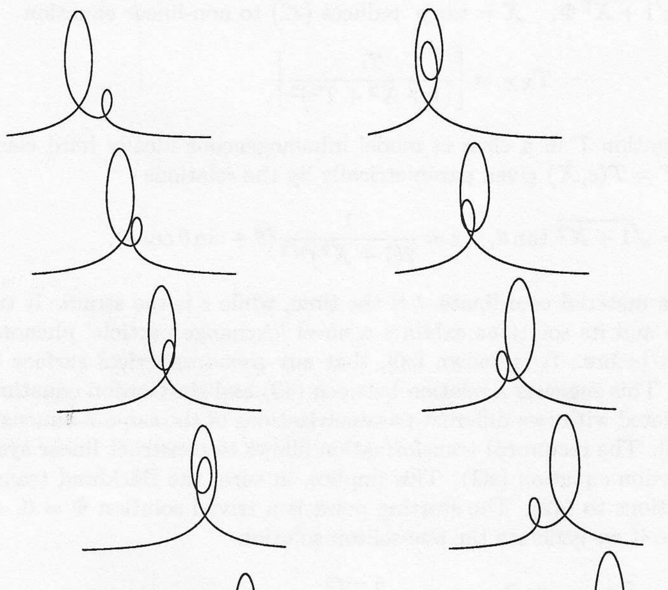

Figure 8: $\hat{\Phi}$ as a function of $\hat{\rho}$ for various times $\hat{t}$: interaction of a loop soliton for $\mu = 2$ with a loop soliton for $\nu = -6$.

The behaviour of two-loop solution described above is quite standard. The non-standard and probably new features are arrived at by consideration of the two soliton solution for $\mu = 2$ i $\nu = 1/2$. As before, two solitons of the same size[4] approach each other with a constant velocity. When they are close to each other an additional figure eight 8 is created which makes the interaction more involved as in the ordinary case. The figure 8 grows when solitons approach and then it disappears when they go away. During the interaction, the two-loop solitons and the figure 8 'exchange' their identities.

[4]The same size is a consequence of seting μ, ν such that $\mu\nu = 1$.

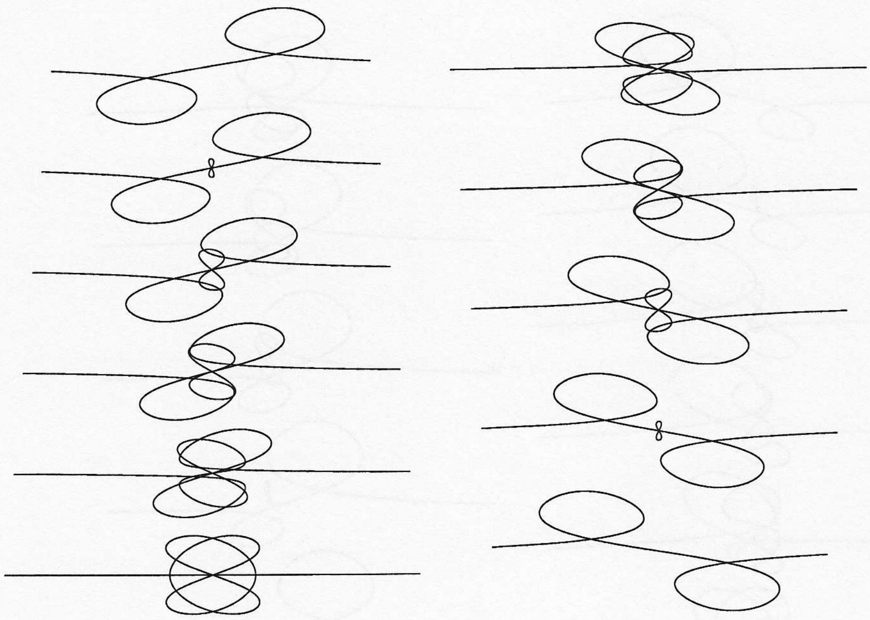

Figure 9: $\hat{\Phi}$ as a function of $\hat{\rho}$ for various times $\hat{t}$: interaction of a loop soliton for $\mu = 2$ with a loop soliton for $\nu = 1/2$.

The analysis of solitons for $\mu = 2$ i $\nu = 3/10$ reveals their interaction in great detail (see figure 10). At the beginning the figure 8 is completely detached from the main curve but eventually joins it to form a single differentiable curve. It is quite evident from the figure 11 that once the particle 8 has merged with the loops, the identities of the latter and the figure 8 start to exchange. To conclude, the figure 8 behaves like a particle which exists only for a finite time and is exchanged between interacting solitons.

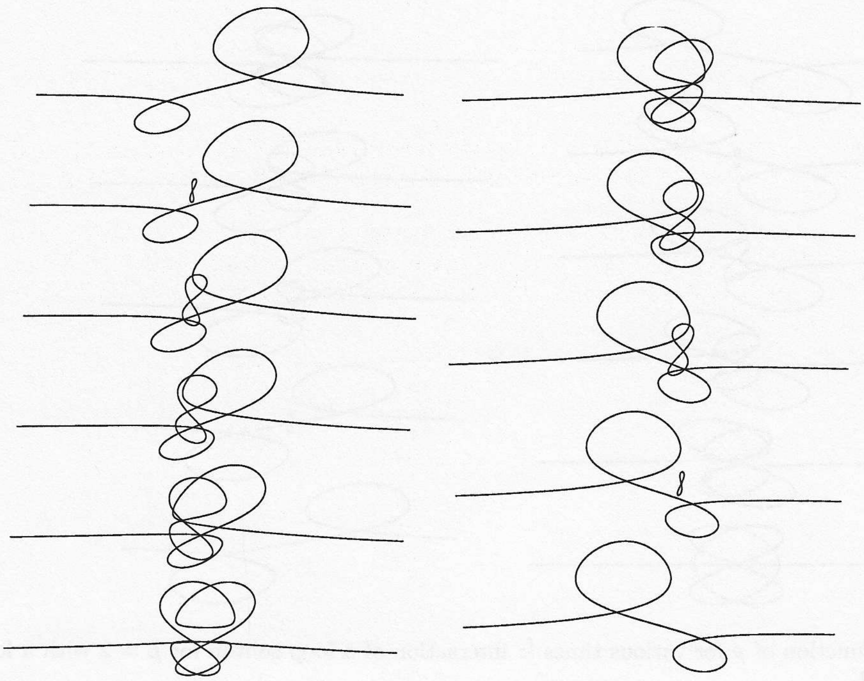

Figure 10: $\hat{\Phi}$ as a function of $\hat{\rho}$ for various times $\hat{t}$: interaction of a loop soliton for $\mu = 2$ with a loop soliton for $\nu = 3/10$. The figure 8 which is initially disconnected forms a single curve with the two loops.

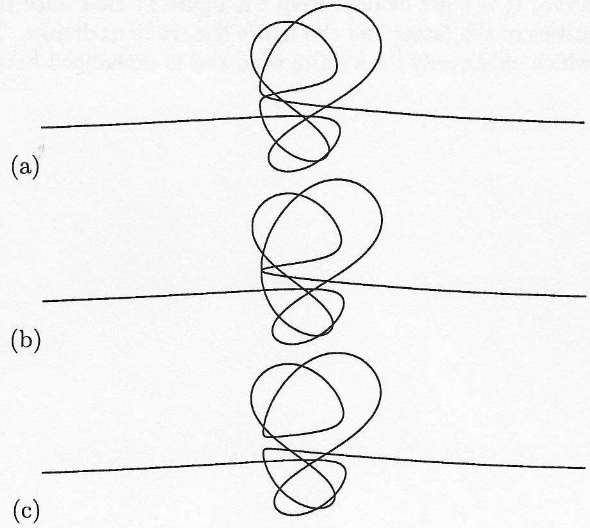

Figure 11: $\hat{\Phi}$ as a function of $\hat{\rho}$ for various times $\hat{t}$: interaction of a loop soliton for $\mu = 2$ with a loop soliton for $\nu = 3/10$. Blow-up of the merging phase: (a) The main curve and the figure 8 intersect transversally; (b) The figure 8 and the main curve intersect with common tangent; (c) The figure 8 and the main curve have merged.

3 Main results

The series of articles [H1]-[H5] include the following main results:

- Geometric and algebraic characterization of membranes with not uniquely determined stress distribution [H1]
- A detailed description of the equilibrium equations (8) supplemented by the Gauss-Mainardi-Codazzi equations (9) as a integrable system [H1]
- Analytic approach to description of L-isothermic surfaces by using linear equation (33) [H1]-[H4]
- Description of Bäcklund transformation in terms of linear equation (33) [H3]
- Analysis of L-minimal surfaces which are also L-isothermic and construction of Weierstrass representation (40)-(41) [H4]
- Description of the solitonic properties of telegraphy equation and discovering a novel solitonic 'exchange particle' phenomenon.

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5. Discussion of the other scientific achievements

1) **A. Szereszewski**, J. Tafel, Integrability of the Rarita-Schwinger equation, *Class. Quantum Grav.* **18** No 18, L129-L132 (2001)

2) **A. Szereszewski**, J. Tafel, Solutions of the Rarita-Schwinger equation in Einstein spaces, *Phys. Lett. A* **297** Issue: 5-6, 359362 (2002)

In article 1) Rarita-Schwinger equation defined on curved manifold and its integrability conditions were considered. It is known that integrability conditions reduce to Einstein equations with cosmological constants in this case. Assuming that the manifold is equipped with a stationary axially symmetric metric Rarita-Schwinger equation reduces to a linear system which integrability conditions consist of Ernst equation. It turned out that this linear system was different form the well known linear system related to complete integrability of Ernst equation. It was concluded that Rarita-Schwinger equation could not be used to examine the complete integrability of Einstein equations.

In article 2) special solutions of the Rarita-Schwinger equation in spacetimes admitting shear free congruences of null geodesics were analyzed. In case when the Rarita-Schwinger spinor decomposes into three 1-valence spinors the constraints on geometry were calculated. This assumption allowed to construct solutions for the Schwarzschild metric, pp waves and conformally flat metrics in explicit form. Just like in 1), the Weyl spinor approach was used.

3) **A. Szereszewski**, J. Tafel, Perfect fluid spacetimes with two symmetries, *Class. Quantum Grav.* **21**, 17551759 (2004)

4) **A. Szereszewski**, J. Tafel, From 2-dimensional surfaces to cosmological solutions, *Gen. Relativ. Gravit.* **37** (2), 257-269 (2005)

In article 3) a new method of solving perfect fluid Einstein equations with two commuting spacelike Killing vectors was introduced. Given a spacelike 2-dimensional surface in the 3-dimensional nonphysical Minkowski space the field equations reduce to a single nonlinear differential equation. By considering a rotational surface in Minkowski space probably new non-tilted cosmological solution of Bianchi type VII_0 was constructed.

In article 4) this method was applied to surfaces invariant under 1-dimensional group of isometries in Minkowski space. As a result, new cosmological perfect fluid solutions of Bianchi II, VI_0 and VII_0 were found. The metrics depend on an arbitrary function of time, which can be further specified in order to satisfy an equation of state.

5) C. Rogers, **A. Szereszewski**, On the Geometry of Complex-Lamellar Magnetohydrodynamics: Universal Motions, *Stud. Appl. Math.* **128** (3), 225-251 (2012)

In article 5) a nonlinear magnetohydrodynamic system of differential equations was considered. Using a geometric formulation the case when the magnetic field is aligned with the direction of the binormal to the streamlines was examined in detail. Assuming the additional geometric constraint which is commonly termed a complex-lamellar motion it was shown that the fluid stramlines are geodesics on generalized helicoids and the magnetic lines are helices thereon. It was established that the key geometric and physical parameters could be determined in terms of the torsion of the streamlines. The superposition principle was also constructed which provided new (not necessarily complex-lamellar) solution to the system.

6) **A. Szereszewski**, J. Tafel, M. Jakimowicz, D-dimensional metrics with $D-3$ symmetries, *Int. J. Theor. Phys.* **51** (5), 1360-1369 (2012)

In article 6) symmetry transformations in a space of D-dimensional vacuum metrics with $D-3$ commuting Killing vectors were considered. The relevant parameters of these transformations which may lead to new vacuum solutions were revealed. The method was applied to special 5-dimensional metrics. As a result, it was discovered that the Kaluza-Klein version of the Reissner-Nordström solution is the symmetry transform of a Gross-Perry metric and that the 5-dimensional plane wave metric is related to the Gross-Perry-Sorkin monopole solution.

7) A. Sym, **A. Szereszewski**, On Darboux's Approach to R-Separability of Variables, *SIGMA* **7**, 095 (2011)
8) **A. Szereszewski**, A. Sym, On Darboux's approach to R-separability of variables. Classification of conformally flat 4-dimensional binary metrics. *J. Phys. A, Math. Theor.* **48** (2015), No. 38, 385201.

In article 7) a novel approach to the problem of R-separability (i.e. separability with a factor R) of variables in the stationary Schrödinger equation on n-dimensional Riemann space admitting orthogonal coordinates was proposed. This method initiated by G. Darboux allows to consider separation with arbitrary (not necessarily maximal) number of constants of separation. The necessary and sufficient conditions for the factor R and the metric were explicitly formulated. An important class of binary metrics (i.e. metrics which satisfy one of the conditions of R-separability) were introduced. The systematic procedure to isolate R-separable metrics were formulated which enables to find and analyze many 3-dimensional examples.

The subject originated in 7) was continued in article 8), where R-separability conditions were generalized to metrics of arbitrary signature. All 4-dimensional binary metrics were classified which required to solve system of nonlinear partial differential equations. It was shown that in the most interesting case of metric type:

$$g = \frac{u_{12}^{\gamma}u_{13}^{\gamma}u_{14}^{\gamma}}{F_1(u^1)}(du^1)^2 + \frac{u_{12}^{\gamma}u_{23}^{\gamma}u_{24}^{\gamma}}{F_2(u^2)}(du^2)^2 + \frac{u_{13}^{\gamma}u_{23}^{\gamma}u_{34}^{\gamma}}{F_3(u^3)}(du^3)^2 + \frac{u_{14}^{\gamma}u_{24}^{\gamma}u_{34}^{\gamma}}{F_4(u^4)}(du^4)^2, \qquad (\star)$$

where $u_{ij} = u^i - u^j$ the constant γ had to be 'qunatized' ($\gamma \in \{-2, -1, 0, 1\}$), while functions $F(u^i)$ had to be appropriate polynomials. It was also proven that all well know separable solution that had been found by E. Kalnins and W. Miller Jr belonged in fact to special subclass of binary metrics. Various examples of non-Stäckel and non-regular R-separation metrics were constructed. The justification for a term 'n-dimensional isothermic metric' introduced in 7) was also given (any 2-dimensional submanifold defined by fixing values of $n-2$ coordinates possesses 2-dimensional metric which is isothermic by standard definition).

The examination of binary metrics is continued and new results will be publised in paper *Diagonal Einstein metrics of special kind*, where binary metrics are considered from the point of view of being conformall to Einstein metrics. It turns out that there exists a family of metrics ($\star$) with $\gamma = -3/2$, which are not conformally flat but satisfy vacuum Einstein equations $R_{\mu\nu} = 0$. In general these metrics do not possesses any Killing vectors, but those which do have symmetries may be classified.

Another scientific project I am involved in concerns discretisation of 2-dimensional surfaces in 3-dimensional real projective space $\mathbb{P}^3$. By considering surfaces parametrized is terms of asymptotic coordinates the standard method of discretisation can be used. Surprisingly the discretisation can be performed on algebraic and geometric level. It is well known that there exists an integrable reduction of the projecive Gauss-Mainardi-Codazzi equations which leads to integrable equations. Remarkably Lie point symmetry reduction of these equations leads to projective-minimal surfaces which arise naturally in the contex of soliton theory and can be classified both geometrically and algebraically. The similar reduction may be performed on discrete level which allows to define discrete projective-minimal surfaces. All these results will be published in article *Discrete projective minimal surfaces: geometry and integrability* which is in preparation with cooperation with prof. W. K. Schief.

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