**Załącznik nr 3**

# Autoreferat

## 1 Name

Andrzej Okołów

## 2 Degrees

1. Doctor of Philosophy in Physical Sciences, Faculty of Physics, University of Warsaw, 2004.
   Title of PhD thesis: *Representations of Quantum Geometry.*

2. Master of Science, Faculty of Physics, University of Warsaw, 1999.
   Title of MSc thesis: *Nowe zmienne samodualne w kanonicznej grawitacji (New self-dual variables for canonical gravity).*

## 3 Employment in academic institutions

1. since October 2004—Assistant professor (adiunkt) in Institute of Theoretical Physics, University of Warsaw;

2. since March 2005 till February 2006—postdoctoral researcher in Department of Physics & Astronomy, Louisiana State University, USA.

## 4 Scientific achievement, in the sense of article 16, paragraph 2 of the Act on academic degrees and academic title and degrees and title in art (Dz. U. nr 65, poz. 595 ze zm.)

### 4.1 Title of the scientific achievement—a monographic series of publications:

Canonical quantization of the Teleparallel Equivalent of General Relativity—Hamiltonian analysis and construction of kinematic quantum states.

### 4.2 The monographic series of publications

[H1] Okołów A, Świeżewski J, 2012 Hamiltonian formulation of a simple theory of the teleparallel geometry. *Class. Quant. Grav.* **29** 045008. `arXiv:1111.5490`.

[H2] Okołów A, 2013 ADM-like Hamiltonian formulation of gravity in the teleparallel geometry. *Gen. Rel. Grav.* **45** 2569–2610. `arXiv:1111.5498`.

[H3] Okołów A, 2014 ADM-like Hamiltonian formulation of gravity in the teleparallel geometry: derivation of constraint algebra. *Gen. Rel. Grav.* **46** 1636. `extedned version: arXiv:1309.4685`.

[H4] Okołów A, 2013 Construction of spaces of kinematic quantum states for field theories via projective techniques. *Class. Quant. Grav.* **30** 195003. `arXiv:1304.6330`.

[H5] Okołów A, 2014 Variables suitable for constructing quantum states for the Teleparallel Equivalent of General Relativity I. *Gen. Rel. Grav.* **46** 1620. `arXiv:1305.4526`.

[H6] Okołów A, 2014 Variables suitable for constructing quantum states for the Teleparallel Equivalent of General Relativity II. *Gen. Rel. Grav.* **46** 1638. `arXiv:1308.2104`.

[H7] Okołów A, 2014 Kinematic quantum states for the Teleparallel Equivalent of General Relativity. *Gen. Rel. Grav.* **46** 1653. `arXiv:1304.6492`.

## 4.3 Descriptions of scientific goal of the monographic series of publications and the results achieved and a description of possible applications of the results

### 4.3.1 Introduction

One of the greatest challenges facing theoretical physics today is to formulate a theory being a coherent synthesis of general relativity (GR) and quantum mechanics (quantum field theory). First attempts to create such a theory called commonly *quantum gravity* took place in the 1930s. Nowadays there are many approaches to quantum gravity [1, 2] like string theory, loop quantum gravity (LQG), spin foams, dynamical triangulations, but no one is free from more or less serious "internal" problems. Moreover, experimental verification of any quantum gravity model is still beyond our abilities. Since we cannot be sure that any of the existing approaches will finally lead to a physically correct quantum gravity it is still worth to formulate new approaches to this problem.

A natural way to construct models of quantum gravity is to quantize GR, that is, to apply a quantization procedure to this classical theory. A particular feature of GR is multitude of its Lagrangian formulations which use distinct sorts of fields as configurations variables e.g. the Hilbert-Einstein action is a functional on the space of Lorentzian metrics, while the Palatini formulation applies a coframe and a connection one-form as configuration variables. The diversity of formulations of GR means a *potential* diversity of quantum gravity models obtained by application of distinct quantization methods to distinct formulations of the theory.

A formulation of GR which so far has not been used as a point of departure for quantization is the Teleparallel Equivalent of General Relativity[1] (TEGR). There are two versions of TEGR: one of them uses a coframe and a connection one-form of zero curvature and non-zero torsion as configuration variables, the other one uses a coframe only.

[1]The newest review paper on this formulation of GR is [3].

The series of publications [H1]-[H7] arose as a part of my research project *aimed at checking whether it is possible to quantize TEGR by means of canonical quantization procedure.* Moreover, taking into account that TEGR is a background independent (diffeomorphism invariant) theory I decided to quantize this theory *in a background independent manner* similarly as it was done in the case of LQG (see [4] and references therein).

Very briefly, *results achieved in publications* [H1]-[H7] can be summarized as follows:

1. I described *Hamiltonian structures* of TEGR and a similar theory which can be used as a toy-model helpful for quantizing TEGR; the descriptions satisfy requirements imposed on them by a procedure of background independent canonical quantization;

2. I generalized a method of constructing spaces of kinematic quantum states for field theories invented by J. Kijowski [5] — the original method requires that the phase space of a theory is equipped with a linear structure, while the generalized method is free of this limitation;

3. by means of the generalized method I constructed in a background independent manner *a space of kinematic quantum states* for TEGR (and for the toy-model mentioned above).

Let me now describe the results in details.

### 4.3.2 Hamiltonian formulation of TEGR

A preparatory step required by the procedure of canonical quantization is a transformation of a theory to a Hamiltonian form. At the moment of the start of the project I knew works [6, 7, 8, 9, 10] describing the Hamiltonian structure of TEGR. According to results of the works TEGR is a Hamiltonian system with constraints being so complicated functions of canonical variables that one can exclude a possibility of obtaining general solutions of the constraints. Therefore I decided to quantize TEGR by means of the Dirac procedure of quantization of constrained systems. Roughly speaking, this procedure consists of two steps: first one neglects constraints and constructs so called *space of kinematic quantum states*, that is, a space of quantum states corresponding to the whole unconstrained phase space of a theory under quantization, then one imposes some conditions (quantum constraints) on the kinematic quantum states as counterparts of the constraints on the phase space.

Taking into account the quantization method I was going to apply I formulated the following criteria to be satisfied by a Hamiltonian formulation of TEGR useful for realization of my project:

C1. the formulation is derived without any gauge fixing (which would unnecessarily limit symmetries of the resulting quantum model);

C2. the phase space is as simple as possible: canonical variables are a coframe pulled back on a space-like slice of a spacetime and a momentum conjugated to it (this assumption is aimed at simplifying a construction of a space of quantum states, actually the assumption makes it possible to construct the space);

C3. a complete set of constraints on the phase space and their algebra are known explicitely (this is a requirement imposed by the Dirac quantization procedure);

C4. the formulation is of the ADM-type [11] i.e. non-dynamical degrees of freedom on the Lagrangian configuration space are described by the lapse function $N$ and the shift vector field $\vec{N}$—then one of constraints is a vector constraint which generates gauge transformations on the phase space corresponding to an action of spatial diffeomorphisms (this requirement is motivated by the desire to quantize TEGR in a background independent manner).

Since no formulation described in [6, 7, 8, 9, 10] satisfies all the criteria it was necessary to derive a new Hamiltonian formulation of TEGR which would meet the criteria. However, it turned out that this task requires rather long and complicated calculations to be carried out. Therefore I decided to do first a preparatory exercise, that is, to derive a Hamiltonian formulation of a theory, (*i*) which possesses the same Lagrangian configuration space as TEGR does and (*ii*) the action of which is simpler than that of TEGR but generates *the same phase space* as that of TEGR. Moreover, I was going to obtain in this way a simple toy-model which could be used to test elements of the quantization procedure before they will be applied to more complicated TEGR. As the simple theory I chose so called *Yang-Mills-type Teleparallel Model* (YMTM) [12] defined by the following action:

$$s[\boldsymbol{\theta}^A] = -\frac{1}{2}\int \boldsymbol{d\theta}^A \wedge \star \boldsymbol{d\theta}_A, \tag{4.1}$$

where $(\boldsymbol{\theta}^A)_{A=0,1,2,3}$ is a coframe on a four-dimensional oriented manifold $\mathcal{M}$, $\star$ is a Hodge dualization operator given by a metric

$$g := \eta_{AB}\, \boldsymbol{\theta}^A \otimes \boldsymbol{\theta}^B, \quad (\eta_{AB}) = \mathrm{diag}(-1,1,1,1)$$

and the index $A$ in $\boldsymbol{\theta}_A$ was "lowered" by means of the matrix $(\eta_{AB})$.

**Publication [H1]** I described the Hamiltonian structure of YMTM in [H1] applying a general Hamiltonian formalism adapted to differential forms taken from [13]. As a part of an analysis necessary for a $3+1$ decomposition of the action (4.1) I found an explicite formula describing the time-like component $(\boldsymbol{\theta}^A_\perp)$ of the coframe as a function of the lapse $N$, the shift $\vec{N}$ and the space-like part $(\underline{\boldsymbol{\theta}}^A)$ of the coframe and derived a formula describing a $3+1$ decomposition of a four-form $\alpha\wedge\star\beta$, where $\alpha, \beta$ are $k$-forms. Then I presented a description of the phase space[2], found a Hamiltonian, a complete set of constraints on the phase space and derived a constraint algebra—the Hamiltonian turned out to be a sum of all the constraints, and all the constraints turned out to be of the first class.

[2]There is a slight mistake in the description: a condition imposed there on "position" variables $(\theta^A)$ is too weak. The mistake was corrected in [H2].

**Publications [H2, H3]** Basing on experiences gained through studying the Hamiltonian structure of YMTM I proceeded to investigate the Hamiltonian structure of TEGR. Results of the investigation were published in [H2] and [H3]—in the former paper I derived constraints and a Hamiltonian and analyzed gauge transformations on the phase space generated by the constraints, in the latter paper I derived a constraint algebra.

As a point of departure for the Hamiltonian analysis of TEGR I chose the following action [14, 15, 16, 17, 18, 19]:

$$S[\boldsymbol{\theta}^A] = \int -\frac{1}{2}(\boldsymbol{d\theta}^A \wedge \boldsymbol{\theta}_B) \wedge \star(\boldsymbol{d\theta}^B \wedge \boldsymbol{\theta}_A) + \frac{1}{4}(\boldsymbol{d\theta}^A \wedge \boldsymbol{\theta}_A) \wedge \star(\boldsymbol{d\theta}^B \wedge \boldsymbol{\theta}_B). \tag{4.2}$$

A phase space of TEGR I obtained is a Cartesian product $P \times \Theta$, where

1. $\Theta$ is a set of all quadruplets $(\theta^A)_{A=0,1,2,3}$ of one-forms defined on a three-dimensional *oriented* manifold[3] $\Sigma$ such that

$$q := \eta_{AB}\, \theta^A \otimes \theta^B \tag{4.3}$$

is a *Riemannian (positive definite)* metric on $\Sigma$,

2. $P$ is a set of all quadruplets $(p_B)_{B=0,1,2,3}$ of two-forms defined on $\Sigma$—the two-form $p_A$ is the momentum conjugate to the one-form $\theta^A$.

A complete set of constraints on the phase space I found consists of a scalar constraint, a vector one and two other constraints—I called one of them boost constraint and the other rotation constraint. According to a constraint algebra I derived all the constraints are of the first class. A Hamiltonian of TEGR obtained from the action (4.2) is a sum of all the constraints.

The vector constraint $V(\vec{M})$ smeared with a vector field $\vec{M}$ on $\Sigma$ generates on the phase space gauge transformations being pull-backs of the forms $(p_A, \theta^B)$ along integral curves of the vector field, that is, an action of spatial diffeomorphisms on the canonical variables. Gauge transformations generated by the boost and rotation constraints are in fact local Lorentz transformations of the canonical variables—a more detailed analysis of the transformations showed that they act on the variables in a rather non-standard way.

The Hamiltonian formulation of TEGR described in [H2, H3] satisfies all the criteria C1–C4, which means that the formulation may be used as a point of departure for background independent canonical quantization *à la Dirac* of the theory.

### 4.3.3 Construction of a space of kinematic quantum states by means of projective techniques

Since I decided to quantize TEGR in *a background independent manner* I should construct likewise a space of kinematic quantum states required by the Dirac quantization. A method of constructing such spaces [20] was elaborated for LQG, but it is not general i.e. it cannot be applied for every field theory, in particular, it is not applicable to TEGR.

[3] The manifold $\Sigma$ is a standard leaf of a foliation defining a $3+1$ decomposition of a spacetime.

An other method of constructing spaces of quantum states for field theories based on some projective techniques was introduced by J. Kijowski [5]. Applicability of this method is limited to theories of special phase spaces—such a phase space should be equipped with a structure of a linear space. The fact, that the phase space of TEGR obtained in [H2] is non-linear (the source of the non-linearity is the condition imposed on the variables $(\theta^A)$) motivated me to search for a generalization of the Kijowski's method which would be applicable to theories of non-linear phase spaces. This search turned out to be doubly successful: using some ideas taken from LQG [4, 21, 22] (*i*) I found an appropriate generalization and (*ii*) managed to construct a space of kinematic quantum states for TEGR using the generalized method.

**Publication [H4]** I described the generalized method of constructing spaces of quantum states in [H4]. An idea of this construction is taken from [5] and reads as follows. A point of departure for the construction is a special directed set $(\Lambda, \geq)$—every element $\lambda$ is (more or less literally) a physical system of finite number of degrees of freedom obtained from a phase space of a field theory, and the relation $\geq$ is chosen in such a way that if $\lambda' \geq \lambda$ then the system $\lambda$ is a *subsystem* of $\lambda'$. In the first step of the construction one "quantizes" every system $\lambda$ by assigning to it a space $\mathcal{D}_\lambda$ of quantum states, in the second step one merges all spaces $\{\mathcal{D}_\lambda\}_{\lambda\in\Lambda}$ into one space $\mathcal{D}$ of quantum states corresponding to the phase space. Let me emphasize that the generalized method, as its archetype presented in [5], is based on the structure of a phase space of a theory without taking into account (possible) constraints imposed on canonical variables. Therefore the space $\mathcal{D}$ is a space of *kinematic* quantum states.

In [H4] I based a set $(\Lambda, \geq)$ on a family of functions on the phase space in a way being a combination of the original Kijowski's method and a construction of an algebra of classical observables [22] known from LQG. I called the functions *elementary degrees of freedom*. Assuming that the phase space is a product $P\times Q$ of a space $P$ of momenta and a configuration space $Q$ (a space of "positions"), I introduced a set of *elementary configurational d.o.f.* as a family of real functions on $Q$. Similarly, I defined a set of *elementary momentum d.o.f.* as a family of real functions on $P$. Then I assumed that every momentum d.o.f. $\varphi$ gives a linear operator $\hat{\varphi}$ acting on functions defined on $Q$ — $\hat{\varphi}$ is defined by means of a Poisson bracket

$$\mathrm{Cyl} \ni \Psi \mapsto \hat{\varphi}\Psi := \{\varphi, \Psi\} \in \mathrm{Cyl} \tag{4.4}$$

or, if necessary, by means of an appropriate regularization of the bracket. In the next step, I introduced a real linear space $\hat{\mathcal{F}}$ spanned by all operators $\{\hat{\varphi}\}$. I defined the directed set $(\Lambda, \geq)$ as one consisting of pairs $(\hat{F}, K)$ such that $\hat{F}$ is a *finite dimensional* linear subspace of $\hat{\mathcal{F}}$ and $K$ is a *finite* set of configurational d.o.f..

According to the idea of the construction of the space $\mathcal{D}$ outlined above every element $\lambda = (\hat{F}, K)$ of the set $\Lambda$ should be interpreted as a finite physical system. And indeed, such an interpretation can be formulated. Namely, configuration d.o.f. constituting the set $K$ define so called *reduced configuration space* $Q_K$—a point of $Q_K$ is a subset of the configuration space $Q$ on which every d.o.f. $\kappa \in K$ is a constant function. The space $Q_K$ plays a role of a configuration space of a finite system, and operators in $\hat{F}$ act naturally on functions defined on $Q_K$ and contain information about both momentum d.o.f. and the Poisson bracket[4].

However, in order to make more solid the interpretation of the set $(\Lambda, \geq)$ as a family of finite physical systems equipped with the relation system–subsystem and in order to guarantee that the set generates a space $\mathcal{D}$ of quantum states I had to require that this set satisfies a number of assumptions—here I will list only three of them:

A1. if $(\hat{F}, K) \in \Lambda$ and $K$ consists of $N$ elements then the reduced configuration space $Q_K$ is in a natural bijection with $\mathbb{R}^N$.

A2. if $(\hat{F}, K) \in \Lambda$ then the "Poisson structure" encoded in operators belonging to $\hat{F}$ is *non-degenerate*—this non-degeneracy is a counterpart of the non-degeneracy of usual Poisson bracket[5] (the present assumption is a natural generalization of one introduced in [5]).

A3. if $(\hat{F}', K') \geq (\hat{F}, K)$ then the system $(\hat{F}', K')$ contains complete information about the system $(\hat{F}, K)$, and the relation between them is linear in the following sense:

   (a) every d.o.f. $\kappa \in K$ is *a linear combination* of d.o.f. in $K'$;

   (b) $\hat{F}$ is a *linear subspace* of $\hat{F}'$.

I showed then, that the set $(\Lambda, \geq)$ generates naturally a space $\mathcal{D}$—I did this by repeating basic steps of the original construction using merely these properties of the set $(\Lambda, \geq)$ which follow from the assumptions imposed by me.

Thus, if $\lambda = (\hat{F}, K) \in \Lambda$ then by virtue of Assumption A1 there exist on $Q_K$ a measure $d\mu_\lambda$ corresponding to the Lebesgue measure on $\mathbb{R}^N$. This measure provides a Hilbert space

$$\mathcal{H}_\lambda := L^2(Q_K, d\mu_\lambda)$$

as a space of "pure quantum states" of the system $\lambda$ and a space $\mathcal{D}_\lambda$ of all density operators on $\mathcal{H}_\lambda$ as a space of "mixed quantum states" of the system.

In the next step I proved that if $\lambda' \geq \lambda$ then it follows from the assumptions imposed on $(\Lambda, \geq)$ that the Hilbert space $\mathcal{H}_{\lambda'}$ splits naturally into two "factors"

$$\mathcal{H}_{\lambda'} = \tilde{\mathcal{H}}_{\lambda'\lambda} \otimes \mathcal{H}_{\lambda'\lambda}$$

such that $\mathcal{H}_{\lambda'\lambda}$ is naturally isomorphic to $\mathcal{H}_\lambda$. This decomposition allows to use a partial trace with respect to $\tilde{\mathcal{H}}_{\lambda'\lambda}$ to define a projection

$$\pi_{\lambda\lambda'} : \mathcal{D}_{\lambda'} \to \mathcal{D}_\lambda.$$

Then I showed that the family $\{\mathcal{D}_\lambda, \pi_{\lambda\lambda'}\}_{\lambda\in\Lambda}$ is a projective family which made it possible to define the space $\mathcal{D}$ as a *projective limit* of the family. Let me emphasize that the space $\mathcal{D}$ of quantum states obtained in this way *is not* a Hilbert space but rather *a convex set of quantum states.*

[4]According to a terminology used in canonical quantization the space $\hat{F}$ and a space of functions on $Q_K$ form *an algebra of elementary classical observables* [22] of a finite physical system.

[5]Poisson bracket is non-degenerate, if it defines a symplectic form.

Both methods of constructing the space $\mathcal{D}$ of quantum states, that is, the generalized method and the original one are founded on linearity of reduced configuration spaces and on linearity of the relation system–subsystem. The *main difference* between the methods is the source of these linearities: in the case of the original method the source are the assumed linearity of a phase space and the choice of elementary d.o.f. as linear functions on the phase space, while in the case of the generalized method the source are the assumptions A1, A3a and A3b which makes it possible to apply the generalized method to theories of non-linear phase spaces.

The list of assumptions the set $(\Lambda, \geq)$ should satisfy in order to generate the space $\mathcal{D}$ and the construction of the space from the set are the main results of [H4]. It should be stressed that these results *do not guarantee* either existence or uniqueness of a space $\mathcal{D}$ for every field theory. Actually, the uniqueness cannot be achieved—I managed to construct two distinct spaces $\mathcal{D}$ for TEGR [H5, H7] (the two spaces are based on distinct families of elementary d.o.f. defined on the same phase space of TEGR).

Let me now describe other important results presented in [H4], that is,

1. a construction of a space $\mathcal{D}$ for so called degenerate Plebański gravity,
2. a collection of some auxiliary propositions which are useful for constructing directed sets of the sort of $(\Lambda, \geq)$ for field theories,
3. a construction of a Hilbert space for a field theory based on some almost periodic functions.

From a practical point of view the main result of [H4] reduces the task of constructing a space $\mathcal{D}$ to one of constructing an appropriate directed set $(\Lambda, \geq)$. Thus it is essential to present a construction of such a set. In [H4] I did this for a simple background independent theory which I introduced in [P7]. Configuration variables of this theory are fields defined on a four-dimensional manifold $\mathcal{M}$: a function $\mathbf{\Psi}$, a one-form $\mathbf{A}$ representing a connection on a trivial principal bundle $\mathcal{M} \times \mathbb{R}$ and a two-form $\boldsymbol{\sigma}$. The action of the theory reads

$$S[\boldsymbol{\sigma}, \mathbf{A}, \mathbf{\Theta}] := \int \boldsymbol{\sigma} \wedge \boldsymbol{d}\mathbf{A} - \frac{1}{2} \mathbf{\Psi}\, \boldsymbol{\sigma} \wedge \boldsymbol{\sigma},$$

and its phase space is a set of all pairs $(\sigma, A)$ of fields defined on a three-dimensional manifold $\Sigma$—$\sigma$ is a two-form playing a role of the momentum conjugate to the one-form $A$. I called this theory *degenerate Plebański gravity* (DPG) since its action arose as a simplification of the self-dual Plebański action [23] and the theory describes so called $1 + 1$ degenerate sector of GR analyzed in [24].

Defining a suitable directed set $(\Lambda, \geq)$ for DPG I used some objects applied in LQG [4, 21, 22]: I chose elementary configurational d.o.f. to be integrals of the one-form $A$ along compact curves (edges) and elementary momentum d.o.f. as integrals over bounded two-dimensional surfaces:

$$\kappa_e(A) := \int_e A, \qquad\qquad \varphi_S(\sigma) := \int_S \sigma,$$

where $e$ is an edge, and $S$ a surface. Since the Poisson bracket $\{\varphi_S, \kappa_e\}$ is not well defined I used so called flux operator [22] as the linear operator $\hat{\varphi}_S$—the flux operator is obtained by a regularization of the bracket $\{\varphi_S, \kappa_e\}$. Given graph $\gamma \subset \Sigma$, I introduced a set $K_\gamma$ of configurational d.o.f. defined by edges of the graph. Then I defined a set $\Lambda$ as a collection of all pairs $\{(\hat{F}, K_\gamma)\}$ of non-degenerate "Poisson structure" (Assumption A2), where $\gamma$ runs through the set of all graphs in $\Sigma$. Next I introduced on $\Lambda$ the following relation:

$$(\hat{F}', K_{\gamma'}) \geq (\hat{F}, K_\gamma) \qquad \text{if and only if} \qquad \begin{cases} \hat{F}' \supset \hat{F} \\ \gamma' \geq \gamma \end{cases}$$

(the latter relation is a standard directing relation on the set of graphs used commonly in LQG). Then I showed that $(\Lambda, \geq)$ is a directed set and that it satisfies all assumptions needed to generate a space $\mathcal{D}$.

Constructing the set $(\Lambda, \geq)$ for DPG and proving that it satisfies the assumptions I concluded that it is worth to formulate and prove in a general manner some propositions used to reach the goals—the propositions formulated and proven in such a way were placed in [H4] (Section 3.6). They were used then in [H7] e.g. two propositions dealing with linear independence of operators belonging to $\hat{\mathcal{F}}$ turned out to be indispensable in proving that a set $(\Lambda, \geq)$ constructed in [H7] for TEGR is indeed a directed set.

While working on the general construction of a space $\mathcal{D}$ I noted that for every field theory for which a space $\mathcal{D}$ can be constructed there exists a Hilbert space. Namely, there exists a directed set $(\mathbb{K}, \geq)$ related to a directed set $(\Lambda, \geq)$ defining the space $\mathcal{D}$—the set $\mathbb{K}$ consists of sets $\{K\}$ used to construct the set $\Lambda$, and the relation $\geq$ corresponds to Assumption A3a. On every reduced configuration space $Q_K$ there exists a set of almost periodic functions which form a (non-separable) Hilbert space $\mathcal{H}_K$. I showed that the family $\{\mathcal{H}_K\}$ can be easily equipped with a structure of an inductive family, which allows to define a new Hilbert space $\mathcal{H}$ as *the inductive limit of the family.*

The Hilbert space $\mathcal{H}$ is a mathematical by-product of the construction of a space $\mathcal{D}$—so far I have not been able to find any physical application for $\mathcal{H}$.

### 4.3.4 Construction of a space of kinematic quantum states for TEGR

Three last papers of the series, that is, [H5], [H6] and [H7] are devoted to a construction of a space of kinematic quantum states for TEGR (and YMTM) by means of the generalized projective method—in [H5] and [H6] I introduced and analyzed new canonical variables for TEGR, finally in [H7] I described a construction of a suitable directed set $(\Lambda, \geq)$ for TEGR founded on the new variables.

**Publication [H5]** I began the paper [H5] by checking whether it is possible to build a suitable set $(\Lambda, \geq)$ for TEGR using elementary d.o.f. defined in a natural way by the canonical variables $(p_A, \theta^B)$ obtained in [H2, H1]:

$$\kappa^A_e(\theta) := \int_e \theta^A, \qquad \varphi^S_B(p) := \int_S p_B, \tag{4.5}$$

where $\theta \equiv (\theta^A)$ and $p \equiv (p_B)$. To this end I showed that for every graph $\gamma$ a set $K_\gamma$ of configurational d.o.f. given by edges of the graph defines a reduced configuration space $\Theta_{K_\gamma}$ which is in a natural bijection with an appropriate $\mathbb{R}^N$ (Assumption A1). This means that using d.o.f. (4.5) one can construct a directed set $(\Lambda, \geq)$ for TEGR which satisfies all needed assumptions—it is enough to repeat all steps of the construction of such a set for DPG modifying them slightly whenever it is necessary. Thus there exists a space $\bar{\mathcal{D}}$ of kinematic quantum states for TEGR generated by the d.o.f. (4.5).

I found however that the space $\bar{D}$ has a serious drawback. Namely, the d.o.f. (4.5) "do not know" anything about the condition defining the configuration space $\Theta$—let me recall that it is required that one-forms $(\theta^A) \in \Theta$ define a Riemannian metric via the formula (4.3). Consequently, quantum states constituting the space $\bar{\mathcal{D}}$ correspond also to those one-forms $(\theta^A)$ which define Lorentzian metrics. This means that the space $\bar{\mathcal{D}}$ is "too large" for the quantization of TEGR and in order to adjust it to my purposes I have to impose on the quantum states a restriction corresponding to that imposed on the variables $(\theta^A)$. I showed that this goal cannot be achieved by means of a family $\{R_\lambda\}_{\lambda\in\Lambda}$ of restrictions such that for each $\lambda \in \Lambda$ the restriction $R_\lambda$ is imposed on elements of the space $\mathcal{D}_\lambda$ used to build $\bar{\mathcal{D}}$. This fact together with high complexity of $\bar{\mathcal{D}}$ makes it very difficult to define the desired restriction. This conclusion meant in practice that it is necessary to find new canonical variables on the phase space of TEGR which would generate a space $\mathcal{D}$ free of the drawback of the space $\bar{\mathcal{D}}$.

In the first step to this end I introduced new variables on the configuration space $\Theta$—in fact, I introduced a family $\{(\xi^I_\iota, \theta^J)\}$ of variables on $\Theta$ labeled by an index $\iota$. For a fixed index

1. $(\xi^I_\iota)_{I=1,2,3}$ is a triplet of real functions on the manifold $\Sigma$,
2. $(\theta^J)_{J=1,2,3}$ is a triplet of one-forms on $\Sigma$ constituting a global coframe on the manifold.

New variables $(\xi^I_\iota, \theta^J)$ are related to the original ones $(\theta^A)$ as follows: the one-forms $(\theta^J)$ belong to the quadruplet $(\theta^A) = (\theta^0, \theta^J)$, while the three functions $(\xi^I_\iota)$ contain information about components of $\theta^0$ given by the coframe $(\theta^J)$. I justified the choice of the new variables $(\xi^I_\iota, \theta^J)$ by invoking to

1. properties of the metric (4.3) expressed in terms of the new variables,
2. a natural interpretation of the variables,
3. properties of elementary configurational d.o.f. defined naturally by the new variables.

Namely, I proved, that the metric (4.3) when expressed as a function of the new variables cannot be Lorentzian even if one gives up the requirement that $(\theta^J)$ form a global coframe on $\Sigma$. It means that if it is possible to construct a space $\mathcal{D}$ for TEGR founded on $(\xi^I_\iota, \theta^J)$ and corresponding momenta then quantum states in $\mathcal{D}$ cannot be related to Lorentzian metrics on $\Sigma$.

Regarding the interpretation of the variables $(\xi^I_\iota)$: in the descriptions of the Hamiltonian structures of TEGR and YMTM presented in [H2, H3, H1] an important role is played by a quadruplet $(\xi^A)_{A=0,1,2,3}$ of functions on $\Sigma$. These functions enable to reconstruct the time-like component $(\boldsymbol{\theta}^A_\perp)$ of the spacetime coframe from the one-forms $(\theta^A) \in \Theta$, the lapse function $N$ and the shift vector field $\vec{N}$:

$$\boldsymbol{\theta}^A_\perp = N\xi^A + \vec{N}\lrcorner\theta^A,$$

the functions $(\xi^A)$ appear also in constraints and Hamiltonians of TEGR and YMTM. The new variables $(\xi^I_\iota)$ are identical (modulo the factor $-1$) to functions $\xi^1, \xi^2, \xi^3$ being elements of the quadruplet $(\xi^A)$.

The new variables $(\xi^I_\iota, \theta^J)$ define naturally the following elementary configurational d.o.f:

$$\kappa^I_y(\theta) := \xi^I_\iota(y), \qquad\qquad \kappa^J_e(\theta) = \int_e \theta^J, \tag{4.6}$$

where $\theta \equiv (\xi^I_\iota, \theta^J)$, and $y$ is a point of the manifold $\Sigma$. I showed that the new variables possess a number of properties which are desirable from the point of view of their future application to a background independent construction of a directed set $(\Lambda, \geq)$ for TEGR. In particular, I proved that a set $K_{u,\gamma}$ of d.o.f. (4.6) given by points belonging to a finite set $u \in \Sigma$ and by edges of a graph $\gamma$ defines a reduced configuration space $Q_{K_{u,\gamma}}$ being in a natural bijection with an appropriate $\mathbb{R}^N$.

At this point I finished the paper [H5].

**Publication [H6]** I continued the analysis of the family $\{(\xi^I_\iota, \theta^J)\}$ of new variables in [H6] obtaining the following results:

1. I found a simple criterion distinguishing differentiable (in the sense of variational calculus) variables $(\xi^I_\iota, \theta^J)$ from non-differentiable ones;

2. I introduced momenta $(\zeta_{\iota I}, r_J)$ conjugate to differentiable variables $(\xi^I_\iota, \theta^J)$—$\zeta_{\iota I}$ is a three-form on $\Sigma$ playing a role of the momentum conjugate to $\xi^I_\iota$, while the two-form $r_J$ is the momentum conjugate to $\theta^J$;

3. I expressed the new canonical variables $(\zeta_{\iota I}, r_J, \xi^K_\iota, \theta^L)$ as functions of the original ones $(p_A, \theta^B)$ and *vice versa*;

4. I expressed constraints (and thereby Hamiltonians) of TEGR and YMTM in terms of the new canonical variables;

5. I showed, that the Hamiltonian structures of TEGR and YMTM described in [H2, H1] *distinguish two collections of variables* $(\zeta_{\iota I}, r_J, \xi^K_\iota, \theta^L)$. Namely, for all other variables of this sort there exists an obstacle for defining quantum constraints: if one can build a space $\mathcal{D}$ of quantum states founded on d.o.f. defined naturally by these variables then in the case of some constraints of TEGR and YMTM found in [H2, H1] there do not exist their quantum counterparts in a form of a family of operators $\{\hat{C}_\lambda\}_{\lambda\in\Lambda}$ such that $\hat{C}_\lambda$ is a constraint operator on the Hilbert space $\mathcal{H}_\lambda$. For the two collections of the variables mentioned above this obstacle does not appear.

**Publication [H7]** The final result of the series [H1]-[H7] is a background independent construction of a space $\mathcal{D}$ of kinematic quantum states for TEGR (and YMTM)—this construction reduces to a construction of an appropriate directed set $(\Lambda, \geq)$ described in [H7].

To construct a set $(\Lambda, \geq)$ for TEGR I used d.o.f. defined by one of the two distinguished collections of variables $(\zeta_{\iota I}, r_J, \xi^K_\iota, \theta^L)$, although this particular construction is valid in the case of any variables of this sort. Configurational d.o.f. applied in the construction are obviously the d.o.f. (4.6), momentum d.o.f. applied are defined as follows:

$$\varphi^V_I(p) := \int_V \zeta_{\iota I}, \qquad\qquad \varphi^S_J(p) := \int_S r_J,$$

where $p \equiv (\zeta_{\iota I}, r_J)$, and $V \subset \Sigma$ is a three-dimensional submanifold of $\Sigma$. An operator $\hat{\varphi}^V_I$ was defined by means of the Poisson bracket (4.4), and a flux operator was chosen to play a role of $\hat{\varphi}^S_J$ similarly as it was done in the case of DPG.

Results obtained in [H5] suggest that a set $\Lambda$ for TEGR should be defined as a collection of all pairs $\{(\hat{F}, K_{u,\gamma})\}$ of non-degenerate "Poisson structure" (let me recall that here $u$ is a finite subset of $\Sigma$ and $\gamma$ is a graph). However, I found it desirable to be able to define on every Hilbert space $\mathcal{H}_\lambda$ used to build the space $\mathcal{D}$ a sort of quantum geometry generated by the Riemannian geometry of $\Sigma$. I showed that such a geometry can be easily defined if a pair $(u, \gamma)$ satisfies some simple conditions. I called *speckled graph* a pair $\dot{\gamma} \equiv (u, \gamma)$ meeting these conditions and proved that all speckled graphs form a directed set.

Finally, I defined the set $\Lambda$ for TEGR as one consisting of all pairs $\{(\hat{F}, K_{\dot{\gamma}})\}$ of non-degenerate "Poisson structure", where $\dot{\gamma}$ runs through a set of all speckled graphs in $\Sigma$. Then I introduced on $\Lambda$ the following relation:

$$(\hat{F}', K_{\dot{\gamma}'}) \geq (\hat{F}, K_{\dot{\gamma}}) \qquad \text{if and only if} \qquad \begin{cases} \hat{F}' \supset \hat{F} \\ \dot{\gamma}' \geq \dot{\gamma} \end{cases},$$

and proved that $(\Lambda, \geq)$ is a directed set which meets all requirements imposed on such a set in [H4]. This means, that $(\Lambda, \geq)$ generates a space $\mathcal{D}$ of kinematic quantum states for TEGR (and YMTM).

Moreover, in [H7] I presented two other important results:

1. I showed that the natural action of diffeomorphisms of $\Sigma$ on the fields $(\zeta_{\iota I}, r_J, \xi^K_\iota, \theta^L)$ induces an action of these diffeomorphisms on the space $\mathcal{D}$ and that this action preserves the space—this result indicates that the space $\mathcal{D}$ can be applied to a background independent quantization of TEGR;

2. I proved that the two distinguished collections of variables $(\zeta_{\iota I}, r_J, \xi^K_\iota, \theta^L)$ generate the same space $\mathcal{D}$.

### 4.3.5 Summary of the results

The series of publications [H1]-[H7] describes results obtained so far while working on background independent canonical quantization of TEGR. In these papers

1. I described Hamiltonian structures of TEGR and YMTM in a way consistent with demands of background independent canonical quantization;
2. I elaborated the generalized method of constructing spaces of kinematic quantum states for field theories by means of projective techniques—the generalization consists in an extension of applicability of the original method to theories of non-linear phase spaces;
3. I carried out the first step of the Dirac's quantization of TEGR and YMTM—applying the generalized method I constructed in a background independent manner the space $\mathcal{D}$ of kinematic quantum states for these theories;
4. I defined the action of diffeomorphisms of a three-dimensional manifold $\Sigma$ on the space $\mathcal{D}$ and proved that this action preserves the space.

To the best of my knowledge

1. the description of the Hamiltonian structure of TEGR presented in [H2] is the first description of this structure obtained by the ADM-like decomposition of a spacetime coframe (that is, the decomposition into the lapse $N$, the shift $\vec{N}$ and the space-like part $(\underline{\boldsymbol{\theta}}^A)$) which satisfies the criteria C1–C3,
2. the space $\mathcal{D}$ constructed for TEGR is the first space of kinematic quantum states built for this formulation of GR.

#### 4.3.6 Applications of the results

The results obtained in the series of publications [H1]-[H7] allow to begin the second stage of the Dirac's quantization of TEGR which consists in defining on the space $\mathcal{D}$ (and solving) quantum constraints as counterparts of classical constraints imposed on the canonical variables. In particular, the action of spatial diffeomorphisms on $\mathcal{D}$ may be used to find solutions of quantum constraints corresponding to the classical vector constraint $V(\vec{M})$—following LQG [21] one can treat as such a solution each element of the space $\mathcal{D}$ preserved by the action of diffeomorphisms (provided such an element exists).

Moreover, the method of constructing spaces of quantum states presented in [H4] seems to be general enough to attempt to apply it to other fields theories or to other formulations of general relativity—constructions of new spaces of quantum states for LQG and loop quantum cosmology refering to papers of J. Kijowski and of mine were presented [25] during a recent conference.

## 5 Description of other scientific achievements

### 5.1 Other publications

[P1] Lewandowski J, Okołów A, 2000 2-Form Gravity of the Lorentzian Signature. *Class. Quant. Grav.* **17** L47–L51. `arXiv:gr-qc/9911121`.

[P2] Okołów A, Lewandowski J, 2003 Diffeomorphism covariant representations of the holonomy-flux *-algebra. *Class. Quant. Grav.* **20** 3543–3567. arXiv:gr-qc/0302059.

[P3] Okołów A, Lewandowski J, 2005 Automorphism covariant representations of the holonomy-flux *-algebra. *Class. Quant. Grav.* **22** 657–679. arXiv:gr-qc/0405119.

[P4] Okołów A, 2005 Hilbert space built over connections with a non-compact structure group. *Class. Quant. Grav.* **22** 1329–1359. arXiv:gr-qc/0406028.

[P5] Lewandowski J, Okołów A, Sahlmann H, Thiemann T, 2006 Uniqueness of diffeomorphism invariant states on holonomy-flux algebras. *Comm. Math. Phys.* **267** 703–733. arXiv:gr-qc/0504147.

[P6] Kamiński W, Lewandowski J, Okołów A, 2006 Background independent quantizations: the scalar field II. *Class. Quant. Grav.* **23** 5547–5586. arXiv:gr-qc/0604112.

[P7] Okołów A, 2009 Quantization of diffeomorphism invariant theories of connections with a non-compact structure group—an example. *Comm. Math. Phys.* **28** 335–382. arXiv:gr-qc/0605138.

[P8] Lewandowski J, Okołów A, 2009 Quantum group connections. *J. Math. Phys.* **50** 123522. arXiv:0810.2992.

[P9] Dziendzikowski M, Okołów A, 2010 New diffeomorphism invariant states on a holonomy-flux algebra. *Class. Quant. Grav.* **27** 225005. arXiv:0912.1278.

## 5.2 Description of the publications above

My first scientific achievement was a description of a canonical structure of general relativity derived from a new action postulated for this theory by J. Lewandowski. A summary of obtained results was published in [P1].

Then I worked on representations of so called holonomy-flux $*$-algebra. This algebra is an essential element of the canonical quantization procedure which leads from general relativity described in terms of the real Ashtekar-Barbero variables [26] to LQG [21, 4]. In the most general sense, a holonomy-flux algebra is a $*$-algebra of "abstract" operators constructed over a phase space which consists of pairs $(E, A)$, where $A$ is a connection on a principal bundle $P(\Sigma, G)$ of the base manifolds $\Sigma$ and the structure group $G$ and the momentum $E$ conjugate to $A$ is an ad-covariant $(\dim \Sigma - 1)$-form on the bundle—in the case of LQG $P = \Sigma \times SU(2)$, where $\dim \Sigma = 3$.

A space of kinematic quantum states used in LQG is a Hilbert space $\mathcal{H}_{AL}$ of wave functions defined on a space of (generalized) $SU(2)$-connections square integrable with respect to the Ashtekar-Lewandowski measure [20]. This model of quantum gravity is background independent (diffeomorphism invariant)—this feature manifests through *diffeomorphism covariance* of a representation of the holonomy-flux algebra on $\mathcal{H}_{AL}$ the model is founded on.

An issue of other representations of holonomy-flux algebras was taken up for the first time by H. Sahlmann [27][6]. In [28] he showed that if a representation of a holonomy-flux algebra based on a $U(1)$-bundle is diffeomorphism covariant then the carrier Hilbert space of the representation is a space of wave functions square integrable with respect to the Ashtekar-Lewandowski measure.

With some help of J. Lewandowski I managed to generalize the Sahlmann proposition to the case of a holonomy-flux algebra founded on a trivial principal bundle $\mathbb{R}^n \times G$, where $G$ is any compact connected Lie group. This result was presented in [P2]. Then I achieved even more general result—I showed that the Sahlmann proposition is true in the case of any principal bundle $P(\Sigma, G)$ of a compact connected structure group $G$. This generalization was described in [P3].

The final result of research on diffeomorphism covariant representations of a holonomy-flux algebra is a *theorem of existence and uniqueness of diffeomorphism invariant state (linear functional)* on the algebra—this state defines via the GNS construction a diffeomorphism covariant representation of the algebra. This theorem was presented and proved in [P5]. My contribution to this result (consisting in some essential elements of the formulation and of the proof of the theorem) was smaller than those of J. Lewandowski and H. Sahlmann.

The theorem proved in [P5] is true provided the holonomy-flux algebra satisfies a requirement. I found an example (also presented in [P5]) of a holonomy-flux algebra which does not meet the requirement and on which there exist two distinct diffeomorphism invariant states. This led naturally to a question about other examples of this sort. Such an example was described in [P9]—a holonomy-flux algebra presented there does not satisfy the requirement mentioned above and there are infinitely many diffeomorphism invariant states on the algebra. An idea of how to construct the algebra and the states come form M. Dziendzikowski. My contribution to [P9] consists in elaborating an essential part of a proof of a theorem which states that a formula proposed by the coauthor defines a diffeomorphism invariant linear functional on the algebra.

A problem related to the problem of existence and uniqueness of diffeomorphism invariant state on a holonomy-flux algebra is one of classification of homeomorphism invariant states on a $*$-algebra of "abstract" operators constructed for a scalar field theory. This problem was investigated by W. Kamiński and J. Lewandowski. One of elements of a solution to this problem found by them was a lemma formulated and proved by me which at that time had not been published yet. Therefore W. Kamiński and J. Lewandowski decided to add my name to the list of authors of [P6] describing their results.

An other issue I worked on was one of constructing a non-commutative counterpart of so called *space of generalized connections*. The space $\overline{\mathcal{A}}$ of generalized connections is an extension of a space $\mathcal{A}$ of connections on a principal bundle $P(\Sigma, G)$ of a compact structure group. This extension was introduced as an element of quantization—the space of generalized $SU(2)$-connections defined over three-dimensional base manifold $\Sigma$ is a quantum configurational space for LQG [29]. The space $\overline{\mathcal{A}}$ can be defined as the Gelfand' spectrum of a commutative $C^*$-algebra called Ashtekar-Isham (AI) algebra. J. Lewandowski noted that

[6]Both works [27] and [28] by H. Sahlmann were published as preprints in 2002, and published in a reviewed journal only in 2011.

the AI algebra can be constructed from the $C^*$-algebra $C^0(G)$ of continuous functions on a Lie group $G$ by means of some inductive techniques and formulated the following problem: is it possible to replace in the construction of the AI algebra the commutative algebra $C^0(G)$ by its non-commutative "deformation", or strictly speaking, by a corresponding Woronowicz's compact quantum group [30] and obtain thereby a non-commutative AI algebra which, according to philosophy of non-commutative geometry, "defines" a space of "non-commutative connections"? I gave an affirmative answer to this question by presenting a non-commutative AI algebra constructed from the quantum group $SU_q(2)$ [31]. This construction was published in [P8].

I was also engaged in an issue of constructing spaces of quantum states for background independent theories of connections with a *non-compact* structure group. The main motivation to work on this issue was a desire to construct a space of quantum states for general relativity expressed in terms of complex Ashtekar variables [32, 33]—one of these variables is an $SL(2,\mathbb{C})$-connection. This is quite an essential problem since inability to construct such a space is a reason for which LQG gravity is based on the Ashtekar-Barbero [26] connection of the structure group $SU(2)$ which means that in this model of quantum gravity the Lorentzian symmetry of general relativity is broken to that of spatial rotations.

Working on this issue I introduced a diffeomorphism invariant positive definite scalar product on a set of functions defined on a space of connections of the structure group $\mathbb{R}$ (the set $\mathbb{R}$ of real numbers equipped with addition is the simplest non-compact Lie group). I showed moreover that there exists an obstacle which in practice makes it impossible to define a representation of the corresponding holonomy-flux algebra on a Hilbert space given by the scalar product and on similar Hilbert spaces like that presented in [34]. These results were published in [P4].

I continued further investigations of this issue taking an advantage of a hint given me by J. Kijowski to attempt to construct spaces of quantum states by means of the projective techniques [5]. Desiring to check whether these techniques are applicable to background independent theories of connections of non-compact structure group I introduced a simple model of such a theory being the degenerate Plebański gravity described earlier. By means of the techniques I managed to construct a space of quantum states for DPG (here I mean a construction essentially different from that presented in [H4]). I found moreover a broad class of physical quantum states (that is, quantum states satisfying all constraints) for this model. A description of these results can be found in [P7].

Let me add finally that the paper [P1] summarizes results of my master thesis while works [P2, P3, P4, P8] are based on my PhD thesis.

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