1 First and last name

Mikołaj Konrad Korzyński

2 Scientific degrees

- Sept 2002: Master degree in theoretical physics, general relativity (honours degree), University of Warsaw. Thesis *"Bach tensor and normal conformal Cartan connection"* written under the supervision of Prof. Jerzy Lewandowski
- March 2008: PhD degree in physics, University of Warsaw. Thesis *"Dynamical aspects of the quasi-local black hole theory"*, supervisor: Prof. Jerzy Lewandowski

3 Employment in academic institutions

- April 2008 – Sept 2010: Postdoc at the Max Plack Institute for Gravitational Physics in Potsdam, Germany
- Oct 2010 – Aug 2012: Postdoc/Assistant professor (*Universitätsassistent*) at the University of Vienna, Faculty of Physics, Gravitational Physics Group
- Sept 2012 – present: Assistant professor (*adiunkt*) at the Center for Theoretical Physics, Polish Academy of Sciences, Warsaw.

4 Scientific accomplishment

a) Title of the scientific accomplishment

Coarse-graining in general relativity (series of publications)

b) List of publications (authors, title, journal, year)

[H1] Mikołaj Korzyński, *"Covariant coarse-graining of inhomogeneous dust flow in General Relativity"*, Classical and Quantum Gravity **27** (2010) 105015, 21 pp.

[H2] Eloisa Bentivegna, Mikołaj Korzyński, *"Evolution of a periodic eight-black-hole lattice in numerical relativity"*, Classical and Quantum Gravity **29** (2012) 165007, 20 pp.

[H3] Eloisa Bentivegna, Mikolaj Korzyński, *"Evolution of a family of expanding cubic black-hole lattices in numerical relativity"*, Classical and Quantum Gravity **30** (2013) 235008, 18 pp.

[H4] Mikołaj Korzyński, *"Backreaction and continuum limit in a closed universe filled with black holes"*, Classical and Quantum Gravity **31** (2014) 085002, 32 pp.

[H5] Mikołaj Korzyński, *"Nonlinear effects of general relativity from multiscale structure"*, Classical and Quantum Gravity **32** (2015) 215013, 32 pp.

[H6] Mikołaj Korzyński, Ian Hinder, Eloisa Bentivegna, *"On the vacuum Einstein equations along curves with a discrete local rotation and reflection symmetry"*, JCAP 1508 (2015) 08, 025

c) Description of the scientific goals of the above mentioned work, obtained results and prospects of applications

The description is organized as follows: in the next section I discuss the coarse-graining and the backreaction problem in general relativity. Then I present in Section 4.2 the state of the art before the papers [H1-H6] appeared and outline three possible approaches to the problem. In Section 4.4 I focus on the specific unsolved questions I addressed in my publications. I discuss the publications in detail in Sections 4.5–4.9 and conclude with a summary and the future prospects.

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4.1 Introduction: Coarse-graining and the backreaction problem in general relativity

Coarse-graining is one of the fundamental ideas of theoretical physics, allowing to describe very complicated systems with a large number of degrees of freedom in an approximate but much simpler and tractable way. The basic idea is very simple: we carefully choose a number of variables describing the state of the system on large scale disregarding the fine details. We then derive the effective equations governing the behaviour of these collective, large-scale variables under certain reasonable assumptions about the physics of the fine scales and its influence on the large-scale variables. If the choice we have made is correct then the effective equations form a closed system and we obtain a good approximation of the physics of the full system.

Rigid body mechanics provides an old and excellent example of the power of the coarse-graining approach: from the microscopic point of view a rigid body is an enormously complicated system composed of a crystalline or amorphous lattice of atoms bound by electromagnetic forces. Nevertheless when the forces in question, the velocities and angular velocities are small enough we may describe its motion completely using surprisingly few parameters: the position of its center of mass, its velocity, body's total mass and moments of inertia, three Euler's angles describing its orientation and its angular velocity. It turns out that the complicated, small scale dynamics of the system doesn't play any significant role as long as we are only interested in the collective motion of all constituents of the body.

This approximation has its limitations of course: it tends to break down if the forces in question are too large or the spin is too fast, although we may still improve it by including the solid mechanics of the system, its oscillatory modes etc. Coarse-graining provides thus more than just a set of effective parameters describing a complicated system and a set of relations between them. It can also provide a framework in which we may find ways to improve the effective description when the system approaches its limits of validity.

Coarse-graining is widespread in many other fields of theoretical physics. The Navier-Stokes equations can be thought of as arising from the coarse-graining of the equations of the kinetic theory of gases. The renormalization group flow in quantum field theory can be interpreted as coarse-graining over very small scales or equivalently over very high energies. One of very few branches of physics where the problem of coarse-graining has not been researched too well is general relativity (GR).

GR is currently the established theory of gravitation and at the same time the theory of the geometry of spacetime. It postulates that the geometry of spacetime is pseudo-Riemannian and that the metric tensor $g_{\mu\nu}$ of the spacetime, describing the way we measure angles and distances, is related to the matter content of the spacetime via a system of second order partial differential equations (PDE's), called the Einstein equations[1]

$$G_{\mu\nu} = 8\pi G\, T_{\mu\nu}. \tag{1}$$

They relate the curvature of the geometry, given by a combination of the second derivatives of the metric tensor and represented by the Einstein tensor $G_{\mu\nu}$, to the stress-energy tensor of the matter $T_{\mu\nu}$. Note that while the left hand side of the Einstein equations is obviously a local, geometric object depending on the metric tensor and its partial derivatives at a given point p, the right hand side is usually understood as a *statistical* object, encoding the information about the average matter content in a small region of spacetime in the vicinity of the point p. This statistical, somewhat nonlocal interpretation of $T_{\mu\nu}$ is widespread in GR textbooks. It is clear that it tacitly involves the coarse-graining of the finest structures of matter: rather than solving the Einstein equations with all the details of the distribution of mass on atomic and subatomic scales we are supposed to represent it by a suitable *local average* in which the inhomogeneities on scales much lower that the scale of the system are absent. The stress-energy tensor is thus a "naïve" average of the matter content, given by the sum of contributions from all matter constituents present in the vicinity of p, while $g_{\mu\nu}$ represents the geometry of spacetime without the distortions caused by the granularity of matter. This interpretation is very useful and prevalent in astrophysical relativity because it allows to solve the Einstein equations for objects like galaxies or galaxy clusters without investigating their small scale structure, but it immediately raises a number of questions.

Firstly, can it be justified rigorously if we assume GR to hold exactly locally, down to the finest scales of the system? Recall the classical Maxwell's theory of electromagnetism, where we represent a complicated motion of electrons or ions by a large-scale, averaged electric current. This procedure is easy to justify in this case because the equations involved are exactly linear. The local Maxwell equations

[1] We assume the light speed $c = 1$ thoroughout the text.

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for complicated electric and magnetic fields on atomic level imply the same equations holding for the averaged, large-scale electric and magnetic fields, provided that we average the sources (i.e. charges and currents) in an appropriate way. But for a nonlinear system of PDE's like (1) it is entirely unclear whether the large-scale equations, obtained by averaging the local metric, will have the form of Einstein equations with the "naïve" average stress-energy tensor on the right hand side. In a generic case we expect those two to be different and the difference between them is often referred to as the *backreaction*. Secondly, on a more fundamental level it is unclear how actually we are supposed to understand the average tensorial quantities like the metric or the stress-energy tensor. In a general spacetime the geometry does not admit any symmetries and consequently we cannot rely on the Euclidean structure of the spacetime to define the coarse-graining in terms of, for example, the Fourier transform. In fact there is no general coordinate-independent way to compare or combine tensorial quantities at two distant points which makes the standard prescription for coarse-graining in physics, integrating over a region and dividing by its volume, impossible to use as it yields a strongly coordinate system-dependent result.

4.2 State of the art

The problem of backreaction and averaging, despite being fundamental in the discussion of applicability of GR to astrophysics, has attracted surprisingly little attention throughout most of the history of mathematical relativity. In 1968 Isaacson considered the propagation of gravitational waves on a general background metric in the perturbative, short wavelength approximation [1, 2]. He calculated the influence of the gravitational waves on the large-scale stress-energy tensor obtained by averaging out the waves and considering the nonlinear relation between the metric and the stress-energy tensor. This way Isaacson derived the expression for nonlinear corrections in coarse-graining over the graviational wave modes.

However, most of research in this direction so far has been done in the context of the backreaction and fitting problems in cosmology [3, 4]. Recall that in the modern cosmological paradigm we assume that the geometry of the Universe in the largest scales is described by a manifold equipped with a homogeneous and isotropic metric whose physical scale evolves in time, called the Friedman-Lemaître-Robertson-Walker metric (FLRW). In the FLRW model the matter is assumed to be distributed in an exactly homogeneous way. Both the fitting problem and the backreaction problem have their origin in the observation that in the physical Universe the matter distribution is far from homogeneous on lower scales and any realistic backreaction model needs to take this fact into account. Ellis and Stoeger [3] noted that this fact affects the observations and introduces an observation bias in cosmology. This path of research led to the study of the direct influence of inhomogeneities on the observations [5, 6, 7, 8, 9]. Another approach was presented later by Buchert and others, when they noted that the small scale structure can potentially influence the large scale dynamics of the Universe in many ways [10, 11, 12, 13].

The topic sparked a hot debate around early 2000's, when some authors suggested that the backreaction terms may explain the apparent accelerated expansion of the universe observed recently in the redshift-luminosity relation for the type I supernovae [13, 14]. The claim was rejected by the mainstream community, but the problem of backreaction in cosmology remained discussed until now. In particular, Buchert's research programme was challenged by Wald and Ishibashi [15]. Their arguments against Buchert's approach and the backreaction idea in general rely on a set of assumptions about the metric tensor of the Universe $g_{\mu\nu}$: they assume that it does not deviate very much from an average, large-scale one together with its derivatives and show that this is consistent with the Universe having very large density contrasts. Under these assumptions they show that the Einstein equations reduce to a simple, linear system for one variable and the nonlinear terms are negligible. In their later papers Green and Wald [16, 17] present a more refined approach, where they effectively assume that the metric is close to the large-scale one, its first derivatives are controlled and satisfy a technical condition concerning its weak limit. They later derive an expression for backreaction as a traceless, positive tensor, and argue that in the context of cosmology it is fairly small.

The arguments presented in [15, 16, 17] rely on the assumption that the deviation of the physical metric from the average one is small everywhere and therefore linearization or a similar type of perturbative approximation is valid. They advocate the approach based on *assuming a priori* that the physical metric is FLRW plus corrections which are small everywhere, adding perturbations, evaluating various observables and fitting the parameters of the model to observations - as it is commonplace in cosmology nowadays.

Unfortunately when the condition of small perturbations is not met, *even locally*, the validity of the whole approach is very much in question. In cosmology and astrophysics we expect this condition to break down for example close to very compact massive objects like neutron stars or black holes. The question

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is thus whether or not the results of naive averaging can be applied in the presence of regions with strong distortions of the metric tensor, which lie beyond the validity of the perturbative expansion, and how should one possibly correct them. Additionally, it is possible that the inhomogeneities in question, while large, mimic the observational properties of the homogeneous model (with small perturbations) [18, 19].

The backreaction discussion has been going on until 2016, with Buchert et al pointing out various problems with the Green-Wald formalism [20], including the issue of its gauge dependence and the correct definition of the average quantities. Green and Wald responded with 2 further papers [21, 22], where they defended the validity of their approach.

The general problem of the impact of inhomogeneities in cosmology can be divided into two related but distinct components:

- The question of the *physical backreaction*: how should we define the coarse-grained field variables, especially the metric tensor? Note that the prescription should be covariant, i.e. it should not depend upon structures introduced ad hoc such as a solution-specific coordinate system. Otherwise the results will be coordinates-dependent and thus difficult to interpret. Given a coarse-graining formalism, how do the Einstein equations behave under the coarse-graining? This part of the backreaction problem belongs more to the realm of mathematical relativity and PDE theory.

- The question of the *observational consequences of the inhomogeneities*: how do the inhomogeneities affect the light propagation and the cosmological observations (redshift-luminosity relation etc.) as well as the data analysis? This problem is obviously related to the previous one, since the physical backreaction effects on the large scale obviously affect also the light propagation. It is important from the point of view of the observational cosmology.

In papers [H1-H6] I am concerned mostly with the first problem. The research presented there belongs therefore to the field of mathematical relativity. The main questions I was addressing in my work can be summarized as follows:

- How large are the backreaction effects in real astrophysical situations? In particular, how do they affect the large scale dynamics in cosmology? These problems are difficult once we leave the linear or perturbative regime in GR. They are difficult to attack in full generality, so we may begin by somewhat simplified questions: how do the backreaction terms depend on the details of the microscopic structure and for which matter distributions can they be significant?

- What is the status of the fluid approximation in GR and cosmology? This is especially interesting for highly discretized matter content, i.e. matter in the form of compact objects (black holes, neutron stars etc.) in the vicinity of which the linear approximation in GR breaks down. Can one prove the existence of a continuum limit for a large number of objects?

4.3 Possible approaches to the backreaction and coarse-graining problem

In the literature one may identify three lines of reasearch approaching the backreaction problem. The first one is the *perturbative approach* in which the metric tensor is expanded using some kind of approximation scheme (short-wave approximation [2, 23], linearized GR [24, 25] or the combination of both [16, 26, 22]). The results obtained this way have a limited regime of validity.

Secondly, one may study the exact solutions of the Einstein equations corresponding to inhomogeneous cosmological models. Unfortunately not too many exact solutions of this type are known. The largest family is the Swiss-cheese family (or cut-and-paste family, as David Wiltshire calls them appropriately), in which some parts of the homogeneous FLRW solutions have been replaced by a Lemaître-Tolman-Bondi metric, the Szekeres metric or the Schwarzschild solution [27, 28, 29, 30]. Models of this type have been investigated by many authors [31, 32, 33]. Their evolution can be performed analitically (up to quadratures) and their basic properties are well known. However they require a very special, simplified geometry of the inhomogeneities. The study of other examples requires numerical techniques. The progress in the numerical relativity in the recent decade, fuelled by the study of the black hole mergers [34, 35], has made it possible to solve within a reasonable time the full 3+1 equations for fairly complex matter distributions with black holes as matter sources at least on a sufficiently powerful computer cluster. This has been done for the first time in [H2].

Rather than considering the full time evolution of the Einstein equation one can focus on the initial data. Recall that the initial data for the Einstein equations consist of a three-manifold with a positive

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metric q_{ij} and the extrinsic curvature tensor K_{ij}. The data are subject to the 4 constraint equations relating them to the matter distribution:

$$\begin{aligned} R^{(3)} + \left(K^i{}_i\right)^2 - K_{ij}\,K^{ij} &= 16\pi G\,T_{00} \\ D_i K^i{}_j - K_{,j} &= 8\pi G\,T_{0j} \end{aligned}$$

These equations have a physical meaning and their solution are sufficient to study backreaction in the T_{0i} and T_{00} components of the stress-energy tensor. They are also significantly easier to work with and many techniques have been developed to solve them, including the conformal decomposition method of Lichnerowicz and York [36, 37]. The study of these examples has been pioneered by Linquist and Wheeler [38], who were first to consider the black hole lattices (BHL's), i.e. arrangements of black holes with a discrete symmetry resembling a crystal lattice. They can be constructed easily using exact expression thanks to the Lichnerowicz ansatz. Later these initial data have been considered in [39, 12].

However the most valuable results one may envisage are exact, analytical results valid for a large family of solutions. They could have the form of exact expressions for backreaction or inequalities estimating it. Due to the difficulty of the problem no results of this kind have appeared in the literature before my papers.

4.4 Specific problems discussed in the series of publications

The papers I present for the habilitation had a common goal of advancing the understanding of the mathematics of coarse-graining in GR, the backreaction effects for inhomogeneous distributions of matter and the continuum limit problem beyond the regime of applicability of simple perturbative techniques. I would like to give here a short overview of the "missing gaps" in the literature I addressed in the papers presented here.

- **Exact results concerning the backreaction.** One obvious gap is the lack of any general but exact results concerning the value of the backreaction. While it is very likely that there is no simple expression for backreaction valid in any situation, one may still consider a physically interesting family of solutions and aim for results in the form of inequalities bounding the backreaction from above. The use of inequalities of various types has a long history in the context of GR, see for example the positive mass theorem or the inequalities concerning the area, angular momentum and mass of a horizon of a black hole [40, 41, 42]. Inequalities, while not as useful or versatile as full, exact expressions for given quantities, still provide us with information about the quantity they estimate. In particular, they may be useful in determining when we need to take the backreaction effects into account. Apart from that the way the bound depends on the properties of the solution has a often deeper physical meaning. This observation leads us in a natural way to the next unsolved problem.

- **Dependence of the backreaction terms on the details of the microscopic matter distribution.** It is not at all clear which details of the microscopic matter distribution matter when we evaluate the backreaction terms. This is obviously a crucial problem if we would like to apply statistical methods to the cosmic structure: it is impossible to apply GR consistently if we do not know which properties of the distribution of small scales give rise to large relativistic effects on larger scales. The backreaction terms in various averaging or coarse-graining schemes have usually a simple expression in terms of the properties of the microscopic metric tensor, but these results are of limited use because the microscopic metric is more difficult to determine form observations made from large distances.

- **Exact results concerning the continuum limit.** Discretized matter distributions with very compact, relativistic sources can be described using the dust or the Einstein-Vlasov collisonless matter approximation. Intuitively we might expect the approximation to work better if the ratio between the macroscopic scale and the scale of compact objects is larger, or equivalently - if the number of the objects diverges while keeping the macroscopic parameters intact. This is known in statistical physics as the *continuum limit.* Does this limit always exist in GR? Does the metric tensor in this case converge to the appropriate large-scale average one and in what sense? Note that due to the presence of the compact objects this problem lies outside the validity regime of linearized Einstein equations.

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- **Discussion of the geometry of the general Linquist-Wheeler spherical universe models without symmetries.** Clifton, Tavakol and Rosquist discussed thoroughly the geometry and other properties of the initial data with regular BHL's on S^3 [12], Wheeler described in detail the 5BH configuration in [39], but these results were limited to configurations with the largest possible groups of isometries. It would be interesting to see what features of these models carry over to BH arrangements without any symmetries imposed.
- **Full 3D numerical simulations of black hole lattices.** As we mentioned, the BHL's offer a simple and excellent example of a matter distribution which is homogeneous on large scales, but microscopically strongly inhomogeneous and relativistic. This makes them one of the most important families of solutions one would like to study in the context of backreaction as well as a perfect testbed for coarse-graining schemes. Many papers have dealt with the evolution of these models, including the seminal Lindquist-Wheeler paper [38], but only using various approximation schemes [43, 44, 45]. However the advancement of the numerical relativity techniques in the early 2000's, including the first reliable 2 BH merger simulation [46] using the moving puncture method, as well as the steady increase of the computational power of the computer cluster, offer a new way to study the time develoment of these models using numerical integration of the full 3D Einstein equation. Studies of this kind were missing until 2012.
- **Covariant coarse-graining of the tensorial part of the Einstein equations and the matter equations of motion.** The Buchert's averaging scheme, probably the most widely used coarse-graining formalism in inhomogeneous cosmology, takes only the scalar part of the equations into account. Ideally we would like the coarse-graining to involve the tensorial parts of the evolution equations as well.

4.5 Paper [H1]: Coarse-graining of inhomogeneous dust flow in general relativity

The goal of the paper [H1] is to provide an alternative to the Buchert's averaging scheme which extends the coarse-graining to the tensorial part of the equations. We consider a solution of the Einstein equations with dust. We assume that we have an a priori given 3+1 splitting of the spacetime (if the dust flow is irrotational then the comoving, orthogonal splitting can serve this purpose). Recall that in the Buchert's averaging scheme we assign the coarse-grained expansion to a comoving volume D of fluid using its physical volume. Buchert assigns the effective scale factor to D via

$$a_{eff}(t) = \left(\frac{V_D(t)}{V_D(t_0)}\right)^{1/3}, \tag{2}$$

where t_0 is a fixed time. The scale factor is then used to define the average expansion rate $\langle\theta\rangle_D$, which satisfies a volume-averaged version of the Raychaudhuri equation and the Hamiltionian constraint equation. This way Buchert defined the coarse-grained scalar quanties obeying the same evolution equation as the local ones plus well-defined backreaction terms. The downside of his approach is that he is only able to do it for the *scalar* part of the equation, while the cosmic flow on fine scales may also exhibit vorticity and shearing, both described by tensorial objects.

The main obstacle one encounters when trying to define the coarse-grained vorticity and shear is the need for covariance. The prescription for the tensorial quantities should be coordinate system-independent, otherwise it will yield results which are difficult to interpret physically. In fact, a coordinate-dependent prescription may assign backreaction to the homogeneous FLRW model seen in perturbed coordinates. But covariance is difficult to achieve in Riemannian or pseudo-Riemannian geometry where no simple prescription for averaging tensor quantities exists.

The main idea behind the paper [H1] is that we do not need the coarse-grained quantities to be literally *averages* with respect to a measure. The assignment of the large-scale flow can be obtained in a different manner as long as it satisfies the commonsense properties of the large-scale variables, for example it coincides with the standard averages in simple case.

Let now $Q^i{}_j = v^i{}_{,j}$ denote the velocity gradient of a non-relativistic fluid in a Euclidean space. Its trace is the expansion of the fluid, its symmetric tracefree part constitutes the shear and its antisymmetric part is the vorticity. The volume average of $Q^i{}_j$ over a region D can be turned into a boundary integral using the Gauss theorem

$$\langle Q^i{}_j\rangle = V_D^{-1}\int_D v^i{}_{,j}\mathrm{d}^3x = V_D^{-1}\int_{\partial D} v^i n_j \mathrm{d}^2\sigma. \tag{3}$$

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Generalizing the first expression into an arbitrary geometry is difficult, but we can equally well generalize the second one. Outside the Euclidean context they won't be exactly equal, but still they can be both used as the basis of the definition of $\langle Q^i{}_j \rangle$. Quantities defined using only the boundary data are known in GR as *quasi-local* [47].

In order to define the quasi-local, coarse-grained shear, expansion and velocity I use a classic result of differential geometry, i.e. the isometric embedding theorem of S^2 surfaces. Recall that given a 2-surface C of S^2 topology and a sufficiently regular positive-definite metric with positive curvature we may find an isometric embedding $f : C \mapsto \mathbf{R}^3$ into the 3D Euclidean space. Moreover the embedding is unique up to reflections and Euclean motions of the image [48].

Consider now the 2-metric $q_{AB}(t)$ induced on ∂D. Assuming it satisfies the assumptions of the isometric embedding theorem at least for some time we may find a one-parameter family of isometric embeddings f_t into $\mathbf{R}^3$. We may calculate the time derivative of the position of each point under the embeddings obtaining a fictitious velocity field v^i in $\mathbf{R}^3$ defined on the image of the embedding. The velocity field can then be used to define the average using equation (3). Due to the ambiguity of the embedding the velocity field is defined only up to the generators of translations and rotations, but this does not affect the value of the shear and expansion. As for the vorticity, I present a slightly different but related method of coarse-graining using the projection of the 4-velocity field of the fluid to the constant time slice.

The coarse-grained $\langle Q^i{}_j \rangle$ is defined in the abstract target space $\mathbf{R}^3$ of the embedding f_t, but in [49] I show how this space can be identified with the tangent space to the constant time slice at the points lying on the boundary of D, thus providing a geometric meaning to the construction above. $\langle Q^i{}_j \rangle$ becomes a tensor field defined everywhere on the boundary of D.

I show in [H1] that the coarse-grained quantities satisfy a generalized version of the Raychaudhuri equation, identical to the local Raychaudhuri equation except a bunch of backreaction terms arising due to the inhomogeneities present inside D. I also prove that it satisfies the basic assumptions about the coarse-graining: in the limit of the domain D shrinking to a point the coarse-grained quantities reproduce the local ones and in a few exact solutions (FLRW, LTB, Gödel) the results agree with the common sense expactations.

The main conclusions of [H1] can be summarized as follows:

- The isometric embedding theorem offers a way to assign coarse-grained expansion, shear and voritcity to a finite portion of the fluid in GR in a covariant way. The quantities in question are quasi-local, i.e. they depend only on the data at the boundary of the coarse-graining domain. The method requires the boundary in question to have the S^2 topology and a positive curvature everywhere.

- The expansion, shear and vorticity satisfy a generalized version of the Raychaudhuri equation with an additional backreaction term due to the inhomogeneities inside the domain of integration. The backreaction consists of a Newtonian term, also present in coarse-graining in purely Newtonian cosmology, and a relativistic term.

- In the limit of a very small coarse-graining domain the coarse-grained quantities agree with the local ones, as expected and the generalized Raychaudhuri equation tends to the local one along a timelike geodesic, also as expected.

4.6 Papers [H2] and [H3]: Numerical evolution of the black hole lattices

Black hole lattices (BHL's), i.e. arrangements of black holes with a discrete symmetry of reflections and rotations, offer a simple and neat example of a discretized matter content which, when viewed on large scales, seems uniform. They have first been investigated by Lindquist and Wheeler using approximate methods [38], later extended by Clifton and Ferreira [45, 43]. Wheeler described in detail the initial data of the 5-BH configuration on a 3-sphere [39]. All configurations on a sphere were later described in [12].

Two main advantages of BHL's are their relative simplicity due to the high symmetry and at the same time high degree of nonlinearity of black hole configurations allowing to proble the nonlinear regime of GR. Recall that in the FLRW solutions three types of spatial slices are possible: flat ($\mathbf{R}^3$), spherical S^3 in case of closed models and hyperbolic H^3. Introducing a black hole lattice breaks the full isometry group of these manifolds to a discrete subgroup. The full isometry group of each of them, i.e. $E(3)$, $SO(4)$ and $SO(1,3)$ respectively, allows for different types of discrete subgroups.

In [H2] we first discuss the construction of the initial data for a BH lattice starting from the flat, spherical or hyperbolic metric using the Lichnerowicz-York conformal method [36, 37, 50]. In the first

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step we take the background, Riemannian 3-metric b_{ij} which is either the metric of a round sphere γ^S_{ij}, hyperbolic space γ^H_{ij} or a flat, Euclidean δ_{ij}. We assume the physical metric to be related to b_{ij} via a conformal transformation

$$q_{ij} = \phi^4 \, b_{ij}, \tag{4}$$

while the extrinsic curvature is given by

$$K_{ij} = \frac{K}{3}\, q_{ij} + A_{ij}, \tag{5}$$

where A_{ij} is traceless. The constraint equations take a simplified form of

$$\tilde{\Delta}\psi - \frac{\tilde{R}^{(3)}}{8}\,\psi - \frac{K^2}{12}\,\psi^5 + \frac{1}{8}\tilde{A}_{ij}\tilde{A}^{ij}\psi^{-7} \quad = \quad -2\pi G\,\psi^5\,\rho \tag{6}$$

$$\tilde{D}_i\tilde{A}^{ij} - \frac{2}{3}\psi^6\tilde{\gamma}^{ij}\tilde{D}_i K \quad = \quad 0 \tag{7}$$

$\tilde{\Delta}$ being the Laplacian operator of the conformal metric b_{ij}, $\tilde{R}^{(3)}$ its Ricci scalar and $\tilde{A}_{ij}$ being related to A_{ij} by $\tilde{A}_{ij} = \psi^2 A_{ij}$. Before we proceed we note that there is a relationship between the curvature of the underlying manifold, the momentary expansion rate of the model and the matter content, reminescent of the relation between the topology, the expansion rate and the matter content in the FLRW models, see [H2]. Namely, in the FLRW spacetimes it follows from the first Friedmann equation,

$$\frac{R^{(3)}}{8} + \frac{K^2}{12} = 2\pi G\rho \tag{8}$$

that if the matter density ρ is positive, then either the expansion rate K doesn't vanish or the curvature $R^{(3)}$ is positive and thus the spatial slice topology is S^3 (or both). It is impossible to have a model that is momentary at rest and at the same time of flat or hyperbolic geometry ($R^{(3)} \geq 0$). This also holds in the BH lattice case: it is impossible to have time-symmetric ($K_{ij} = 0$) initial data in which the underlying lattice geometry is flat or hyperbolic. We are thus left with two distinct cases: the spherical time-symmetric case or the non-time-symmetric cases (spherical, flat or hyperbolic). In [H2] we discuss thoroughly the first case while in [H3] we consider the latter case with a flat, cubic lattice.

The time-symmetric spherical lattice corresponds to $K_{ij} = 0$ and $b_{ij} = \gamma^S_{ij}$ being the metric of a round, unit 3-sphere. Fix the coordinate system $(\lambda, \theta, \varphi)$ on S^3. The vacuum vector constraints are satified trivially and we are only left with the Hamiltonian constraint equation in the form of a linear, elliptic equation for the conformal factor:

$$\tilde{\Delta}\,\psi - \frac{3}{4}\,\psi = 0. \tag{9}$$

It has no regular solutions, but it does have a Green's function with a puncture-type singularity $\frac{1}{\lambda}$:

$$\psi(\lambda) = \frac{1}{\sin\lambda/2}. \tag{10}$$

An arbitrary number of solutions of this kind, centered at points $x_1, \ldots, x_N$ may be superimposed:

$$\psi(x) = \sum_{i=1}^{N} \frac{\alpha_i}{\sin\frac{\Lambda(x_i,x)}{2}}, \tag{11}$$

where $\Lambda(x, y)$ denotes the geodesic distance on S^3 and α_i are real coefficients. For positive $\alpha_i's$ and $N \geq 3$ this initial data correspond to the distorted original sphere with N asymptotically flat ends joined through throats with minimal surfaces. The geometry of the throats is very close to the Schwarzschild geometry (for $N = 2$ it *is* the Schwarzschild geometry), so we may think of the punctures as N BH's in a spherical, vacuum cosmological model.

In [H2] and [H3] I considered only BHL's whose cells are regular polyhedra with BH's located at the center. It turns out that on S^3 there exist only 6 arrangements of this type, corresponding to the 6 possible tessellations of an S^3 sphere [12]. The tessellations have the symmetry group of a regular, 4–dimensional polytope inscribed in the 3–sphere. The black holes in this case are positioned at the

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vertices of the polytope. The arrangements have 5, 8, 16, 24, 120 or 600 BH's in a lattice. Note that there is no possibility of varying the size of a cell. Instead of one or more parameter families of crystaline lattices of different sizes we know in the Euclidean space we only have a finite number of fixed geometries to work with.

In [H2] we consider the 8BH lattice on S^3 and perform the full 3D numerical simulations of the evolution of the model. The tools for the numerical solving of the Einstein equations with black holes are publicly available since 2011 [51]. They were developed for the purpose of modelling the binary black hole mergers in order to study the gravitational radiation emitted in the process [35]. Detailed examination of the merger waveform was a crucial step in the gravitational wave detection by LIGO and other GW observatories.

When we attempt to use the BH merger codes to simulate a BHL we immediately run into a problem. The codes were written assuming asymptotically flat boundary conditions, while the BHL cells require more complicated, periodic boundary conditions. These are relatively easy to impose for flat, cubic lattices, where the shape of the cell is perfectly cubic and the walls meet at right angles, but much harder in general [52]. Fortunately we have found a simple workaround which allows to evolve the data immediately without explicitly imposing the periodicity conditions.

The trick is based on the observation that one may regard the initial data described above as the standard, asymptotically flat Lichnerowicz initial data constructed on $\mathbf{R}^3$. If we consider the 3-sphere as a unit sphere in $\mathbf{R}^3$ given by $(X^1)^2 + (X^2)^2 + (X^3)^2 + (X^4)^2 = 1$ then the stereographic projection P given by

$$x^i = \frac{2X^i}{1 - X^4}, \tag{12}$$

where x^i are the Cartesian coordinates in $\mathbf{R}^3$, is conformal, i.e. the pushforward of γ^S_{ij} satisfies

$$P_* \gamma^S_{ij} = \left(|\vec{x}|^2/4 + 1\right)^{-2} \delta_{ij}. \tag{13}$$

Assume now that one of the punctures, say α_1, is located exactly at $X = (1, 0, 0, 0)$. This point is not in the domain of P (it has been "mapped to infinity"). In this situation the physical metric γ_{ij} projected down to $\mathbf{R}^3$ takes the form of

$$\begin{aligned} \gamma^S_{ij} &= \psi^4 \gamma^S_{ij} = (\alpha_1)^4 \, \tilde{\psi}^4 \, \delta_{ij} \\ \tilde{\psi}(\vec{x}) &= 1 + \sum_{i=2}^{N} \frac{2\alpha_i \sqrt{1 + |\vec{n}_i|^2/4}}{\alpha_1} \cdot \frac{1}{|\vec{x} - \vec{n}_i|} \end{aligned} \tag{14}$$

i.e. it has the form of the standard, asymptotically flat Lichnerowicz initial data with $N - 1$ black holes, located at $\vec{n}_i$, up to a constant rescaling. This rescaling can be removed by rescaling the Cartesian coordinates[2].

The Lichnerowicz initial data with punctures can easily be handled by EinsteinToolkit without any modifications. The corresponding $\mathbf{R}^3$ initial data consists of 7 punctures located at points

$$\begin{aligned} \vec{N}_2 &= (0,0,0)\,, \\ \vec{N}_3 &= (2,0,0), \\ \vec{N}_4 &= (-2,0,0), \\ \vec{N}_5 &= (0,2,0)\,, \\ \vec{N}_6 &= (0,-2,0)\,, \\ \vec{N}_7 &= (0,0,2), \\ \vec{N}_8 &= (0,0,-2) \end{aligned} \tag{15}$$

with mass parameters $m_2 = 4M$ and $m_3, \ldots, m_7 = 4\sqrt{2}M$, M being a mass and length. The solution together with minimal surfaces enclosing the punctures is given in Figure 1. Note that the black hole at α_1, at first glance absent from (14), did not disappear from the solution. Its presence may be detected when we search for minimal surfaces on the spatial slice: it will show up as a minimal surface encompassing all other ones. While it is not immediately clear from this form of the initial data that all BH's are identical and can be exchanged using an isometry, it does turn out to be true if we consider the conformal factor (14).

[2] Note missprints in equations (21), (23) and (24) in [H2]: 4 in the denominator should be replaced by $4A_1^2$.

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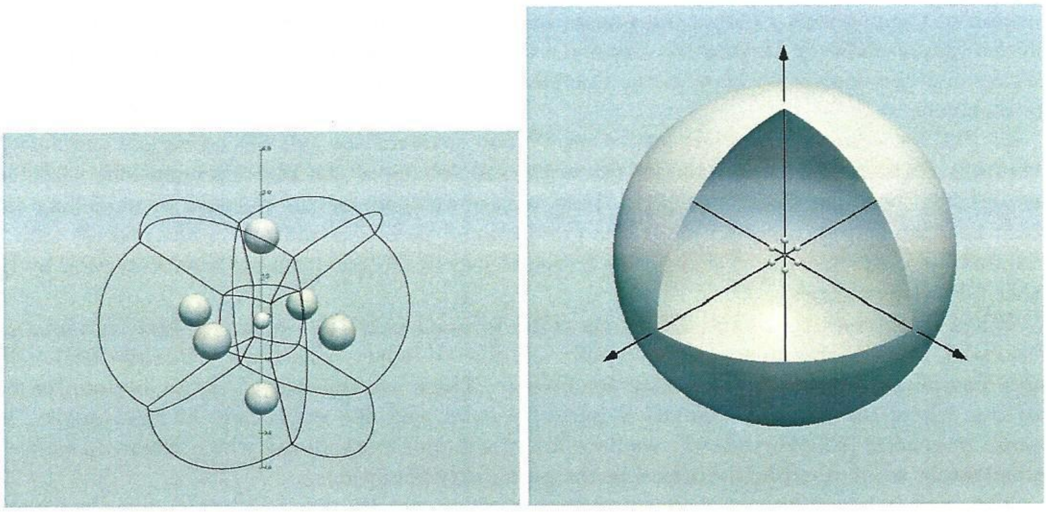

Figure 1: Left: The initial position of the 7 BH's represented by the minimal surfaces encompassing the punctures and the edges of the elementary cells for the 8-black-hole configuration. The quasi-cubes are projected to $\mathbf{R}^3$. Note that the 8 cubical lattice cells are isometric after the conformal rescaling. Right: The 8th minimal surface encompassing the configuration.

We evaluated both the kinematical and the dynamical effects of backreaction. We have taken the geodesic length of a cell edge D_{edge} as the measure of the size of the model and used it, together with its first derivative at $t = 0$, to fit an FLRW model. The kinematical effects have the form of total mass correction of the model: the effective mass obtained from the fit is 25% larger than the sum of the ADM masses of the individual black holes, in consistence with the results from [12], where the fit is performed differently.

The results of the numerical evolution is given in Figure 7 of [H2]. Note that the model is symmetric with respect to the time reversal $t \to -t$, so evolving it forward in time is sufficient to understand the full evolution of this model. The numerical evolution we performed lasted until the coordinate time of around $t = 150M$ in the time units consistent with the equations. Later the evolution froze in the vicinity of the black holes due to the properties of the gauge conditions employed. This is not very surprising, because the gauge condition used in the simulations, taken from [53], has been developed for the purpose of investigating BH mergers and is known to slow down the evolution at the punctures to a complete stall and at the same keep it going far away. Later experiments with modifying the gauge conditions done by E. Bentivegna and I. Hinder allowed to push the evolution somewhat further, but the results have not been published yet.

Even before the freezing, around $t = 80M$, the numerical error grew significantly and consequently we do not include further points in the plot. Up until that time the evolution of D_{edge}, expressed in the proper time of Gaussian observers τ, follows the FLRW evolution with 1% accuracy, which is close to the numerical error of the simulations. We did not observe thus any dynamical effects of backreaction in this model within our limits of precission.

Apart from that we have followed the evolution of the mariginally outer trapped surfaces (MOTS) representing the outer boundary of the black hole. We also measured the geodesic distance D_{hor} between the MOTS's of 2 neighbouring BH's. This quantity may look at first like another promising candidate for the measure of the scale, along with D_{edge}, but we have found out that it cannot be used for that purpose. Quite surprisingly it has a non-vanishing first derivative at $t = 0$ despite the fact that the initial data is time-symmetric and the black holes are initally at rest. This result looks like a contradiction with the time reversal symmetricity of the solution. The resolution of this paradox lies in the fact that the MOTS's we were looking at were *future* MOTS's. The initial data surface of this solution contains mariginally trapped surfaces that are bifurcation surfaces analogous to the bifurcation surface in the Schwarzschild geometry. The future and past outer trapped horizons cross there in perfect symmetry with each other, see Figure C2 from [H2]. This way the distance between the two future MOTS decreases with the velocity of 2 even at $t = 0$. This behaviour has to be contrasted with the behaviour of the scale factor of the FLRW solution which has a maximum at $t = 0$.

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[H2] is to my knowledge the very first paper discussing the full numerical evolution of a BHL. In this sense it was a pioneering work in the field of numerical relativity applied to inhomogeneneous cosmology. This route was then followed by others [54, 55, 56, 57] (see also [58] using the Regge calculus instead of the full 3D simulations). [H2] has been cited by essentially all subsequent papers about the numerical evolution of BHL's and many papers discussing other inhomogeneneous cosmological models. It was also cited in a number of review papers about numerical relativity and inhomogeneneous cosmolgical models [35, 34, 59]. Its main conclusions can be summarized as follows:

- The main effect of backreaction in the regular 8BH lattice is the kinematic backreaction in the form of mass renormalization. On the other hand the evolution is consistent with a dust FLRW to a high degree.
- Unlike the edge length, the distance between the black holes, defined as the distance between the corresponding MOTS, is a poor measure of the size of this universe as its behaviour at the moment of the largest expansion is quite different from the behaviour of the scale factor.

In the next paper devoted to the BHL's [H3] we looked at a cubic lattice of BH's. As we noted above, the lattice cannot be stationary. The construction of the initial data requires a non-vanishing extrinsic curvature and therefore, unlike the situation in [H2], the Hamiltionian constraint equation is not a linear, elliptic PDE any more. The initial data was constructed numerically, using methods borrowed from [60], with a few improvements [61]. In the initial data we considered the extrinsic curvature was positive and constant near the cell faces, it vanishes inside a sphere around the cell center, with a transitional zone in between. In the center we placed a puncture-type singularity of the type of $\frac{m}{r}$. The initial data construction allows for choosing the mass parameter m of the central black hole, the value of the mean extrinsic curvature near the cell faces and the length of the cell edge. The code solves the nonlinear elliptic equation for the conformal factor by the relaxation method. Note that the conformal factor rescales all the fixed quantities, so their physical values in the initial data turn out to be slightly different from the assumed ones.

In [H3] we followed the evolution of 4 sets of initial data with 4 different values of the mass m. The evolution lasted for a fairly long time, but at some point the code seems to have lost the numerical accuracy. The evolution required a modification of the standard β-tracker gauge condition. The simulations were performed in three runs with increasing resolutions. The comparison of the runs showed first-order convergence up until a moment depending on the black hole mass, later the convergence of the results became problematic.

We discussed in [H3] the time evolution of the edge length and the geometry of the solution. The edge length seems to follow the dust FLRW evolution with a high accuracy save for quick oscillations. The oscillations are probably caused by excited gravitational wave modes in the cell. This is a fairly common occurence for the initial data constructed using the conformal method in the multi-black hole merger studies [62]. They did not appear in [H2] because the initial data had a simpler geometry and was time-symmetric. Note that in the BH merger simulations these modes are fairly quickly radiated away to infinity. This is not the case in simulations with periodic boundary condtions like [H3], where the excited waves remain closed in the cubic cells.

Unlike [H2] this paper was not the first one to report a numerical simulation of a cubic BHL. Two weeks before it appeared on the arXiv Yoo, Okawa and Nakao posted their paper [54]. Our analysis of the results is nevertheless more thorough then in the aforementioned paper.

The main conclusions of [H3] can be summarized as follows:

- The main effect of backreaction in cubic lattices of BH's with initial data constructed by the prescription from [60] is the kinematical effect of mass renormalization.
- The time evolution of the solution is fairly close to the corresponding dust FLRW model apart from quick oscillations superimposed on the continuous expansion, caused probably by excited gravitational wave modes within the lattice cells.
- The effective pressure of the solution has the form of quick oscillations around 0, with magnitude comparable with the matter density, but which average out to 0 in long term.

4.7 Paper [H4]: Continuum limit and backreaction in the spherical Linquist-Wheeler model with BH's

This is the most mathematically involved paper of the series and at the same time the most innovative in its approach. Recall that there are only 6 possible regular lattices on a 3-sphere. This is insufficient if we

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would like to proble the continuum limit, i.e. check what happens if the number of black holes becomes so large that they can be treated as a continuum, just like a fluid in the kinetic theory. Therefore I considered there the time-symmertric Lindquist-Wheeler initial data from (11) with an *arbitrary* number of black holes and with *arbitrary* masses and positions. I addressed two main questions:

1. In what sense and under what conditions does the 3-metric of the initial data converge to the metric of a closed FLRW in the moment of largest expansion?
2. What is the difference between the total mass of this model, i.e. the sum of the ADM masses of all BH's, and the effective mass one obtains by assigning a coarse-grained FLRW model to the initial data by averaging the data in a suitable sense?

The averaging and the FLRW fitting in this model can be done easily by averaging the conformal factor ψ using the standard volume form η on the 'round' S^3:

$$\langle\psi\rangle = \frac{1}{2\pi^2}\int_{S^3}\psi\,\eta,$$

$2\pi^2$ being the volume of S^3. Note that the singularities in the Green's function (11) are of the type $\frac{1}{\lambda}$, i.e. integrable in 3 dimensions, so the integral above converges.

The key result of the paper is contained in two inequalities. The first one, Theorem 3.1 in [H4], bounds the deviation of the conformal factor (and thus of the whole 3-metric q_{ij}) from the average one at a given point

$$\frac{|\Phi(p) - \langle\Phi\rangle|}{\langle\Phi\rangle} \le C_\varepsilon\, U_\varepsilon(E, \lambda_{\min}) \tag{16}$$

The second one, Theorem 3.2 in the same paper, estimates the dimensionless mass deficit, i.e. the difference between the total mass and effective mass inferred from the average metric divided by the same effective mass[3]

$$\frac{|M_{eff} - M_{tot}|}{M_{eff}} \le C_\varepsilon\, W_\varepsilon\left(E, \frac{\alpha_{\max}}{\alpha}, \delta_{\max}, \delta_{\min}\right) + \frac{\alpha_{\max}}{\alpha} \tag{17}$$

The estimates depend on a handful of parameters describing the microscopic details of the black hole distribution. In particular, the function $U(E, \lambda_{\min})$ from (16) depends on the distance from the nearest puncture $\lambda_{\min}$ and the modified cap discrepancy E. The latter measures how evenly the black holes are distributed on the sphere. It is a modification of the standard cap discrepancy, already known in the literature []. It provides global bounds on how the standard volume of a spherical cap differs from the volume measured by a measure concentrated at the punctures. For sufficiently even arrangements of BH's it converges to 0 with the number of black holes diverging. The bound depends on E via a positive power and on $\lambda_{\min}$ via a negative one, i.e. it bounds the deviation more far away from the nearest BH and less in their vicinity.

The function $W_\varepsilon\left(E, \frac{\alpha_{\max}}{\alpha}, \delta_{\max}, \delta_{\min}\right)$ from (17) depends again on E, but also on the ratio between the largest parameter α_i and the sum of all α_i as well as the largest and the smallest distance between any two pairs of BH's. The dependence on the latter is again via a negative power, so the bound is weaker when any of the black holes get too close. This way the convergence of the relative difference between the two masses to 0 depends on the convergence rates of E and $\delta_{\min}$ to 0: the latter needs to converge sufficiently slowly with $N \to \infty$.

I also present in [H4] the construction of a sequence of solutions with a growing N in which you can easily estimate the convergence rate of E and $\delta_{\min}$, use the inequalities above to understant the geometry of the solution for very large N and prove that in this limit the left hand side of (17) vanishes (asymptotic additivity of masses). The same solution with a slight modification (a pair of close BH's replacing each BH) can be shown to have a very similar geometry, but on the other hand the BH masses are not asymptotically additive. Indeed, $\delta_{\min}$ in the modified solutions converges much faster to 0 and (17) yields no bound on the mass deficit.

This is to my knowledge the first paper proving an exact, non-perturbative result concerning the backreaction and the continuum limit in GR in the form of inequalities. Its pioneering nature lies also in the methods of the proof. First, we observe that both σ_M and σ_Φ can be expressed as a sum of a few terms which are easy to estimate and a term of the form of a difference between the volume average

[3] Note the missprint in the denominator of equations (24) and the definition of σ_M on page 8, M_{tot} should read M_{eff}.

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of the Green's function (10) and the weighted average of its values over the punctures. The weighted average is quite similar to the partial sum in the definition of a Riemann integral and thus one may intuitively expect this difference to converge to 0 as the number of points diverges.

The problem of estimating expressions of this kind is known in mathematics in the context of the quasi-Monte Carlo integration theory [63, 64, 65, 66]. It is fairly easy for regular functions, but rather hard if we allow (integrable) singularities in the integrand [63, 67, 68, 69]. None of the results available in the literature seemed to work in my case, so the proof is original. It is based on a version of the Koksma-Hlawka identity for unbounded functions. The result may be of interest even outside the field of the backreaction problem and general relativity, in the theory of the quasi-Monte Carlo integration.

[H4] attracted a considerable interest in the backreaction community. Shortly afterwards Clifton presented somewhat related results in BH configurations obtained by the method of images [70]. Liu and Williams considered later BHL using various approximation schemes and the Regge calculus [71, 58], citing [H4]. Moreover, the results from [H4] were mentioned by both sides in the backreaction debate between Green and Wald on one hand and Buchert et al on the other [17, 20]. Note also that the ball bearing example from [17] and its discussion bears some similarity to the solutions I considered. The paper [H4] was included by the Editioral Board of the Classical and Quantum Gravity in the CQG Highlights 2013/2014 in cosmology.

The main conclusions of [H4] can be summarized as follows:

- Infinite sequences of time-symmetric, vacuum initial data on S^3 with BH's as the only sources of gravitational field may have a continuum limit as the number of black holes diverges. The limit has the form of a spherical FLRW model with a continuous, homogeneous distribution of matter. Convergence requires the BH's to be distributed uniformly over the S^3. The precise criterion is given in terms of the discrepancy between the standard measure on the round S^3 and the measure concentrated in the punctures.

- The convergence of the metric tensor to the continuum limit is not pointwise, but rather more compicated: the metric tensor remains strongly distorted for arbitrary large N in the vicinity of each black hole, i.e. in the region within a finite number of Schwarzschild radii from them. As the number of BH's N grows the black holes and the distorted regions cover densely the solution. However in the region far away from the nearest black hole the metric still converges to the FLRW one. Moreover, it is the faraway regions that asymptotically take up the whole volume of the S^3, thus making the distorted region negligible.

- The existence of the continuum limit does not imply automatically the vanishing of backreaction, i.e. the relative difference between the total mass of the corresponding FLRW solution and the sum of the ADM masses of the BH's. It is necessary to add a condition on the microscopic distribution of the BH's, namely we need to make sure that the BH's do not get too close to each other. In [H4] we demand that the minimal distance between a pair of punctures doesn't decrease too quickly.

- It is the clustering of matter on small scales that gives rise to backreaction. It is easy to understand the physical reason for that: clustering of matter means larger binding energy, which in GR needs to be taken into account as a source of the gravitational fields on larger scale ("gravity gravitates").

- It is possible to prove inequalities estimating the backreaction terms in GR using techniques taken from the measure theory and the quasi-Monte Carlo integration theory.

4.8 Paper [H5]: Nested structures

The large-scale matter distribution in the Universe is known to be nested: large walls are made of smaller fillaments, each of them made of galaxy clusters, which in turn are made of individual galaxies. This situation is quite different from the simplified examples discussed in [H2-H4], where we had a well-defined, macroscopic homogeneity scale, a microscopic scale defined by very small compact objects (black holes etc.) and virtually no structures in between. In [H5] I therefore consider the situation when the matter distribution is homogeneous on the homogenerity scale R_{hom}, but below that the structure is present on all scales down to the fixed smallest ripples scale R_{min}. The goal of the paper is to understand how the backreaction effects depend on the matter distribution in this case, when exacly we should expect them to be significant and which parameters characterizing the matter distribution play an important role in estimating them.

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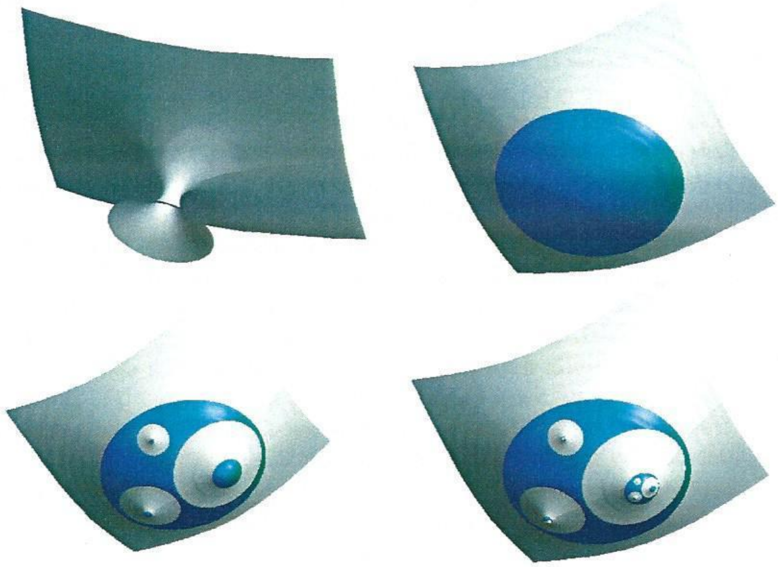

Figure 2: Construction of the multiscale foam with one dimension suppressed. The Schwarzschild solution in the form of the Flamm's paraboloid (top-left). The paraboloid is then cut along a circle and the remaining hole is filled with an appropriately fitted blue spherical cap producing the solution q_0 (top-right). We then fit a number of funnel-shaped sections of Schwarzschild to a number disks excised from the cap. The funnels are then cut along appropriately chosen circles and the interior replaced again with matching caps (bottom-left). The positions of the cuts on the funnels are chosen carefully so that the small caps covered the same solid angle as the original cap, yielding q_1. The whole step is then repeated for each of the small caps producing q_2. Note the self-similarity of the construction.

Just like [H4], [H5] is a careful analysis of an exact solution of the Einstein equations with a simplified geometry. Unlike the models from [H2-H4] it is not vacuum, but it is filled with matter in the form of presureless dust. It belongs to the generalized Swiss cheese class of models, obtained by matching a constant density FLRW solution with a vacuum Schwarzschild solution along a sphere. The construction proceeds as follows: we begin by an *outer* Schwarzschild solution matched to an *inner* homogeneous ball of dust with a closed FLRW geometry (see Figure 2). This matching is fairly standard and has been described in detail in many textbooks [72, 73]. We then excise inside the ball a number of spherical voids, i.e. regions of appropriately matched *inner* Schwarzschild metrics. Inside these voids we match again spherical overdensities in the form of *internal* FLRW solutions with a larger matter density. The size of the inner balls of matter has been chosen in such a way that the ball is a perfectly scaled down copy of the orginal ball of dust. The construction may then be repeated on each small ball of dust. We may iterate the last step as many times as we want, obtaining this way an almost self-similar, fractal distribution of both matter and the gravitational field.

Let N denote the nesting level of the construction, i.e. the number of iterative steps of the construction. We may compare the ADM mass of the solution M_{ADM}, measured at a distance from the initial ball of matter and the total mass M_{tot}, defined, as usual, as the integral of the mass density of the dust. The construction of the voids and overdensities has no influence whatsoever on M_{ADM}, but it does change the total mass, which can be expressed as a series

$$M_{tot}^{(N)} = M_{tot}^{(0)} + \Delta M_{tot}^{(1)} + \cdots + \Delta M_{tot}^{(N)}, \tag{18}$$

in which $\Delta M_{tot}^{(k)}$ is the correction introduced by the k-th step of the construction. Due to the self-similarity of the metric the series above is geometric and can be summed exactly, see the equation (37) from [H5]. The relative mass deficit $x = \frac{M_{tot}-M_{ADM}}{M_{ADM}}$, which is simply the integrated backreaction of the T_{00} component, can be expressed exactly as a function of the parameters describing the construction: the dimensionless compactness parameter ε of the ball of matter defined as the ratio between its mass (expressed in the geometric units) and its size, the total volume fraction α excised from the uniform ball of dust at each step and finally the nesting level N.

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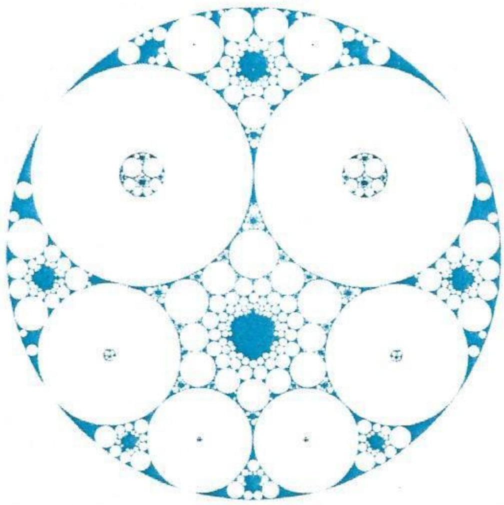

Figure 3: The solution with a large backreaction despite very weakly relativistic inhomogeneities. The voids constitute and Appollonian packing of a 3-sphere.

In the Newtonian approximation the mass deficit must vanish, because the mass measured far away from the body is related to the integral of the mass density via the Gauss law. Any difference between the two is therefore a non-linear GR effect. In objects without a complicated nested structure, like a uniform ball of matter, it is proportional in the leading order to the compactness parameter ε:

$$x(\varepsilon) = C_1\,\varepsilon + O(\varepsilon^2), \tag{19}$$

C_1 being a constant determined by the geometry of the system. Note that due to the self-similarity of the solution the compactness of the initial ball of matter ε is equal to the compactness of any smaller overdensity. ε is thus a universal constant characterizing how stongly relativistic the whole structure is.

In [H5] I presented a configuration of overdensities and voids (see Figure 3) in which the mass deficit takes the form of

$$x = C_2\,N\varepsilon + O(\varepsilon^2), \tag{20}$$

where N is the nesting level and C_2 is again a constant. Obvioulsy no matter how small ε is, it can be compensated by a sufficiently large value of N. This happens despite the fact that for small ε each of the voids and inhomogeneities seems to lie within the reach of the Newtonian approximation, where the backreaction effects are negligible. The reason for their amplification is the accumulation of small contributions from various intermediate scales where inhomogeneities are present. The deficit from (20) can be re-expressed in the following form:

$$x = C_3\,\varepsilon D + O(\varepsilon^2) \tag{21}$$

where D is the dimensionless *depth of structure*, i.e. the logarithm of the ratio of the homogeneity scale and the scale of the smallest ripples:

$$D = \ln\frac{R_{hom}}{R_{min}}. \tag{22}$$

We conclude that D is an important parameter characterizing the backreaction in matter distributions with nested structues.

I prove in [H5] that these results are both time- and gauge-independent. It is interesting to contrast them with the claims of Ishibashi and Wald from [15]. The authors dismiss there the idea that the backreaction may explain the accelerated expansion of the Universe and state explicitly in the abstract the reason why: *"We point out that our universe appears to be described very accurately on all scales by a Newtonianly perturbed FLRW metric. (This assertion is entirely consistent with the fact that we commonly encounter* [the density contrast of] $\delta\rho/\rho > 10^{30}$ *.) If the universe is accurately described by*

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a Newtonianly perturbed FLRW metric, then the back-reaction of inhomogeneities on the dynamics of the universe is negligible." Obvioulsy the toy model discussed in [H5] shows that the bare fact that the inhomogeneities on all scales seem to be well described by a Newtonian perturbation does not guarantee that total backreaction is negligible. A more refined analysis, which includes the depth of the cosmological structure, is needed to give a reasonalbe estimate of the nonlinear GR effects present. The paradoxical properties of the toy model illustrate the fundamental difference between the coarse-graining of nested, multiscale structures and the inhomogeneities of a given scale.

[H5] was cited by Buchert et al in their rebuttal paper [20] as an example where the Green-Wald formalism fails to capture the backreaction effects. It was also mentioned by other authors discussing the backreaction in inhomogeneneous Universe [25, 74, 32], as well as the review paper on the Hubble law by MacCallum [75].

The main conclusions of the paper [H5] can be summarized as follows:

- In nested structures, in which overdensities of a given size contain smaller structures, the total backreaction in the form of mass deficit is the sum of contributions from all intermediate scales from the scale of the smallest inhomogeneities up to the homogeneity scale.
- The magnitude of the contribution is proportional to the dimensionless compactness parameter ε of each of the structures on a given scale. If ε is small then the structures, *when considered in isolation from the rest of the solution*, can be described using the Newtonian approximation and the backreaction effects are small. Nevertheless if the nesting level is large enough the backreaction may turn out to be substantial, because small contributions may accumulate to a large value. This may happen despite the fact that on each scale the matter distribution seems very weakly relativistic. More than that, in my example the solution is everywhere isometric to a scaled-down copy of a very weakly relativistic solution with a uniform ball of matter. Large backreaction is therefore a global, emergent effect which cannot be explained by the presence of strong gravitational fields in any of the overdensities or voids.
- In the leading order of expansion in ε the backreaction in proportional to the product of ε and the depth of structure D. The latter is therefore an important dimensionless parameter scaling the magnitude of the backreaction effects in nested structures.
- The backreaction problem in nested structures lies within the reach of the perturbative approximation applied together with the renormalization group approach.

4.9 Paper [H6]: Evolution of the BHL along the curves of local discrete rotational and reflection symmetry (LDRRS)

The paper [H6] is a reaction to the paper written by Clifton, Gregoris, Rosquist and Tavakol in 2013 [56], in which the authors proposed a way to investigate the time development of the regular lattices of black holes on S^3. The idea is to consider the Einstein equations along certain special curves on the manfold. The curves are the axes of a discrete rotational symmetry of the solution, i.e. an axial symmetry with respect to a finite subgroup of the full $U(1)$ group of rotations around it. Additionally the solution exhibits a number of reflection symmetries with respect to several hyperplanes passing through these curves. In [56] the authors derive the vacuum Einstein equations along the curves of local discrete rotational and reflection symmetry (LDRRS). According to the authors the vacuum Einstein equations at the points on a LDRRS curve turn out to decouple entirely from the rest of the solution. Namely, the equations reduce to local ODE's when expressed in the Gaussian normal coordinates. The resulting ODE's turn out to be solvable by quadratures. This way the time evolution of these models can be followed to arbitrary long times using a desktop computer at least along the edges of the cells as well as some of the cell diagonals. It is also straightforward to investigate the basic optical properties of these lattices along the LDRRS curves, i.e. the redshift-luminosity relation for objects and observers lying on them. Obviously this result, when confirmed, would simplify tremendously the study of the time evolution of the BH lattices in vacuum.

At the same time my collaborators were testing the new code for tracing null geodesics and investigating the optical properties of spacetimes concurrently with their numerical evolution on a cluster. As one of the test cases we have chosen the regular lattices of BH's on a sphere. We have noticed immediately a discrepancy between the results of the full 3D simulations and those coming from the simple code implementing the CGRT equations. Having eliminated the possiblity of bugs in both codes we turned to the CGRT equations themselves. We decided to rederive them carefully using the ADM formulation

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of the Einstein equations (the authors of [56] used a less known, othonormal frame-based formulation of van Elst and Uggla [76]).

After a few weeks of fairly complex equation manipulations we realized that the CGRT result is incorrect. The authors imposed the LDRRS conditions on all geometric objects along the curve and showed that all vectors and tensors in question must take a special, reduced form. Additionally due to the reflection symmetry at these points the curl of all tensorial objects must vanish. In order to derive the CGRT equations one has to differentiate the ADM equations with respect to time and derive the equation for the time derivative of the 3-dimensional Ricci tensor. This equation taken together with the ADM rest of the equations form a closed system except one term proportional to the curl of the magnetic Weyl tensor H_{ij}. The term has no evolution equation of its own at this order of differentiation with respect to t. Clifton and collaborators use then the reflection symmetry argument to prove that the problematic term vanishes. This is however incorrect, because H_{ij} is not a tensor, but a pseudotensor, obtained by contracting the Weyl with the volume 3-form. H_{ij} can be in fact expressed as the curl of the extrinsic curvature tensor. Now while a curl of a tensor has to vanish along the plane of symmetry, the curl of a curl of a tensor does not. The problematic term doesn't vanish a priori and the CGRT equation do not close.

Quite interestingly the anomalous term vanishes together with its first 2 time derivatives in our time-symmetric initial data. Thus the difference between the true solution and the solution of the CGRT equation grows very slowly, like t^6. In order to confirm the hypothesis that the magnetic part of Weyl affects the local evolution along the LDRRS curves we compared the anomaly in the CGRT equations in the full 3D simulations with the curl of the magnetic Weyl. We obtained a good numerical match. On top of that we calculated the analytic expression for the first non-vanishing derivative of the offending term in the initial data and compared the result with the numerically evaluated derivative in the numerical solution[4]. Again the values matched very well.

The main conclusions of the paper [H6] can be summarized as follows:

- The full Einstein equations along the LDRRS curves simplify to a system of ODE's which in general includes the curl of the magnetic Weyl tensor H_{ij}. This curl does not have to vanish along an LDRRS curve, as was reported in [56], and thus the ODE's do not form a closed system.
- In case of the 8BH model the magnetic Weyl vanishes initially and the correction term with respect to the simplified, closed system with the curl of H_{ij} set to 0 is of the order of t^6 for short times.
- It is nevertheless not possible to obtain any conclusions concerning the long-time behaviour of the BHL's without the full 3D numerical simulations.

4.10 Summary and future prospects

The papers [H1-H6], despite very different methods, represent various approaches to the same backreaction problem and to the issue of coarse-graining in GR. They should also be considered voices in the decade-long backreaction debate among cosmologists and relativists. Out of this list [H4] and [H5] are probably the most important ones in terms of the relevance of the results and the novelty of the mathematical methods. They both concern fundamental issues connected with the application of the Einstein equations to inhomogeneous matter distributions. [H4] clarifies the notions of the continuum limit and the fluid approximation in GR and introduces mathematical precission and rigor to the problem. It shows how the "best fit FLRW metric" arises in a natural although complicated way in the continuum limit, providing an illustration of the possible solution of the Ellis and Stroeger's "fitting problem" in cosmology. The example is a more precise illustration of the origin of the smooth FLRW metric than the ball bearing example from [17] (it has in fact been noted in that paper and some subsequent ones). It also explains that backreaction, in the form of mass renormalization, is a distinct problem from the fitting question or the validity of the continuum approximation. [H5] on the other hand highlights the collective, emergent character of the backreaction effect: it is there even though inhomogeneities on every scale, considered in isolation from the rest, are very well described using the Newtionian approximation.

The numerical studies discussed in papers [H2, H3, H6], although based on very highly simplified models, constitutes an important step towards fully relativistic 3+1 numerical simulations of inhomogeneous cosmological models and the structure formation. This type of research is likely to dominate the field of comsology in the next decade, providing the basis for the interpretation of the observational

[4]Since the result contained many thousands of terms the calculation was done using Mathematica. The reduction of the result to a simpler form using various symmetries and identities took several hours.

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data of increasing precission. Hopefully it will provide as much important input for cosmology as the BH merger simulations did for the gravitational wave astronomy in the last decade. In my future work I plan to continue my collaboration with numerical relativists on this topic.

The results and methods presented here open up new directions for research in both mathematical and numerical relativity. An obvious continuation of [H4] is the generalization of the results to more general situations, for example general, asymptotically flat initial data (time symmetric, constant mean curvature or more complicated). In both [H4] and [H5] I would like to extend the formalism to include the time evolution of the initial data and the pressure backreaction effects, proceeding in the similar direction as [24], but in a more general setting.

Finally the results of [H1] suggest the use of quasi-local quantities constructed from the geometry of the boundary of the coarse-graining region to obtain the full large-scale Einstein equations, not only the matter flow.

5 Other research accomplishments

a) Bibliometric data according to Web of Science (as for April 18th, 2016)

number of papers: 14 + 1 preprint

h-index (Hirsch index): 7

number of citations: 106

number of citations with self-citations excluded: 95

total impact factor (the sum of the two-year journal impact factors from the publication year, 2014 data used for publications from 2015): 40.454

b) Research not contributing to the habilitation

Research published before the PhD degree

- **Normal conformal Cartan connection.** This is the topic of my MSc project, done under the supervision of J. Lewandowski, in the field of conformal geometry, i.e. the geometry of manifolds equipped with metric given up to a conformal rescaling. We calculated the Yang-Mills current of the normal conformal Cartan connection, an object associated with the conformal geometry of a manifold. In dimension 4 we proved that the current, which is an invariant of the conformal geometry, has the only non-vanishing components proportional to the Bach tensor, another known invariant of the conformal geometry. The vanishing of the Bach tensor is a necessary but not sufficient condition for a metric to be conformally equivalent to an Einstein metric. We then considered geometries with a reducible normal conformal Cartan connection in signature $(-,+,+,+)$, arising from metrics with a null Killing vector and of Petrov type N, called the Fefferman metrics. We found all examples of Fefferman metrics with a vanishing Bach tensor [77]. Metrics of this type are of interest for mathematicians because they provide examples of metrics for which the Bach tensor vainshes, but which are *not* conformal to an Einstein metric, thus proving the necessity of another condition for a metric to be conformally Einstein (note however that our example was not the first one [78]).
- **Isolated and dynamical horizons.** This is the topic of my PhD project. I worked together with J. Lewandowski and T. Pawłowski on the multidimensional generalization of the notion of an isolated horizon, i.e. a generalization of the standard event horizon as the boundary of a Kerr or Schwarzschild black hole. We managed to generalize the first law of black hole mechanics to multidimensional isolated horizons [79]. In the standard dimension and signature 3+1 I introduced a formalism describing the geometry of an isolated and dynamical horizon using a single set of variables [80]. This is non-trivial as the isolated horizons, representing the boundary of a black hole not interacting with its environment at the moment, are null surfaces, while the dynamical ones, representing black holes accreting matter or interacting gravitationally with the surroundings, are spacelike. I also introduced the notion of angular momentum of an isolated or dynamical horizon without an axial symmetry [81]. It is defined using the conformal decomposition of the 2-metric of a section of the horizon into a round metric and the conformal factor (the uniformization of a S^2 surface). The angular momentum is a quasi-local quantity arising from the Hamiltonian formulation of general relativity as the generator of an appropriately defined rotation of the horizon.

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- **Rotation curves in fully relativistic models of galaxies.** In 2005 I wrote the first paper pointing out the error in the relativistic galaxy model proposed by Cooperstock and Tieu [82]. Their proposal sparked a short but hot debate among the researchers working on the galactic dynamics and dark matter. Their model was supposed to explain the flat rotation curves in galaxies by general relativistic corrections in gravitationally bound systems, without the need of dark matter of any kind. I showed in [83] that the model they proposed contains an overlooked sheet of exotic matter through the galactic plane which modifies the rotation curves, thus invalidating their proposal. The original Cooperstock-Tieu paper was not published and thus neither was my rebuttal paper. The result appeared later in a proceedings paper [84].

Research conducted or published after the PhD

- **Tsodyks-Markram model of synaptic depression.** Project in theoretical biology. I worked in collaboration with J. Mazurkiewicz (J. Jędrzejewska-Szmek) and J. Żygierewicz on the regularization of the ODE's describing the response of a synapes to stimuli. In particular, we focused on the short-term synaptic depression described by the the Tsodyks-Markram model. I helped to find the analytical form of solutions corresponding to the response to a time-localized, Dirac delta-like stimulus [85].

- **Einstein-Vlasov equations: variational principle and axisymmetric solutions.** In collaboration with a nuber of relativists from Germany I worked on the Einstein equations coupled with matter in the form of collisionless dust (described by the relativistic Vlasov equations). Main results: deriving the equations from a variational principle and successful numerical search for stationary, axisymmetric solutions (unpublished so far).

- **Numerical methods for finding the isometric embedding of S^2 surfaces with a positive metric of positive curvature in $\mathbf{R}^3$.** In collaboration with M. Jasiulek we proposed a numerical method to find the isometric embedding of a surface of S^2 topology equipped with a Riemannian geometry with a positive curvature in the Euclean space [86]. It is known that this embedding always exists and that it is unique (up to rigid motions and reflections) [48]. Our algorithm is based on the original proof of the existence of the embedding by Nirenberg [87]. It finds the embedding for surfaces which do not deviate very much from a sphere. The application of the code involves the visualization of various surfaces in numerical relativity as well as coarse-graining of the fluid flow in cosmology [H1], [49].

c) Prizes and awards

- Project *"Nonlinear effects of general relativity in the coarse-graining of inhomogeneous gravitational fields and matter sources"* submitted to the European Research Council for the ERC Starting 2015 programme (panel PE1-Mathematics) reaches the second step of the review, i.e. the interview. Obtains the final degree B (not financed).

- The paper [H4] has been selected by the Editorial Board of Classical and Quantum Gravity as a CQG Highlight 2013/2014 in the field of cosmology, i.e. one of the best papers based on criteria of interest, significance and novelty. The author was also asked to write a short introduction to the paper for the CQG plus webpage.

- Juliusz Łukasiewicz prize for the best student of the Stefan Batory High School in the field of science in 1998

- Reached the 3rd stage of the Polish Physics Olympiad (1997) and Polish Mathematics Olympiad (1998) for high school students

- Awarded once the 2nd and twice the 3rd prize in the Warsaw Technical University Physics Competition for high school students (1996-1998)

d) Directing research projects

- 2012-2014: Principal Investigator of the project *"The role of small-scale inhomogeneities in general relativity and cosmology"* funded by the Foundation for Polish Science (FNP) within the HOMING PLUS program, co-funded by the European Union

- 2006: One-year grant from the Polish Ministry of Science and Informatization for graduate students *(grant promotorski)*

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e) Participation in research projects

- 2007: Participated in the MASTER grant (MISTRZ) of the supervisor Prof. J. Lewandowski, funded by the Foundation for Polish Science (FNP)

f) Talks

Invited conference talks

- July 2016: 14th Marcel Grossmann Meeting, Rome. *"Nonlinear effects and coarse-graining in general relativity"* - 20-minute opening talk on the parallel session on inhomogeneous cosmological models.

Contributed conference talks (selection)

- Nov 2015: 2nd Conference of the Polish Society on Relativity (POTOR), Warsaw, *"Coarse-graining of the Einstein equations: recent results"*
- July 2014: 1st Conference of the Polish Society on Relativity (POTOR), *"Nonlinear effects of general relativity from multi-scale structure"*
- March 2014: 4th Central European Relativity Seminar, ESI Vienna, *"Backreaction and continuum limit in a closed universe filled with black holes"*
- July 2013: GR20/Amaldi 10, Warsaw, *"Periodic lattices of black holes as inhomogeneous cosmological models"*
- Dec 2012: Follow-up workshop "Dynamics in General Relativity: Black Holes and Asymptotics", ESI Vienna, *"Black hole lattices on S^3 and on $\boldsymbol{R}^3$: initial data and evolution"*
- Feb 2012: Relativity workshop at Jagiellonian University, Kraków, talk about the 8-black hole model of the Universe investigated with Eloisa Bentivegna
- Jul 2011: Relativity workshop at the Erwin Schrödinger Institute in Vienna, collaboration with researchers from other institutes including Eloisa Bentivegna and Lars Andersson
- Sep 2010: Relativity workshop in Edinburgh, 45min talk about covariant coarse-graining of inhomogeneous dust flow in GR
- Jul 2009: 12th Marcel Grossman Meeting in Paris. 12min talk about covariant coarse-graining of inhomogeneous dust flow in GR
- Jun 2009: Invisible Universe 2009 Conference, UNESCO Paris. 15min talk about covariant coarse-graining of inhomogeneous dust flow in GR
- Mar 2009: 49th Kraków School of Theoretical Physics, Zakopane, March 2009. A 45min talk on covariant averaging in cosmology and the backreaction problem
- Jul 2006: 11th Marcel Grossmann Meeting in Berlin. Talk about isolated and dynamical horizons
- Jul 2006: IRGAC 2006, Barcelona. 2nd International Conference on Quantum Theories and Renormalization Group in Gravity and Cosmology. Talk about Cooperstock-Tieu model of galaxies
- Jun 2003: Conference "Gravitation: A Decennial Perspective", Center for Gravitational Physics and Geometry, PennState University, State College PA, USA. 30-minute talk about the results of the M. Sc. thesis
- Sep 2002: Astro-Particle Physics Workshop during the Polish-German Student Summer Academy in Krzyżowa. One-hour talk about supernovae and neutrinos (together with S. Lewicka)

Invited talks on seminars (selection)

- Numerous talks at the Relativity Seminar at the Faculty of Physics, University of Warsaw, including Apr 2016: *"Backreaction and continuum limit in a closed universe filled with black holes"*
- Oct 2009: Cosmology seminar, Institute for Theoretical Physics, University of Heidelberg, *"Covariant coarse-graining of Einstein equations and the backreaction problem in cosmology"*

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- Oct 2013: Astronomical Observatory, Jagiellonian University, Kraków, *"Numerical evolution of regular black hole lattices"*
- Oct 2013: Institute of Physics, Jagiellonian University, Kraków, *"Backreaction and continuum limit in a closed universe filled with black holes"*
- Apr 2014: Max Planck Institute for Gravitational Physics (Albert Einstein Institute), Potsdam, *"Nonlinear effects of general relativity from multi-scale structure"*
- Feb 2015: Centre de Recherche Astrophysique de Lyon École Normale Supérieure de Lyon, *"Nonlinear effects and coarse-graining in general relativity"*
- May 2015: University of Helsinki, Department of Physics, Division of Particle Physics and Astrophysics *"Nonlinear effects and coarse-graining in general relativity"*
- Oct 2015: Institute of Physics, Jagiellonian University, Kraków, *"Nonlinear effects and coarse-graining in general relativity"*

g) National and international collaboration

- 2001: Research project with prof. J. Stepaniak from the Institute of Nuclear Problems (IPJ), Warsaw, within the VASA collaboration in high energy physics and cyclotron physics, Warsaw University/Faculty of Physics and The Svedberg Laboratory, Uppsala, Sweden - writing a code for the data analysis during the calibration runs for the particle detectors as an undergraduate student
- 2007–2008: Collaboration with biophysicists J. Jędrzejewska-Szmek (Mazurkiewicz) and J. Żygierewicz from University of Warsaw on a project in theoretical biology
- 2008–2010: Collaboration with L. Andersson, M. Ansorg, J. Hennig (Max Planck Institute for Gravitational Physics, Potsdam) and G. Rein (University of Bayreuth, Germany), working on the Einstein equations with collisonless matter.
- 2010–present: Collaboration with numerical relativists Eloisa Bentivegna (Max Plack Institute for Gravitational Physics, Potsdam and University of Catania) and later also with Ian Hinder (Max Plack Institute for Gravitational Physics, Potsdam) on inhomogeneous cosmological models, especially the black hole lattices, their evolution and optical properties. Papers from this collaboration have been included in the habilitation series [H2, H3, H6].

References

[1] Richard A. Isaacson. Gravitational radiation in the limit of high frequency. I. The linear approximation and geometrical optics. *Physical Review*, 166(5):1263, 1968.

[2] Richard A. Isaacson. Gravitational radiation in the limit of high frequency. II. Non–linear terms and the effective stress tensor. *Physical Review*, 166(5):1272, 1968.

[3] G.F.R. Ellis and W. Stoeger. The 'fitting problem' in cosmology. *Class. Quantum Grav.*, 4:1697–1729, 1987.

[4] Chris Clarkson, G.F.R. Ellis, Julien Larena, and Umeh Obinna. Does the growth of structure affect our dynamical models of the universe? the averaging, backreaction, and fitting problems in cosmology. *Rep. Prog. Phys.*, 74:112901, 2011.

[5] Syksy Rasanen. Light propagation in statistically homogeneous and isotropic dust universes. *JCAP*, 0902:011, 2009.

[6] Syksy Rasanen. Light propagation in statistically homogeneous and isotropic universes with general matter content. *JCAP*, 1003:018, 2010.

[7] Andrzej Krasiński and Krzysztof Bolejko. Exact Inhomogeneous Models and the Drift of Light Rays Induced by Nonsymmetric Flow of the Cosmic Medium. In *Proceedings, 13th Marcel Grossmann Meeting on Recent Developments in Theoretical and Experimental General Relativity, Astrophysics, and Relativistic Field Theories (MG13)*, pages 922–934, 2015.

M. Korzyński

[8] Andrzej Krasinski and Krzysztof Bolejko. Drift of Light Rays Induced by Nonsymmetric Cosmic Flow: an observational test of homogeneity of the Universe + a few general comments on inhomogeneous models. In *Proceedings, 13th Marcel Grossmann Meeting on Recent Developments in Theoretical and Experimental General Relativity, Astrophysics, and Relativistic Field Theories (MG13)*, 2012. [Int. J. Mod. Phys.D22,0013(2013)].

[9] Krzysztof Bolejko and Pedro G. Ferreira. Ricci focusing, shearing, and the expansion rate in an almost homogeneous Universe. *JCAP*, 1205:003, 2012.

[10] Roustam Zalaletdinov. The Averaging Problem in Cosmology and Macroscopic Gravity. *Int.J.Mod.Phys.*, A23:1173–1181, 2008.

[11] Johan Brannlund, Robert van den Hoogen, and Alan Coley. Averaging geometrical objects on a differentiable manifold. *Int.J.Mod.Phys.*, D19:1915–1923, 2010.

[12] Timothy Clifton, Kjell Rosquist, and Reza Tavakol. An Exact quantification of backreaction in relativistic cosmology. *Phys. Rev.*, D86:043506, 2012.

[13] Thomas Buchert. Dark Energy from Structure: A Status Report. *Gen.Rel.Grav.*, 40:467–527, 2008.

[14] Syksy Rasanen. Dark energy from backreaction. *JCAP*, 0402:003, 2004.

[15] Akihiro Ishibashi and Robert M. Wald. Can the acceleration of our universe be explained by the effects of inhomogeneities? *Class.Quant.Grav.*, 23:235–250, 2006.

[16] Stephen R. Green and Robert M. Wald. A new framework for analyzing the effects of small scale inhomogeneities in cosmology. *Phys.Rev.*, D83:084020, 2011.

[17] Stephen R. Green and Robert M. Wald. How well is our universe described by an FLRW model? *Class.Quant.Grav.*, 31:234003, 2014.

[18] Andrzej Krasiński. Accelerating expansion or inhomogeneity? a comparison of the ΛCDM and lemaître-tolman models. *Phys. Rev. D*, 89:023520, Jan 2014.

[19] Andrzej Krasiński. Accelerating expansion or inhomogeneity? II. Mimicking acceleration with the energy function in the Lemaître-Tolman model. *Phys. Rev.*, D90(2):023524, 2014.

[20] T. Buchert et al. Is there proof that backreaction of inhomogeneities is irrelevant in cosmology? 2015.

[21] Stephen R. Green and Robert M. Wald. Comments on Backreaction. 2015.

[22] Stephen R. Green and Robert M. Wald. A Simple, Heuristic Derivation of our "No Backreaction" Results. 2016.

[23] Gregory A. Burnett. The high-frequency limit in general relativity. *Journal of Mathematical Physics*, 30(1):90–96, 1989.

[24] Daniel Baumann, Alberto Nicolis, Leonardo Senatore, and Matias Zaldarriaga. Cosmological Non-Linearities as an Effective Fluid. *JCAP*, 1207:051, 2012.

[25] Viraj A. A. Sanghai and Timothy Clifton. Post-Newtonian Cosmological Modelling. *Phys. Rev.*, D91:103532, 2015.

[26] Stephen R. Green and Robert M. Wald. Examples of backreaction of small scale inhomogeneities in cosmology. *Phys. Rev.*, D87(12):124037, 2013.

[27] Krzysztof Bolejko, Marie-Noelle Celerier, and Andrzej Krasinski. Inhomogeneous cosmological models: Exact solutions and their applications. *Class. Quant. Grav.*, 28:164002, 2011.

[28] Andrzej Krasiński. *Inhomogeneous Cosmological Models.* Cambridge University Press, 1997. Cambridge Books Online.

[29] Roberto A. Sussman. Scalar Averaging in Szekeres Models. *Springer Proc. Phys.*, 157:407–414, 2014.

M. Korzyński

[30] Roberto A. Sussman. Back-reaction and effective acceleration in generic LTB dust models. *Class. Quant. Grav.*, 28:235002, 2011.

[31] Sebastian J. Szybka. On light propagation in Swiss-Cheese cosmologies. *Phys. Rev.*, D84:044011, 2011.

[32] Mikko Lavinto and Syksy Rasanen. CMB seen through random Swiss Cheese. *JCAP*, 1510(10):057, 2015.

[33] Krzysztof Bolejko and Marie-Noelle Celerier. Szekeres Swiss-Cheese model and supernova observations. *Phys. Rev.*, D82:103510, 2010.

[34] Ulrich Sperhake. The numerical relativity breakthrough for binary black holes. *Class. Quant. Grav.*, 32(12):124011, 2015.

[35] Vitor Cardoso, Leonardo Gualtieri, Carlos Herdeiro, and Ulrich Sperhake. Exploring New Physics Frontiers Through Numerical Relativity. *Living Rev. Relativity*, 18:1, 2015.

[36] A. Lichnerowicz. L'integration des équations de la gravitation relativiste et le probleme des n corps. *J. Math. Pures Appl.*, 23(37), 1944.

[37] Jr. York, James W. Gravitational degrees of freedom and the initial-value problem. *Phys. Rev. Lett.*, 26:1656–1658, 1971.

[38] Richard W. Lindquist and John A. Wheeler. Dynamics of a lattice universe by the schwarzschild-cell method. *Rev. Mod. Phys.*, 29(3):432–443, Jul 1957.

[39] John Wheeler. The geometrostatic lattice cell. *Foundations of Physics*, 13:161–173, 1983. 10.1007/BF01889418.

[40] Richard Schoen and Shing Tung Yau. On the proof of the positive mass conjecture in general relativity. *Comm. Math. Phys.*, 65(1):45–76, 1979.

[41] Edward Witten. A new proof of the positive energy theorem. *Comm. Math. Phys.*, 80(3):381–402, 1981.

[42] Sergio Dain. Geometric inequalities for black holes. *Gen. Rel. Grav.*, 46:1715, 2014.

[43] Timothy Clifton and Pedro G. Ferreira. Archipelagian Cosmology: Dynamics and Observables in a Universe with Discretized Matter Content. *Phys. Rev. D*, 80:103503, 2009.

[44] Jean-Philippe Bruneton and Julien Larena. Dynamics of a lattice universe: the dust approximation in cosmology. *Classical and Quantum Gravity*, 29(15):155001, 2012.

[45] Timothy Clifton, Pedro G. Ferreira, and Kane O'Donnell. An Improved Treatment of Optics in the Lindquist-Wheeler Models. *Phys. Rev.*, D85:023502, 2012.

[46] Frans Pretorius. Evolution of binary black hole spacetimes. *Phys. Rev. Lett.*, 95:121101, 2005.

[47] László B. Szabados. Quasi-local energy-momentum and angular momentum in gr: A review article. *Living Reviews in Relativity*, 7(4):10, 2004.

[48] Michael Spivak. *A Comprehensive Introduction to Differential Geometry*, volume 5. Publish or Perish, Berkeley, 1979.

[49] Mikolaj Korzynski. Coarse-graining of inhomogeneous dust flow in General Relativity via isometric embeddings. *AIP Conf. Proc.*, 1241:973–980, 2010.

[50] Niall Ó Murchadha and James W. York. Existence and uniqueness of solutions of the hamiltonian constraint of general relativity on compact manifolds. *Journal of Mathematical Physics*, 14(11):1551–1557, 1973.

[51] Frank Löffler, Joshua Faber, Eloisa Bentivegna, Tanja Bode, Peter Diener, Roland Haas, Ian Hinder, Bruno C. Mundim, Christian D. Ott, Erik Schnetter, Gabrielle Allen, Manuela Campanelli, and Pablo Laguna. The Einstein Toolkit: A Community Computational Infrastructure for Relativistic Astrophysics. *Class. Quantum Grav.*, 29(11):115001, 2012.

M. Korzyński

[52] Lee Lindblom, Bela Szilagyi, and Nicholas W. Taylor. Solving Einstein's equation numerically on manifolds with arbitrary spatial topologies. *Phys. Rev.*, D89(4):044044, 2014.

[53] Miguel Alcubierre, Bernd Bruegmann, Peter Diener, Michael Koppitz, Denis Pollney, Edward Seidel, and Ryoji Takahashi. Gauge conditions for long term numerical black hole evolutions without excision. *Phys. Rev.*, D67:084023, 2003.

[54] Chul-Moon Yoo, Hirotada Okawa, and Ken-ichi Nakao. Black Hole Universe: Time Evolution. *Phys.Rev.Lett.*, 111:161102, 2013.

[55] Chul-Moon Yoo and Hirotada Okawa. Black hole universe with a cosmological constant. *Phys. Rev.*, D89(12):123502, 2014.

[56] Timothy Clifton, Daniele Gregoris, Kjell Rosquist, and Reza Tavakol. Exact Evolution of Discrete Relativistic Cosmological Models. *JCAP*, 11:010, 2013.

[57] Timothy Clifton, Daniele Gregoris, and Kjell Rosquist. Applications of black hole lattices in relativistic cosmology. *AIP Conf. Proc.*, 1693:070006, 2015.

[58] Rex G. Liu and Ruth M. Williams. Regge calculus models of closed lattice universes. *Phys. Rev.*, D93(2):023502, 2016.

[59] Lars Andersson. Cosmological models and stability. *Fundam. Theor. Phys.*, 177:277–303, 2014.

[60] Chul-Moon Yoo, Hiroyuki Abe, Ken-ichi Nakao, and Yohsuke Takamori. Black Hole Universe: Construction and Analysis of Initial Data. *Phys.Rev.*, D86:044027, 2012.

[61] Eloisa Bentivegna. Solving the Einstein constraints in periodic spaces with a multigrid approach. *Class. Quant. Grav.*, 31:035004, 2014.

[62] Nigel Bishop, Denis Pollney, and Christian Reisswig. Initial data transients in binary black hole evolutions. *Class. Quant. Grav.*, 28:155019, 2011.

[63] Art B. Owen. Quasi-monte carlo for integrands with point singularities at unknown locations. In Harald Niederreiter and Denis Talay, editors, *Monte Carlo and Quasi-Monte Carlo Methods 2004*, pages 403–417. Springer Berlin Heidelberg, 2006.

[64] I.M. Sobol'. On quasi-monte carlo integrations. *Mathematics and Computers in Simulation*, 47(2–5):103 – 112, 1998.

[65] Michael Drmota and Robert F. Tichy. *Sequences, Discrepancies and Applications.* Springer–Verlag, 1997.

[66] I.M. Sobol. *Mathematics and Computers in Simulations*, 47:103–112, 1998.

[67] Jürgen Hartinger, Reinhold F. Kainhofer, and Robert F. Tichy. Quasi-monte carlo algorithms for unbounded, weighted integration problems. *Journal of Complexity*, 20(5):654 – 668, 2004. Dagstuhl 2002 - Festschrift for the 70th Birthday of Joseph F. Traub.

[68] Elise de Doncker and Yuqiang Guan. Error bounds for the integration of singular functions using equidistributed sequences. *Journal of Complexity*, 19(3):259 – 271, 2003. Oberwolfach Special Issue.

[69] I.M. Sobol'. Calculation of improper integrals using uniformly distributed sequences. *Soviet Mathematics Doklady*, 14:734–738, 1973.

[70] Timothy Clifton. The Method of Images in Cosmology. *Class.Quant.Grav.*, 31:175010, 2014.

[71] Rex G. Liu. Lindquist-Wheeler formulation of lattice universes. *Phys. Rev.*, D92(6):063529, 2015.

[72] C. W. Misner, K. S. Thorne, and J. A. Wheeler. *Gravitation*, chapter 23, page 607. 1973.

[73] R.M. Wald. *General Relativity*, chapter 6. University of Chicago Press, 2010.

[74] Sebastian J. Szybka and Michał J. Wyrębowski. Backreaction for Einstein-Rosen waves coupled to a massless scalar field. 2015.

M. Korzyński

[75] Malcolm A. H. MacCallum. Milestones of general relativity: Hubble's law (1929) and the expansion of the universe. *Class. Quant. Grav.*, 32(12):124002, 2015.

[76] Henk van Elst and Claes Uggla. General relativistic 1+3 orthonormal frame approach. *Classical and Quantum Gravity*, 14(9):2673, 1997.

[77] Mikołaj Korzyński and Jerzy Lewandowski. The Normal conformal Cartan connection and the Bach tensor. *Class. Quant. Grav.*, 20:3745–3764, 2003.

[78] Pawel Nurowski and Jerzy F Plebanski. Non-vacuum twisting type-n metrics. *Classical and Quantum Gravity*, 18(2):341, 2001.

[79] Mikołaj Korzyński, Jerzy Lewandowski, and Tomasz Pawłowski. Mechanics of multidimensional isolated horizons. *Class. Quant. Grav.*, 22:2001–2016, 2005.

[80] Mikołaj Korzyński. Isolated and dynamical horizons from a common perspective. *Phys. Rev.*, D74:104029, 2006.

[81] Mikołaj Korzyński. Quasi-local angular momentum of non-symmetric isolated and dynamical horizons from the conformal decomposition of the metric. *Class. Quant. Grav.*, 24:5935–5944, 2007.

[82] Fred I. Cooperstock and S. Tieu. General relativity resolves galactic rotation without exotic dark matter. *Submitted to: Astrophys. J.*, 2005.

[83] Mikołaj Korzyński. Singular disk of matter in the cooperstock and tieu galaxy model. 2005.

[84] Mikołaj Korzyński. Can dark matter in galaxies be explained by relativistic corrections? *J. Phys.*, A40:7087–7092, 2007.

[85] J. Mazurkiewicz, J. Żygierewicz, and M. Korzyński. Short term synaptic depression model–analytical solution and analysis. *J. Theor. Biol.*, 254(1):82–88, Sep 2008.

[86] Michael Jasiulek and Miko Korzynski. Isometric embeddings of 2-spheres by embedding flow for applications in numerical relativity. *Class. Quant. Grav.*, 29:155010, 2012.

[87] Louis Nirenberg. The Weyl and Minkowski problems in differential geometry in the large. *Communications on pure and applied mathematics*, 6:337–394, 1953.

M. Korzyński