TULCZYJEW TRIPLES IN MECHANICS AND FIELD THEORY

Author's-abstract

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(1) Katarzyna Grabowska, Janusz Grabowski, Paweł Urbański, *Geometrical Mechanics on algebroids*, Int. J. Geom. Meth. Mod. Phys. **3** (2006), 559-575.

(2) Katarzyna Grabowska, Janusz Grabowski, P. Urbański, *AV-differential geometry: Euler-Lagrange equations*, J. Geom. Phys., **57** (2007), 1984-1998

(3) Katarzyna Grabowska, Janusz Grabowski *Variational calculus with constraints on general algebroids* J. Phys. A: Math. Theor. **41** (2008) 175204 (25pp);

(4) Katarzyna Grabowska, *Lagrangian and Hamiltonian formalism in field theory: a simple model*, J. Geom. Mech., **2** (2010), 375-395;

(5) Katarzyna Grabowska, Janusz Grabowski *Dirac Algebroids in Lagrangian and Hamiltonian Mechanics*, J. Geom. Phys., **61** (2011), 2233-2253;

(6) Katarzyna Grabowska, *A Tulczyjew triple for classical fields*, J. Phys. A: Math. Theor. **45** (2012) 145207;

In the above publications new geometric methods in Lagrangian and Hamiltonian formalisms of mechanics and field theories have been proposed. Following Tulczyjew ideas of a variational calculus, we have constructed generalizations of the classical Tulczyjew triple that can be used for a simple and consistent description of many different systems: with or without constraints, with regular or singular Lagrangians, reduced with respect to symmetries, etc.

The main advantage of our approach is the completeness of the theory. We do not concentrate on the Euler-Lagrange equations as critical points of an action functional, but we perform also nontrivial variations on the boundary, from where we get the information about momenta. We construct phase spaces and phase dynamics. The systematic approach to the Lagrangian formalism leads at the same time to a deeper understanding of the Hamiltonian formalism for various types of systems.

In our works concerning analytical mechanics we develop new tools associated with the concept of a Lie algebroid and its more general versions. Mathematical structures needed

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in the Lagrangian and Hamiltonian formulations of mechanics can be understood much better in this framework. The most important objects in this picture turned out to be double structures such as double vector bundles, double vector-affine bundles, and finally double affine bundles. We believe that the concept of the affine duality can be used also in a future work concerning a Lagrangian description of a field theory with Lagrangians depending on jets of higher order.

Main results

1. The Tulczyjew triple in analytical mechanics

In analytical mechanics there are two commonly accepted formalisms: Lagrangian and Hamiltonian [1]. In the Hamiltonian formalism we derive phase equations, i.e., differential equations for curves in the space of momenta of our system. The Lagrangian formalism, in the version introduced by Klein [19], is used to derive the Euler-Lagrange equations without the use of the variational calculus by means of purely geometric tools such as canonical structures on tangent and cotangent bundles.

Let Q denote the differential manifold of configurations of the system. If there are no constraints, then the tangent bundle $\mathsf{T}Q$ is the space of velocities, while the cotangent bundle T^*Q is the space of momenta or the *the phase space*. The Hamiltonian formalism is associated with the phase space, and the Lagrangian formalism with the velocity space. In the following, τ_Q will denote the canonical projection $\tau_Q : \mathsf{T}Q \to Q$ that assigns the point $q \in Q$ to the vector $v \in \mathsf{T}_qQ$, $\pi_Q : \mathsf{T}^*Q \to Q$ is the corresponding projection of the cotangent bundle. The cotangent bundle is canonically a symplectic manifold. The canonical symplectic form will be denoted by ω_Q.

The Euler-Lagrange equation for a systems with a first-order Lagrangian, i.e., a Lagrangian being a function on the velocity space, is a second-order differential equation. It can be represented as a subset in the space T^2Q of tangent elements of the second order. For practical reasons we often prefer to treat second-order differential equations as first-order differential equations of certain type on the velocity space. From the point of view of geometry it means that T^2Q is embedded in the iterated tangent bundle $\mathsf{TT}Q$. Elements of T^2Q are identified with those vectors $v \in \mathsf{TT}Q$ which satisfy the condition $\mathsf{T}\tau_Q(v) = \tau_{\mathsf{T}Q}(v)$. Such vectors are called *holonomic*. We usually expect that the Euler-Lagrange equation will be represented by a vector field on $\mathsf{T}Q$ with values in the space of holonomic vectors.

The traditional understanding of the Lagrangian formalism in analytical mechanics is the method of obtaining the Euler-Lagrange equation for a given Lagrangian by using canonical structures of the tangent and cotangent bundle. Any function $L : \mathsf{T}Q \to \mathbb{R}$ defines a map $\lambda : \mathsf{T}Q \to \mathsf{T}^*Q$ given by the differential $\mathrm{d}L$ restricted to the fibre of τ_Q. We observe that vectors tangent to the fibre of τ_Q can be identified with elements of the fibre itself, because every fibre is a vector space. When we restrict $\mathrm{d}L(v_q)$ to vectors tangent to the fibre at v_q, we get an element of T^*_qQ that we denote with $\lambda(v_q)$. The map λ is called *the Legendre map* or sometimes the *Legendre transformation*. For us, however, the Legendre transformation will be the process of passing from the Lagrangian to the Hamiltonian formalism that is more complicated that defining just one map. The traditional Lagrangian formalism can be used when the above Legendre map is a local diffeomorphism. In such a case we call the Lagrangian *regular*. The more comfortable situation is

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when λ is a global diffeomorphism, i.e., in the case of a *hyperregular* Lagrangian. Using the map λ, we define a 2-form

$$\omega_L = \lambda^* \omega_Q.$$

If the Lagrangian is regular, then ω_L is a symplectic form that can be used to obtain Hamiltonian vector fields from functions on $\mathsf{T}Q$. For the function $E(v) = \langle \lambda(v), v \rangle - L(v)$, we get the vector field X_E which is the vector field representing the Euler-Lagrange equation. One can check that X_E takes values in the space of holonomic vectors. It is usually done by means of the *vertical endomorphism* $S : \mathsf{TT}Q \to \mathsf{TT}Q$ which assigns to any vector X_v tangent to $\mathsf{T}Q$ at v the vertical lift to the point v of the tangent projection $\mathsf{T}\tau_Q(X_v)$. A vector field X on $\mathsf{T}Q$ represents a second-order differential equation if $S(X(v)) = \nabla(v)$, where $\nabla(v)$ is the value of the Liouville (Euler) vector field at v, i.e., the vertical lift of v to the point v. The vertical endomorphism S and the Liouville vector field ∇ are canonical elements of the structure of $\mathsf{T}Q$. The traditional Lagrangian formalism works well only for regular Lagrangians. Another disadvantage is that this cannot be easily generalized to more advanced setting, as e.g. Lie algebroids. There are also serious complications in the case of constraints.

By the Hamiltonian formalism we mean deriving the phase equation from a function on the phase space called the *Hamiltonian function*. The phase equation is given by the Hamiltonian vector field X_H obtained from H by means of the canonical symplectic structure of T^*Q. In the case of a hyperregular Lagrangian, we can get the Hamiltonian function from the energy function E by the composition with λ^{-1}: $H(p) = E(\lambda^{-1}(p))$. In such a case one can show that $\lambda_* X_E = X_H$. The traditional Lagrangian formulation of mechanics is therefore equivalent to the Hamiltonian formulation if the Lagrangian function is hyperregular.

It is difficult to use the above traditional theories in the case of a constrained system or a system with a singular Lagrangian. Note that even such a simple mechanical system as a free particle in the Minkowski space-time is described by a singular Lagrangian. We obtain also serious difficulties trying to include external forces into the theory.

An alternative approach to Lagrangian and Hamiltonian mechanics was proposed by W. M. Tulczyjew and published in a series of papers [27, 30, 31, 32, 33, 34, 35, 36, 37] and the book [38]. The Tulczyjew formulation is elegant, simple, and very general. It can be easily used for generalizations to the algebroid setting. All the concepts and constructions used in the Tulczyjew formulation of mechanics come from the variational calculus, more precisely from the variational calculus of statics. Similar constructions for dynamics and field theories can also be used but they need a proper interpretation. In the following, we present only those elements of the Tulczyjew theory which will be needed for our purposes.

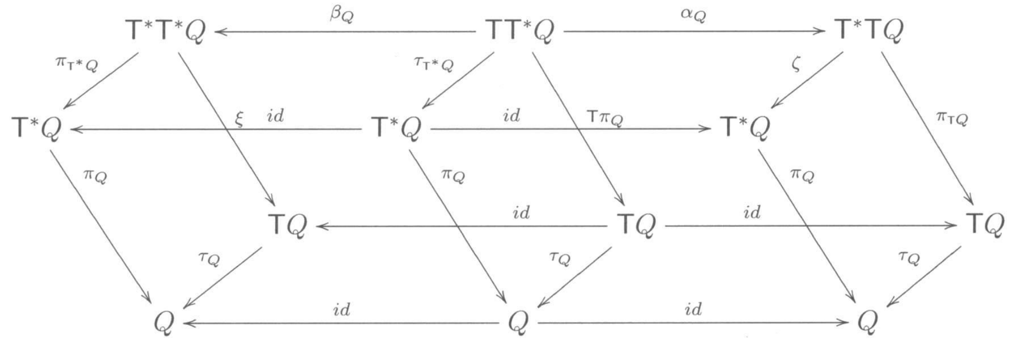
Let us concentrate on the simplest case of autonomous analytical mechanics. The phase space (the space of momenta) is the cotangent bundle T^*Q of the configuration manifold Q. The space of velocities $\mathsf{T}Q$ is in this case called the *space of infinitesimal configurations*. The content of both, the Lagrangian and Hamiltonian formalism, depends on obtaining phase equations from a Lagrangian or a Hamiltonian. Phase equations are differential equations for curves in the phase space. The most important tool is a certain diagram called the *Tulczyjew tripe*. In this diagram one can find encoded the complete geometric structure of the tangent and the cotangent bundle. In a simplified version, the diagram reads as

(1) $$\mathsf{T}^*\mathsf{T}^*Q \xleftarrow{\beta_Q} \mathsf{TT}^*Q \xrightarrow{\alpha_Q} \mathsf{T}^*\mathsf{T}Q\ .$$

The right-hand side of the triple is related to the Lagrangian formalism. The map α_Q is the map dual to the canonical involution $\kappa_Q : \mathsf{T}\mathsf{T}Q \to \mathsf{T}\mathsf{T}Q$ which is an isomorphism of the two different vector bundle structures, $\mathsf{T}\tau_Q$ and $\tau_{\mathsf{T}Q}$, existing in $\mathsf{T}\mathsf{T}Q$. Both spaces, $\mathsf{T}^*\mathsf{T}Q$ and $\mathsf{T}\mathsf{T}^*Q$, are *double vector bundles* that means that they have two 'compatible' vector bundle structures [28, 20, 14], just like $\mathsf{T}\mathsf{T}Q$. The map α_Q is a morphism of double vector bundles. Both spaces carry canonical symplectic structures. On $\mathsf{T}^*\mathsf{T}Q$ we have $\omega_{\mathsf{T}Q}$ as on any cotangent bundle. On $\mathsf{T}\mathsf{T}^*Q$ the symplectic structure is lifted from T^*Q, i.e., it is $\mathrm{d}_{\mathsf{T}}\omega_Q$. The map α_Q is additionally a symplectomorphism.

The left-hand side of the triple is related to the Hamiltonian formulation. The map β_Q comes from the canonical symplectic form ω_Q, precisely, $\beta_Q(v) = \omega(v, \cdot)$. The map β_Q is a diffeomorphism, a morphism of double vector bundles, and a symplectomorphism. The Tulczyjew triple with all three double vector bundle structures reads as

(2)



A mechanical system is described by a first-order differential equation on the phase space. The equation can be given by a vector field, but as well it can have an implicit form, i.e., it can be given by a subset of $\mathsf{T}\mathsf{T}^*Q$ not being the image of a vector field. Such a subset will be called the *phase dynamics* and denoted $\mathcal{D}$. In many cases, $\mathcal{D}$ is a Lagrangian submanifold with respect to the symplectic structure $\mathrm{d}_{\mathsf{T}}\omega_M$. In the simplest case, if the system is described by a Lagrangian $L : \mathsf{T}Q \to \mathbb{R}$ and there are no constraints, the phase dynamics is $\mathcal{D} = \alpha_Q^{-1}(\mathrm{d}L(\mathsf{T}Q))$, i.e., it is an inverse image by α_Q of the Lagrangian submanifold $N_L = \mathrm{d}L(\mathsf{T}Q)$ generated in $\mathsf{T}^*\mathsf{T}Q$ by the Lagrangian. One should stress that it is not important here if the Lagrangian is regular or not.

The same dynamics can be generated in the Hamiltonian way as the inverse image by β_Q of a certain Lagrangian submanifold $N_H \subset \mathsf{T}^*\mathsf{T}^*Q$. If $\mathcal{D}$ is not an image of a vector field, then N_H is not an image of a differential of any Hamiltonian. In such a case we need a more complicated generating object than just one function on the phase space. It can be a function on a submanifold or a family of functions (the so called *Morse family*). There exists always a particular family of functions generating N_H, namely

$$h : \mathsf{T}^*Q \times_Q \mathsf{T}Q \to \mathbb{R}, \quad h(p, v) = \langle p, v \rangle - L(v)\,.$$

Functions in the above family are parameterized by velocities. For some Lagrangians this generating family can be simplified. In [41] one can find examples of physical systems with singular Lagrangians for which the correct Hamiltonian description was found. The dynamics $\mathcal{D} \subset \mathsf{T}\mathsf{T}^*Q$ can be used for generating the Euler-Lagrange equation as well. The Euler-Lagrange equations are here second-order equations for curves in the manifold Q. The geometric representation of the Euler Lagrange equation is a subset $E_L \subset \mathsf{T}^2Q$.

Denoting with $\mathsf{T}^2\pi_Q$ the canonical projection $\mathsf{T}^2\pi_Q : \mathsf{T}^2\mathsf{T}^*Q \to \mathsf{T}^2Q$, we can write

$$E_L = \mathsf{T}^2\pi_Q(P\mathcal{D}),$$

where $P\mathcal{D} = \mathsf{T}D \cap \mathsf{T}^2\mathsf{T}^*Q$. The Lagrangian formulation of analytical mechanics proposed by Tulczyjew is not only more general than the traditional one, but also more elegant and simpler. It is based on well-defined geometrical ideas that come from variational calculus. These ideas make it possible to construct several generalizations of the theory.

The main content of my postdoctoral thesis are constructions of the Tulczyjew triple in situations more general than described above.

For instance, in paper (1) we construct the Tulczyjew triple for analytical mechanics on algebroids, while in paper (2) we discuss the variational calculus associated with that triple. Analyzing the variational approach, we find an appropriate definition and description of systems with different types of constraints.

It happens in many situations in classical mechanics that the affine structures are needed to achieve the frame-independence of the theory. This is the case e.g. of one particle moving in the Newtonian space-time. In paper (2) we construct the affine version of the Tulczyjew triple. For, we need an affine version of differential geometry where, instead of functions on a manifold, sections of one-dimensional affine bundles are used.

In paper (5) we define a notion of a *Dirac algebroid* as a linear variant of a Dirac structure. This concept is then used to construct the Tulczyjew triple that can be used in description of a very broad class of systems. The point is that within the same formalism we can describe unconstrained systems and systems with nonholonomic constraints.

The papers (4) and (6) are devoted to the construction of the Tulczyjew triple in field theories. In (4) we present the simple case where fields are just mappings from $\mathbb{R}^n$ to a manifold M. In (6) we describe the general situation of fields being sections of an arbitrary fibration. In the general case the affine framework is essential. The results contained in papers (1)-(6) will be presented in a more detail.

2. Lagrangian reductions – mechanics on algebroids

In [46] Weinstein posed the problem of finding a formulation of analytical mechanics (i.e., of deriving the Euler-Lagrange equations) in the case when Lagrangian is a function on a Lie algebroid. Such a formulation is needed because reductions with respect to symmetries usually lead us out of the framework of the tangent bundle. This is a Lagrangian version of the reduction of Hamiltonian systems that leads to non-symplectic Poisson brackets. A widely known example of such a situation is the mechanics of a rigid body. The configuration space of a rigid body is SO(3) group. Since the free Lagrangian $L : \mathsf{T}\mathrm{SO}(3) \to \mathbb{R}$ does not depend on the configuration, after a reduction we deal with a system on the Lie algebra $\mathfrak{so}(3)$ of the group, a simple example of a Lie algebroid.

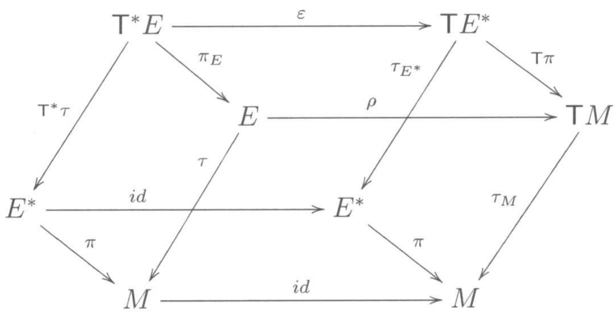
According to the traditional presentation, a *Lie algebroid* is a vector bundle $\tau : E \to M$ equipped with a Lie bracket $[\cdot,\cdot]$ on the space of its sections. There is also a vector bundle morphism $\rho : E \to \mathsf{T}M$ covering the identity on M, called the *anchor*, which satisfies

$$[X, fY] = f[X,Y] + \rho(X)(f)Y$$

for all sections X, Y of the bundle τ and any smooth function f on M. The canonical example of a Lie algebroid is of course the tangent bundle $\mathsf{T}M$ with the Lie bracket of vector fields and the identity map as the anchor. Another example is a Lie algebra $\mathfrak{g}$ over a one-point manifold with the constant map equal to zero as the anchor. The example known in the gauge theory is the so called *Atiyah algebroid* associated with the principal fibration.

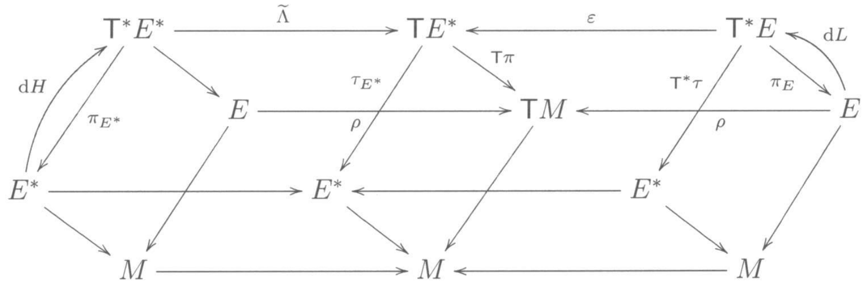
The problem posed by Weinstein (and to some extent by Liebermann [21]) was then studied by many mathematicians and physicists. There are for example numerous papers by Martínez [23, 24, 25], where an interesting version of Lagrangian mechanics on Lie algebroids is proposed. This version is based on the Klein method and therefore it uses more complicated geometrical structures. Instead of the original Lie algebroid and its dual bundle, one has to use prolongations of both bundles. The reason is that the Klein formalism uses the vertical endomorphism S which is not present on an arbitrary Lie algebroid.

In the paper [15] one can find an alternative definition of a Lie algebroid and its generalization called a *general algebroid*. This idea is based on the observation that the structure of a Lie algebroid on a bundle E can be encoded in a particular double vector bundle morphism $\varepsilon : \mathsf{T}^*E \longrightarrow \mathsf{T}E^*$ over the identity on E^*. The diagram for ε reads



In the above diagram the map ρ is the anchor map. It is well known that if there is a Lie algebroid structure on a bundle $\tau : E \to M$, then on the dual bundle $\pi : E^* \to M$ there is a uniquely defined linear Poisson structure Λ. Using the alternative definition of a Lie algebroid, we obtain the Poisson structure composing ε with the canonical isomorphism $\mathcal{R}_E : \mathsf{T}^*E^* \to \mathsf{T}E^*$. The composition is a map $\tilde{\Lambda} = \varepsilon \circ \mathcal{R}_E$ corresponding to Λ. Any morphism ε of double vector bundles T^*E and $\mathsf{T}E^*$ over the identity on E^* is then viewed as an *algebroid* structure on E. The algebroid is called *skew-symmetric* if the corresponding $\tilde{\Lambda}$ comes from a bivector field. If additionally the bivector field is Poisson, we actually deal with a Lie algebroid. It means that in this new language an algebroid structure is encoded not in a bracket of sections but in a double vector bundle morphism. In the example of the tangent bundle $\mathsf{T}M$ this morphism is α_M^{-1}, and the corresponding Poisson structure is the canonical one coming form the symplectic structure ω_M or from β_M^{-1}. Both mappings were crucial for the Lagrangian and Hamiltonian formalisms in the Tulczyjew's approach. The paper (1) is devoted to the Lagrangian and Hamiltonian formalisms on a general algebroid. The Tulczyjew triple we have obtained reads as

(3)



A Lagrangian $L : E \to \mathbb{R}$ generates the Lagrangian submanifold N_L of T^*E. In unconstrained case the submanifold N_L is an image of the differential of L, $N_L = \mathrm{d}L(E)$. The phase dynamics $\mathcal{D} = \varepsilon(N_L) \subset \mathsf{T}E^*$ is a *differential inclusion*, i.e., an implicit differential equation for phase curves. The phase curves are now curves in E^*. Like in the case of the tangent bundle, any Lagrangian defines the Legendre map that associates momenta to infinitesimal configurations,

$$\lambda : E \to E^*, \quad \lambda(e) = \mathsf{T}^*\tau(\mathrm{d}L(e)) = \tau_{E^*}(\varepsilon(\mathrm{d}L(e)))\,.$$

The Legendre map is a vertical differential of the Lagrangian.

The Hamiltonian side of the Tulczyjew triple is based on the map $\widetilde{\Lambda}$. The dynamics $\mathcal{D}$ is obtained from a function $H : E^* \to \mathbb{R}$ as the image of $N_H = \mathrm{d}H(E^*)$ with respect to $\widetilde{\Lambda}_\varepsilon$. The dynamics is therefore the image of a Hamiltonian vector field of H created by means of Λ_ε. We can ask if, for a given dynamics, there exists a Hamiltonian description. This means that we ask if the dynamics is the image of a Hamiltonian vector field.

Like in the classical case, we can generalize the concept of the Hamiltonian description of phase dynamics by looking for more general generating object than just one function on E^*. There is, however, much more freedom in choosing the generating object because of the possible degeneracy of Λ. We have still the following (Lemma 1, p. 569).

Theorem 1. If the Lagrangian L is hyperregular, then the Lagrange submanifold $N_L = dL(E)$ in T^*E corresponds, under the canonical isomorphism $\mathcal{R}_E$, to the Lagrange submanifold $N_H = dH(E)$ in T^*E^*, where $H : E^* \to E$,

$$H(\varphi) = \langle(\varphi, \lambda^{-1}(\varphi)\rangle - L(\lambda^{-1}(\varphi))\,. \tag{4}$$

It means that if the Lagrangian L is hyperregular, then the Lagrangian description is equivalent to the Hamiltonian description with the Hamiltonian function given by (4).

In the algebroid framework we have also an analog of the Euler-Lagrange equation. The Euler- Lagrange equation is now the equation for curves in E, not for curves on M like in the classical situation. We have considered two equations. The first equation reads

$$E^1_L = \mathsf{T}\lambda^{-1}(\mathcal{D})\,.$$

A curve γ in E is a solution of the above equation if $\lambda \circ \gamma$ is a solution of the phase dynamics. The equation E^1_L corresponds to the construction proposed by de Leon and Lacomba in [22]. The second equation reads

$$E^2_L = \{v \in \mathsf{T}E : \ \mathsf{T}(\varepsilon \circ \mathrm{d}L)(v) \in \mathsf{T}^2E^*\} = \mathsf{T}\lambda^{-1}(\mathsf{T}^2E^*)\,.$$

A curve γ in E is a solution of the equation E^2_L if the tangent prolongation of $\lambda \circ \gamma$ is equal to $\varepsilon \circ \mathrm{d}L \circ \lambda$. It is clear that E^2_L is more restrictive than E^1_L, i.e., $E^2_L \subset E^1_L$. In local coordinates (x^a, y^i) on E, where (x^a) are coordinates on M, and (y^i) being linear coordinates in the fibres, we get

$$(5) \qquad \frac{\mathrm{d}x^a}{\mathrm{d}t} = \rho^a_k(x)y^k, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial y^j}(x,y)\right) = c^k_{ij}(x)y^i\frac{\partial L}{\partial y^k}(x,y) + \sigma^a_j(x)\frac{\partial L}{\partial x^a}(x,y)\,,$$

in the full correspondence with the equations proposed in [24, 25, 46]. Here, $\rho^a_k(x)$ are the coordinates of the anchor $\rho : E \to \mathsf{T}M$, and $\sigma^a_j(x)$ i $c^k_{ij}(x)$ form the rest of the 'structural functions' of the algebroid. Note that the equation $\frac{\mathrm{d}x^a}{\mathrm{d}t} = \rho^a_k(x)y^k$ means that the solutions $\gamma(t)$ of E^2_L are automatically *admissible*, i.e., the tangent vector $\dot{\underline{\gamma}}(t)$ to the projection $\underline{\gamma}(t) = \tau \circ \gamma(t)$ on M equals $\rho(\gamma(t))$. In the case $E = \mathsf{T}M$ it means that γ is the tangent prolongation of $\underline{\gamma}$. Since in this case $y = \dot{x}$, the equations (5) reduce to the standard Euler-Lagrange equations of the second order for curves $x(t)$ in the configuration manifold M,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}^a}\right) = \frac{\partial L}{\partial x^a}\,.$$

If a Lagrangian is hyperregular, equations E^2_L and E^1_L coincide. In paper (1) one can find also a version of the Noether theorem in the algebroid setting.

3. Variational calculus on algebroids

In paper (3) we discuss the variational foundations of the Lagrangian formalism in the case of a Lagrangian being a function on a general algebroid $\tau : E \to M$. We can then include constraints to our theory. To define a variational problem on an algebroid we have to specify a manifold $\mathcal{M}$ of paths, whose tangent space $\mathsf{T}\mathcal{M}$ represents all possible variations, and an action functional W on $\mathcal{M}$. Then, we have to choose a submanifold $\mathcal{N}$ of admissible paths and a set (generalized distribution) $\mathcal{D} \subset \mathsf{T}\mathcal{M}_{|\mathcal{N}}$ of admissible variations of admissible paths.

In the case $E = \mathsf{T}M$ without constraints, we take smooth curves in $\mathsf{T}M$ defined for $t \in [t_0, t_1]$ as paths and curves in $\mathsf{TT}M$ as variations. The admissible paths are tangent prolongations of curves in M, while admissible variations come from homotopies of curves in M, i.e., maps $\chi : \mathbb{R}^2 \to M$. In the construction of admissible variations we use the canonical flip $\kappa_M : \mathsf{TT}M \to \mathsf{TT}M$. For a curve $\delta\gamma : \mathbb{R} \to \mathsf{T}M$ such that $\tau_M \circ \delta\gamma = \gamma$, we define the admissible variation of the admissible curve $\dot{\gamma} : \mathbb{R} \to \mathsf{T}M$ as $\kappa \circ \dot{\delta\gamma} : \mathbb{R} \to \mathsf{TT}M$.

In the algebroid setting admissible curves are curves $\gamma : \mathbb{R} \to E$ which satisfy $\dot{\underline{\gamma}} = \rho \circ \gamma$, where $\underline{\gamma} = \tau \circ \gamma$. Note that the concept of a homotopy of an admissible curve is closely related to the problem of integrating Lie algebroids [5]. In the case of a general algebroid we are forced to use a different method. The construction we use is similar to the classical one in the sense that the crucial role is played by the algebroid analog of the morphism κ_M. Since ε is not an isomorphism, its dual $\kappa_\varepsilon : \mathsf{T}E \longrightarrow\!\!\triangleright\, \mathsf{T}E$ is a linear relation only. Admissible variations are constructed by means of κ_ε from 'vertical variations' of admissible curves, i.e., vertical vector fields along admissible curves. In the case of a Lie algebroid we obtain standard infinitesimal homotopies of admissible curves. Using the defined elements of the structure, we derive the Euler-Lagrange equations which happen to be the same as E^2_L obtained in (1).

It happens quite often in mathematics and physics that considering generalizations of certain theories we gain some new knowledge about their classical versions. Exactly that happened, while we considered constrained systems in the algebroid setting. It turned out that the correct description of a constrained system requires more than defining the set of admissible infinitesimal configurations (constraints), namely also a set of admissible

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variations. We would even say that constraints are put on variations rather than on configurations. Conditions that configurations must fulfill are consequences of that put on variations.

There are two standard methods of assigning admissible variations to the subset $\Sigma \subset E$ of configurations. We can use all variations that are tangent to Σ or we can construct variations out of vertical variations tangent to Σ by means of κ_ε. In the case $E = \mathsf{T}M$, the first method refers to *vakonomic constraints* and the second to *nonholonomic constraints*. If variations obtained in a nonholonomic way are tangent to Σ, we say that we have *holonomic constraints*. For all three types of constraints we derive the Euler-Lagrange equations in a geometric way.

One of the important observations coming from the constructions done in the algebroid setting is that, even in the case $E = \mathsf{T}M$, it is the structure of an algebroid that matters and is the main tool in constructing the phase dynamics and the Euler-Lagrange equations. Working on the tangent bundle we do not pay much attention to this fact, because the structures of the tangent and cotangent bundles are regarded as natural ingredients of the theory. It turns out, however, that the most important from the point of view of mechanics is the map α_M (or, equivalently, κ_M).

The theory of constrained systems is illustrated with few examples. As the example of vakonomic system we describe the differential version of the Pontryagin maximum Principle, the basic tool in the theory of optimal control. The nonholonomic example is a ball moving on a rotating table.

4. The frame independent description

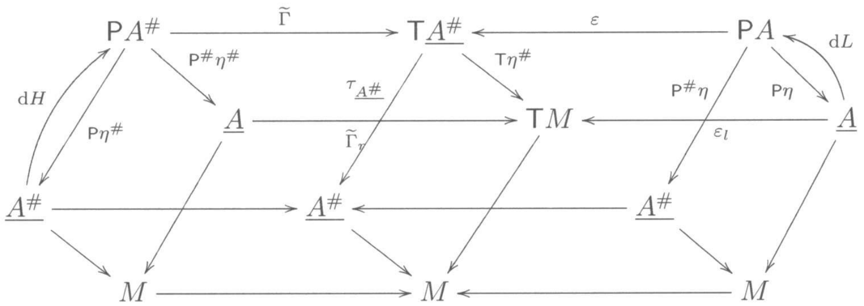
In many situations of classical mechanics the frame independent formulation is possible only when constructions of affine geometry, understood as the geometry of affine bundles, are used. This is the case of nonautonomous mechanics, mechanics of a charged particle, and even the dynamics of one particle moving in the Newtonian space-time. To observe that the affine geometry appears naturally, it is enough to see that e.g. the momentum of a particle in the Newtonian space-time transforms in an affine way under the change of an inertial observer. The mathematical representative of the momentum should therefore be of an affine nature rather than of a vector one. The affine geometry as a tool for such constructions appeared in papers of Tulczyjew and Urbański [44, 42, 39, 45]. The frame independent formulation of analytical mechanics in the Newtonian space-time can be found in [13]. Our paper (2) is devoted to constructing the Tulczyjew triple in the affine setting. In the earlier paper [12] we built a theory called the *geometry of affine values*. The main idea of the theory is to replace functions on a manifold with sections of a one-dimensional affine bundle modeled on a trivial vector bundle $pr_1 : M \times \mathbb{R} \to M$. Sections of the affine bundle play the role of functions with affine values, therefore the bundle itself was called the *bundle of affine values* (*AV-bundle* for short).

In the geometry of affine values the cotangent bundle is replaced by the *phase bundle*. Let $\zeta : Z \to M$ denote an AV-bundle. In the set of pairs (x, σ), where x is a point in M and σ is a section of ζ, we define an equivalence relation according to which (x, σ) and (x', σ') are equivalent if and only if $x = x'$ and $\mathrm{d}(\sigma - \sigma')(x) = 0$. Note that the difference of two sections of an AV-bundle is a function on the base manifold, so we can calculate the differential of that function. The set $\mathsf{P}Z$ of equivalence classes with respect to the above relation carries a natural structure of a manifold. There is a canonical projection from $\mathsf{P}Z$ to the base manifold M. As a bundle over M, the bundle $\mathsf{P}Z$ is an affine bundle modeled on the cotangent bundle T^*M. Moreover, $\mathsf{P}Z$ is, exactly like T^*M, equipped with a canonical symplectic structure.

In the geometry of affine values we use also *special affine bundles*, i.e., affine bundles modeled on vector bundles equipped with a distinguished nowhere-vanishing section. In the category of special affine bundles there is a natural concept of duality. Namely, let $\eta : A \to M$ be a special affine bundle with the distinguished section v_A. We denote by $A^\dagger$ the bundle of affine functionals on A. The fibres of $A^\dagger$ are spaces of affine functions on fibres of A. We will also denote with $\mathsf{V}(A)$ the model vector bundle for A. We call the *affine dual bundle* $A^\#$ of A the subbundle of $A^\dagger$ of all affine functions on fibres of A whose linear part equals 1 while evaluated on v_A. The affine dual $A^\#$ is an affine bundle modeled on the subbundle of affine functions on A with the linear part vanishing on v_A. The model vector bundle has the distinguished section: the constant function 1_A equal to 1. This makes $A^\#$ into a special affine bundle. Every special affine bundle has a certain AV-bundle associated with the quotient $\underline{A} = A/\langle v_A \rangle$.

Let us now assume that a Lagrangian is not a function but a section of the AV-bundle associated with some special affine bundle A. We can find such situations e.g. in the Newtonian mechanics. An appropriate tool for describing such a system is the following affine Tulczyjew triple:

(6)



The bundles $\mathsf{P}A$ and $\mathsf{P}A^\#$ are double affine bundles while $\mathsf{T}\underline{A^\#}$ is a double affine-vector bundle. The map ε defines on A a structure of a *special affgebroid*, that can be equivalently described as an affine-linear bracket

$$[\cdot,\cdot]_\varepsilon : \mathsf{Sec}(A) \times \mathsf{Sec}(\mathsf{V}(A)) \to \mathsf{Sec}(\mathsf{V}(A)),$$

together with two morphisms: $\varepsilon_l : A \to \mathsf{T}M$ and $\varepsilon_r : \mathsf{V}(A) \to \mathsf{T}M$ which are the left and the right anchors and fulfill the following conditions. First, the bracket is *special*, i.e.,

$$[a, X + v_A]_\varepsilon = [a + v_A, X]_\varepsilon = [a, X]_\varepsilon .$$

Then,

$$[a, gX]_\varepsilon = g[a, X]_\varepsilon + (\varepsilon_l \circ a)(g)X ,$$
$$[a + fY, X]_\varepsilon = (1 - f)\,[a, X]_\varepsilon + f\,[a + Y, X]_\varepsilon - (\varepsilon_r \circ X)(f)Y ,$$

where a is a section of A, X, Y are sections of $\mathsf{V}(A)$, and f, g are functions on M.

Let us observe, that the diagram (6) is an affine version of the diagram (3). The map $\widetilde{\Gamma}$ is conencted to the affine tensor $\Gamma \in \mathsf{Sec}(\widetilde{\mathsf{T}}A^\# \otimes_{\underline{A^\#}} \mathsf{T}\underline{A^\#})$. The tensor Γ gives rise to the bracket between sections of the AV-bundle associated to $A^\#$ and functions on $\underline{A^\#}$, with values in functions on $\underline{A^\#}$.

The bundle $\widetilde{\mathsf{T}}Z$ appearing above is another example of a natural construction in the geometry of affine values. Namely, every AV-bundle $\zeta : Z \to M$ can be equivalently defined as a principal bundle with the structure group $(\mathbb{R}, +)$ of additive reals. Then,

$\widetilde{\mathsf{T}}Z = \mathsf{T}Z/\mathbb{R} \to M$ is a special vector bundle. The distinguished section comes from the fundamental vector field for the action of $\mathbb{R}$ on Z.

In the affine version we find similar scheme for the description of dynamics as in the linear case. A Lagrangian section defines the Lagrangian submanifold $N_L \subset \mathsf{P}A$ which is the image of $\mathrm{d}L$. The dynamics $\mathcal{D}$ is the image of N_L with respect to the map ε. The phase dynamics, as a subset of the tangent bundle, can be interpreted as a first-order differential equation (possibly implicit) on curves in the affine phase space $\underline{A^\#}$. An affine Legendre map

$$\lambda : \underline{A} \to \underline{A^\#}, \quad \lambda = \mathsf{P}^\#\eta \circ \mathrm{d}L$$

associates momenta to the infinitesimal configurations. In the case when λ is a diffeomorphism, i.e., when the Lagrangian is hyperregular, the dynamics is the image of a Hamiltonian vector field for a Hamiltonian being a section of the AV-bundle associated to $A^\#$. In all other cases we can look for more complicated generating object.

In the affine setting we can construct the Euler-Lagrange equation as well. It is an equation for curves in $\underline{A}$, therefore it can be represented by a subset of $\mathsf{T}\underline{A}$. If $\widetilde{\lambda}$ denotes the composition $\widetilde{\lambda} = \varepsilon \circ \mathrm{d}L$, the Euler-Lagrange equation can be written as

$$E_L = \mathsf{T}\widetilde{\lambda}^{-1}(\mathsf{T}^2\underline{A^\#}).$$

The solution of the above equation is a curve γ in $\underline{A}$ such that the tangent prolongation of the curve $\lambda \circ \gamma$ equals $\widetilde{\lambda} \circ \gamma$. To illustrate the above theory we described the mechanics of charged particle and dynamics of one particle in the Newtonian space-time.

5. Nonholonomic constraints and Dirac algebroids

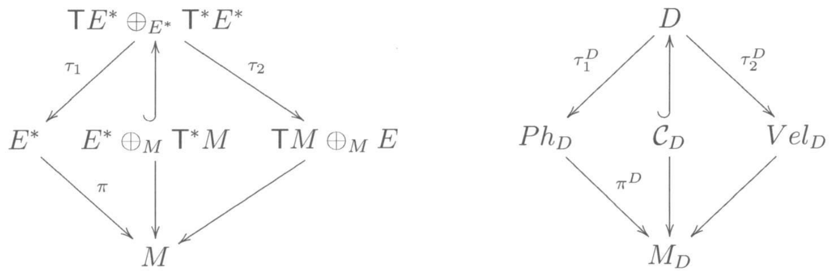
In paper (5) we continue our work on Lagrangian and Hamiltonian formulation of mechanics. We introduce the concept of a *Dirac algebroid* which is a natural generalization of the Dirac structure on a manifold M, defined by Dorfman [6] and studied by Courant [4], as well as the algebroid structure on a vector bundle $\tau : E \to M$. Originally, the Dirac structure was proposed as a tool in the theory of integrable systems with constraints. In analytical mechanics this concept was used lately in the context of singular Lagrangians [47].

The definition of a Dirac algebroid is based on the following observations. Studying complicated systems in mechanics and field theory we were more and more convinced that the proper tool one should use is not associated with maps, like α_M, β_M or maps coming from Poisson structures, but with relations which are compatible with natural bundle structures. An example of such important relation is $\kappa_\varepsilon : \mathsf{T}E \longrightarrow\!\!\triangleright \mathsf{T}E$ which plays the role of the canonical flip κ_M in the algebroid setting. The definition of the canonical isomorphism of the double vector bundles T^*E and T^*E is based on some symplectic relation generated by a canonical evaluation between a vector bundle and its dual. In the affine setting, when we define an isomorphism between $\mathsf{P}A^\#$ and $\mathsf{P}A$, relations are even more visible. We use relations also in the construction of the Tulczyjew triple for field theory.

The second observation is that linearity of different structures, e.g. symplectic, Poisson, connection, etc., can be expressed in an elegant way in the language of double vector bundle morphism [20]. In particular, an algebroid is a certain morphism of double vector bundles.

For a vector bundle $\tau : E \to M$, let $\mathcal{T}E = \mathsf{T}E^* \oplus_{E^*} \mathsf{T}^*E^*$ be the Whitney sum of T^*E^* and $\mathsf{T}E^*$. The bundle $\mathcal{T}E$, sometimes called a *Pontryagin bundle*, is a double vector bundle over E^* and $\mathsf{T}M \oplus_M E$ equipped additionally with a canonical symmetric form. A *Dirac algebroid* is a maximal isotropic subbundle D of $\mathcal{T}E$ which is a double vector subbundle. It means that D is a subbundle of the bundle over E^* and over $\mathsf{T}M \oplus_M E$ simultaneously. Here, we have to use the concept of a subbundle in a sense of [14] as a subbundle supported on a submanifold. If D is a Dirac subbundle, i.e., is closed with respect to the *Courant bracket*, then the structure is called a *Dirac-Lie algebroid*. The following diagrams describe the structure of the double vector bundle $\mathcal{T}E$ and its duble subbundle D, being a Dirac algebroid:

(7)

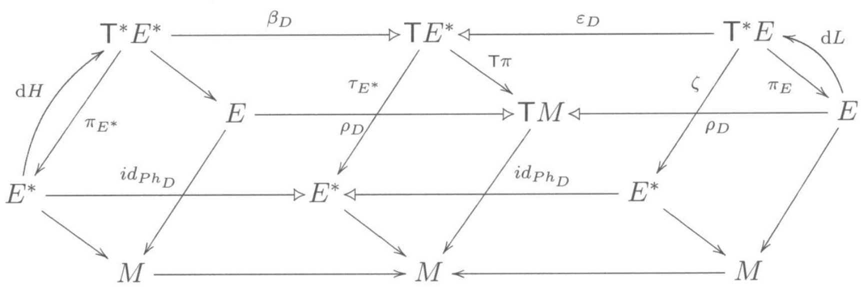


An example of a Dirac algebroid is the graph of the map $\widetilde{\Pi}$ that corresponds to any linear bivector field Π on a vector bundle E^*. If the field is a Poisson tensor, then we get a Dirac-Lie algebroid. Similarly, if $\widetilde{\omega} : \mathsf{T}E^* \to \mathsf{T}^*E^*$ is the map associated to a linear 2-form on a vector bundle, then the graph of $\widetilde{\omega}$ is a Dirac algebroid. If additionally ω is closed (i.e., is a presymplectic structure), then the graph of $\widetilde{\omega}$ is a Dirac-Lie algebroid. The graph of β_M is a special case of both examples.

The base bundles Ph_D and Vel_D of D are called the *phase bundle* and the *velocity bundle* (or the *anchor relation*), respectively. Indeed, if a Dirac algebroid is given by a skew algebroid (linear bivector field), then Vel_D is the graph of the anchor map. In paper (5) we study thoroughly the structure of a Dirac algebroid. This structure turns out to be unexpectedly rich. For example, it follows that the bundle $C_D \subset \mathsf{T}^*M \oplus_M E^*$ being the core of D is the anihilator of Vel_D. We construct also local coordinates adapted to the structure of a Dirac algebroid.

A Dirac algebroid can be regarded as a relation $\beta_D : \mathsf{T}^*E^* \relbar\!\rhd \mathsf{T}E^*$ which plays the role of β_M in the construction of the Tulczyjew triple. The composition of β_D with $\mathcal{R}_E$ gives the relation $\varepsilon_D : \mathsf{T}^*E \relbar\!\rhd \mathsf{T}E^*$ for the Lagrangian side of the triple. The Tulczyjew triple built on the Dirac algebroid is represented by the diagram

(8)



which is similar to (3), but some maps are replaced with relations.

The process of generating the dynamics from a given Lagrangian or Hamiltonian is ideologically the same as in previous cases. The only change is that we have to compose relations. The dynamics given by a Lagrangian L is a subset of $\mathsf{T}E^*$ equal to $\varepsilon_D(\mathrm{d}L(E))$.

The Legendre relation (an analog of the Legendre map) we obtain as a composition $\lambda_D = \tau_{E^*} \circ \varepsilon_D \circ \mathrm{d}L$. It turns out that λ_D is actually a map when restricted to its the domain. The domain of λ_D is called the *Euler-Lagrange domain*, because in this set we find curves being solutions of the appropriate Euler-Lagrange equation. The dynamics given by a Hamiltonian is generated in a similar way: $\mathcal{D} = \beta_D(\mathrm{d}H(E^*))$. In concrete physical situations the same dynamics can be generated out of a Lagrangian or a Hamiltonian. For the obvious reasons we have more freedom in the choice of a generating object than in previous cases.

Let $\widetilde{L}_D$ denote the composition $\widetilde{L}_D = \varepsilon_D \circ \mathrm{d}L$. The Euler-Lagrange equation one can get as the inverse image of the set of holonomic vectors on E^* with respect to the tangent prolongation of the relation inverse to $\widetilde{L}_D$, i.e.,

$$E_L = \mathsf{T}\widetilde{L}_D^{-1}(\mathsf{T}^2 E^*)\,.$$

The solutions of the above equation are curves in the Euler-Lagrange domain and such that they are in the relation $\widetilde{L}_D$ with some admissible curve in $\mathsf{T}E^*$. Admissible curves in $\mathsf{T}E^*$ are tangent prolongations of curves in E^*.

In the hyperregular case, i.e., when the vertical differential of L is a diffeomorphism, the dynamics generated by the Lagrangian has also a Hamiltonian description with the Hamiltonian function $H(p) = \langle p, \lambda^{-1}(p)\rangle - L(\lambda^{-1}(p))$ as a generating object.

The main advantage of the Tulczyjew triple based on Dirac algebroid is its universality and the possibility to include nonholonomic constraints in the geometry of the system. Using a Dirac algebroid, we can describe most of known mechanical systems, moreover the basic description does not change when we impose nonholonomic constraints. In such a case we have to change a Dirac algebroid used, but we stay within the same setting. If, for example, we start from a given Dirac algebroid D, then nonholonomic constraints are given by a subbundle V of the bundle Vel_D. The Dirac algebroid *induced by constraints* is given by

$$D^V = (\tau_2^D)^{-1}(V) + V^0\,,$$

where $V^0 \subset E^* \oplus_M \mathsf{T}^*M$ is the anihilator of the subbundle V. We get the constrained phase dynamics and the constrained Euler-Lagrange equations when we use D^V instead of D.

In paper (5) we describe also an affine analog of a Dirac algebroid and discuss affine constraints. The universality of our method is illustrated by several examples. We present a system on a skew-algebroid, Pontryagin maximum principle as an example of vakonomic constraints, a presymplectic system, a system with Lagrangian depending on time, and a system with nonholonomic constraints. We make concrete calculations for the example of a disc rolling without slipping. All the above and very different examples can be described by means of the Tulczyjew triple based on the same type of a structure – a Dirac algebroid. Note that reduction with respect to symmetries does not require any change of the language used.

6. Classical field theory

The papers (4) and (6) are devoted to the construction of the Tulczyjew triple in field theories. Variational calculus is a natural language for describing statics of mechanical systems. All mathematical objects that are used in statics have direct physical interpretations. Moreover, similar mathematical tools are also widely used in other theories, like dynamics of particles or field theories. In classical mechanics and field theory the variational calculus is traditionally used only for deriving the Euler–Lagrange equations.

The alternative point of view of the role of the variational calculus for mechanics adopted by Tulczyjew led to the new formulation of Lagrangian and Hamiltonian formalisms. We have generalized it further to the case of mechanics on algebroids and on Dirac algebroids. According to Tulczyjew, the concepts of momentum and phase space have also variational origins. In papers (4) and (6) we use the same philosophy in constructing the Lagrangian and Hamiltonian description of a classical field theory. In (6) we analyze the simplest topological case which can be called statics of multidimensional objects. Instead od curves (as in dynamics), we consider maps from a disc $D \subset \mathbb{R}^n$ into a manifold M. Some generalization of the symplectic approach to the case of multidimensional objects was considered earlier by Günther [16]. The mathematics he used can be found in [2, 3].

Our approach to the Lagrangian and Hamiltonian description of field theory is different. Like in mechanics, we do not use Klein's formalism, so that we can avoid difficulties with singular Lagrangians. We construct systematically all mathematical objects following the ideas coming from the variational description of statics. To simplify the notation we work with the case $n = 2$.

We start with identifying momenta and the phase space by analyzing variations of the action functional on the boundary of the disc D. The phase space is $\mathsf{T}^*M \otimes (\mathbb{R}^2)^* \simeq \mathsf{T}^*M \times_M \mathsf{T}^*M$, that we denote simply by $\overset{2}{\mathsf{T}}{}^*M$. The phase space is a vector bundle. The bundle of infinitesimal configurations is dual to the phase space and reads $\mathsf{T}M \times_M \mathsf{T}M = \overset{2}{\mathsf{T}} M$. The simplified topological structure allows us to use double vector bundles and the vector duality in the construction of the triple. This is not the case in a more general setting where we have to use affine geometry. In the place of κ_M that is used in mechanics in construction of admissible variations, we have here

$$\kappa : \mathsf{T}\,\overset{2}{\mathsf{T}} M \to \overset{2}{\mathsf{T}}\,\mathsf{T} M\,,$$

which is a double vector bundle isomorphism. From the variational calculus we get also the evaluation between the space of first jets of mappings from $\mathbb{R}^2$ to the phase space ($\overset{2}{\mathsf{T}}\overset{2}{\mathsf{T}}{}^*M$) and the space of infinitesimal variations of configurations $\mathsf{T}\,\overset{2}{\mathsf{T}} M$. The evaluation is degenerate, therefore we expect that the relation α, dual to κ, is not an isomorphism. The relation α building the Lagrangian side of the triple turns out to be a map

$$\alpha : \overset{2}{\mathsf{T}}\overset{2}{\mathsf{T}}{}^*M \to \mathsf{T}^*\mathsf{T}^*M$$

and a double vector bundle morphism.

The Hamiltonian formulation of mechanics is based on the existence of the symplectic structure on the cotangent bundle. Here, the phase space is not a symplectic bundle any more, nevertheless one can construct a map β that forms the Hamiltonian side of the triple. There are, in principle, two ways of defining the map β. The first method uses the canonical isomorphism $\mathcal{R}$ between $\mathsf{T}^*\,\overset{2}{\mathsf{T}}{}^*M$ and $\mathsf{T}^*\,\overset{2}{\mathsf{T}} M$ which is a particular example of an isomorphism between cotangent bundles of the dual pair of vector bundles. The map β is a composition $\beta = \alpha \circ \mathcal{R}$. The above definition of β is related to the idea that the Hamiltonian formulation is just an alternative way of generating the dynamics. We therefore look for some cotangent bundle isomorphic to the the bundle that we found on the Lagrangian side [43]. Another method of constructing β is related to the canonical structures on the phase bundle. Instead of the symplectic form, we have a tensor

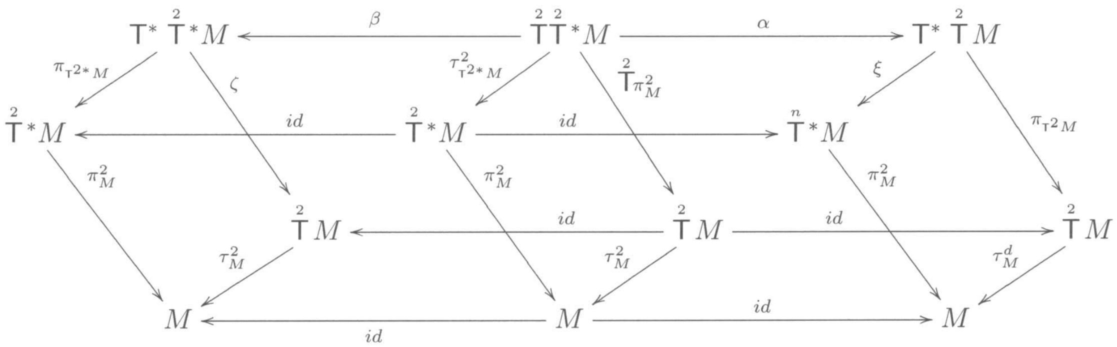
$$\overset{2}{\omega}_M \in \mathsf{Sec}(\overset{2}{\mathsf{T}}{}^*\,\overset{2}{\mathsf{T}}{}^*M \otimes \mathsf{T}^*\,\overset{2}{\mathsf{T}}{}^*M)\,.$$

Since there is an isomorphism

$$\overset{2}{\mathsf{T}}{}^*\,\overset{2}{\mathsf{T}}{}^*M \otimes \mathsf{T}^*\,\overset{2}{\mathsf{T}}{}^*M \simeq \mathsf{T}^*\,\overset{2}{\mathsf{T}}{}^*M \otimes \mathsf{T}^*\,\overset{2}{\mathsf{T}}{}^*M \times_{\mathsf{T}^{2*}M} \mathsf{T}^*\,\overset{2}{\mathsf{T}}{}^*M \otimes \mathsf{T}^*\,\overset{2}{\mathsf{T}}{}^*M\,,$$

the above tensor can be represented as a pair of symplectic forms. The map β is therefore related to a *polysymplectic structure*. The Tulczyjew triple constructed in (4) reads

(9)



The partial differential equation (usually in an implicit form) for phase maps

$$p : \mathbb{R}^2 \to \overset{2}{\mathsf{T}}{}^*M$$

can be obtained from Lagrangian

$$\overset{2}{\mathsf{T}}\overset{2}{\mathsf{T}}{}^*M \supset \mathcal{D} = \alpha^{-1}(\mathrm{d}L(\overset{2}{\mathsf{T}}M))\,,$$

or from Hamiltonian

$$\overset{2}{\mathsf{T}}\overset{2}{\mathsf{T}}{}^*M \supset \mathcal{D} = \beta^{-1}(\mathrm{d}H(\overset{2}{\mathsf{T}}{}^*M))\,.$$

As an illustration of the theory we have chosen an example associated to the bosonic string theory proposed by Nambu.

In paper (6) we construct the Tulczyjew triple for a classical field theory in a very general setting, where fields are sections of a general fibration $\zeta : E \to M$ without any additional structure assumed. As always, all the geometric constructions are based on the variational calculus with nonvanishing boundary terms. We pay a special attention in choosing a proper mathematical language for physical concepts and quantities.

The classical field theory is usually associated with a multisymplectic geometry. The literature concerning multisymplectic structure is very rich. For the first time, this concept appeared in papers of Tulczyjew, Szczyrba, Gawędzki and Kijowski [7, 17, 18, 40]. Then, it was developed by Gotay, Isennberg, Marsden, and others in [8, 9, 10, 11]. Our construction of the Lagrangian and Hamiltonian description of the field theory is different. We do not use directly the multisymplectic formalism but we construct the Tulczyjew triple using, on one hand, the variational calculus, and on the other, our experience in working with double structures. We do not concentrate on the Euler-Lagrange equations being interested rather in phase dynamics. The Euler-Lagrange equations appear as consequences of the phase dynamics. The Hamiltonian side of the triple is based on affine structures.

The starting point of the construction is a locally trivial fibration $\zeta : E \to M$ over a manifold M of dimension m. Sections of ζ are fields. The bundle of first jets J^1E of sections of ζ is the space of infinitesimal configurations on which a Lagrangian is defined. Te bundle of first jets is an affine bundle over E. A Lagrangian is a map $L : \mathsf{J}^1E \to \Omega^m$, where $\Omega^k := \bigwedge^k \mathsf{T}^*M$ is the bundle of k-forms on M. The phase space for our system is $\mathcal{P} = \mathsf{V}^*E \otimes_E \zeta^*(\Omega^{m-1})$. Here, V^*E is the bundle dual to the subbundle $\mathsf{V}E$ of vectors tangent to E and vertical with respect to the projection onto M. The symbol $\zeta^*(\Omega^{m-1})$ denotes the pull-back of the bundle of multi-covectors with respect to ζ. To simplify the notation we omit in the following all symbols of the pull-back, writing simply $\mathcal{P} = \mathsf{V}^*E \otimes_E \Omega^{m-1}$. The Lagrangian side of the triple is based on the map

$$\alpha : \mathsf{J}^1\mathcal{P} \longrightarrow \mathsf{V}^*\mathsf{J}^1E \otimes_{\mathsf{J}^1E} \Omega^m .$$

The map α is a morphism of double vector-affine bundles associated with the projections on $\mathcal{P}$ and J^1E. The vertical differential $\mathrm{d}L$ is a section of

$$\mathsf{V}^*\mathsf{J}^1E \otimes_{\mathsf{J}^1E} \Omega^m \to \mathsf{J}^1E$$

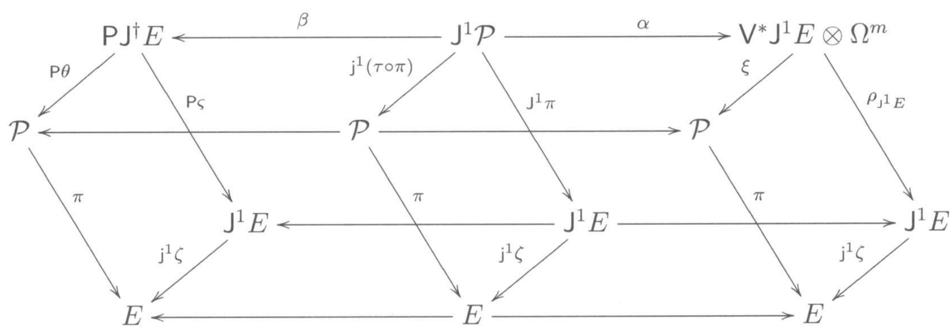
and the phase dynamics is a subset of $\mathsf{J}^1\mathcal{P}$, given as the inverse image of $\mathrm{d}L(\mathsf{J}^1E)$ with respect to α, i.e. $\mathcal{D} = \alpha^{-1}(\mathrm{d}L(\mathsf{J}^1E))$.

The Hamiltonian side of the triple is the map

$$\beta : \mathsf{J}^1\mathcal{P} \longrightarrow \mathsf{PJ}^\dagger E ,$$

where $\mathsf{J}^\dagger E$ is the dual to J^1E in the affine sense, i.e., elements of $\mathsf{J}^\dagger E$ are affine maps form fibres J^1_eE to $\Omega^m_{\zeta(e)}$. Here, $\mathsf{PJ}^\dagger E$ denotes the affine phase bundle for the AV-bundle $\theta : \mathsf{J}^\dagger E \to \mathcal{P}$. The difference with respect to AV-bundles used previously is that now one-dimensional fibres of the bundle are modeled on the appropriate fibres of Ω^m. Elements of $\mathsf{PJ}^\dagger E$ are equivalence classes of sections of θ. The affine phase bundle is an analog of the cotangent bundle. A Hamiltonian is a section of the bundle θ. Its affine differential is then a map $\mathrm{d}H : \mathcal{P} \to \mathsf{PJ}^\dagger E$. The phase dynamics is given as the inverse image with respect to β of the image of $\mathrm{d}H$. The spaces $\mathsf{PJ}^\dagger E$ and $\mathsf{V}^*\mathsf{J}^1E \otimes_{\mathsf{J}^1E} \Omega^m$ are canonically isomorphic. The maps α and β are not isomorphisms. We can construct the triple based on isomorphisms by passing to a quotient space of $\mathsf{J}^1\mathcal{P}$ loosing, however, the natural interpretation od the dynamics as a first-order partial differential equation on sections of the phase bundle. The diagram of the Tulczyjew triple for classical field theory reads as

(10)



The spaces $\mathsf{PJ}^\dagger E$ and $\mathsf{V}^*\mathsf{J}^1E \otimes_{\mathsf{J}^1E} \Omega^m$ carry the canonical 2-forms with values in Ω^m that restricted to fibres over M are symplectic. The canonical structure of $\mathsf{J}^1\mathcal{P}$ is a presymplectic form with values in Ω^m. The phase space also has some canonical structure. It is a one-form with values in Ω^{m-1} that plays the role of the Liouville one-form on T^*M.

(5) Other publications after the PhD:

(1) Janusz Grabowski, Katarzyna Grabowska, *The Lie algebra of a Lie algebroid* In: Lie algebroids and related topics in differential geometry (Warsaw, 2000), 43-50, Banach Center Publ., **54**, Polish Acad. Sci., Warsaw, 2001.

(2) Jerzy Kijowski, Katarzyna Grabowska, *Canonical Gravity and Gravitational Energy*, Differential Geometry and its applications, Ed.: O. Kowalski, D. Krupka, J. Slovak, Opava, (2001) 261–274.

(3) Katarzyna Grabowska, Janusz Grabowski, Paweł Urbański, *Lie Brackets on Affine Bundles*, Annals of Global Analysis and Geometry **24**, (2003), 101–130.

(4) Katarzyna Grabowska, Janusz Grabowski, Paweł Urbański, *AV-differential geometry: Poisson and Jacobi structures*, J. Geom. Phys., **52** (2004), 398-446.

(5) Katarzyna Grabowska, Janusz Grabowski, Paweł Urbański, *Frame-independent mechanics: geometry on affine bundles*, Travaux mathématiques, **16** (2005), 107–120.

(6) Katarzyna Grabowska, Paweł Urbański, *AV-differential geometry and Newtonian mechanics*, Rep. Math. Phys.,**58** (2006), 21-40.

(7) Katarzyna Grabowska, Paweł Urbański, *AV-differential geometry and calculus of variations*, XV International Workshop on Geometry and Physics, 144-152, Publ. R. Soc. Mat. Esp. **11** R. Soc. Mat. Esp., Madrid, (2007).

(8) Katarzyna Grabowska, Janusz Grabowski, Paweł Urbański, *The Schrödinger operator as a generalized Laplacian*, J. Phys. A: Math. Theor. **41** (2008) 145204 (20pp).

The summary of results

One of the classical theorems in differential geometry is the result of Pursell and Shanks [29] which states that the Lie algebra of all compactly supported smooth vector fields on a smooth manifold M determines the smooth structure of M, i.e., the corresponding Lie algebras of vector fields on M_1 and M_2 are isomorphic if and only if M_1 and M_2 are diffeomorphic. There are similar results in special geometric situations (Hamiltonian, contact, group invariant vector fields, etc.) In paper (1) we analyze a similar problem in the context of Lie algebroids. Sections of a Lie algebroid bundle form a (possibly infinite-dimensional) Lie algebra with properties, generically, close to that of the Lie algebras of vector fields. In the paper we describe ideals and maximal finite-codimensional subalgebras, together with some Shanks-Pursell's type theorems in the case when a Lie algebroid satisfies some non-singularity conditions of its anchor map.

Paper (2) is devoted to a concept of the gravitational energy. According to the standard approach to the Legendre transformation in variational theories, the Hamiltonian in the theory of gravity equals zero modulo the boundary terms. On the other hand, the boundary terms in variational theories are usually neglected. The solutions of this problem proposed previously, e.g. by imposing additional conditions on the energy functional or adding some corrections to the Lagrangian are not satisfactory. They are not universal and not well justified. In paper (2) we propose a solution based on a profound analysis of the Einstein equations. We use such a version of the theory of gravity where boundary terms are not neglected. In our approach, the gravitational energy is a quasi-local quantity contained within a generic two-dimensional compact boundary. The total energy can be obtained by a limiting procedure, where the boundary goes to infinity spatial or null. The results do not depend on a variational principle chosen. One can use either a Hilbert-Einstein Lagrangian or an affine Lagrangian.

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In papers (3), (4), (5), (6), and (7) the geometry of affine values is developed. It is an affine version of differential geometry in which functions on a manifold M are replaced by sections of a one-dimensional affine bundle $\zeta : Z \to M$ modeled on the trivial vector bundle $pr_1 : M \times \mathbb{R} \to M$. Almost all constructions from classical differential geometry can be transformed into an affine version. For example, instead of the cotangent bundle, we obtain the phase bundle $\mathsf{P}Z \to M$. Affine structures are natural in theories like dynamics of a charged particle [45], nonautonomous mechanics, and frame independent formulation of the Newtonian mechanics.

In (3) we define an affine analog of the Lie bracket on an affine space and an affine bundle. We introduce an affine Poisson structure and the concept of a *Lie affgebroid* which is an affine analog of a Lie algebroid. We analyze the correspondence between affine and vector structures. Using the idea of affine-vector duality we prove the theorem that states that every Lie affgebroid embeds canonically as a subbundle in some Lie algebroid. This observation enables us to construct a Cartan-like calculus of affine forms.

In paper (4) we continue the works started in (3). We discuss systematically the notion of a bundle of affine values (AV-bundle), together with canonical examples. We introduce the concept of a special vector and a special affine bundle and develop the idea of an affine duality in the category of special affine bundles. One of the sections is devoted to a profound analysis of canonical bundles that appear in the geometry of affine values, i.e., the phase bundle, the bundle of contact elements, and the reduced tangent bundle. We come back also to the affine Lie brackets, affine Poisson, and affine Jacobi brackets. The theory is completed by a short presentation of some possible applications in mechanics.

Papers (5) and (6) are devoted to concrete applications of the geometry of affine values. We formulate a Lagrangian and Hamiltonian description of nonautonomous analytical mechanics, and mechanics in the Newtonian space-time in homogeneous and inhomogeneous versions. In paper (7) we discuss variational aspects of affine theories. The Lagrangian description of a system with Lagrangian being a section of an AV-bundle is related to the affine variational calculus.

The knowledge we gained while working with affine structures in applications to frame independent description of different classical systems, turned out to be very useful also in quantum mechanics, especially in the interpretation of the Schrödinger equation. It is well known that the wave function which is a solution of the Schrödinger equation does not transform under the change of inertial reference frame as ‘ordinary function’. In paper (8) we present the idea of treating wave functions as sections of some one-dimensional complex vector bundle associated with an $U(1)$-principal bundle with distinguished set of trivializations. Those trivializations are associated with inertial frames in the Newtonian space-time. The distinguished set of trivializations is an important element of the structure of the bundle. For the principal $U(1)$-bundle equipped with a set of trivializations we propose the name a *principal Schrödinger bundle*. On this bundle there exists a natural differential calculus of wave forms closely related to the corresponding Atiyah algebroid. This leads to a generalization of the concept of the Laplace-Beltrami operator associated with a metric. The free Schrödinger operator turns out to be the Laplace-Beltrami operator associated with an invariant pseudo-metric on the principal Schrödinger bundle. In (8) we show also the correspondence of the above theory with the frame independent description of the classical mechanics in the Newtonian space-time, especially with the Hamilton-Jacobi theory.

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