1. Yang-Baxter maps and their role in discrete nonlinear integrable systems theory

What does elastic collisions of mass particles, discrete analytic functions and rotating axis-symmetric black holes have in common? Surely they can be considered, in a broader sense, as mathematical and physical entities, but there is something more tangible and fundamental that connects them.

Consider for instance three mass particles which pairwise undergo elastic collisions. Remember that two point particles collide elastically, if the total momentum and energy are the same prior and after the collision. If one measures carefully the velocities of the particles after each particle collides elastically with the remaining ones, will figure out that these velocities are independent of the order that the elastic interactions take place. More precise, the outgoing velocities after the interaction of the 1st particle with the 2nd, followed by the interaction of the 1st with the 3rd and finally of the 2nd particle with the 3rd, are the same if the order of the three interactions is reversed (see Figure 1). The mathematical manifestation of the phenomenon described above is the celebrated **Yang-Baxter** relation that is a property that is commonly shared by many and seemingly different fundamental objects in mathematics and in physics. The same property can be found in discrete analytic functions theory, in the problem of solving Einstein equations describing gravitational field outside the axis-symmetric distribution of matter (referred to as Ernst equation) or in exactly solvable models of statistical mechanics and quantum field theories as Yang and independently Baxter showed in the 70s.

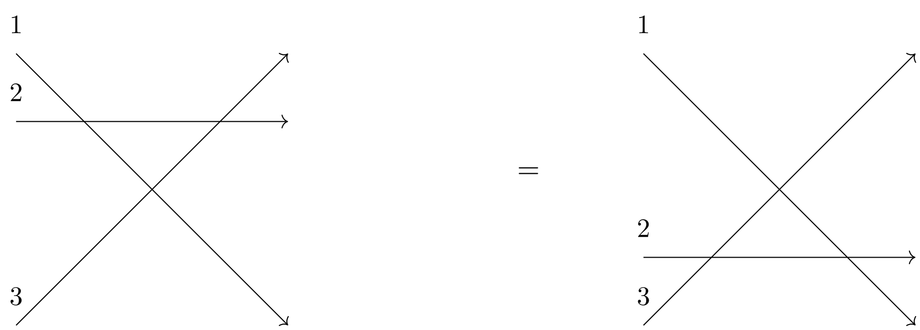

Figure 1. The Yang-Baxter equation

Furthermore, the physical phenomenon of the elastic collisions of particles, is a prototypical example of an **integrable** difference equation. This difference equation is a linear one that is why in some sense it can be regarded trivial. A standard mathematical trick to make a problem less trivial is to replace commuting variables with non-commuting ones. And here comes our main goal of this project, that is, in a systematic way to extend difference equations that exhibit the Yang-Baxter property, into variables that do not commute and in such a way that integrability is preserved! Abandoning the assumption of commutability of variables, a new world arises. For instance, we can prove that even the simple linear Yang-Baxter system associated with elastic collisions, when extended to non-commuting variables, yields nonlinear integrable difference systems of equations that can be recognized as the non-linear superposition of the Ernst equation. And what happens if we consider less trivial solutions of the Yang-Baxter equation? We will report the answer to this question soon.