Moment maps in multisymplectic geometry

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Outline

Symplectic manifold

Definition

A *symplectic* manifold is a manifold M equipped with a differential 2-form ω such that

- ω is non-degenerate, i.e., $\forall p \in M, v_p \in T_pM$
$$\iota_{v_p}\omega = 0 \iff v_p = 0.$$
- ω is closed, i.e.,
$$d\omega = 0.$$

Examples

- Any orientable 2-dimensional manifold M with an area form ω.
- $\mathbb{R}^{2n}$ with coordinates $(q, p) = (q_1, ..., q_n, p_1, ..., p_n)$ equipped with the form $\omega = \sum_{i=1}^{n} dq_i \wedge dp_i$.
- Cotangent bundle T^*M of any manifold M with $\omega = -d\theta$, where

$$\theta_\alpha(v) := \alpha(\pi_* v),$$

where $\pi : T^*M \to M, v \in T_\alpha T^*M$.

Hamiltonian vector fields

A non-degenerate ω provides an isomorphism $\bar{\omega} : TM \to T^*M$. So, we can associate a unique vector field to a given function:

Definition

A vector field v_f such that

$$df = -i_{v_f}\omega$$

for a given $f \in C^\infty(M)$ is called *the Hamiltonian vector field* of f.

Example

Consider a particle moving in $\mathbb{R}^3$, with the phase space ($\mathbb{R}^6$, $\omega = \sum_{i=1}^{3} dq_i \wedge dp_i$). For a Hamiltonian $H(q,p)$, Hamilton's equations for this particle are:

$$\dot{q}_i(t) = \frac{\partial H}{\partial p_i}(q(t), p(t))$$
$$\dot{p}_i(t) = -\frac{\partial H}{\partial q_i}(q(t), p(t))$$

The vector field $v_H := (-\frac{\partial H}{\partial p_1}, -\frac{\partial H}{\partial p_2}, -\frac{\partial H}{\partial p_3}, \frac{\partial H}{\partial q_1}, \frac{\partial H}{\partial q_2}, \frac{\partial H}{\partial q_3})$ is the Hamitonian vector field corresponding to $H(q,p)$.
Hamilton's equations can be written as

$$(\dot{q}(t), \dot{p}(t)) = -v_H(q(t), p(t)).$$

The Poisson algebra $C^\infty(M)$ of observables

Definition

Let (M, ω) be a symplectic manifold. The *Poisson algebra of observables* on M is $C^\infty(M)$ equipped with the following bracket

$$\{f, g\} = \omega(v_f, v_g),$$

where v_f and v_g are the Hamiltonian vector fields corresponding to f and g.

Note: this is a Lie algebra.

Moment map

Definition

Let a Lie group G act on (M, ω), and let v_x be the infinitesimal generator of the action corresponding to $x \in \mathfrak{g}$. A *(co)moment map* for G is a Lie algebra morphism

$$\mu : \mathfrak{g} \to C^\infty(M)$$

such that

$$d(\mu(x)) = -i_{v_x}\omega.$$

$$\begin{array}{ccc} & & C^\infty(M) \\ & \overset{\mu}{\nearrow} & \downarrow \\ \mathfrak{g} & \longrightarrow & \mathfrak{X}_{Ham}(M) \end{array}$$

n-plectic manifolds

Definition

An *n-plectic* manifold is a manifold M equipped with a differential $(n+1)$-form ω such that

- ω is non-degenerate, i.e., $\forall p \in M, v_p \in T_pM$

$$\iota_{v_p}\omega = 0 \iff v_p = 0.$$

- ω is closed, i.e.,

$$d\omega = 0.$$

Examples

- Symplectic manifolds for $n = 1$
- An orientable manifold M of dimension $n + 1$ together with a volume form.
- $\wedge^n T^*M$ with $\omega = -d\theta$, where θ is the canonical n-form defined by:

$$\theta|_\alpha(v_1, ..., v_n) = \alpha(\pi_* v_1, ..., \pi_* v_n).$$

Examples: multiphase space

- Consider a vector bundle $\pi : E \to M$, where $\dim M = n$, and $i : \wedge^n_1 T^*E \hookrightarrow \wedge^n T^*E$ defined by $(\wedge^n_1 T^*E)_e = \{\alpha \in (\wedge^n T^*E)_e : \iota_v \iota_u \alpha = 0, \forall v, u \in V_e E\}$. Then the pullback $i^*\omega$ of the canonical n-plectic form ω on $\wedge^n T^*E$ is an n-plectic structure on $\wedge^n_1 T^*E$.
- Let J^1E be the first jet bundle of E, i.e., for $m = \pi(e)$,

$$J^1_e E = \{\gamma \in L(T_mM, T_eE) : \pi_* \circ \gamma = Id_{T_mM}\}.$$

Let J^1E^* be the *affine dual* of J^1E, i.e.,

$$J^1_e E^* = \{\text{affine maps} : J^1_e E \to \wedge^n T^*_m M\}.$$

J^1E^* is isomorphic to $\wedge^n_1 T^*E$ (as vector bundles over E), so acquires an n-plectic structure via this isomorphism.

Hamiltonian vector fields and $(n-1)$-forms

Definition

Let (M, ω) be an n-plectic manifold. An $(n-1)$-form $\alpha \in \Omega^{n-1}(M)$ is *Hamiltonian* iff there exists a vector field $v_\alpha \in \mathfrak{X}(M)$ such that

$$d\alpha = -\iota_{v_\alpha}\omega.$$

The vector field v_α is the *Hamiltonian vector field* corresponding to α.

We will denote the set of Hamiltonian $(n-1)$-forms by $\Omega^{n-1}_{Ham}(M)$.

n-plectic geometry: Lie algebra of observables??

Candidate: Hamiltonian $(n-1)$-forms
Can try: for $\alpha, \beta \in \Omega^{n-1}_{Ham}(M)$

$$\{\alpha, \beta\} = \iota_{v_\beta}\iota_{v_\alpha}\omega.$$

What works:

- $d\{\alpha, \beta\} = -\iota_{[v_\alpha, v_\beta]}\omega$
- skew-symmetry

What does not work: Jacobi identity!

$$\{\alpha, \{\beta, \gamma\}\} + \{\beta, \{\gamma, \alpha\}\} + \{\gamma, \{\alpha, \beta\}\} = -d\iota_{v_\gamma}\iota_{v_\beta}\iota_{v_\alpha}\omega$$

What to do? L_∞-algebras!

L_∞-algebras

Definition (Lada, Stasheff [5])

An L_∞*-algebra* is a graded vector space L equipped with a collection

$$\{[\ ,\dots,\ ]_k : L^{\otimes k} \to L | 1 \leq k < \infty\}$$

of graded skew-symmetric linear maps (also called *multibrackets*) of degree $|[\ ,\dots,\ ]_k| = 2-k$ satisfying the *higher Jacobi identities*.

- $[\ ]_1$ squares to 0 and is of degree 1, i.e., is a differential, and an L_∞-algebra is, in particular, a cochain complex. We denote $[\ ]_1$ by d.
- d is a graded derivation of $[\ ,\ ]_2$.
- $[\ ,\ ,\ ]_3$ satisfies:

$$[[x,y]_2,z]_2 \pm [[x,z]_2,y]_2 \pm [[y,z]_2,x]_2 = \pm d([x,y,z]_3) \pm [d(x),y,z]_3 \pm [d(y),x,z]_3 \pm [d(z),x,y]_3,$$

Examples

- A cochain complex (L, d)

$$\cdots \xrightarrow{d} L_{i-1} \xrightarrow{d} L_i \xrightarrow{d} L_{i+1} \cdots$$

- A differential graded Lie algebra $(L, d, [\ ,\ ]_2, [\ ,\ ,\ ]_3 = 0)$

$$\cdots \xrightarrow{d} L_{i-1} \xrightarrow{d} L_i \xrightarrow{d} L_{i+1} \cdots$$

such that

$$d[x, y] = [d(x), y] - (-1)^{|x||y|}[dy, x]$$

and

$$(-1)^{|x||z|}[x, [y, z]] + (-1)^{|y||x|}[y, [z, x]] + (-1)^{|z||y|}[z, [x, y]] = 0.$$

Note: when L is concentrated in degree 0, and $d = 0$, this becomes a Lie algebra.

L_∞-algebras as differential graded co-algebras

There is a correspondence

$$\{L_\infty - \text{algebras}\} \longrightarrow \{\text{Differential graded co-algebras}\}$$
$$(L, [,...,]_k) \longrightarrow (C(L), D)$$

Then

$$\{\text{The higher Jacobi identities}\} \Leftrightarrow \{D^2 = 0\}.$$

Definition

An L_∞-morphism between $(L, [,...,]_k)$ and $(L', [,...,]'_k)$ is a co-algebra morphism $F : C(L) \to C(L')$ of graded co-algebras such that

$$F \circ D = D' \circ F.$$

This translates to: a collection of (graded) skew-symmetric maps $f_k : L^{\otimes k} \to L', \quad k \geq 1$ of degree $1 - k$, that are "compatible with the brackets".

L_∞-algebra of observables of an n-plectic manifold

Theorem (Rogers, [7])

Given an n-plectic manifold, there is a corresponding L_∞-algebra $(L, \{[\ ,\ \dots\ ,\ ]_k\})$ with the underlying cochain complex

$$C^\infty(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{n-2}(M) \xrightarrow{d} \Omega^{n-1}_{Ham}(M)$$

with Ω^{n-1}_{Ham} in degree 0 and $C^\infty(M)$ in degree $1-n$, and maps $\{[\ ,\ \dots\ ,\ ]_k : \Omega^{n-1}_{Ham}(M)^{\otimes k} \to \Omega^{n+1-k}(M)\}$ for $k > 1$,

$$[\alpha_1,\ \dots\ , \alpha_k]_k = -(-1)^{\frac{k(k+1)}{2}} \iota(v_{\alpha_1} \wedge \dots \wedge v_{\alpha_k})\omega$$

where v_{α_i} is the Hamiltonian vector field associated to α_i, and $i(\dots)$ denotes contraction with a multivector field:
$\iota(v_{\alpha_1} \wedge \dots \wedge v_{\alpha_k})\omega = \iota_{v_{\alpha_k}} \dots \iota_{v_{\alpha_1}} \omega.$

Examples: 1-plectic and 2-plectic manifolds

- If (M, ω) is a 1-plectic (symplectic) manifold, $L_\infty(M, \omega)$ has
$$C^\infty(M)$$
as the underlying vector space, concentrated in degree 0.
The multibracket $[\ ,\ ]$ is given by
$$[\alpha_1, \alpha_2] = \omega(v_{\alpha_1}, v_{\alpha_2}).$$
- If (M, ω) is a 2-plectic manifold, $L_\infty(M, \omega)$ has
$$C^\infty(M) \xrightarrow{d} \Omega^1_{Ham}(M)$$
as the cochain complex, with $C^\infty(M)$ in degree -1, and $\Omega^1_{Ham}(M)$ in degree 0.
The multibrackets $[\ ,\ ]$, $[\ ,\ ,\ ]$ are given by
$$[\alpha_1, \alpha_2] = \omega(v_{\alpha_1}, v_{\alpha_2})$$
$$[\alpha_1, \alpha_2, \alpha_3] = -\omega(v_{\alpha_1}, v_{\alpha_2}, v_{\alpha_3}).$$

Homotopy moment map

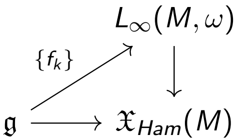
Definition (Callies, Fregier, Rogers, Zambon, [2])

Let $\mathfrak{g} \to \mathfrak{X}(M), x \mapsto v_x$ be a Lie algebra action on an n-plectic manifold (M, ω) by Hamiltonian vector fields. A *homotopy moment map* for this action is an L_∞-morphism

$$\{f_k\} : \mathfrak{g} \to L_\infty(M, \omega)$$

such that

$$-i_{v_x}\omega = d(f_1(x)) \qquad \forall x \in \mathfrak{g}.$$



Homotopy moment map: restatement

- Consider the k-th Lie algebra homology differential $\delta_k : \wedge^k \mathfrak{g} \to \wedge^{k-1}\mathfrak{g}$ given by

$$\delta_k : x_1 \wedge \ldots \wedge x_k \mapsto \sum_{1 \le i < j \le k} (-1)^{i+j}[x_i, x_j] \wedge x_1 \wedge \ldots \widehat{x_i} \wedge \ldots \wedge \widehat{x_j} \wedge \ldots x_k.$$

- For Lie algebra $\mathfrak{g}$ acting on M, consider $p = x_1 \wedge x_2 \wedge \cdots \wedge x_k \in \wedge^k \mathfrak{g}$, and let v_{x_i} be the vector field associated to x_i via the $\mathfrak{g}$-action. The multivector field

$$v_p := v_{x_1} \wedge v_{x_2} \wedge \cdots \wedge v_{x_k} \in \Gamma(\wedge^k TM)$$

is the *multivector field corresponding to p*. This extends linearly to all $p \in \wedge^k \mathfrak{g}$.

Homotopy moment map: restatement

Definition

A *homotopy moment map* for the action of $\mathfrak{g}$ on an n-plectic manifold (M, ω) is a collection of linear maps $f_k : \wedge^k \mathfrak{g} \to \Omega^{n-k}(M)$, such that for $1 \leq k \leq n+1$ and all $p \in \wedge^k \mathfrak{g}$:

$$-f_{k-1}(\delta_k(p)) = df_k(p) - (-1)^{\frac{k(k+1)}{2}} \iota_{v_p} \omega,$$

where v_p is the multivector field corresponding to p, and f_0 and f_{n+1} are defined to be zero: $f_0 = f_{n+1} = 0$.

Examples

- For $n = 1$, a homotopy moment map is the moment map introduced earlier.
- Consider a Lie group G acting on an n-plectic manifold (M, ω), and let $\omega = d\alpha$ for G-invariant α. Then $\{f_k\} : \mathfrak{g} \to L_\infty(M, \omega)$ given by

$$f_k(x_1, ..., x_k) = (-1)^{k-1}(-1)^{\frac{k(k+1)}{2}}\iota(v_{x_1} \wedge \cdots \wedge v_{x_k})\alpha$$

  is a homotopy moment map for the action of G.
  Note: in particular, this includes $(M = \wedge^n T^*N, \omega = -d\theta)$ introduced earlier (see [2]).

Weak (homotopy) moment map

Definition (J. Herman, [4])

Let $\mathfrak{g} \to \mathfrak{X}(M), x \mapsto v_x$ be a Lie algebra action on an n-plectic manifold (M, ω). A *weak (homotopy) moment map* is a collection of linear maps $f_k : P_{k,\mathfrak{g}} \to \Omega^{n-k}(M)$, where $1 \leq k \leq n$, satisfying

$$df_k(p) = (-1)^{\frac{k(k+1)}{2}} \iota_{v_p}\omega$$

for $k \in 1, ..., n$ and all $p \in P_{k,\mathfrak{g}}$, where $P_{k,\mathfrak{g}} \subset \wedge^k \mathfrak{g}$ is the k-th Lie kernel of $\mathfrak{g}$, i.e., the kernel of $\delta_k : \wedge^k \mathfrak{g} \to \wedge^{k-1} \mathfrak{g}$.

Comparing the two maps

By comparing the two definitions, it is clear that a homotopy moment map, when it exists, gives a weak moment map by restricting the f_k to $P_{k,\mathfrak{g}}$, i.e.,

Existence of homotopy moment map $\Rightarrow$ Existence of weak moment map

Question Is the converse true?

Comparing the two maps: preliminary notions

- Consider

$$\omega_k : \wedge^k \mathfrak{g} \to \Omega^{n+1-k}$$
$$x_1 \wedge \cdots \wedge x_k \mapsto \iota(v_{x_1} \wedge \cdots \wedge v_{x_k})\omega$$

- Consider

$$\widetilde{\omega} := \sum_{k=1}^{n+1} (-1)^{k-1} \omega_k$$

and

$$\widehat{\omega} := \sum_{k=1}^{n+1} (-1)^{k-1} (\omega_k|_{P_{k,\mathfrak{g}}}).$$

Characterization in terms of double complexes

Consider the following complexes:

- The total complex $(\widetilde{C}, \widetilde{d}_{tot})$ of the double complex $(\wedge^{\geq 1}\mathfrak{g}^* \otimes \Omega(M), d_{\mathfrak{g}}, d)$, with the Chevalley-Eilenberg differential $d_{\mathfrak{g}}$ on $\wedge^{\geq 1}\mathfrak{g}^* := \bigoplus_{k=1} \wedge^k \mathfrak{g}^*$ and the de Rham differential on $\Omega(M)$.
  Here $\widetilde{d}_{tot} := d_{\mathfrak{g}} \otimes 1 + 1 \otimes d$.
- The total complex $(\widehat{C}, \widehat{d}_{tot})$ of the double complex $(P^*_{\geq 1,\mathfrak{g}} \otimes \Omega(M), 0, d)$ with zero differential on $P^*_{\geq 1,\mathfrak{g}} := \bigoplus_{k=1} P^*_{k,\mathfrak{g}}$ and the de Rham differential on $\Omega(M)$.
  Here $\widehat{d}_{tot} := 1 \otimes d$.

Characterization in terms of double complexes

Theorem (Fregier, Laurent-Gengoux, Zambon[3], Ryvkin, Wurzbacher [8])

There exists a homotopy moment map for the action of $\mathfrak{g}$ on (M,ω) if and only if $[\widetilde{\omega}] = 0 \in H^{n+1}(\widetilde{C})$.

Theorem (M., Ryvkin, [6])

There exists a weak moment map for the action of $\mathfrak{g}$ on (M,ω) if and only if $[\widehat{\omega}] = 0 \in H^{n+1}(\widehat{C})$.

Existence result for homotopy moment maps

Theorem (M., Ryvkin, [6])

Let (M, ω) be an n-plectic manifold, and let $\mathfrak{g}$ act on (M, ω) by preserving ω. The following statements are equivalent:

- *The action of $\mathfrak{g}$ on (M, ω) admits a homotopy moment map*
- *The action of $\mathfrak{g}$ on (M, ω) admits a weak moment map and $\phi \in P^*_{n+1,\mathfrak{g}} \otimes C^\infty(M)$ defined by*

$$\begin{aligned} \phi : P_{n+1,\mathfrak{g}} &\to C^\infty(M) \\ p &\mapsto \iota_{v_p}\omega \end{aligned}$$

  vanishes identically.

Elements of the proof

- Construct a map $H^{n+1}(\widetilde{C}) \to H^{n+1}(\widehat{C})$ that maps $[\widetilde{\omega}]$ to $[\widehat{\omega}]$. If this map is injective, then
  $[\widehat{\omega}] = 0 \Rightarrow [\widetilde{\omega}] = 0$
- The sequence

$$0 \to P_{k,\mathfrak{g}} \xrightarrow{i} \wedge^k \mathfrak{g} \xrightarrow{\delta^k} \wedge^{k-1} \mathfrak{g}$$

and its dual

$$0 \leftarrow P^*_{k,\mathfrak{g}} \xleftarrow{\pi} \wedge^k \mathfrak{g}^* \xleftarrow{d_{\mathfrak{g}}^{k-1}} \wedge^{k-1} \mathfrak{g}^*$$

are exact.

Thus,

$$P^*_{k,\mathfrak{g}} = \wedge^k \mathfrak{g}^* / im(d_{\mathfrak{g}}^{k-1}) \hookleftarrow ker(d_{\mathfrak{g}}^k)/im(d_{\mathfrak{g}}^{k-1}) = H^k(\mathfrak{g}),$$

Elements of the proof

- Apply the Künneth theorem:

$$H^{n+1}(\widetilde{C}) = H^{n+1}(\wedge^{\geq 1}\mathfrak{g}^*\otimes\Omega(M)) = \bigoplus_{k\geq 1} H^k\mathfrak{g}\otimes H^{n+1-k}(M)$$

and

$$H^{n+1}(\widehat{C}) = H^{n+1}(P^*_{\geq 1,\mathfrak{g}}\otimes\Omega(M)) = \bigoplus_{k\geq 1} P^*_{k,\mathfrak{g}}\otimes H^{n+1-k}(M)$$

Examples

- Consider $\mathbb{R}^{2n}$ acting on $(\mathbb{R}^{2n}, \omega = \sum_{i=1}^{n} dq_i \wedge dp_i)$ by translations. This action does not admit a moment map, but admits a weak moment map.
- Let G be a compact simple Lie group, θ^L the Maurer-Cartan form given by $\theta^L_g : T_g G \to T_e G, v \mapsto L_{g^{-1}*} v$, and $\langle , \rangle$ an Ad-invariant inner product on $\mathfrak{g}$. Then (G, ω), where $\omega := \langle \theta^L, [\theta^L, \theta^L] \rangle$ is a 2-plectic manifold (see [1]). The action of G on itself by (left) translations does not admit a homotopy moment map, but admits a weak moment map.

Strict extensions

Question: Given a weak moment map and assuming $\phi \equiv 0$, does there always exist a homotopy moment map that restricts to the given weak moment map?

Proposition (M., Ryvkin, [6])

Let $\widehat{f}$ be a weak moment map, and $\phi = 0$. There exists a well-defined class $[\gamma]_{\widetilde{d}_{tot}} \in H^{n+1}(\widetilde{C})$ such that the following are equivalent:

- $[\gamma]_{\widetilde{d}_{tot}} = 0$ *and* γ *admits a* $\widetilde{d}_{tot}$*-primitive in* $\bigoplus_{k=1}^{n} d_{\mathfrak{g}}(\Lambda^k \mathfrak{g}^*) \otimes \Omega^{n-k-1}(M)$
- *There exists a homotopy moment map* $\widetilde{f}$*, such that* $\widetilde{f}|_{P_{\mathfrak{g}}} = \widehat{f}$.

Thank you!

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